On modulations of weakly nonlinear water waves

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Abstract

The authors recently studied rotational effects on the modulations of water waves (Colin, Dias & Ghidaglia [1995]). In the process of deriving the amplitude equations, they found a new singularity for which the derivation failed. In this note, the nature of this singularity is discussed. Moreover, it is shown that the well-known resonant interaction between a short capillary–gravity wave and a long gravity wave is present only for one-dimensional modulations.

1 Introduction

The authors recently studied the modulational stability of Stokes' waves in the context of rotational free-surface Euler equations (Colin, Dias & Ghidaglia [1995], referred to as CDG below). Their primary interest was to study rotational effects on the stability of water waves and we refer to their conclusion (Section 5 of CDG) for a discussion of these effects. However, in the process of deriving the amplitude equations, they found a new singularity for which the derivation failed. This singularity is not related to the introduction of rotational effects and is also present in the classical potential case, although it was never mentioned before. In this note, the nature of this singularity is discussed. The motivation for this discussion was raised by Pierce [1995], who believes that this singularity is artificial because we neglected the second order homogeneous solutions. The singularity appeared as we were trying to show (and not to assume, as is usually done) that the mean flow depends on the slow space and time variables x_1 and t_1 through $(x_1 - c_g t_1)$, where c_g is the group velocity. In this note, we show that indeed one can neglect the second order homogeneous solutions. Therefore, our proof that the mean flow only depends on $(x_1 - c_g t_1)$ still holds away from the singular case. However, the singularity found previously disappears if one takes the average of the amplitude equations with respect to $(x_1 + c_g t_1)$. In other words, the Davey–Stewartson equations are still valid in the singular case, but the mean flow must be replaced by its average value with respect to $(x_1 + c_g t_1)$.

In this note, we also show that the well-known resonant interaction between a short capillary–gravity wave and a long gravity wave is present only for one-dimensional modulations.

2 Formulation of the problem

In this section, we recall the problem and the ansatz used in order to derive the Davey–Stewartson equations. We work in the general framework of rotational motions, but, as mentioned in the introduction, the questions we want to address are independent of the fact that the flow is rotational or not.

The three-dimensional Euler equations describing the motion of an inviscid and incompressible liquid layer of mean depth h read as follows. Denoting by $\Omega(t)$ the volume occupied at time t by the liquid,

$$\Omega(t) = \left\{ X = (x, y, z) \in \mathbb{R}^3, -h \leq z \leq \eta(x, y, t) \right\} , \tag{2.1}$$

and by $\underline{V} = (u, v, w)$ the velocity vector field, we have

$$\underline{V}_t + (\underline{V} \cdot \nabla)\underline{V} + \nabla p = \underline{g} \quad \text{in } \ \Omega(t) , \tag{2.2}$$

$$\nabla \cdot \underline{V} = 0 \quad \text{in } \ \Omega(t) , \tag{2.3}$$

together with the boundary conditions

$$\begin{aligned} w &= 0 , \quad \text{for } \ z = -h , & (2.4)\\ w &= \frac{\partial \eta}{\partial t} + u \frac{\partial \eta}{\partial x} + v \frac{\partial \eta}{\partial y} , \quad \text{for } \ z = \eta(x, y, t) , & (2.5)\\ p - p_0 &= -T\kappa , \quad \text{for } \ z = \eta(x, y, t) , & (2.6) \end{aligned}$$

where $\eta(x, y, t)$ is the elevation of the free surface, κ its mean curvature:

$$\kappa = \frac{\eta_{xx}(1 + \eta_y^2) - 2\eta_x\eta_y\eta_{xy} + \eta_{yy}(1 + \eta_x^2)}{(1 + \eta_x^2 + \eta_y^2)^{3/2}} , \tag{2.7}$$

p the pressure, p_0 the pressure above the liquid, g the acceleration due to gravity and T the surface tension per unit density of the liquid.

The state of rest ($\underline{V} = 0$, $p = p_0 - gz$, $\eta = 0$) is solution to (2.2)–(2.6). Its linear stability is governed by the following set of equations on the strip

$$S = \left\{ X = (x, y, z) \in \mathbb{R}^3, -h \leq z \leq 0 \right\} :$$

$$\begin{aligned} \tilde{\underline{V}}_t + \nabla \tilde{p} &= 0 , & (2.8)\\ \nabla \cdot \tilde{\underline{V}} &= 0 , & (2.9)\\ \tilde{w} &= 0 , \quad \text{for } \ z = -h , & (2.10)\\ \tilde{w} &= \frac{\partial \tilde{\eta}}{\partial t} , \quad \text{for } \ z = 0 , & (2.11)\\ -g\tilde{\eta} + \tilde{p} &= -T\Delta\tilde{\eta} , \quad \left(\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) , \quad \text{for } \ z = 0 , & (2.12) \end{aligned}$$

where

$$\underline{V} = \tilde{\underline{V}} + \cdots, p = p_0 - gz + \tilde{p} + \cdots, w = \tilde{w} + \cdots, \eta = \tilde{\eta} + \cdots .$$

The existence of plane wave solutions to (2.8)–(2.12) of the form

$$(\tilde{\underline{V}}, \tilde{p}, \tilde{w}, \tilde{\eta}) = \mathrm{e}^{i(kx-\omega t)}(\hat{V}(z), \hat{p}(z), \hat{w}(z), \hat{\eta}) \tag{2.13}$$

is decided upon the dispersion relation

$$\omega^2 = k(g + Tk^2)\tanh(kh)\,. \tag{2.14}$$

Hence solutions to (2.8)–(2.12) are bounded and the state of rest is linearly stable. It is not asymptotically stable; therefore the study of the nonlinear stability of the state of rest cannot be decided at this stage.

In order to address the problem of nonlinear interactions of small amplitude waves, we assume that the initial data for (2.2)–(2.6) have the following form:

$$\underline{V}(t=0) = \sum_{n=1}^{3} \epsilon^n \underline{V}_n^0(x, \epsilon x, \epsilon^2 x, \epsilon y, \epsilon^2 y, z) + o(\epsilon^3)\,, \tag{2.15}$$

$$\eta(t=0) = \sum_{n=1}^{3} \epsilon^n \eta_n^0(x, \epsilon x, \epsilon^2 x, \epsilon y, \epsilon^2 y) + o(\epsilon^3)\,. \tag{2.16}$$

That is, the initial data are of size ϵ and depend also on the slow variables $(\epsilon x, \epsilon^2 x, \epsilon y, \epsilon^2 y)$. At this point we make the following ansatz on $\underline{V}$, p and η. For that, it is convenient to introduce several temporal scales: $t_0 = t, t_1 = \epsilon t, t_2 = \epsilon^2 t$ and we assume that $\underline{V}, \eta, p$ have uniformly valid asymptotic expansions in terms of ϵ:

$$\underline{V} = \sum_{n=1}^{3} \epsilon^n \underline{V}_n(x_0, x_1, x_2, y_1, y_2, z; t_0, t_1, t_2) + o(\epsilon^3)\,, \tag{2.17}$$

$$p = p_0 - gz + \sum_{n=1}^{3} \epsilon^n p_n(x_0, x_1, x_2, y_1, y_2, z; t_0, t_1, t_2) + o(\epsilon^3)\,, \tag{2.18}$$

$$\eta = \sum_{n=1}^{3} \epsilon^n \eta_n(x_0, x_1, x_2, y_1, y_2; t_0, t_1, t_2) + o(\epsilon^3)\,, \tag{2.19}$$

where $\underline{V} = (u, v, w)$, $\underline{V}_n = (u_n, v_n, w_n)$ and

$$x_0 = x,\ x_1 = \epsilon x,\ x_2 = \epsilon^2 x,\ y_1 = \epsilon y,\ y_2 = \epsilon^2 y, \tag{2.20}$$

denote the spatial scales introduced in (2.15). In the next section, we will apply the derivative expansion method to this problem (Davey & Stewartson [1974], Kawahara [1975], Djordjevic & Redekopp [1977]).

3 Three-dimensional flows: amplitude equations

In order to obtain the amplitude equations, we substitute the ansatz (2.17)–(2.19) into (2.2)–(2.6) and we expand (2.5)–(2.6) around $z = 0$. Equating coefficients of equal powers of ϵ yields three sets of equations up to the third order.

As solutions to the first order linear equations, we take the plane waves introduced in Section 2. More precisely:

$$
\begin{aligned}
u_1 &= \mathrm{i}A(x_1,x_2,y_1,y_2;t_1,t_2)\cosh[k(z+h)]\mathrm{e}^{i\theta} + \text{c.c.}\,, & (3.1)\\
v_1 &= 0\,, & (3.2)\\
w_1 &= A(x_1,x_2,y_1,y_2;t_1,t_2)\sinh[k(z+h)]\mathrm{e}^{i\theta} + \text{c.c.}\,, & (3.3)\\
p_1 &= \frac{\mathrm{i}\omega}{k}A(x_1,x_2,y_1,y_2;t_1,t_2)\cosh[k(z+h)]\mathrm{e}^{i\theta} + \text{c.c.}\,, & (3.4)\\
\eta_1 &= \frac{\mathrm{i}}{\omega}A(x_1,x_2,y_1,y_2;t_1,t_2)\sinh(kh)\mathrm{e}^{i\theta} + \text{c.c.}\,, & (3.5)
\end{aligned}
$$

where $\theta = kx_0 - \omega t_0$ and ω satisfies the dispersion relation (2.14)

$$\omega^2 = k(g+Tk^2)\tanh(kh)\,,$$

and c.c. denotes the complex conjugate of the preceding terms. The complex amplitude A depends only upon the slow scales x_1, x_2, y_1, y_2, t_1 and t_2 and the aim of the expansion is to determine the differential equation satisfied by A. We now substitute this first order solution (3.1)–(3.5) into the second order equations and we look for a solution in the form

$$
\begin{aligned}
\underline{V}_2 &= \mathrm{e}^{2i\theta}\underline{V}_{22}(x_1,x_2,y_1,y_2,z;t_1,t_2) + \mathrm{e}^{i\theta}\underline{V}_{21}(x_1,x_2,y_1,y_2,z;t_1,t_2) + \text{c.c.}\\
&\quad + \underline{V}_{20}(x_1,x_2,y_1,y_2,z;t_1,t_2)\,, & (3.6)\\
p_2 &= \mathrm{e}^{2i\theta}p_{22}(x_1,x_2,y_1,y_2,z;t_1,t_2) + \mathrm{e}^{i\theta}p_{21}(x_1,x_2,y_1,y_2,z;t_1,t_2) + \text{c.c.}\\
&\quad + p_{20}(x_1,x_2,y_1,y_2,z;t_1,t_2)\,, & (3.7)\\
\eta_2 &= \mathrm{e}^{2i\theta}\eta_{22}(x_1,x_2,y_1,y_2;t_1,t_2) + \mathrm{e}^{i\theta}\eta_{21}(x_1,x_2,y_1,y_2;t_1,t_2) + \text{c.c.}\\
&\quad + \eta_{20}(x_1,x_2,y_1,y_2;t_1,t_2)\,. & (3.8)
\end{aligned}
$$

One finds for the coefficient of $(\mathrm{e}^{i\theta})^0$:

$$
\left\{
\begin{aligned}
p_{20} &= -|A|^2\cosh[2k(z+h)] + \zeta(x_1,x_2,y_1,y_2;t_1,t_2)\,,\\
w_{20} &= 0\,,\\
\eta_{20} &= \tfrac{1}{g}(\zeta - |A|^2)\,.
\end{aligned}
\right. \tag{3.9}
$$

For the term in $(\mathrm{e}^{i\theta})^1$, one obtains

$$
\left\{
\begin{aligned}
u_{21} &= (z+h)\sinh[k(z+h)]\tfrac{\partial A}{\partial x_1}\,,\\
v_{21} &= \tfrac{1}{k}\cosh[k(z+h)]\tfrac{\partial A}{\partial y_1}\,,\\
w_{21} &= -\mathrm{i}(z+h)\cosh[k(z+h)]\tfrac{\partial A}{\partial x_1}\,,\\
p_{21} &= \tfrac{\omega}{k}\Big((z+h)\sinh[k(z+h)]\tfrac{\partial A}{\partial x_1} - \tfrac{1}{k}\cosh[k(z+h)]\tfrac{\partial A}{\partial x_1} - \tfrac{1}{\omega}\cosh[k(z+h)]\tfrac{\partial A}{\partial t_1}\Big)\,,\\
\eta_{21} &= \tfrac{1}{\omega}\Big(h\cosh(kh)\tfrac{\partial A}{\partial x_1} + \tfrac{\sinh(kh)}{\omega}\tfrac{\partial A}{\partial t_1}\Big)\,,
\end{aligned}
\right. \tag{3.10}
$$

together with the non-secularity condition

$$\frac{\partial A}{\partial t_1} + c_g\frac{\partial A}{\partial x_1} = 0\,, \tag{3.11}$$

where c_g denotes the group velocity $d\omega/dk$. Moreover the higher harmonic terms are given by

$$\left\{\begin{array}{rcl} u_{22} & = & -\mu A^2\cosh[2k(z+h)]\,, \\ v_{22} & = & 0\,, \\ w_{22} & = & \mathrm{i}\mu A^2\sinh[2k(z+h)]\,, \\ p_{22} & = & -\mu\frac{\omega}{k}A^2\cosh[2k(z+h)]+\frac{1}{2}A^2\,, \\ \eta_{22} & = & -\frac{k}{2\omega^2}A^2\sinh(2kh)-\mu\frac{1}{2\omega}A^2\sinh(2kh)\,, \end{array}\right. \tag{3.12}$$

where

$$\mu=\frac{3k}{2\omega}\frac{g(1-\sigma^2)+Tk^2(3-\sigma^2)}{g\sigma^2+(\sigma^2-3)Tk^2}\,,\quad \sigma=\tanh(kh)\,.$$

The denominator of μ vanishes when $g\sigma^2+(\sigma^2-3)Tk^2=0$, which corresponds to the well-known second-harmonic resonance.

Note that, at this stage, one could add to the above given solution $(u_{21},v_{21},w_{21},p_{21},\eta_{21})$ an homogeneous solution (Pierce [1995]), i.e. terms like

$$\begin{array}{c} \mathrm{i}D(x_1,x_2,y_1,y_2;t_1,t_2)\cosh[k(z+h)]\,, \\ 0\,, \\ D(x_1,x_2,y_1,y_2;t_1,t_2)\sinh[k(z+h)]\,, \\ \frac{\mathrm{i}\omega}{k}D(x_1,x_2,y_1,y_2;t_1,t_2)\cosh[k(z+h)]\,, \\ \frac{\mathrm{i}}{\omega}D(x_1,x_2,y_1,y_2;t_1,t_2)\sinh(kh)\,. \end{array} \tag{3.13}$$

In CDG, we did not add these terms; however, the parameter ϵ that was introduced in order to apply the derivative expansion method is artificial and has no precise physical meaning. It is therefore possible to use $\tilde{A}=A+\epsilon D$ as a new amplitude. If we use this new amplitude, the non-secularity condition obtained for the term $e^{\mathrm{i}\theta}$ of the second order solution yields the equivalent of (3.11), that is

$$\frac{\partial\tilde{A}}{\partial t_1}+c_g\frac{\partial\tilde{A}}{\partial x_1}=0\,.$$

This last equation yields with (3.11)

$$\frac{\partial D}{\partial t_1}+c_g\frac{\partial D}{\partial x_1}=0\,. \tag{3.14}$$

Note that Djordjevic & Redekopp [1977] take into account the second order homogeneous solution, but they assume *a priori* that D satisfies (3.14). Again, we show that this is not an hypothesis.

Introducing (3.12) into the third order perturbation equations yields the following set of amplitude equations:

$$2\mathrm{i}\sinh(kh)\left(\frac{\partial A}{\partial t_2}+c_g\frac{\partial A}{\partial x_2}\right)+\sinh(kh)\frac{d^2\omega}{dk^2}\frac{\partial^2 A}{\partial\xi^2}+c_g\frac{\sinh(kh)}{k}\frac{\partial^2 A}{\partial y_1^2}$$

$$=\frac{k\omega}{g\cosh(kh)}A\zeta+\frac{2k^2A}{\cosh(kh)}\int_{-h}^{0}\cosh[2k(z+h)]u_{20}(z)\,dz-\nu|A|^2A\,, \tag{3.15}$$

$$
\begin{aligned}
\frac{\partial \zeta}{\partial x_1} + \frac{\partial u_{20}}{\partial t_1} &= 0 \quad \text{in } S\,, && (3.16)\\
\frac{\partial \zeta}{\partial y_1} + \frac{\partial v_{20}}{\partial t_1} &= 0 \quad \text{in } S\,, && (3.17)\\
\frac{\partial w_{30}}{\partial z} + \frac{\partial v_{20}}{\partial y_1} + \frac{\partial u_{20}}{\partial x_1} &= 0 \quad \text{in } S\,, && (3.18)\\
w_{30} &= 0 \quad \text{at } z=-h\,, && (3.19)\\
w_{30} - \frac{1}{g}\left(c_g \frac{\partial |A|^2}{\partial \xi} + \frac{\partial \zeta}{\partial t_1}\right) - \frac{\sinh(2kh)}{\omega}\frac{\partial |A|^2}{\partial \xi} &= 0 \quad \text{at } z=0\,, && (3.20)
\end{aligned}
$$

and A depends on x_1 and t_1 only through $\xi = x_1 - c_g t_1$. Note that there is a contribution of D to equation (3.15), and this contribution is

$$
2\mathrm{i}\sinh(kh)\left(\frac{\partial D}{\partial t_1} + c_g\frac{\partial D}{\partial x_1}\right),
$$

which is zero according to (3.14).

The coefficient ν is given by

$$
\begin{aligned}
\nu \;=\; & \tfrac{1}{2}k^3\sigma^2\Big[g^3(9-12\sigma^2+13\sigma^4-2\sigma^6) + k^2g^2(36-62\sigma^2+33\sigma^4-6\sigma^6)T \\
& + gk^4(33-55\sigma^2+30\sigma^4-6\sigma^6)T^2 + k^6(6-14\sigma^2+10\sigma^4-2\sigma^6)T^3\Big]\cosh(kh)\times \\
& \times[g\omega^3(\sigma^2-1)(g\sigma^2-3Tk^2+k^2\sigma^2T)]^{-1}\,.
\end{aligned}
$$

This value of ν corresponds to that given in Djordjevic & Redekopp [1977] after a correction by a multiplication factor due to the fact that we do not take the same linear solution as they do. The set of equations (3.15)–(3.20) governs the modulations of a small amplitude wave train.

Let us conclude this section by saying that the introduction of the term (3.13) does not have any influence on the amplitude equations (3.15)–(3.20).

The next step is to simplify this system and to clarify the situation concerning the singularity found in CDG.

4 On the singularity $gh = \frac{g^2\sinh^2(2kh)}{\omega^2}$.

For the convenience of the reader, we recall here how the Davey–Stewartson equations were derived from (3.15)–(3.20) in CDG, and how the new singularity appeared. We will show that the functions ζ and $\int_{-h}^0 \cosh[2k(z+h)]u_{20}(z)\,dz$ depend on x_1 and t_1 only through $\xi = x_1 - c_g t_1$, except for a singular case. We first apply the differential operator

$$
\left(\frac{\partial}{\partial t_1} + c_g\frac{\partial}{\partial x_1}\right)\frac{\partial}{\partial t_1}
$$

on (3.15) and the fact that $\frac{\partial A}{\partial t_1} + c_g\frac{\partial A}{\partial x_1} = 0$ yields

$$
k\omega\left(\frac{\partial}{\partial t_1} + c_g\frac{\partial}{\partial x_1}\right)\frac{\partial \zeta}{\partial t_1} + 2gk^2\int_{-h}^{0}\left(\frac{\partial}{\partial t_1} + c_g\frac{\partial}{\partial x_1}\right)\frac{\partial u_{20}}{\partial t_1}\cosh[2k(z+h)]\,dz = 0\,. \qquad (4.1)
$$

Equation (3.16) gives

$$\frac{\partial \zeta}{\partial x_1} = -\frac{\partial u_{20}}{\partial t_1}\,.$$

Therefore, (4.1) leads to

$$\left(\frac{\partial}{\partial t_1} + c_g \frac{\partial}{\partial x_1}\right)\left(\frac{\partial}{\partial t_1} - \frac{g\sinh(2kh)}{\omega}\frac{\partial}{\partial x_1}\right)\zeta = 0\,.$$

Hence ζ can be written in the form

$$\zeta = f_1(x_1 - c_g t_1, y_1) + f_2(x_1 - \beta t_1, y_1)\,, \tag{4.2}$$

where $\beta = -g\sinh(2kh)/\omega$ and f_1, f_2 are arbitrary functions. We have dropped the dependences in x_2, y_2, t_2 which are considered as parameters in this calculation. On the other hand, (3.16) and (3.17) imply

$$\frac{\partial^2 \zeta}{\partial x_1^2} = -\frac{\partial^2 u_{20}}{\partial t_1 \partial x_1} \quad \text{and} \quad \frac{\partial^2 \zeta}{\partial y_1^2} = -\frac{\partial^2 v_{20}}{\partial t_1 \partial y_1}\,, \tag{4.3}$$

while $\frac{\partial(3.18)}{\partial t_1}$ gives

$$\frac{\partial^2 w_{30}}{\partial z \partial t_1} = -\frac{\partial^2 v_{20}}{\partial t_1 \partial y_1} - \frac{\partial^2 u_{20}}{\partial t_1 \partial x_1}\,. \tag{4.4}$$

Equations (4.3) and (4.4) lead to

$$\frac{\partial^2 w_{30}}{\partial t_1 \partial z} = \Delta_{x_1,y_1}\zeta\,.$$

The boundary condition (3.19) yields

$$\frac{\partial w_{30}}{\partial t_1} = (z+h)\Delta_{x_1,y_1}\zeta\,, \tag{4.5}$$

since ζ does not depend on z. We substitute (4.5) in $\frac{\partial(3.20)}{\partial t_1}$:

$$h\Delta_{x_1,y_1}\zeta - \frac{1}{g}\left(-c_g^2\frac{\partial^2 |A|^2}{\partial \xi^2} + \frac{\partial^2 \zeta}{\partial t_1^2}\right) + \frac{\sinh(2kh)}{\omega}c_g\frac{\partial^2 |A|^2}{\partial \xi^2} = 0\,. \tag{4.6}$$

Combining (4.2) and (4.6) and denoting by Ξ the quantity $x_1 - \beta t_1$ yields

$$\begin{aligned} & h\frac{\partial^2 f_1}{\partial \xi^2} + h\frac{\partial^2 f_1}{\partial y_1^2} + h\frac{\partial^2 f_2}{\partial \Xi^2} + h\frac{\partial^2 f_2}{\partial y_1^2} + \frac{c_g^2}{g}\frac{\partial^2 |A|^2}{\partial \xi^2} \\ & -\frac{c_g^2}{g}\frac{\partial^2 f_1}{\partial \xi^2} - \frac{\beta^2}{g}\frac{\partial^2 f_2}{\partial \Xi^2} + \frac{c_g\sinh(2kh)}{\omega}\frac{\partial^2 |A|^2}{\partial \xi^2} \quad = \quad 0\,. \end{aligned} \tag{4.7}$$

It follows that

$$\frac{\partial^2 f_2}{\partial \Xi^2}\left(h - \frac{\beta^2}{g}\right) + h\frac{\partial^2 f_2}{\partial y_1^2} = \alpha(y_1)\,. \tag{4.8}$$

Since $\alpha(y_1) \to 0$ as $\Xi \to \infty$, it follows that $\alpha(y_1) \equiv 0$ and (4.8) reads

$$\frac{\partial^2 f_2}{\partial \Xi^2}\left(1 - \frac{\beta^2}{gh}\right) + \frac{\partial^2 f_2}{\partial y_1^2} = 0\,. \tag{4.9}$$

Three cases can occur:

(i) <u>If $gh - \beta^2 > 0$</u>: taking the Fourier transform in Ξ, y_1, we obtain that f_2 has to be linear in Ξ, y_1 and since f_2 tends to zero at infinity, $f_2 \equiv 0$.

(ii) <u>If $gh - \beta^2 < 0$</u>: f_2 satisfies a wave equation in the variables Ξ, y_1, hence it can be written as

$$f_2 = f_{2+}(\Xi + \mathcal{C}y_1) + f_{2-}(\Xi - \mathcal{C}y_1)\,,$$

where $\mathcal{C} = \sqrt{\beta^2/gh - 1}$. It follows that $f_{2+} = f_{2-} = 0$, since f_2 has to decay at infinity.

(iii) <u>If $gh - \beta^2 = 0$</u>: it is a singular case and one cannot conclude. This singularity occurs when

$$gh = \frac{g^2 \sinh^2(2kh)}{\omega^2}\,. \tag{4.10}$$

In both cases (i) and (ii), we obtain that ζ depends on x_1 and t_1 only through ξ. Now, (3.15) implies that

$$k\omega\zeta + 2gk^2 \int_{-h}^{0} \cosh[2k(z+h)]u_{20}\,dz$$

depends on x_1 and t_1 only through ξ and we denote its value by $2gk^2\mu(\xi, y_1)$. Hence

$$\int_{-h}^{0} \cosh[2k(z+h)]u_{20}\,dz = \mu(\xi, y_1) - \frac{\omega}{2gk}\zeta\,. \tag{4.11}$$

Differentiating (4.11) with respect to t_1 leads to

$$\int_{-h}^{0} \cosh[2k(z+h)]\frac{\partial u_{20}}{\partial t_1}(z)\,dz = -c_g\frac{\partial \mu}{\partial \xi} + \frac{c_g\omega}{2gk}\frac{\partial \zeta}{\partial \xi}\,. \tag{4.12}$$

Moreover, multiplying (3.16) by $\cosh[2k(z+h)]$ and integrating on $[-h;0]$ leads to

$$\int_{-h}^{0} \cosh[2k(z+h)]\frac{\partial u_{20}}{\partial t_1}(z)\,dz = -\frac{\partial \zeta}{\partial x_1}\frac{\sinh(2kh)}{2k}\,. \tag{4.13}$$

Combining (4.13) and (4.12) yields

$$\frac{1}{c_g}\left(\frac{c_g\omega}{2gk} + \frac{\sinh(2kh)}{2k}\right)\frac{\partial \zeta}{\partial \xi} = \frac{\partial \mu}{\partial \xi}\,.$$

It follows that

$$\mu = \frac{\zeta}{c_g}\left(\frac{c_g\omega}{2gk} + \frac{\sinh(2kh)}{2k}\right)\,.$$

Hence (4.11) reads

$$\omega\zeta + 2gk\int_{-h}^{0} \cosh[2k(z+h)]u_{20}(z)\,dz = \frac{2gk}{c_g}\left(\frac{c_g\omega}{2gk} + \frac{\sinh(2kh)}{2k}\right)\zeta\,,$$

and (3.15) becomes

$$2\mathrm{i}\sinh(kh)\left(\frac{\partial A}{\partial t_2}+c_g\frac{\partial A}{\partial x_2}\right)+\sinh(kh)\frac{d^2\omega}{dk^2}\frac{\partial^2 A}{\partial \xi^2}+\frac{c_g\sinh(kh)}{k}\frac{\partial^2 A}{\partial y_1^2}$$

$$=\frac{2k^2}{c_g\cosh(kh)}\left(\frac{c_g\omega}{2gk}+\frac{\sinh(2kh)}{2k}\right)\zeta A-\nu|A|^2A\,. \tag{4.14}$$

and

$$(gh-c_g^2)\frac{\partial^2\zeta}{\partial x_1^2}+gh\frac{\partial^2\zeta}{\partial y_1^2}=-c_g^2\left(1+\frac{g\sinh(2kh)}{c_g\omega}\right)\frac{\partial^2|A|^2}{\partial\xi^2}\,. \tag{4.15}$$

Equations (4.14) and (4.15) form the Davey–Stewartson system.

Concerning the new singularity that we have found, we notice that we can proceed following the idea of Pierce & Knobloch [1994]. Let us introduce the variable

$$\xi_1=x_1+c_gt_1.$$

Using ξ and ξ_1 instead of x_1 and t_1, and taking the mean value of equations (3.15)-(3.20) with respect to ξ_1 yields the following set of equations:

$$2\mathrm{i}\sinh(kh)\left(\frac{\partial A}{\partial t_2}+c_g\frac{\partial A}{\partial x_2}\right)+\sinh(kh)\frac{d^2\omega}{dk^2}\frac{\partial^2 A}{\partial \xi^2}+c_g\frac{\sinh(kh)}{k}\frac{\partial^2 A}{\partial y_1^2}$$

$$=\frac{k\omega}{g\cosh(kh)}A\langle\zeta\rangle+\frac{2k^2A}{\cosh(kh)}\int_{-h}^{0}\cosh[2k(z+h)]\langle u_{20}\rangle(z)\,dz-\nu|A|^2A\,, \tag{4.16}$$

$$\begin{aligned}
\frac{\partial\langle\zeta\rangle}{\partial\xi}-c_g\frac{\partial\langle u_{20}\rangle}{\partial\xi} &= 0 \quad\text{in } S\,, && (4.17)\\
\frac{\partial\langle\zeta\rangle}{\partial y_1}-c_g\frac{\partial\langle v_{20}\rangle}{\partial\xi} &= 0 \quad\text{in } S\,, && (4.18)\\
\frac{\partial\langle w_{30}\rangle}{\partial z}+\frac{\partial\langle v_{20}\rangle}{\partial y_1}+\frac{\partial\langle u_{20}\rangle}{\partial\xi} &= 0 \quad\text{in } S\,, && (4.19)\\
\langle w_{30}\rangle &= 0 \quad\text{at } z=-h\,, && (4.20)\\
\langle w_{30}\rangle-\frac{1}{g}\left(c_g\frac{\partial|A|^2}{\partial\xi}-c_g\frac{\partial\langle\zeta\rangle}{\partial\xi}\right)-\frac{\sinh(2kh)}{\omega}\frac{\partial|A|^2}{\partial\xi} &= 0 \quad\text{at } z=0\,, && (4.21)
\end{aligned}$$

where $\langle . \rangle$ denotes the mean value in the ξ_1 component. It is easy to see that the same proof as above shows that A and $\langle\zeta\rangle$ satisfies the D–S equations.

In order to conclude these computations, let us note that in the case when $gh\neq\frac{g^2\sinh^2(2kh)}{\omega^2}$, we know that $\langle\zeta\rangle=\zeta$, and therefore A and ζ satisfy the D–S equations (4.14)-(4.15). But when $gh=\frac{g^2\sinh^2(2kh)}{\omega^2}$, we do not know whether $\langle\zeta\rangle=\zeta$ or not; however, A and $\langle\zeta\rangle$ still satisfy the D–S equations, i.e.

$$2\mathrm{i}\sinh(kh)\left(\frac{\partial A}{\partial t_2}+c_g\frac{\partial A}{\partial x_2}\right)+\sinh(kh)\frac{d^2\omega}{dk^2}\frac{\partial^2 A}{\partial \xi^2}+\frac{c_g\sinh(kh)}{k}\frac{\partial^2 A}{\partial y_1^2}$$

$$= \frac{2k^2}{c_g \cosh(kh)} \left(\frac{c_g \omega}{2gk} + \frac{\sinh(2kh)}{2k} \right) \langle \zeta \rangle A - \nu |A|^2 A \,. \tag{4.22}$$

Since ζ depends only on ξ, (4.6) yields

$$(gh - c_g^2) \frac{\partial^2 \langle \zeta \rangle}{\partial x_1^2} + gh \frac{\partial^2 \langle \zeta \rangle}{\partial y_1^2} = -c_g^2 \left(1 + \frac{g \sinh(2kh)}{c_g \omega} \right) \frac{\partial^2 |A|^2}{\partial \xi^2} \,. \tag{4.23}$$

That is why, one can say that when $gh = \frac{g^2 \sinh^2(2kh)}{\omega^2}$ the system is not resonant, but it is singular since the analysis is different for this value of the parameters.

5 On the long wave-short wave resonance

Djordjevic & Redekopp [1977] noticed that when $c_g^2 = gh$ the modulation equations that they found are singular. However, the form of the Davey–Stewartson equations that we have found here (equations (4.14)-(4.15)) are not singular when $c_g^2 = gh$. Indeed, they become

$$2\mathrm{i} \sinh(kh) \left(\frac{\partial A}{\partial t_2} + c_g \frac{\partial A}{\partial x_2} \right) + \sinh(kh) \frac{d^2 \omega}{dk^2} \frac{\partial^2 A}{\partial \xi^2} + \frac{c_g \sinh(kh)}{k} \frac{\partial^2 A}{\partial y_1^2}$$

$$= \frac{2k^2}{c_g \cosh(kh)} \left(\frac{c_g \omega}{2gk} + \frac{\sinh(2kh)}{2k} \right) \zeta A - \nu |A|^2 A \,, \tag{5.1}$$

and

$$gh \frac{\partial^2 \zeta}{\partial y_1^2} = -c_g^2 \left(1 + \frac{g \sinh(2kh)}{c_g \omega} \right) \frac{\partial^2 |A|^2}{\partial \xi^2} \,. \tag{5.2}$$

These equations are similar to that obtained for pure capillary waves by Djordjevic and Redekopp [1977] (see their equations (A7) and (A8)), when exchanging the roles of the x and y variables. Therefore, there is no long wave-short wave resonance in 3D. That is why Grimshaw [1977], who studied the modulations of an internal gravity wave packet and considered the special case of the resonant interaction with a long wave, obtained in the 3D case the same amplitude equations for the resonance as that obtained by Djordjevic and Redekopp [1977] in the 2D resonant case! Of course, in the 2D case, when the functions do not depend on the variable y, equation (5.2) implies that $|A|^2$ does not depend on ξ, i.e. the train of wave is not modulated, and there is indeed a resonance. Therefore a different scaling must be introduced in order to study modulations near that resonance.

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