Relaxation approximation of the Kerr model for the impedance initial-boundary value problem

Gilles Carbou and Bernard Hanouzet

Institut de Mathématiques de Bordeaux
CNRS UMR 5251
Université Bordeaux 1

1. Physical context
2. Relaxation framework
3. Semilinear behaviour for Kerr-Debye
4. Convergence result
5. Proof of the convergence

1. Physical context

Propagation of electromagnetic waves in a homogeneous isotropic nonlinear material (cristal)

Maxwell's equations

$$\partial_t D - \text{curl } H = 0$$
$$\partial_t B + \text{curl } E = 0$$

$$\text{div } D = \text{div } B = 0.$$

E : electric field
H : magnetic field
D : electric displacement
B : magnetic induction

1. Physical context

Propagation of electromagnetic waves in a homogeneous isotropic nonlinear material (cristal)

$$\partial_t D - \text{curl } H = 0$$
$$\partial_t B + \text{curl } E = 0$$

$$\text{div } D = \text{div } B = 0.$$

Kerr Model : instantaneous response

$$B = H \text{ and } D = (1 + |E|^2)E$$

Kerr-Debye Model : finite response time

$$B = H \text{ and } D = (1 + \chi)E$$

with

$$\partial_t \chi + \frac{1}{\varepsilon}\chi = \frac{1}{\varepsilon}|E|^2$$

ε : delay time

Physical experiments : cristal excited by a laser beam

Initial Boundary Value Problem

- Cristal: $\Omega = \mathbb{R}^+ \times \mathbb{R}^2, \quad \Gamma = \partial\Omega$
- For $t \leq 0$, $B = H = 0$, $\chi = 0$
- impedance condition on the boundary $\Gamma = \{0\} \times \mathbb{R}^2$:

$$H \times n + A((E \times n) \times n) = \varphi \text{ for } (t, x) \in \mathbb{R} \times \Gamma$$

A positive endomorphism on Γ.

φ : source term modeling the laser impact, supp $\varphi \subset \mathbb{R}_t^+ \times \Gamma$

References:

Y.- R. Shen, The Principles of Nonlinear Optics, Wiley Interscience, 1994.

R.-W. Ziolkowski, *The incorporation of microscopic material models into FDTD approach for ultrafast optical pulses simulations*, IEEE Transactions on Antennas and Propagation, **45** (1997).

2. Relaxation Framework

Kerr-Debye is a relaxation model of Kerr

G.Q. Chen, C. D. Levermore, T.-P. Liu, *Hyperbolic conservation laws with stiff relaxation terms and entropy*, Comm. Pure Appl. Math. **47** (1994).

R. Natalini, *Recent results on hyperbolic relaxation problems. Analysis of systems of conservation laws (Aachen, 1997)*, Chapman & Hall/CRC Monogr. Surv. Pure Appl. Math.,1999.

Kerr model	Kerr-Debye Model
$D = (1+\|E\|^2)E$	$D = (1+\chi)E$
	$\partial_t \chi = \frac{1}{\varepsilon}(\|E\|^2 - \chi)$

Equilibrium manifold:

$$\mathcal{V} = \{(D, H, \chi), \chi = |E|^2 = (1+\chi)^{-2}|D|^2\}$$

Reduced system:

Kerr is the reduced system of Kerr-Debye on the equilibrium manifold

Entropy relations:

$$\mathcal{E}_K(D,H) = \frac{1}{2}(|E|^2 + |H|^2 + \frac{3}{2}|E|^4)$$

$$\mathcal{E}_{KD}(D,H,\chi) = \frac{1}{2}(1+\chi)^{-1}|D|^2 + \frac{1}{2}|H|^2 + \frac{1}{4}\chi^2$$

On the equilibrium manifold,

$$\mathcal{E}_{KD}(D,H,\chi(D)) = \mathcal{E}_K(D,H)$$

3. Semi linear behaviour of Kerr-Debye

Kerr-Debye is quasilinear hyperbolic totally linearly degenerated

Theorem 1. Semilinear behaviour for KD
In the one dimensional and the 2d-TE models, if T_ε^* is the lifespan of smooth solutions, if $T_\varepsilon^* < +\infty$ then

$$\lim_{T \to T_\varepsilon^*} \|(D, H, \chi)\|_{L^\infty(\Omega)} = +\infty$$

KD does not develop shock waves

G. Carbou, B. Hanouzet, *Comportement semi-linéaire d'un système hyperbolique quasi-linéaire: le modèle de Kerr-Debye*, C. R. Math. Acad. Sci. Paris, **343** (2006)

3. Semi linear behaviour of Kerr-Debye

Theorem 1. Semilinear behaviour for KD
In the one dimensional and the 2d-TE models, if T_ε^* is the lifespan of smooth solutions, if $T_\varepsilon^* < +\infty$ then

$$\lim_{T \to T_\varepsilon^*} \|(D, H, \chi)\|_{L^\infty(\Omega)} = +\infty$$

Proof: we assume that the solution is bounded in $L^\infty([0, T_\varepsilon^*[\times\Omega)$.
We will prove that it is bounded in $\mathcal{H}^4(\Omega_{T_\varepsilon^*})$:

$$\|W\|_{\mathcal{H}^4(\Omega_t)} = \sum_{i=0}^{4} \|\partial_t^i W\|_{L^\infty(0,t;H^{4-i}(\Omega))}$$

Olivier Guès, *Problème mixte hyperbolique quasi-linéaire caractéristique,* Comm. Partial Differential Equations, **15**, 1990.

2d-TE model

$$E(x_1,x_2,x_3) = {}^t(0, E_2(x_1,x_3), 0),$$
$$H(x_1,x_2,x_3) = {}^t(H_1(x_1,x_3), 0, H_3(x_1,x_3)).$$

$U = (E_2, H_1, H_3)$

$$\begin{pmatrix} (1+\chi) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \partial_t U + \begin{pmatrix} 0 & -\partial_3 & \partial_1 \\ -\partial_3 & 0 & 0 \\ \partial_1 & 0 & 0 \end{pmatrix} U = \begin{pmatrix} -\partial_t \chi E_2 \\ 0 \\ 0 \end{pmatrix}.$$

$$\partial_t \chi + \chi = (E_2)^2$$

- Symmetrized system for U
- Variational estimates for $\partial_t^i U$: $\partial_t \chi$ bounded in L^∞
- Classical div-curl lemma to estimate $\|U\|_{H^k(\Omega)}$
- We solve the e.d.o. to estimate χ

4. Convergence result

For the Cauchy Problem :
$t \geq 0$, $x \in I\!R^3$, $E(t=0) = E_0$, $H(t=0) = H_0$, $\chi(t=0) = \chi_0$.

Convergence proved for the smooth solutions by Hanouzet-Huynh using the results of Yong

B. Hanouzet, P. Huynh, *Approximation par relaxation d'un système de Maxwell non linéaire,* C. R. Acad. Sci. Paris Sér. I Math., **330**, 2000.

W.-A. Yong, *Singular perturbations of first-order hyperbolic systems with stiff source terms,* J. Differential Equations, **155**, 1999.

Non compatibility between the initial data and the equilibrium condition $\Rightarrow$ Boundary layer in time

4. Convergence result

For the Cauchy Problem : convergence and boundary layers in time

For the Initial Boundary Value Problem:

Theorem 2 . Convergence result
let (D_0, H_0) be a smooth solution of Kerr on $[0, T^*[$.
$(D_\varepsilon, H_\varepsilon, \chi_\varepsilon)$ smooth solution of Kerr-Debye on $[0, T_\varepsilon^*[$.
We fix $T < T^*$. For ε small enough, $T_\varepsilon^* > T$ and

$$\|D_\varepsilon - D_0\| + \|H_\varepsilon - H_0\| \leq C\varepsilon$$

null initial data, same boundary condition $\Rightarrow$ **no boundary layer**

the boundary is characteristic

5. Proof of the convergence

A. Symmetrization with the entropic variables

$$\partial_D \mathcal{E}_{KD} = E \qquad \partial_H \mathcal{E}_{KD} = H \qquad \partial_\chi \mathcal{E}_{KD} = \tfrac{1}{2}(\chi - |E|^2) := v \qquad W_\varepsilon = \begin{pmatrix} E_\varepsilon \\ H_\varepsilon \\ v_\varepsilon \end{pmatrix}$$

$$A_0(W_\varepsilon)\partial_t W_\varepsilon + \sum_{j=1}^{3} A_j \partial_j W_\varepsilon = \frac{1}{\varepsilon} Q(W_\varepsilon)$$

with

$$A_0(W_\varepsilon) = \begin{pmatrix} (|E_\varepsilon|^2 + 2v_\varepsilon + 1)I_3 + 2E_\varepsilon\, {}^tE_\varepsilon & 0 & 2E_\varepsilon \\ 0 & I_3 & 0 \\ 2\,{}^tE_\varepsilon & 0 & 2 \end{pmatrix}$$

$$\sum_{j=1}^{3} A_j \partial_j = \begin{pmatrix} 0 & -\mathrm{curl} & 0 \\ \mathrm{curl} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad Q(W_\varepsilon) = \begin{pmatrix} 0 \\ 0 \\ -2v_\varepsilon \end{pmatrix}$$

- The system is symmetrized
- The boundary condition is linear for the variables $(E_\varepsilon, H_\varepsilon)$

$$H_\varepsilon \times n + A((E_\varepsilon \times n) \times n) = 0 \text{ on } \mathbb{R}_t^+ \times \Gamma_x$$

- The equilibrium manifold is linearized (see Hanouzet-Natalini)

$$\Big\{(D_\varepsilon, H_\varepsilon, \chi_\varepsilon),\ \chi_\varepsilon = |E_\varepsilon|^2\Big\} = \Big\{(E_\varepsilon, H_\varepsilon, v_\varepsilon),\ v_\varepsilon = 0\Big\}$$

and the relaxation term is linear

B. Hilbert expansion

null initial data, same boundary condition $\Rightarrow$ No Boundary layer

$$\begin{aligned} E_\varepsilon &= E_0 + \varepsilon R_\varepsilon \\ H_\varepsilon &= H_0 + \varepsilon S_\varepsilon \\ v_\varepsilon &= \varepsilon s_1 + \varepsilon s_\varepsilon \end{aligned} \qquad \rho_\varepsilon = \begin{pmatrix} R_\varepsilon \\ S_\varepsilon \\ s_\varepsilon \end{pmatrix}$$

$$A_0(t,x)\partial_t\rho_\varepsilon + \sum_{j=1}^{3} A_j\partial_j\rho_\varepsilon + L(t,x)\rho_\varepsilon + B(t,x)$$
$$+G(\varepsilon,t,x,\rho_\varepsilon,\partial_t\rho_\varepsilon) = -\frac{2}{\varepsilon}\begin{pmatrix} 0 \\ 0 \\ s_\varepsilon \end{pmatrix}$$

$$\rho_\varepsilon(t=0) = 0$$

$$R_\varepsilon \times n + A((S_\varepsilon \times n) \times n) = 0 \text{ on } \mathbb{R}_t^+ \times \Gamma$$

- $A_0(t,x) = A_0(W_0) = \begin{pmatrix} (|E_0|^2+1)I_3 + 2E_0\,{}^tE_0 & 0 & 2E_0 \\ 0 & I_3 & 0 \\ 2\,{}^tE_0 & 0 & 2 \end{pmatrix}$
- $L(t,x)$ linear, $B(t,x)$ source term
- $G(t,x,\varepsilon\rho_\varepsilon,\varepsilon\partial_t\rho_\varepsilon)$ non linear part

$T < T^*$ is fixed.

We perform estimates on $[0, T_\varepsilon]$

$$T_\varepsilon = \sup\left\{ t \le T, \|\rho_\varepsilon\|_{\mathcal{H}^4(\Omega_t)} \le \frac{1}{\sqrt{\varepsilon}} \right\}$$

$$\|\rho\|_{\mathcal{H}^4(\Omega_t)} = \sum_{i=0}^{4} \|\partial_t^i \rho\|_{L^\infty(0,t;H^{4-i}(\Omega))}$$

$$G(t, x, \varepsilon\rho_\varepsilon, \varepsilon\partial_t\rho_\varepsilon) = \frac{1}{\varepsilon} P(\varepsilon\rho_\varepsilon, \varepsilon\partial_t\rho_\varepsilon)$$

On $[0, T_\varepsilon[$, the nonlinear term is bounded

C. Variational estimates for the tangential derivatives

L^2**-estimate** : we take the inner product of the system with ρ_ε:

$$A_0(t,x)\partial_t\rho_\varepsilon + \sum_{j=1}^{3} A_j\partial_j\rho_\varepsilon + L(t,x)\rho_\varepsilon + B(t,x) + G(\varepsilon,t,x,\rho_\varepsilon,\partial_t\rho_\varepsilon) = -\frac{2}{\varepsilon}\begin{pmatrix} 0 \\ 0 \\ s_\varepsilon \end{pmatrix}$$

$$\rho_\varepsilon(t=0) = 0$$

$$R_\varepsilon \times n + A((S_\varepsilon \times n) \times n) = 0 \text{ on } \mathbb{R}_t^+ \times \Gamma$$

C. Variational estimates for the tangential derivatives

L^2**-estimate** : we take the inner product of the system with ρ_ε:

$$A_0(t,x)\partial_t\rho_\varepsilon + \sum_{j=1}^3 A_j\partial_j\rho_\varepsilon + L(t,x)\rho_\varepsilon + B(t,x)$$
$$+G(\varepsilon,t,x,\rho_\varepsilon,\partial_t\rho_\varepsilon) = -\frac{2}{\varepsilon}\begin{pmatrix} 0 \\ 0 \\ s_\varepsilon \end{pmatrix}$$

$$\int_\Omega A_0(t,x)\partial_t\rho_\varepsilon \cdot \rho_\varepsilon = \frac{1}{2}\frac{d}{dt}\int_\Omega A_0(t,x)\rho_\varepsilon\cdot\rho_\varepsilon - \frac{1}{2}\int_\Omega(\partial_t A_0)\rho_\varepsilon\cdot\rho_\varepsilon$$

$$\gamma\|\rho_\varepsilon\|^2_{L^2(\Omega)} \le \int_\Omega A_0(t,x)\rho_\varepsilon\cdot\rho_\varepsilon$$

C. Variational estimates for the tangential derivatives

L^2**-estimate** : we take the inner product of the system with ρ_ε:

$$A_0(t,x)\partial_t\rho_\varepsilon + \sum_{j=1}^3 A_j\partial_j\rho_\varepsilon + L(t,x)\rho_\varepsilon + B(t,x) + G(\varepsilon,t,x,\rho_\varepsilon,\partial_t\rho_\varepsilon) = -\frac{2}{\varepsilon}\begin{pmatrix} 0 \\ 0 \\ s_\varepsilon \end{pmatrix}$$

$$\int_\Omega \sum_{j=1}^3 A_j\partial_j\rho_\varepsilon \cdot \rho_\varepsilon = \int_\Gamma A(R_\varepsilon)_T \cdot (R_\varepsilon)_T \geq 0$$

C. Variational estimates for the tangential derivatives

L^2**-estimate** : we take the inner product of the system with ρ_ε:

$$A_0(t,x)\partial_t\rho_\varepsilon + \sum_{j=1}^{3} A_j\partial_j\rho_\varepsilon + L(t,x)\rho_\varepsilon + B(t,x) + G(\varepsilon, t, x, \rho_\varepsilon, \partial_t\rho_\varepsilon) = -\frac{2}{\varepsilon}\begin{pmatrix} 0 \\ 0 \\ s_\varepsilon \end{pmatrix}$$

$$\int_\Omega G(\varepsilon, t, x, \rho_\varepsilon, \partial_t\rho_\varepsilon)\cdot\rho_\varepsilon \leq C\|\rho\|_{L^2(\Omega)}$$

C. Variational estimates for the tangential derivatives

L^2**-estimate** : we take the inner product of the system with ρ_ε:

$$A_0(t,x)\partial_t\rho_\varepsilon + \sum_{j=1}^{3} A_j\partial_j\rho_\varepsilon + L(t,x)\rho_\varepsilon + B(t,x)$$
$$+G(\varepsilon, t, x, \rho_\varepsilon, \partial_t\rho_\varepsilon) = -\frac{2}{\varepsilon}\begin{pmatrix} 0 \\ 0 \\ s_\varepsilon \end{pmatrix}$$

Good sign!

C. Variational estimates for the tangential derivatives

L^2**-estimate** : Taking the inner product of the system with ρ_ε:
On $[0, T^\varepsilon]$,

$$\frac{1}{2}\frac{d}{dt}\int_\Omega A_0\rho_\varepsilon\cdot\rho_\varepsilon + \frac{1}{\varepsilon}\int_\Omega |s_\varepsilon|^2 \leq C(1+\|\rho_\varepsilon\|^2_{L^2})$$

C. Variational estimates for the tangential derivatives

Estimates on the tangential derivatives:

We can derivate the IBVP for KD with respect to the tangential derivatives : for $i \neq 1$,

- same initial data and same boundary condition for $\partial_i \rho_\varepsilon$
- same form for the relaxation term
- sufficient bounds for the nonlinear terms on $[0, T^\varepsilon]$

$$\varphi_\varepsilon^2(t) = \|\rho_\varepsilon(t)\|_{L^2(\Omega)}^2 + \sum_{i \neq 1} \|\partial_i \rho_\varepsilon(t)\|_{L^2(\Omega)}^2 + \ldots + \sum_{i,j,k,l \neq 1} \|\partial_{ijkl} \rho_\varepsilon(t)\|_{L^2(\Omega)}^2$$

$$\frac{d}{dt} \int_\Omega (A_0 \rho_\varepsilon \cdot \rho_\varepsilon + \sum_{i \neq 1} A_0 \partial_i \rho_\varepsilon \cdot \rho_\varepsilon + \ldots + \sum_{i,j,k,l \neq 1} A_0 \partial_{ijkl} \rho_\varepsilon \cdot \partial_{ijkl} \rho_\varepsilon) \leq C(1 + \varphi_\varepsilon^2)$$

$\Rightarrow \varphi_\varepsilon$ **is uniformly bounded on** $[0, T^\varepsilon]$**.**

D. Estimates on the normal derivatives

- From the curl

$$\partial_t S_\varepsilon + \text{curl } R_\varepsilon = 0$$

$$\partial_1 R_\varepsilon^3 = \partial_3 R_\varepsilon^1 + \partial_t S_\varepsilon^2$$

$\Rightarrow$ estimate on $\partial_1 R_\varepsilon^3$

In the same way, we estimate $\partial_1 R_\varepsilon^2$, $\partial_1 S_\varepsilon^2$, $\partial_1 S_\varepsilon^3$.

D. Estimates on the normal derivatives

• From the curl: estimates on $\partial_1 R_\varepsilon^2$, $\partial_1 R_\varepsilon^3$, $\partial_1 S_\varepsilon^2$, $\partial_1 S_\varepsilon^3$.

• From the divergence free condition on H:

div $H_\varepsilon = 0 \Rightarrow$ div $S_\varepsilon = 0$, that is $\partial_1 S_\varepsilon^1 = -\partial_2 S_\varepsilon^2 - \partial_3 S_\varepsilon^3$

$\Rightarrow$ estimate on $\partial_1 S_\varepsilon^1$.

D. Estimates on the normal derivatives

- From the curl: estimates on $\partial_1 R_\varepsilon^2$, $\partial_1 R_\varepsilon^3$, $\partial_1 S_\varepsilon^2$, $\partial_1 S_\varepsilon^3$.
- From the divergence free condition on H: estimate on $\partial_1 S_\varepsilon^1$.
- From the divergence free condition on D:

$$\partial_1 R_\varepsilon^1 + \frac{2E_0^1}{1 + |E_0|^2 + 2(E_0^1)^2} = K_1$$

D. Estimates on the normal derivatives

- From the curl: estimates on $\partial_1 R_\varepsilon^2$, $\partial_1 R_\varepsilon^3$, $\partial_1 S_\varepsilon^2$, $\partial_1 S_\varepsilon^3$.
- From the divergence free condition on H: estimate on $\partial_1 S_\varepsilon^1$.
- $\partial_1 R_\varepsilon^1 + \dfrac{2E_0^1}{1+|E_0|^2+2(E_0^1)^2} = K_1$
- E.d.o. for $\partial_1 s$

$$\partial_t \partial_1 s_\varepsilon + \frac{1}{\varepsilon}\partial_1 s_\varepsilon = -\partial_t(E_0^1 \partial_1 R_\varepsilon^1) + \partial_t(K_2)$$

$$\partial_t(h\partial_1 s_\varepsilon) + \frac{1}{\varepsilon}\partial_1 s_\varepsilon = \partial_t b$$

D. Estimates on the normal derivatives

- From the curl: estimates on $\partial_1 R_\varepsilon^2$, $\partial_1 R_\varepsilon^3$, $\partial_1 S_\varepsilon^2$, $\partial_1 S_\varepsilon^3$.
- From the divergence free condition on H: estimate on $\partial_1 S_\varepsilon^1$.
- $\partial_1 R_\varepsilon^1 + \dfrac{2E_0^1}{1+|E_0|^2+2(E_0^1)^2} = K_1$
- E.d.o. for $\partial_1 s$

$$\partial_1 s_\varepsilon = -\frac{b}{h} + \frac{1}{h}\int_0^t \frac{b(\sigma)}{h(\sigma)} \frac{1}{\varepsilon} e^{-\int_\sigma^t \frac{1}{\varepsilon h(\tau)} d\tau} d\sigma$$

$$\frac{1}{3} \leq h = \frac{1+|E_0|^2}{1+|E_0|^2+2(E_0^1)^2} \leq 1$$

$\Rightarrow$ Bound for $\partial_1 s_\varepsilon$ and therefore for $\partial_1 R_\varepsilon^1$

D. Estimates on the normal derivatives

- From the curl: estimates on $\partial_1 R_\varepsilon^2$, $\partial_1 R_\varepsilon^3$, $\partial_1 S_\varepsilon^2$, $\partial_1 S_\varepsilon^3$.
- From the divergence free condition on H: estimate on $\partial_1 S_\varepsilon^1$.
- $\partial_1 R_\varepsilon^1 + \dfrac{2E_0^1}{1+|E_0|^2+2(E_0^1)^2} = K_1$
- E.d.o. for $\partial_1 s$: estimate on $\partial_1 s_\varepsilon$ and on $\partial_1 R_\varepsilon^1$

$\Rightarrow$ **Uniform bound for** $\|\rho_\varepsilon\|_{\mathcal{H}^4(\Omega_{T_\varepsilon})}$

E. Conclusion

$T < T^*$ is fixed.

$$T_\varepsilon = \sup \left\{ t \leq T, \|\rho_\varepsilon\|_{\mathcal{H}^4(\Omega_t)} \leq \frac{1}{\sqrt{\varepsilon}} \right\}$$

There exists C s.t. for all $\varepsilon > 0$

$$\|\rho_\varepsilon\|_{\mathcal{H}^4(\Omega_{T_\varepsilon})} \leq C$$

$\Rightarrow$ for ε small enough $T_\varepsilon = T$ and uniform bound for the remainder term.

Perspectives

- Semilinear behaviour for Kerr Debye: 2d TM case, 3d case ?

- Global existence in 3d:

For Kerr:

R. Racke, Lectures on nonlinear evolution equations. Initial value problems. Aspects of Mathematics, E19. Friedr. Vieweg & Sohn, Braunschweig, 1992

For Kerr-Debye ?

- Global existence for KD in 1d

Perturbation of a constant non zero field (Shizuta Kawashima condition)

B. Hanouzet, R. Natalini, *Global existence of smooth solutions for partially dissipative hyperbolic systems with a convex entropy*, Arch. Ration. Mech. Anal. **169** (2003).

General case ?