On rotational effects in the modulations of weakly nonlinear water waves over finite depth

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Abstract

The modulational stability of Stokes' waves has been investigated in the context of potential Euler (free-surface) equations. In this paper, the amplitude equations for three-dimensional rotational flows are derived. It is shown that there are indeed rotational effects. Moreover, except for the classical ansatz on the expansion of the solutions, we derive the set of amplitude equations without making any extra assumption.

Résumé

La stabilité modulationnelle des ondes de Stokes a été étudiée dans le contexte des équations d'Euler pour un écoulement potentiel en présence d'une surface libre. Dans cet article, les équations d'amplitude pour les écoulements rotationnels tri-dimensionnels sont obtenues. En outre, sauf pour l'ansatz classique utilisé pour le développement des solutions, nous dérivons le système des équations d'amplitude sans faire d'hypothèse supplémentaire.

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1 Introduction

An essential part in the study of water waves is their stability. However the notion of stability by itself is not precise enough and that is why the kind of perturbations to which a wave train is stable or not must be defined. The general problem of the stability of a finite amplitude, progressive, periodic wave (Stokes' wave), written in the form

$$\eta(x,t) = \sum_0^\infty A_n \cos[nk(x-ct)]\,,$$

consists in studying the stability of this wave to arbitrary infinitesimal three-dimensional perturbations of the form

$$\eta'(x,y,t) = \exp\{\mathrm{i}[pk(x-ct)+qky-\sigma t]\} \sum_{-\infty}^{\infty} a_n \exp[\mathrm{i}nk(x-ct)] + \text{c.c.}\,,$$

where c.c. denotes the complex conjugate. The symbols used are x for the horizontal direction along which the wave propagates, k for the carrier wavenumber, c for the wave speed, y for the horizontal direction perpendicular to the wave, and t for the time. Substituting the Stokes' wave with its perturbation into the governing equations (Euler equations) and the boundary conditions, and linearizing the perturbation around the basic state results in an eigenvalue problem with σ as eigenvalue. The perturbation wavenumbers p and q are arbitrary. Since σ occurs in complex conjugate pairs, instability corresponds to Im $\sigma \neq 0$. This eigenvalue problem must generally be investigated numerically. However, the case p and q small can be addressed analytically. Historically, the latter case was studied first. In this paper, we will concentrate on that case, which can be thought of as the problem of long-wave instability of Stokes' waves (the perturbation wavenumbers are small) or as the problem of the side-band instability of Stokes' waves (the instability arises from quartet interaction between the waves $(k,0)$ (counted twice) and the waves $(k+kp, kq)$ and $(k-kp,-kq)$). A description of the link between four-wave resonances and modulational stability is given in the review article by Hammack & Henderson (1993).

The long-wave instability of weakly nonlinear travelling capillary–gravity waves has been studied extensively. It is well-known that deep-water gravity waves are unstable to side-band perturbations. Lighthill (1965) provided a geometric condition for wave instability, which is valid when mean flow effects can be neglected, such as in deep water, and essentially obtained what is now called the Benjamin–Feir instability. Benjamin & Feir (1967) showed the result analytically and announced in their paper the extension of the result to finite depth. The analysis for finite depth was given by Benjamin (1967). Gravity waves are unstable if $kh > 1.363$ and stable if $kh < 1.363$ (h denotes the mean water depth). Whitham (1967) essentially obtained the same result independently by using an average Lagrangian approach, which is well explained in his book (1974). It turns out that, as kh decreases through the value 1.363, the type of the system of equations governing the problem changes from elliptic to hyperbolic. At the same time, Benney & Newell (1967) and Zakharov (1968) obtained the same instability result in deep water and derived the cubic nonlinear Schrödinger equation in the context of the modulational stability of water waves. The next essential step in the derivation of model equations was achieved by Benney & Roskes (1969). They extended the stability analysis to three-dimensional disturbances and essentially obtained the equations which are now called the Davey–Stewartson equations. Chu & Mei (1971) extended Whitham's equations by taking into account higher order dispersive effects. The link between the cubic nonlinear Schrödinger and the other approaches was provided by Davey (1972) in infinite depth and independently by Hasimoto & Ono (1972) in finite depth. As already mentioned, Davey & Stewartson (1974) extended the results obtained by Hasimoto & Ono (1972) by using the method of multiple scales in time and in space to three-dimensional perturbations. A description of the steps of the analysis is provided in the book by Mei (1989).

The extension to capillary–gravity waves was given independently by Kawahara (1975), who just considered two-dimensional perturbations, and Djordjevic & Redekopp (1977). Pierce & Knobloch (1994) extended the analysis to waves with $O(2)$ symmetry, therefore including standing waves as well as travelling waves. Surface-tension effects can modify considerably the stability (see for example figure 1 in Djordjevic & Redekopp). In the space of parameters kh and T, where T is some capillary number, there are several regions of stability and instability. The transitions are given by different physical phenomena, corresponding to either the vanishing of certain coefficients or to the blowing-up of other coefficients: extremum of the group velocity, resonance between the fundamental mode and the second harmonic, vanishing of the coefficient of the cubic term (i.e. vanishing of the nonlinearity at the order considered – this is what happens at the critical value $kh = 1.363$ for gravity waves), resonant interaction between long waves and short waves ($c_g^2 = gh$, where c_g is the group velocity and g the acceleration due to gravity). In neighborhoods of all these special cases, the analysis fails and one must either go to higher order or consider a different scaling. The long-wave short-wave resonant interaction was considered by Djordjevic & Redekopp (1977). The second harmonic resonance has been studied a lot, but most of the analyses are restricted to second order. The 1:2 resonance is a strong quadratic resonance and most studies have retained only the quadratic terms (see for example the book by Craik (1985) for a review and an analysis of the resulting equations). If one goes to third order, one obtains two coupled nonlinear Schrödinger equations governing the modulations in space and in time of each of the two modes. To our knowledge, these equations were first obtained by Jones (1992) by the method of multiple scales. While analytical solutions to the cubic nonlinear Schrödinger equation are well-documented (see for example Peregrine (1983) for a review), analytical solutions to Jones' coupled equations have not been studied yet. Jones (1992) only solved them for special cases (periodic solutions). Ablowitz & Segur (1979) analyzed the Davey–Stewartson equations and described the phenomenon of 'focusing' of waves with strong enough surface tension.

In the same spirit, the stability of interfacial waves has also been studied (see for example Nayfeh (1976), Grimshaw & Pullin (1985) for analytical results based on a multiple scale expansion, and Dixon (1990), Zhou et al. (1992) for results based on a Zakharov equation).

The cubic nonlinear Schrödinger equation, which is obtained to $O(\epsilon^3)$ in the wave steepness, describes well the evolution of wave trains and packets at the early stages. At later stages, the agreement is not so good anymore and various attempts have been made to pursue the perturbation analysis one step further, to order $O(\epsilon^4)$. The first such study appears to have been made by Dysthe (1979) for gravity waves and by Hogan (1985) for capillary–gravity waves. One of the higher order effects in infinite depth is the influence of the wave-induced mean flow. It is claimed that the fourth-order terms greatly improve the agreement with numerical results on the full Euler equations. Janssen (1983) investigated the effect of the mean flow on the long time behavior of the Benjamin–Feir instability. Experiments show asymmetric group splitting and frequency downshifting. Lo & Mei (1985) were able to predict these by integrating numerically Dysthe's equation. However, their solutions are modulated periodically and do not account fully for the experimental observations. Brinch–Nielsen & Jonsson (1986) extended the results to finite depth. The true origin of the frequency downshifting is still an open problem. Several explanations have been proposed (including wind-effect, dissipation, wave-breaking) but none is fully satisfactory.

The common ansatz used in the derivation of the Davey–Stewartson equations is that the velocity potential and η, the elevation of the free surface, have uniformly valid asymptotic expansions in terms of a small parameter ϵ (the dimensionless amplitude of the wave, kA, for example). One writes

$$\eta = \sum_{n=1}^{3} \epsilon^n \eta_n(x_0, x_1, x_2, y_1, y_2; t_0, t_1, t_2) + o(\epsilon^3)\,, \tag{1.1}$$

where

$$x_0 = x,\ x_1 = \epsilon x,\ x_2 = \epsilon^2 x,\ y_1 = \epsilon y,\ y_2 = \epsilon^2 y,\ t_0 = t,\ t_1 = \epsilon t,\ t_2 = \epsilon^2 t\,. \tag{1.2}$$

The order one component of η is

$$\eta_1 = \frac{\mathrm{i}}{\omega} A(x_1, x_2, y_1, y_2; t_1, t_2) \sinh(kh) \exp[\mathrm{i}(kx - \omega t)] + \text{c.c.}\,, \tag{1.3}$$

where ω denotes the frequency of the wave. Applying the derivative method leads to the Davey–Stewartson equations for the evolution of the complex amplitude A of the wave,

$$\begin{aligned} 2\mathrm{i}\sinh(kh)\left(\frac{\partial A}{\partial t_2} + c_g \frac{\partial A}{\partial x_2}\right) + \sinh(kh)\frac{d^2\omega}{dk^2}\frac{\partial^2 A}{\partial \xi^2} + \frac{c_g \sinh(kh)}{k}\frac{\partial^2 A}{\partial y_1^2} \\ = \frac{k}{c_g \cosh(kh)}\left(\frac{c_g \omega}{g} + \sinh(2kh)\right)\zeta A - \nu |A|^2 A\,, \end{aligned} \tag{1.4}$$

and for the evolution of the mean flow ζ,

$$(gh - c_g^2)\frac{\partial^2 \zeta}{\partial x_1^2} + gh\frac{\partial^2 \zeta}{\partial y_1^2} = -c_g^2\left(1 + \frac{g\sinh(2kh)}{c_g \omega}\right)\frac{\partial^2 |A|^2}{\partial \xi^2}\,. \tag{1.5}$$

Recall that c_g denotes the group velocity $d\omega/dk$. The derivation of the Davey–Stewartson is tedious and several assumptions have been made by various authors, in addition to the ansatz, in the process of deriving the equations. The main purpose of the present paper is to rederive as rigorously as possible the D–S equations. It is appropriate to say a few words here on the original problem that motivated our study. The water wave problem is governed by the Euler equations. Existence proofs of solutions for all times have been given but only in the special case when time can be removed from the equations by looking for solutions which are steady in a frame of reference moving with the wave (travelling waves, either space periodic or solitary). Concerning the Cauchy problem, i.e. arbitrarily time-dependent solutions, we refer to Section 2. Our long-term motivation is to obtain rigorously an amplitude equation at leading order but that goal has so far eluded us. The ansatz given above has not been justified yet.

In this paper, we start with the ansatz (1.1) and perform the calculations symbolically without making any extra assumption all along. In order to investigate the influence of rotationality, we *do not* make the assumption that the flow is potential. The Euler equations are solved for the three velocity components and for the pressure. The main conclusion is that, for two-dimensional as well as three-dimensional perturbations, the amplitude equations are the same as for the irrotational case. However, the velocity field has a second order rotational component, see (4.16).

Our analysis also reveals a new singularity in the water-wave problem, which, to our knowledge, has never been studied before. Note that this singularity is not associated with the lack of irrotationality. A careful derivation of the D–S equations, even in the irrotational case, clearly reveals the singularity. It occurs (see (4.10)) when

$$gh = \frac{g^2}{\omega^2}\sinh^2(2kh)\,. \tag{1.6}$$

The above relation is shown in parameter space in figure 1 (solid line). The singularity corresponds to a resonance between a long wave and a wave with speed $(-g\sinh(2kh)/\omega)$. In order to understand the origin of the singularity, we recall the equation for the mean flow ζ:

$$\left(\frac{\partial}{\partial t_1} + c_g\frac{\partial}{\partial x_1}\right)\left(\frac{\partial}{\partial t_1} - \frac{g\sinh(2kh)}{\omega}\frac{\partial}{\partial x_1}\right)\zeta = 0\,. \tag{1.7}$$

In previous derivations, it has been assumed that the mean flow depends on x_1 and t_1 through $(x_1 - c_g t_1)$. Brinch–Nielsen & Jonsson (1986) write that the assumption is not generally valid because it implies a connection between the mean water level and the mean Eulerian velocity $\partial\zeta/\partial x$, which is an unphysical and really not necessary restriction. What the above equation shows is that the so-called "assumption" is not an assumption but comes from the equations as long as one can show that the $(x_1 + gt_1\sinh(2kh)/\omega)$ dependence of the mean flow is zero. It can be shown indeed except at the singularity (1.6). Therefore, in addition to the well-known singularities described above where the derivation of the D–S equations fails, there is another singularity. An interesting consequence of our study is that the fact that our equations without the assumption of irrotationality are the same as the D–S equations is closely related to the fact that the mean flow depends on x_1 and t_1 only through $(x_1 - c_g t_1)$. Whether rotational effects come into the picture at the new singularity is an open question. Near the singularity, a different scaling should be introduced. We leave it for future work. Note that nonlinear Schrödinger equations have been derived to describe the modulations of gravity waves in nonpotential flows. See for example the modulations of an internal gravity-wave packet studied by Grimshaw (1977) or by Shrira (1981).

In Section 2, the problem is formulated and the ansatz is introduced. In Section 3, the system of amplitude equations is obtained by using the derivative method. In Section 4, the Davey–Stewartson equations are derived and the new singularity is revealed. Finally, in Section 5, some of the implications of the rotational effects are discussed.

2 Formulation of the problem

The three-dimensional Euler equations describing the motion of an inviscid and incompressible liquid layer of mean depth h read as follows. Denoting by $\Omega(t)$ the volume occupied at time t by the liquid,

$$\Omega(t) = \Big\{ X = (x,y,z) \in \mathrm{I\!R}^3, -h \le z \le \eta(x,y,t) \Big\} \,, \tag{2.1}$$

and by $\underline{V} = (u,v,w)$ the velocity vector field, we have

$$\underline{V}_t + (\underline{V}\cdot\nabla)\underline{V} + \nabla p = \underline{g} \quad \text{in } \Omega(t)\,, \tag{2.2}$$

$$\nabla\cdot\underline{V} = 0 \quad \text{in } \Omega(t)\,, \tag{2.3}$$

together with the boundary conditions

$$w = 0\,, \quad \text{for } z = -h\,, \tag{2.4}$$

$$w = \frac{\partial\eta}{\partial t} + u\frac{\partial\eta}{\partial x} + v\frac{\partial\eta}{\partial y}\,, \quad \text{for } z = \eta(x,y,t)\,, \tag{2.5}$$

$$p - p_0 = -T\kappa\,, \quad \text{for } z = \eta(x,y,t)\,, \tag{2.6}$$

where $\eta(x,y,t)$ is the elevation of the free surface, κ its mean curvature:

$$\kappa = \frac{\eta_{xx}(1+\eta_y^2) - 2\eta_x\eta_y\eta_{xy} + \eta_{yy}(1+\eta_x^2)}{(1+\eta_x^2+\eta_y^2)^{3/2}}\,, \tag{2.7}$$

p the pressure, p_0 the pressure above the liquid, g the acceleration due to gravity and T the surface tension per unit density of the liquid.

The state of rest ($\underline{V} = 0$, $p = p_0 - gz$, $\eta = 0$) is solution to (2.2)–(2.6). Its linear stability is governed by the following set of equations on the strip

$$S = \Big\{ X = (x,y,z) \in \mathrm{I\!R}^3, -h \le z \le 0 \Big\} \,:$$

$$\tilde{\underline{V}}_t + \nabla\tilde{p} = 0\,, \tag{2.8}$$

$$\nabla\cdot\tilde{\underline{V}} = 0\,, \tag{2.9}$$

$$\tilde{w} = 0\,, \quad \text{for } z = -h\,, \tag{2.10}$$

$$\tilde{w} = \frac{\partial\tilde{\eta}}{\partial t}\,, \quad \text{for } z = 0\,, \tag{2.11}$$

$$-g\tilde{\eta} + \tilde{p} = -T\Delta\tilde{\eta}\,, \quad \left(\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\,, \quad \text{for } z = 0\,, \tag{2.12}$$

where

$$\underline{V} = \underline{\tilde{V}} + \cdots, p = p_0 - gz + \tilde{p} + \cdots, w = \tilde{w} + \cdots, \eta = \tilde{\eta} + \cdots .$$

The existence of plane wave solutions to (2.8)–(2.12) of the form

$$(\underline{\tilde{V}}, \tilde{p}, \tilde{w}, \tilde{\eta}) = \mathrm{e}^{i(kx+ly-\omega t)}(\hat{V}(z), \hat{p}(z), \hat{w}(z), \hat{\eta}) \tag{2.13}$$

is decided on the dispersion relation

$$\omega^2 = K(g + TK^2)\tanh(Kh)\,, \quad K = \sqrt{k^2 + l^2}\,. \tag{2.14}$$

Hence solutions to (2.8)–(2.12) are bounded and the state of rest is linearly stable. It is not asymptotically stable; therefore the study of the nonlinear stability of the state of rest cannot be decided at this stage.

Remark 2.1: Equations (2.2)–(2.6) are invariant under rotations around the z-axis, thus it is sufficient to consider the case $l = 0$ in (2.13).

In order to address the problem of nonlinear interactions of small amplitude waves, we assume that the initial data for (2.2)–(2.6) have the following form:

$$\underline{V}(t=0) = \sum_{n=1}^{3} \epsilon^n \underline{V}_n^0(x, \epsilon x, \epsilon^2 x, \epsilon y, \epsilon^2 y, z) + o(\epsilon^3)\,, \tag{2.15}$$

$$\eta(t=0) = \sum_{n=1}^{3} \epsilon^n \eta_n^0(x, \epsilon x, \epsilon^2 x, \epsilon y, \epsilon^2 y) + o(\epsilon^3)\,. \tag{2.16}$$

That is, the initial data are of size ϵ and depend also on the slow variables $(\epsilon x, \epsilon^2 x, \epsilon y, \epsilon^2 y)$. At this point we make the following ansatz on $\underline{V}$, p and η. For that, it is convenient to introduce several temporal scales: $t_0 = t, t_1 = \epsilon t, t_2 = \epsilon^2 t$ and we assume that $\underline{V}, \eta, p$ have uniformly valid asymptotic expansions in terms of ϵ:

$$\underline{V} = \sum_{n=1}^{3} \epsilon^n \underline{V}_n(x_0, x_1, x_2, y_1, y_2, z; t_0, t_1, t_2) + o(\epsilon^3)\,, \tag{2.17}$$

$$p = p_0 - gz + \sum_{n=1}^{3} \epsilon^n p_n(x_0, x_1, x_2, y_1, y_2, z; t_0, t_1, t_2) + o(\epsilon^3)\,, \tag{2.18}$$

$$\eta = \sum_{n=1}^{3} \epsilon^n \eta_n(x_0, x_1, x_2, y_1, y_2; t_0, t_1, t_2) + o(\epsilon^3)\,, \tag{2.19}$$

where $\underline{V} = (u, v, w)$, $\underline{V}_n = (u_n, v_n, w_n)$ and

$$x_0 = x,\ x_1 = \epsilon x,\ x_2 = \epsilon^2 x,\ y_1 = \epsilon y,\ y_2 = \epsilon^2 y, \tag{2.20}$$

denote the spatial scales introduced in (2.15). In the next section, we will apply the derivative expansion method to this problem (Davey & Stewartson (1974), Kawahara (1973,1975), Djordjevic & Redekopp (1977)).

About existence of solutions

The problem of existence (or non-existence) of solutions to the Cauchy problem for the free-surface Euler equations is still widely open. However some results are available. As expected all these results are valid on a finite (small) time interval and some of them impose restrictions either on the size or on the smoothness of the initial data. More precisely, Shinbrot (1976) shows existence of solutions in the case of analytic initial data on a 2D domain which is assumed to be initially a horizontal slab. We also point out that the flow is not necessarily irrotational in his work. Then Reeder & Shinbrot (1976) generalize the previous result to the 3D case while Reeder & Shinbrot (1979) give a further generalization. Namely it is no more assumed that the initial volume occupied by the water is a slab but instead a uniformly analytic domain.

The (non-physical) analyticity assumption on the initial data was first removed by Nalimov (1974) who considers 2D irrotational flows over infinite depth but only allows for small initial data in Sobolev spaces. Then Yosihara (1982) shows the same type of result in the case of finite depth (he considers that the bottom is almost horizontal). Let us point out that Yosihara only considers the case of irrotational motions.

About rigorous asymptotics

Very few papers deal with the proof of a fully rigorous derivation of nonlinear amplitude equations. It is clear that such a derivation must rely on a rigorous existence theory for the free-surface Euler equations. In this direction, we can mention Kano & Nishida (1979) who derive the nonlinear shallow water equations and Kano & Nishida (1986) who derive the Kortweg–de Vries and Boussinesq equations. In both articles, *analyticity* of the initial data is assumed. Then Craig (1985) gives a rigorous proof of the Boussinesq and Korteweg–de Vries limits (together with an existence theory for the free-surface Euler equations) in the case of small initial data lying in Sobolev spaces. These three papers deal with the case of shallow water, and assume again irrotationality of the motion.

Finally, let us mention the only paper we know which deals with rigorous results in the case of finite depth. In Craig, Sulem and Sulem (1992), the authors are able to check a precise consistency of the solution, whose envelope is governed by the cubic nonlinear Schrödinger equation, with the free-surface Euler equations.

3 Three-dimensional flows: amplitude equations

In order to obtain the amplitude equations, we substitute the ansatz (2.17)–(2.19) into (2.2)–(2.6) and we expand (2.5)–(2.6) around $z = 0$. Equating coefficients of equal powers of ϵ yields the following sets of equations up to the third order:

Order ϵ:

$$\frac{\partial u_1}{\partial t_0} + \frac{\partial p_1}{\partial x_0} = 0 \quad \text{in } S\,, \tag{3.1}$$

$$\frac{\partial v_1}{\partial t_0} = 0 \quad \text{in } S\,, \tag{3.2}$$

$$\frac{\partial w_1}{\partial t_0} + \frac{\partial p_1}{\partial z} = 0 \quad \text{in } S\,, \tag{3.3}$$

$$\frac{\partial w_1}{\partial z} + \frac{\partial u_1}{\partial x_0} = 0 \quad \text{in } S\,, \tag{3.4}$$

$$w_1 = 0 \quad \text{at } z = -h\,, \tag{3.5}$$

$$-w_1 + \frac{\partial \eta_1}{\partial t_0} = 0 \quad \text{at } z = 0\,, \tag{3.6}$$

$$-g\eta_1 + p_1 + T\frac{\partial^2 \eta_1}{\partial x_0^2} = 0 \quad \text{at } z = 0\,, \tag{3.7}$$

Order ϵ^2:

$$\frac{\partial u_1}{\partial t_1} + \frac{\partial u_2}{\partial t_0} + w_1\frac{\partial u_1}{\partial z} + \frac{\partial p_1}{\partial x_1} + \frac{\partial p_2}{\partial x_0} + u_1\frac{\partial u_1}{\partial x_0} = 0 \quad \text{in } S\,, \tag{3.8}$$

$$\frac{\partial v_1}{\partial t_1} + \frac{\partial v_2}{\partial t_0} + w_1\frac{\partial v_1}{\partial z} + \frac{\partial p_1}{\partial y_1} + u_1\frac{\partial v_1}{\partial x_0} = 0 \quad \text{in } S\,, \tag{3.9}$$

$$\frac{\partial w_1}{\partial t_1} + \frac{\partial w_2}{\partial t_0} + w_1\frac{\partial w_1}{\partial z} + \frac{\partial p_2}{\partial z} + u_1\frac{\partial w_1}{\partial x_0} = 0 \quad \text{in } S\,, \tag{3.10}$$

$$\frac{\partial w_2}{\partial z} + \frac{\partial v_1}{\partial y_1} + \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_0} = 0 \quad \text{in } S\,, \tag{3.11}$$

$$w_2 = 0 \quad \text{at } z = -h\,, \tag{3.12}$$

$$-w_2 + \frac{\partial \eta_1}{\partial t_1} + \frac{\partial \eta_2}{\partial t_0} + u_1\frac{\partial \eta_1}{\partial x_0} - \eta_1\frac{\partial w_1}{\partial z} = 0 \quad \text{at } z = 0\,, \tag{3.13}$$

$$-g\eta_2 + p_2 + T\left(2\frac{\partial^2 \eta_1}{\partial x_0 \partial x_1} + \frac{\partial^2 \eta_2}{\partial x_0^2}\right) + \eta_1\frac{\partial p_1}{\partial z} = 0 \quad \text{at } z = 0\,, \tag{3.14}$$

and

Order ϵ^3:

$$\frac{\partial u_1}{\partial t_2} + \frac{\partial u_2}{\partial t_1} + \frac{\partial u_3}{\partial t_0} + w_2\frac{\partial u_1}{\partial z} + w_1\frac{\partial u_2}{\partial z} + v_1\frac{\partial u_1}{\partial y_1} + \frac{\partial p_1}{\partial x_2} + \frac{\partial p_2}{\partial x_1} + \frac{\partial p_3}{\partial x_0} + u_2\frac{\partial u_1}{\partial x_0} + u_1\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_0}\right) = 0 \quad \text{in } S\,, \tag{3.15}$$

$$\frac{\partial v_1}{\partial t_2} + \frac{\partial v_2}{\partial t_1} + \frac{\partial v_3}{\partial t_0} + w_2\frac{\partial v_1}{\partial z} + w_1\frac{\partial v_2}{\partial z} + \frac{\partial p_1}{\partial y_2} + \frac{\partial p_2}{\partial y_1} + v_1\frac{\partial v_1}{\partial y_1} + u_2\frac{\partial v_1}{\partial x_0} + u_1\left(\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_0}\right) = 0 \quad \text{in } S\,, \tag{3.16}$$

$$\frac{\partial w_1}{\partial t_2} + \frac{\partial w_2}{\partial t_1} + \frac{\partial w_3}{\partial t_0} + w_2\frac{\partial w_1}{\partial z} + w_1\frac{\partial w_2}{\partial z} + \frac{\partial p_3}{\partial z} + v_1\frac{\partial w_1}{\partial y_1} + u_2\frac{\partial w_1}{\partial x_0} + u_1\left(\frac{\partial w_1}{\partial x_1} + \frac{\partial w_2}{\partial x_0}\right) = 0 \quad \text{in } S\,, \tag{3.17}$$

$$\frac{\partial w_3}{\partial z}+\frac{\partial v_1}{\partial y_2}+\frac{\partial v_2}{\partial y_1}+\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}+\frac{\partial u_3}{\partial x_0} = 0 \quad \text{in } S\,, \tag{3.18}$$

$$w_3 = 0 \quad \text{at } z=-h\,, \tag{3.19}$$

$$-w_3+\frac{\partial \eta_1}{\partial t_2}+\frac{\partial \eta_2}{\partial t_1}+\frac{\partial \eta_3}{\partial t_0}+v_1\frac{\partial \eta_1}{\partial y_1}+u_1\left(\frac{\partial \eta_1}{\partial x_1}+\frac{\partial \eta_2}{\partial x_0}\right)$$
$$+\frac{\partial \eta_1}{\partial x_0}\left(u_2+\eta_1\frac{\partial u_1}{\partial z}\right)-\eta_2\frac{\partial w_1}{\partial z}-\eta_1\frac{\partial w_2}{\partial z}-\frac{1}{2}\eta_1^2\frac{\partial^2 w_1}{\partial z^2} = 0 \quad \text{at } z=0 \tag{3.20}$$

$$-g\eta_3+p_3+T\frac{\partial^2 \eta_1}{\partial y_1^2}-\frac{3}{2}T\left(\frac{\partial \eta_1}{\partial x_0}\right)^2\frac{\partial^2 \eta_1}{\partial x_0^2}+T\left(\frac{\partial^2 \eta_1}{\partial x_1^2}\right.$$
$$\left.+2\frac{\partial^2 \eta_1}{\partial x_0 \partial x_2}+2\frac{\partial^2 \eta_2}{\partial x_0 \partial x_1}+\frac{\partial^2 \eta_3}{\partial x_0^2}\right)+\eta_2\frac{\partial p_1}{\partial z}+\eta_1\frac{\partial p_2}{\partial z}+\frac{1}{2}\eta_1^2\frac{\partial^2 p_1}{\partial z^2} = 0 \quad \text{at } z=0 \tag{3.21}$$

Note that these expansions have been obtained symbolically by using the formal computation software MATHEMATICA.

As solutions to the linear equations (3.1)–(3.7), we take the plane waves introduced in Section 2. More precisely:

$$u_1 = \mathrm{i}A(x_1,x_2,y_1,y_2;t_1,t_2)\cosh[k(z+h)]\mathrm{e}^{i\theta}+\text{c.c.}\,, \tag{3.22}$$
$$v_1 = 0\,, \tag{3.23}$$
$$w_1 = A(x_1,x_2,y_1,y_2;t_1,t_2)\sinh[k(z+h)]\mathrm{e}^{i\theta}+\text{c.c.}\,, \tag{3.24}$$
$$p_1 = \frac{\mathrm{i}\omega}{k}A(x_1,x_2,y_1,y_2;t_1,t_2)\cosh[k(z+h)]\mathrm{e}^{i\theta}+\text{c.c.}\,, \tag{3.25}$$
$$\eta_1 = \frac{\mathrm{i}}{\omega}A(x_1,x_2,y_1,y_2;t_1,t_2)\sinh(kh)\mathrm{e}^{i\theta}+\text{c.c.}\,, \tag{3.26}$$

where $\theta = kx_0-\omega t_0$ and ω satisfies the dispersion relation (2.14)

$$\omega^2 = k(g+Tk^2)\tanh(kh)\,,$$

and c.c. denotes the complex conjugate of the preceding terms. The complex amplitude A depends only upon the slow scales x_1, x_2, y_1, y_2, t_1 and t_2 and the aim of the expansion is to determine the differential equation satisfied by A. We now substitute this first order solution (3.22)–(3.26) into (3.8)–(3.14) and we look for a solution in the form

$$\underline{V}_2 = \mathrm{e}^{2i\theta}\underline{V}_{22}(x_1,x_2,y_1,y_2,z;t_1,t_2)+\mathrm{e}^{i\theta}\underline{V}_{21}(x_1,x_2,y_1,y_2,z;t_1,t_2)+\text{c.c.}$$
$$+\underline{V}_{20}(x_1,x_2,y_1,y_2,z;t_1,t_2)\,, \tag{3.27}$$
$$p_2 = \mathrm{e}^{2i\theta}p_{22}(x_1,x_2,y_1,y_2,z;t_1,t_2)+\mathrm{e}^{i\theta}p_{21}(x_1,x_2,y_1,y_2,z;t_1,t_2)+\text{c.c.}$$
$$+p_{20}(x_1,x_2,y_1,y_2,z;t_1,t_2)\,, \tag{3.28}$$
$$\eta_2 = \mathrm{e}^{2i\theta}\eta_{22}(x_1,x_2,y_1,y_2;t_1,t_2)+\mathrm{e}^{i\theta}\eta_{21}(x_1,x_2,y_1,y_2;t_1,t_2)+\text{c.c.}$$
$$+\eta_{20}(x_1,x_2,y_1,y_2;t_1,t_2)\,. \tag{3.29}$$

One finds for the coefficient of $(\mathrm{e}^{i\theta})^0$:

$$\left\{\begin{array}{rcl} p_{20} &=& -|A|^2\cosh[2k(z+h)]+\zeta(x_1,x_2,y_1,y_2;t_1,t_2)\,,\\ w_{20} &=& 0\,,\\ \eta_{20} &=& \frac{1}{g}(\zeta-|A|^2)\,. \end{array}\right. \tag{3.30}$$

For the term in $(\mathrm{e}^{i\theta})^1$, one obtains

$$\left\{\begin{array}{rcl} u_{21} &=& (z+h)\sinh[k(z+h)]\frac{\partial A}{\partial x_1}\,, \\ v_{21} &=& \frac{1}{k}\cosh[k(z+h)]\frac{\partial A}{\partial y_1}\,, \\ w_{21} &=& -\mathrm{i}(z+h)\cosh[k(z+h)]\frac{\partial A}{\partial x_1}\,, \\ p_{21} &=& \frac{\omega}{k}\left((z+h)\sinh[k(z+h)]\frac{\partial A}{\partial x_1} - \frac{1}{k}\cosh[k(z+h)]\frac{\partial A}{\partial x_1} - \frac{1}{\omega}\cosh[k(z+h)]\frac{\partial A}{\partial t_1}\right)\,, \\ \eta_{21} &=& \frac{1}{\omega}\left(h\cosh(kh)\frac{\partial A}{\partial x_1} + \frac{\sinh(kh)}{\omega}\frac{\partial A}{\partial t_1}\right)\,, \end{array}\right. \tag{3.31}$$

together with the non-secularity condition

$$\frac{\partial A}{\partial t_1} + c_g\frac{\partial A}{\partial x_1} = 0\,, \tag{3.32}$$

where c_g denotes the group velocity $d\omega/dk$. Moreover the higher harmonic terms are given by

$$\left\{\begin{array}{rcl} u_{22} &=& -\mu A^2\cosh[2k(z+h)]\,, \\ v_{22} &=& 0\,, \\ w_{22} &=& \mathrm{i}\mu A^2\sinh[2k(z+h)]\,, \\ p_{22} &=& -\mu\frac{\omega}{k}A^2\cosh[2k(z+h)] + \frac{1}{2}A^2\,, \\ \eta_{22} &=& -\frac{k}{2\omega^2}A^2\sinh(2kh) - \mu\frac{1}{2\omega}A^2\sinh(2kh)\,, \end{array}\right. \tag{3.33}$$

where

$$\mu = \frac{3k}{2\omega}\frac{g(1-\sigma^2)+Tk^2(3-\sigma^2)}{g\sigma^2+(\sigma^2-3)Tk^2}\,, \qquad \sigma = \tanh(kh)\,.$$

The denominator of μ vanishes when $g\sigma^2+(\sigma^2-3)Tk^2 = 0$, which corresponds to the well-known second-harmonic resonance (see dotted line in figure 1). Introducing (3.33) into the third order perturbation equations yields the following set of amplitude equations:

$$2\mathrm{i}\sinh(kh)\left(\frac{\partial A}{\partial t_2} + c_g\frac{\partial A}{\partial x_2}\right) + \sinh(kh)\frac{d^2\omega}{dk^2}\frac{\partial^2 A}{\partial \xi^2} + c_g\frac{\sinh(kh)}{k}\frac{\partial^2 A}{\partial y_1^2}$$

$$= \frac{k\omega}{g\cosh(kh)}A\zeta + \frac{2k^2A}{\cosh(kh)}\int_{-h}^{0}\cosh[2k(z+h)]u_{20}(z)\,dz - \nu|A|^2A\,, \tag{3.34}$$

$$\frac{\partial\zeta}{\partial x_1} + \frac{\partial u_{20}}{\partial t_1} = 0 \quad \text{in } S\,, \tag{3.35}$$

$$\frac{\partial\zeta}{\partial y_1} + \frac{\partial v_{20}}{\partial t_1} = 0 \quad \text{in } S\,, \tag{3.36}$$

$$\frac{\partial w_{30}}{\partial z} + \frac{\partial v_{20}}{\partial y_1} + \frac{\partial u_{20}}{\partial x_1} = 0 \quad \text{in } S\,, \tag{3.37}$$

$$w_{30} = 0 \quad \text{at } z=-h\,, \tag{3.38}$$

$$w_{30} - \frac{1}{g}\left(c_g\frac{\partial|A|^2}{\partial\xi} + \frac{\partial\zeta}{\partial t_1}\right) - \frac{\sinh(2kh)}{\omega}\frac{\partial|A|^2}{\partial\xi} = 0 \quad \text{at } z=0\,, \tag{3.39}$$

and A depends on x_1 and t_1 only through $\xi = x_1 - c_g t_1$. The coefficient ν is given by

$$\begin{aligned}\nu &= \tfrac{1}{2}k^3\sigma^2\Big[g^3(9-12\sigma^2+13\sigma^4-2\sigma^6)+k^2g^2(36-62\sigma^2+33\sigma^4-6\sigma^6)T \\ &\quad +gk^4(33-55\sigma^2+30\sigma^4-6\sigma^6)T^2+k^6(6-14\sigma^2+10\sigma^4-2\sigma^6)T^3\Big]\cosh(kh)\times \\ &\quad \times[g\omega^3(\sigma^2-1)(g\sigma^2-3Tk^2+k^2\sigma^2T)]^{-1}\,.\end{aligned}$$

This value of ν corresponds to that given in Djordjevic & Redekopp (1977) after a correction by a multiplication factor due to the fact that we do not take the same linear solution as they do. As said above, all these computations have been performed using MATHEMATICA. The set of equations (3.34)–(3.39) governs the modulations of a small amplitude wave train. The next step is to simplify this system.

4 Derivation of the Davey–Stewartson equations

The goal of this section is to derive the Davey–Stewartson equations (Davey & Stewartson (1974), Djordjevic & Redekopp (1977)) from (3.34)–(3.39). We show that the functions ζ and $\int_{-h}^{0}\cosh[2k(z+h)]u_{20}(z)\,dz$ depend on x_1 and t_1 only through $\xi = x_1 - c_g t_1$. We insist on the fact that this is not a hypothesis, but it is a property of the system (3.34)–(3.39) except for a resonant case. We first apply the differential operator

$$\left(\frac{\partial}{\partial t_1}+c_g\frac{\partial}{\partial x_1}\right)\frac{\partial}{\partial t_1}$$

on (3.34) and the fact that $\frac{\partial A}{\partial t_1}+c_g\frac{\partial A}{\partial x_1}=0$ yields

$$k\omega\left(\frac{\partial}{\partial t_1}+c_g\frac{\partial}{\partial x_1}\right)\frac{\partial\zeta}{\partial t_1}+2gk^2\int_{-h}^{0}\left(\frac{\partial}{\partial t_1}+c_g\frac{\partial}{\partial x_1}\right)\frac{\partial u_{20}}{\partial t_1}\cosh[2k(z+h)]\,dz=0\,. \quad (4.1)$$

Equation (3.35) gives

$$\frac{\partial\zeta}{\partial x_1}=-\frac{\partial u_{20}}{\partial t_1}\,.$$

Therefore, (4.1) leads to

$$\left(\frac{\partial}{\partial t_1}+c_g\frac{\partial}{\partial x_1}\right)\left(\frac{\partial}{\partial t_1}-\frac{g\sinh(2kh)}{\omega}\frac{\partial}{\partial x_1}\right)\zeta=0\,.$$

Hence ζ can be written in the form

$$\zeta=f_1(x_1-c_g t_1,y_1)+f_2(x_1-\beta t_1,y_1)\,, \quad (4.2)$$

where $\beta = -g\sinh(2kh)/\omega$ and f_1, f_2 are arbitrary functions. We have dropped the dependences in x_2, y_2, t_2 which are considered as parameters in this calculation. On the other hand, (3.35) and (3.36) imply

$$\frac{\partial^2\zeta}{\partial x_1^2}=-\frac{\partial^2 u_{20}}{\partial t_1\partial x_1}\quad\text{and}\quad\frac{\partial^2\zeta}{\partial y_1^2}=-\frac{\partial^2 v_{20}}{\partial t_1\partial y_1}\,, \quad (4.3)$$

while $\frac{\partial(3.37)}{\partial t_1}$ gives

$$\frac{\partial^2 w_{30}}{\partial z \partial t_1} = -\frac{\partial^2 v_{20}}{\partial t_1 \partial y_1} - \frac{\partial^2 u_{20}}{\partial t_1 \partial x_1}\,. \tag{4.4}$$

Equations (4.3) and (4.4) lead to

$$\frac{\partial^2 w_{30}}{\partial t_1 \partial z} = \Delta_{x_1,y_1}\zeta\,.$$

The boundary condition (3.38) yields

$$\frac{\partial w_{30}}{\partial t_1} = (z+h)\Delta_{x_1,y_1}\zeta\,, \tag{4.5}$$

since ζ does not depend on z. We substitute (4.5) in $\frac{\partial(3.39)}{\partial t_1}$:

$$h\Delta_{x_1,y_1}\zeta - \frac{1}{g}\left(-c_g^2\frac{\partial^2|A|^2}{\partial\xi^2} + \frac{\partial^2\zeta}{\partial t_1^2}\right) + \frac{\sinh(2kh)}{\omega}c_g\frac{\partial^2|A|^2}{\partial\xi^2} = 0\,. \tag{4.6}$$

Combining (4.2) and (4.6) and denoting by Ξ the quantity $x_1 - \beta t_1$ yields

$$\begin{aligned} h\frac{\partial^2 f_1}{\partial\xi^2} + h\frac{\partial^2 f_1}{\partial y_1^2} + h\frac{\partial^2 f_2}{\partial\Xi^2} + h\frac{\partial^2 f_2}{\partial y_1^2} + \frac{c_g^2}{g}\frac{\partial^2|A|^2}{\partial\xi^2} & \\ -\frac{c_g^2}{g}\frac{\partial^2 f_1}{\partial\xi^2} - \frac{\beta^2}{g}\frac{\partial^2 f_2}{\partial\Xi^2} + \frac{c_g\sinh(2kh)}{\omega}\frac{\partial^2|A|^2}{\partial\xi^2} &= 0\,. \end{aligned} \tag{4.7}$$

It follows that

$$\frac{\partial^2 f_2}{\partial\Xi^2}\left(h - \frac{\beta^2}{g}\right) + h\frac{\partial^2 f_2}{\partial y_1^2} = \alpha(y_1)\,. \tag{4.8}$$

Since $\alpha(y_1)\to 0$ as $\Xi\to\infty$, it follows that $\alpha(y_1)\equiv 0$ and (4.8) reads

$$\frac{\partial^2 f_2}{\partial\Xi^2}\left(1 - \frac{\beta^2}{gh}\right) + \frac{\partial^2 f_2}{\partial y_1^2} = 0\,. \tag{4.9}$$

Three cases can occur:

(i) <u>If $gh - \beta^2 > 0$</u>: taking the Fourier transform in Ξ, y_1, we obtain that f_2 has to be linear in Ξ, y_1 and since f_2 tends to zero at infinity, $f_2 \equiv 0$.

(ii) <u>If $gh - \beta^2 < 0$</u>: f_2 satisfies a wave equation in the variables Ξ, y_1, hence it can be written as

$$f_2 = f_{2+}(\Xi + \mathcal{C}y_1) + f_{2-}(\Xi - \mathcal{C}y_1)\,,$$

where $\mathcal{C} = \sqrt{\beta^2/gh - 1}$. It follows that $f_{2+} = f_{2-} = 0$, since f_2 has to decay at infinity.

(iii) <u>If $gh - \beta^2 = 0$</u>: it is a resonant case and one cannot conclude. This resonance occurs when

$$gh = \frac{g^2\sinh^2(2kh)}{\omega^2}\,. \tag{4.10}$$

It is shown in parameter space in figure 1.

In both cases (i) and (ii), we obtain that ζ depends on x_1 and t_1 only through ξ. Now, (3.34) implies that

$$k\omega\zeta + 2gk^2 \int_{-h}^{0} \cosh[2k(z+h)] u_{20}\, dz$$

depends on x_1 and t_1 only through ξ and we denote its value by $2gk^2\mu(\xi, y_1)$. Hence

$$\int_{-h}^{0} \cosh[2k(z+h)] u_{20}\, dz = \mu(\xi, y_1) - \frac{\omega}{2gk}\zeta\,. \tag{4.11}$$

Differentiating (4.11) with respect to t_1 leads to

$$\int_{-h}^{0} \cosh[2k(z+h)] \frac{\partial u_{20}}{\partial t_1}(z)\, dz = -c_g \frac{\partial \mu}{\partial \xi} + \frac{c_g \omega}{2gk} \frac{\partial \zeta}{\partial \xi}\,. \tag{4.12}$$

Moreover, multiplying (3.35) by $\cosh[2k(z+h)]$ and integrating on $[-h;0]$ leads to

$$\int_{-h}^{0} \cosh[2k(z+h)] \frac{\partial u_{20}}{\partial t_1}(z)\, dz = -\frac{\partial \zeta}{\partial x_1} \frac{\sinh(2kh)}{2k}\,. \tag{4.13}$$

Combining (4.13) and (4.12) yields

$$\frac{1}{c_g}\left(\frac{c_g\omega}{2gk} + \frac{\sinh(2kh)}{2k}\right)\frac{\partial \zeta}{\partial \xi} = \frac{\partial \mu}{\partial \xi}\,.$$

It follows that

$$\mu = \frac{\zeta}{c_g}\left(\frac{c_g\omega}{2gk} + \frac{\sinh(2kh)}{2k}\right)\,.$$

Hence (4.11) reads

$$\omega\zeta + 2gk \int_{-h}^{0} \cosh[2k(z+h)] u_{20}(z)\, dz = \frac{2gk}{c_g}\left(\frac{c_g\omega}{2gk} + \frac{\sinh(2kh)}{2k}\right)\zeta\,,$$

and (3.34) becomes

$$\begin{aligned} 2\mathrm{i}\sinh(kh)\left(\frac{\partial A}{\partial t_2} + c_g \frac{\partial A}{\partial x_2}\right) + \sinh(kh)\frac{d^2\omega}{dk^2}\frac{\partial^2 A}{\partial \xi^2} + \frac{c_g \sinh(kh)}{k}\frac{\partial^2 A}{\partial y_1^2} \\ = \frac{2k^2}{c_g\cosh(kh)}\left(\frac{c_g\omega}{2gk} + \frac{\sinh(2kh)}{2k}\right)\zeta A - \nu |A|^2 A\,. \end{aligned} \tag{4.14}$$

Since ζ depends only on ξ, (4.6) yields

$$(gh - c_g^2)\frac{\partial^2 \zeta}{\partial x_1^2} + gh\frac{\partial^2 \zeta}{\partial y_1^2} = -c_g^2\left(1 + \frac{g\sinh(2kh)}{c_g\omega}\right)\frac{\partial^2 |A|^2}{\partial \xi^2}\,. \tag{4.15}$$

Equations (4.14) and (4.15) form the Davey–Stewartson system. The degenerate case $gh = c_g^2$, which is shown by the dashed line in figure 1, corresponds to the well-known

Figure 1: Resonances in the capillary–gravity water-wave problem: the solid line corresponds to the newly discovered singularity given by equation (4.10), the dotted line corresponds to the second-harmonic resonance, the dashed line corresponds to the long-wave short-wave resonance.

short-wave long-wave resonance first noted by Djordjevic & Redekopp (1977). Note that one cannot tell whether u_{20} and v_{20} depend on z or not. This point is important since the curl of the velocity vector field up to the second order is

$$\epsilon^2 \begin{pmatrix} -\frac{\partial v_{20}}{\partial z} \\ \frac{\partial u_{20}}{\partial z} \\ 0 \end{pmatrix} . \tag{4.16}$$

Therefore, we have a rotational solution of the Euler equations, but its amplitude equation is the same as for an irrotational motion.

5 Conclusion

Although the water-wave problem is governed by the Euler equations which allow for *rotational* motions, up to now the problem of stability was only addressed in their irrotational (i.e. potential) version. Besides analytical reasons (i.e. the latter are simpler to handle than the former), we believe that this fact may be explained by the following property. The linearization of the rotational Euler equations around the state of rest (see (2.8)–(2.12)) shows that the rotational component of the perturbation (i.e. $\nabla \times \underline{\tilde{V}}$) is steady since $\nabla \times \underline{\tilde{V}}_t \equiv 0$. Since we deal with a *linear* problem, one can split the perturbation $\underline{\tilde{V}}$ into two parts: $\underline{\tilde{V}} = \underline{\tilde{V}}_{rot} + \underline{\tilde{V}}_{pot}$ where $\nabla \times \underline{\tilde{V}}_{pot} = 0$ and $\underline{\tilde{V}}_{pot}$ satisfies the linear equation which is obtained by linearizing the potential Euler equations. Stability or instability is then decided independently on $\underline{\tilde{V}}_{rot}$ and $\underline{\tilde{V}}_{pot}$, but since $\underline{\tilde{V}}_{rot}$ does not evolve, one is led to dealing only with $\underline{\tilde{V}}_{pot}$. This investigation is precisely the problem of stability of the state of rest in the context of potential flows.

The linear stability for the rotational problem is decided on the dispersion relation and indeed the state of rest is linearly stable. Since it is not linearly asymptotically stable, nonlinear stability is not decided at this stage. We have developed in Section 2 the study of nonlinear interactions of small amplitude waves. This study can be performed either on the potential or on the rotational version of the Euler equations. Obviously, if instability against potential perturbations is shown, instability against general (rotational) perturbations is demonstrated. In general, the converse is not true. Our analysis (Section 3), which is devoted to the rotational case, leads to the *same* amplitude equation as for the potential case and therefore establishing that stability indeed does not depend on the irrotationality assumption. However, the velocity of the flow is not potential (see (4.16)). Whether such an effect might be observed experimentally is not clear, because in any experiment vorticity is introduced at the solid boundaries and at the free surface by viscous and surface-film effects. These boundary contributions might obscure the contribution from the wave's velocity field.

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