

# An initial-boundary-value problem for the Korteweg-de Vries equation posed on a finite interval

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*Résumé.* La propagation unidirectionnelle d'ondes de faible amplitude et de grande longueur d'onde est décrite, dans de nombreux systèmes physiques, par l'équation de Korteweg-de Vries. L'objet de ce travail est de proposer un problème mixte bien posé lorsque le domaine spatial est borné. Plus précisément nous établissons l'existence de solutions locales en temps pour des données initiales dans l'espace de Sobolev  $H^1$  ainsi que l'existence de solutions globales pour données initiales petites dans cet espace. Dans un second temps, nous mettons en évidence des effets régularisants globaux forts.

*Abstract.* The Korteweg-de Vries equation occurs as a model for unidirectional propagation of small amplitude long waves in numerous physical systems. The aim of this work is to propose a well posed mixed initial -boundary- values problem when the spacial domain is of finite extent. More precisely, we establish local existence of solutions for arbitrary initial data in the Sobolev space  $H^1$  and global existence for small initial data in this space. In a second step we show global strong regularizing effects.

## 1 Setting of the problem and main results

In 1895 Korteweg and de Vries [18] have introduced the equation that bears their names in the context of uni-directional water waves propagating in an infinite channel. These authors have considered a layer of incompressible fluid over a flat bottom and assumed that the flow was irrotational (see Colin-Dias and Ghidaglia [9] and [10] for rotational flows). Under the influence of gravity, the motion of surface waves leads to a system of p.d.e.'s in the variables  $\eta(x, t)$  and  $v(x, t)$  where  $\eta$  is the wave height measured from an undisturbed water level and  $v(x, t)$  is the spatial derivative of the restriction to the surface of the velocity potential. Considering waves with wavelength  $l$  and amplitude  $a$  which satisfy  $a = O(\varepsilon)$  and

$l = O(\varepsilon^{-1/2})$  (where  $\varepsilon$  denotes a small parameter) and in the hypothesis of uni-directional waves, one gets, at the leading order in  $\varepsilon$  and after scaling transforms the following equation for  $\eta$  :

$$\eta_t + h\eta_x + \frac{\varepsilon h^3}{12}\eta_{xxx} + \frac{3}{2}\eta\eta_x = 0. \quad (1.1)$$

Here  $h$  denotes the (finite) depth of the fluid. In the case of an infinite channel (i.e.  $x \in \mathbb{R}$ ) it is possible to perform further scaling and change in the independent variables  $x$  and  $t$  in order to obtain the classical form the KdV equation namely

$$u_t + uu_x + u_{xxx} = 0. \quad (1.2)$$

However in case of a domain which is not invariant by translation with respect to  $x$  the term in  $\eta_x$  cannot be removed and the relevant equation reads now as (still after scaling)

$$u_t + u_x + uu_x + u_{xxx} = 0. \quad (1.3)$$

In ([3]) and ([4]), Bona and Winther have considered the initial-boundary - value problem for (1.3) in the quarter plane  $\{(x, t), x > 0, t > 0\}$ . As boundary condition, they have prescribed the value of  $u$  at the boundary  $x = 0$ , namely

$$u(0, t) = g(t), t \geq 0. \quad (1.4)$$

As  $x$  goes to  $+\infty$ , Bona and Winther have imposed a kind of decay by looking for solutions which are continuous (*w.r.* to time) functions from  $[0, +\infty[$  into  $H^m(0, \infty)$ , the Sobolev-space of order  $m \geq 0$  constructed on  $L^2(0, \infty)$ .

In this paper we are interested in the case where (1.3) is posed on the strip  $]0, +\infty[ \times ]0, L[$  where  $L$  is a finite positive number. Namely we would like to model the case where a wave-maker is putting energy in a finite length channel from the left ( $x = 0$ ) while the right end ( $x = L$ ) of the channel is free.

As it well-known (1.3), when it is posed on the whole line, has an hamiltonian structure associated to the hamiltonian

$$H(u) = \frac{1}{2} \int_{-\infty}^{+\infty} (u_x^2 - u^2 - u^3/3) dx \quad (1.5)$$

(in fact the structure of (1.3) is even much richer and we refer to Olver [19] for further elements pertaining to the hamiltonian structure see also Ablowitz and Segur [1] concerning exact integrability of (1.3)).

On the other hand,  $\int_{-\infty}^{+\infty} u^2 dx$  is also conserved and the conservation of  $H(u)$  and  $\int_{-\infty}^{+\infty} u^2 dx$  are indeed sufficient for the proof of *a priori* estimates on the  $H^1$ -norm of  $u(\cdot, t)$  which lead to existence results (see Temam [20] and Bona and Smith [2]) in  $H^m$ -spaces for  $m \geq 1$ . The pure initial value problem for (1.3) has been very much investigated and sharp (and quite surprising) results have been obtained e.g. Kenig, Ponce and Vega [17] have proved local in time existence results for initial data in  $H^{-3/4+\varepsilon}(\mathbb{R})$ , a space of distribution ! Bourgain [5] has proven global existence results for small initial data in  $L^2(\mathbb{R})$ .

All these results, for data with very weak regularity, are based on local regularizing effects of the Airy equation and are possible because the problem is posed on  $\mathbb{R}$ .

Concerning the problem on a finite interval, we first try to construct solutions by the classical energy method. It is therefore necessary to consider the evolution of the two integrals  $\int_0^L u^2 dx$  and  $\int_0^L (u_x^2 - u^2 - u^3/3) dx$ . This leads us to propose two boundary conditions at  $x = L$  that will indeed allow us to obtain suitable *a priori* estimates that can be viewed as perturbations of  $\frac{d}{dt}H(u) = 0$  and  $\frac{d}{dt} \int_{-\infty}^{+\infty} u^2 dx = 0$ , see (2.10), (2.11), (3.10) and (3.11).

These boundary conditions read as

$$\frac{\partial u}{\partial x}(L, t) = h(t), \quad \frac{\partial^2 u}{\partial x^2}(L, t) = k(t), \quad t \geq 0. \quad (1.6)$$

In a second step we address the question of smoothing effects. These are not possible on a finite interval with periodic boundary condition since the problem is then time-reversible. In our case we do not have time reversibility since changing  $t$  into  $-t$  amounts to exchange the left and right boundaries. Actually we are able to show global smoothing effects and therefore construct solutions for  $L^2$  initial data.

The results in this paper concern the following IBVP for the KdV equation. Given  $g, h, k$  from  $[0, T[$  into  $\mathbb{R}$  and  $u_0$  from  $[0, L]$  into  $\mathbb{R}$ , find  $u$  from  $[0, L] \times [0, T[$  into  $\mathbb{R}$  such that

$$u_t + u_x + uu_x + u_{3x} = 0, \quad \text{for } (x, t) \in ]0, L[ \times ]0, T[, \quad (1.7)$$

$$u(0, t) = g(t), \quad \text{for } t \in [0, T[, \quad (1.8)$$

$$\frac{\partial u}{\partial x}(L, t) = h(t), \quad \text{for } t \in [0, T[, \quad (1.9)$$

$$\frac{\partial^2 u}{\partial x^2}(L, t) = k(t), \quad \text{for } t \in [0, T[, \quad (1.10)$$

$$u(x, 0) = u_0(x), \quad \text{for } x \in [0, L]. \quad (1.11)$$

Here  $L > 0$  and  $T \in ]0, +\infty]$ .

We are going to establish the following results.

**Theorem 1.1** (*Local existence*). *Let  $u_0 \in H^1(0, L)$  and  $g, h, k \in C^1([0, +\infty[)$  satisfy the compatibility condition  $u_0(0) = g(0)$ . Then there exists  $T > 0$  and a function  $u \in L^\infty(0, T; H^1(0, L)) \cap \mathcal{C}([0, T], L^2(0, L))$  with traces at  $x = L$  :  $u_x(L, \cdot) \in H^{-1}(0, T)$ ,  $u_{xx}(L, \cdot) \in H^{-2}(0, T)$  which solves (1.7) to (1.11) in the distribution sense and (1.11) for almost every  $x \in [0, L]$ .*

By imposing conditions on the size of the data  $u_0, g, h$  and  $k$  it is actually possible to show a global existence result. We refer to Section 3 for the precise meaning of the condition on the data (see (3.12) and (3.15)) and state :

**Theorem 1.2** (*Global existence*). *Let  $u_0, g, h, k$  be as in Theorem 1 and satisfy the smallness assumptions (3.12) and (3.15). Then we can take  $T = +\infty$  in the statement of Theorem 1.1.*

On the other hand, we show that the Airy equation :

$$\frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} = 0, \quad 0 < x < L, \quad t \geq 0,$$

$$u(0, t) = \frac{\partial u}{\partial x}(L, t) = \frac{\partial^2 u}{\partial x^2}(L, t) = 0, \quad t \geq 0$$

$$u(x, 0) = u_0(x), \quad 0 < x < L,$$

has some parabolic type smoothing effects :

$$\int_0^L x u^2(x, t) dt + 3 \int_0^t \int_0^L \left( \frac{\partial u}{\partial x} \right)^2(x, s) dx ds = \int_0^L x |u_0(x)|^2 dx.$$

Using this property, we prove :

**Theorem 1.3** (*Smoothing effects*). *Let  $u_0 \in L^2(0, L)$ , there exists a unique maximal weak solution  $u \in \mathcal{C}([0, T \max[; L^2) \cap L^2_{loc}([0, T \max[; H^1)$  to (1.7) - (1.11) with  $g = h = k \equiv 0$ .*

See definition 4.3 for the precise meaning of a weak solution to (1.7).

The paper is organized as follows. In the next section we address what we call the homogeneous case i.e. the case where  $h \equiv 0$  and  $k \equiv 0$  (but still  $g \not\equiv 0$ ). This allows us to prove Theorems 1.1 and 1.2 in details without too much technicalities. The method of the proof is based on the approach of Bona and Smith [2] namely the introduction of a regularized equation which has its own interest (see (2.41)). Then Section 3 is devoted to the proof of Theorems 1.1 and 1.2 in full generality. In Section 4 we prove global smoothing effects of the Airy equation and give the proof of Theorem 1.3.

Finally in Section 5 we list open questions and directions of further investigations.

We have announced in [11] and [12] the results of this work. Some technical proofs have been omitted but can be found in [13].

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## 2 The homogeneous case

In this Section we consider the case where  $k \equiv 0$  and  $h \equiv 0$  namely we address the following IBVP. Find  $u = u(x, t)$ ,  $x \in [0, L]$  and  $t \geq 0$  solution to

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} + u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0, \quad 0 < x < L, \quad t > 0, \quad (2.1)$$

$$u(0, t) = g(t), \quad t \geq 0, \quad (2.2)$$

$$\frac{\partial u}{\partial x}(L, t) = 0, \quad \frac{\partial^2 u}{\partial x^2}(L, t) = 0, \quad t \geq 0, \quad (2.3)$$

$$u(x, 0) = u_0(x), \quad 0 < x < L. \quad (2.4)$$

First, as usual, we will establish (in Section 2.1) *a priori* estimates on the solutions to (2.1)-(2.4). Then we will construct suitable approximations  $u^\varepsilon$  to  $u$  for  $\varepsilon > 0$ ,  $\varepsilon$  small (Section 2.2) via a regularized equation. Finally Section 2.3 will be devoted to the construction of a solution to (2.1)-(2.4) by letting  $\varepsilon$  going to zero.

## 2.1 Two a priori estimates

In this section, we do not care about regularity and perform formal estimates on the solutions to (2.1) - (2.4). More precisely we assume that  $u$  (and therefore  $u_0$  and  $g$ ) are so smooth that all the computations made hereafter are justified.

Let  $v$  be defined by

$$v(x, t) := u(x, t) - g(t). \quad (2.5)$$

Then (2.1) - (2.4) reads as

$$\frac{\partial v}{\partial t} + (1 + g) \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial x} + \frac{\partial^3 v}{\partial x^3} = -\frac{dg}{dt}, \quad (2.6)$$

$$v(0, t) = 0, \quad t \geq 0, \quad (2.7)$$

$$\frac{\partial v}{\partial x}(L, t) = 0, \quad \frac{\partial^2 v}{\partial x^2}(L, t) = 0, \quad t \geq 0, \quad (2.8)$$

$$v(x, 0) = v_0(x) \equiv u_0(x) - g(0), \quad 0 < x < L. \quad (2.9)$$

We multiply (2.6) by  $v$  and integrate on  $[0, L]$  the resulting identity and, after some integrations by parts, it follows that

$$\begin{aligned} & \frac{d}{dt} \left( \int_0^L v^2(x, t) dx \right) + (1 + g(t))v^2(L, t) + \frac{2v^3(L, t)}{3} + v_x^2(0, t) \\ &= -2g_t(t) \int_0^L v(x, t) dx, \end{aligned} \quad (2.10)$$

where subscripts denote partial differentiation.

Since the left hand side of (2.6) can be written as  $\frac{\partial v}{\partial t} + \frac{\partial}{\partial x} \left\{ (1 + g)v + \frac{v^2}{2} + \frac{\partial^2 v}{\partial x^2} \right\}$  we deduce after multiplication by  $(1 + g)v + v^2/2 + v_{xx}$  that

$$\begin{aligned} & \frac{d}{dt} \left( \int_0^L (v_x^2 - (1 + g)v^2 - \frac{v^3}{3}) dx \right) + v_{xx}^2(0, t) = \\ & \left( (1 + g)v(L, t) + \frac{v^2}{2}(L, t) \right)^2 - 2g_t(t)v_x(0, t) + 2g_t(t)(1 + g(t)) \int_0^L v(x, t) dx. \end{aligned} \quad (2.11)$$

## 2.2 A local in time estimate

Our first goal is to deduce from (2.10) and (2.11) an estimate on the  $H^1$ -norm of  $v$  for arbitrarily large data  $L$ ,  $g$  and  $u_0$ . To that aim we introduce the time dependent functions

$$X(t) = \int_0^t |v(\cdot, s)|_\infty^4 ds, \quad (2.12)$$

$$Y(t) = \int_0^L \left\{ v_x^2(x, t) - (1 + g(t))v^2(x, t) - \frac{v^3(x, t)}{3} \right\} dx, \quad (2.13)$$

where  $\|\cdot\|_\infty$  denote the  $L^\infty$ -norm and more generally  $\|\cdot\|_p$  the  $L^p$ -norm for  $p \in ]1, \infty[$ .

All the calculations rely on the classical Young inequality and on the interpolation inequality :

$$w^4(x) \leq 4 \int_0^L w^2(y) dy \int_0^L w_x^2(y) dy, \quad \forall x, \quad (2.14)$$

which holds for function vanishing at  $x = 0$ .

Time integration of (2.10) leads then to

$$|v|_2^2(t) + \int_0^t v_x^2(0, s) ds \leq \gamma_1(t) X^{1/2} + \frac{2t^{1/4}}{3} X^{3/4} + 2L\gamma_2(t) X^{1/4} + |v_0|_2^2 \quad (2.15)$$

where the  $\gamma_i$  depend on  $g$  :

$$\begin{aligned} \gamma_1(t) &= \left( \int_0^t \{(1+g)_-(s)\}^2 ds \right)^{1/2}, \quad \gamma_2(t) = \left( \int_0^t |g_t(s)|^{4/3} ds \right)^{3/4}, \\ \gamma_3(t) &= \left( \int_0^t g_t^2(s) ds \right)^{1/2} \end{aligned} \quad (2.16)$$

and  $x_- = \text{Min}(x, 0)$ ,  $x \in \mathbb{R}$ .

On the other hand, time integration of (2.11) leads in its turn to

$$\begin{aligned} Y(t) &\leq \left( \gamma_4(t) X^{1/4} + \frac{X^{1/2}}{2} \right)^2 + \int_0^t v_x^2(0, s) ds \\ &+ \gamma_3^2(t) + 2L\gamma_5(t) X^{1/4} + Y(0) \end{aligned} \quad (2.17)$$

where

$$\gamma_4(t) = \left( \int_0^t (1+g(s))^4 ds \right)^{1/4} \quad \text{and} \quad \gamma_5(t) = \left( \int_0^t |g_t(s)|^{4/3} |1+g(s)|^{4/3} ds \right)^{3/4}. \quad (2.18)$$

Next we observe that, by definition,  $\frac{dX}{dt} = |v|_\infty^4$  and therefore by (2.14)

$$\frac{dX}{dt} \leq 4|v|_2^2 |v_x|_2^2. \quad (2.19)$$

Clearly,  $|v(t)|_2^2$  is estimated thanks to (2.15). In order to estimate  $|v_x(\cdot, t)|_2^2$  we are going to make use of  $Y(t)$  as follows now. We write (at time  $t$ )

$$|v_x|_2^2 = Y + (1+g)|v|_2^2 + \frac{1}{3} \int_0^L v^3 dx \quad (2.20)$$

and making use of (2.14) :  $\left| \int_0^L v^3 dx \right| \leq 2^{1/2} |v|_2^{5/2} |v_x|_2^{1/2}$ .

Hence it follows from (2.14), (2.20) and Young's inequality that

$$|v_x|_2^2 \leq \frac{4Y}{3} + \frac{4}{3}(1+g)_+ |v|_2^2 + \left(\frac{2^{1/2}}{3}\right)^{4/3} |v|_2^{10/3}. \quad (2.21)$$

Combining (2.15) and (2.17) we have

$$\begin{aligned} Y &\leq Y(0) + |v_0|_2^2 + \gamma_3^2 + 2L(\gamma_2 + \gamma_5)X^{1/4} \\ &+ (\gamma_1 + \gamma_4^2)X^{1/2} + \left(\gamma_4 + \frac{2t^{1/4}}{3}\right)X^{3/4} + X/4. \end{aligned} \quad (2.22)$$

Clearly, (2.21) and (2.22) gives with (2.15) an explicit estimate of  $|v_x|_2^2$  in terms of  $X^\sigma$ ,  $\sigma > 0$  and of the data  $L$ ,  $g$  and  $u_0$ .

Finally replacing this last estimate on  $|v_x|_2^2$  in (2.19), we obtain

$$\begin{aligned} \frac{1}{4} \frac{dX}{dt} &\leq \left\{ |v_0|_2^2 + 2L\gamma_2 X^{1/4} + \gamma_1 X^{1/2} + \frac{2t^{1/4}}{3} X^{3/4} \right\} \\ &\left\{ \frac{4}{3} \left( (\gamma_1 + \gamma_4^2) X^{1/2} + \left(\gamma_4 + \frac{2t}{3}\right)^{1/4} X^{3/4} + \frac{1}{4} X + \gamma_3^2 + 2L\gamma_5 X^{1/4} + Y(0) + |v_0|_2^2 \right) \right. \\ &+ \frac{4}{3}(1+g)_+ \left( |v_0|_2^2 + 2L\gamma_2 X^{1/4} + \gamma_1 X^{1/2} + \frac{2t^{1/4}}{3} X^{3/4} \right) \\ &\left. + \frac{4^{1/3}}{3^{4/3}} \left( |v_0|_2^2 + 2L\gamma_2 X^{1/4} + \gamma_1 X^{1/2} + \frac{2t^{1/4}}{3} X^{3/4} \right)^{5/3} \right\}. \end{aligned} \quad (2.23)$$

By several applications of Young's inequality, it follows from (2.23) that

$$\frac{dX}{dt} \leq \alpha(t) + (1 + t^{2/3})X^2 \quad (2.24)$$

where  $\alpha$  depends only on the data through the  $\gamma_i$ 's,  $L$ ,  $|v_0|_2$  and  $|v_{0x}|_2$ .

Let  $T \in ]0, \infty[$  be given, we have

$$X(t) \leq X^*(t), \quad 0 \leq t < T^* \quad (2.25)$$

where  $X^*$  is the solution to

$$\begin{cases} \frac{dX^*}{dt} = \alpha_T + (1 + T^{2/3})(X^*)^2, \\ X^*(0) = 0, \end{cases} \quad (2.26)$$

here  $\alpha_T = \sup_{0 \leq t \leq T} \alpha(t)$  and  $T^*$  denotes the life span of the solution to (2.26),  $T^* = \frac{\pi}{2\sqrt{\alpha_T(1 + T^{2/3})}}$ .

Hence for  $t \leq T^*/2 = \frac{\pi}{4\sqrt{\alpha_T(1 + T^{2/3})}}$ , we deduce a bound on  $X(t)$  :

$$X(t) \leq (\alpha_T/(1 + T^{2/3}))^{1/2}, \quad 0 \leq t \leq \frac{\pi}{4\sqrt{\alpha_T(1 + T^{2/3})}}. \quad (2.27)$$

Then (2.15), (2.17) and (2.21) allow us to bound the  $H^1$ -norm of  $v : \int_0^L (v^2 + v_x^2)(x, t) dx$  for  $t$  in the previous range.

### 2.2.1 A global in time estimate

Here we want to show that the previous estimates, which are local in time for arbitrarily large data, can be turned into global ones for small data.

Our first assumption is that

$$\inf_{t \geq 0} (1 + g(t)) \equiv \alpha_0 > 0. \quad (2.28)$$

As it will be seen in the sequel, the following constants appears naturally in the estimates :

$$\alpha_1 = \int_0^\infty |g_t|(t) dt, \quad \alpha_2 = \sup_{t \geq 0} (1 + g(t)), \quad \alpha_3 = \int_0^\infty |g_t|^2(t) dt. \quad (2.29)$$

Let us then denote by  $M_0$  and  $M_1$  the function of the data :

$$M_0 = |v_0|_2^2 + 2\alpha_1\sqrt{L}(\sqrt{L}\alpha_1 + |v_0|_2), \quad (2.30)$$

$$\begin{aligned} M_1 &= |v_{0x}|_2^2 + \alpha_3 + 2\alpha_1\alpha_2\sqrt{L}(\sqrt{L}\alpha_1 + |v_0|_2) \\ &+ M_0\left(1 + \frac{17}{32}\alpha_0 + \frac{7}{4}\alpha_2 + \frac{2\alpha_2^2}{\alpha_0}\right). \end{aligned} \quad (2.31)$$

We observe that  $|v_0|_2^2 \leq M_0$  and  $|v_{0x}|_2^2 \leq M_1$  so that if impose on the initial data

$$M_0M_1 \leq 2^{-6}\alpha_0^4 \quad (2.32)$$

then by (2.14) we have

$$|v_0|_\infty^4 \leq 4M_0M_1 \leq 2^{-4}\alpha_0^4. \quad (2.33)$$

It follows by continuation that

$$|v(\cdot, t)|_\infty \leq 3\alpha_0/4 \text{ for } 0 \leq t \leq T \quad (2.34)$$

and we cannot take  $T = +\infty$  only if  $|v(\cdot, t)|_\infty$  indeed reach the value  $\frac{3\alpha_0}{4}$ . Actually we are going to show that, thanks to (2.34), we have

$$|v(\cdot, t)|_2^2 \leq M_0 \text{ and } |v_x(\cdot, t)|_2^2 \leq M_1, \quad 0 \leq t \leq T. \quad (2.35)$$

Hence by (2.14),  $|v(\cdot, t)|_\infty \leq \alpha_0/2$  which shows that  $T = +\infty$  and therefore

$$|v(\cdot, t)|_\infty \leq \alpha_0/2, |v(\cdot, t)|_2^2 \leq M_0, |v_x(\cdot, t)|_2^2 \leq M_1, \forall t \geq 0. \quad (2.36)$$

It remains to show that as long as (2.34) holds true, the estimates (2.35) are satisfied. The first step relies on the observation that (2.34) implies that  $(1+g)v^2(L, t) + \frac{2v^3(L, t)}{3} \geq \alpha_0 v^2(L, t)/2$  so that it transpires from (2.10) the following inequality :

$$\frac{d}{dt}|v|_2^2 + \frac{\alpha_0}{2}v^2(L, t) + v_x^2(0, t) \leq 2\sqrt{L}|v|_2|g_t|. \quad (2.37)$$

Then we deduce readily that

$$|v(\cdot, t)|_2 \leq |v_0|_2 + \sqrt{L}\alpha_1 \quad (2.38)$$

and

$$|v(\cdot, t)|_2^2 + \frac{\alpha_0}{2} \int_0^t v^2(L, s)ds + \int_0^t v_x^2(0, s)ds \leq M_0. \quad (2.39)$$

Hence the first part of (2.35) is established.

The second step make use of (2.11). Let us first observe that  $(1+g)v_0^2(x, t) - v_0^3(x, t)/3 \geq (\alpha_0 - \frac{3\alpha_0}{4} \cdot \frac{1}{3})v_0^2(x, t) \geq 0$ .

Integrating (2.11) with respect to time leads then to (after utilization of (2.34) and (2.38))

$$\begin{aligned} \int_0^L (v_x^2 - (1+g)v^2 - \frac{v^3}{3})(x, t)dx &\leq |v_{0x}|_2^2 + \alpha_2^2 \int_0^t v^2(L, s)ds \\ &+ \alpha_2 \frac{3\alpha_0}{4} \int_0^t v^2(L, s)ds + \left(\frac{3}{4}\right)^2 (\alpha_0^2/4) \int_0^t v^2(L, s)ds \\ &+ \int_0^t |g_t(s)|^2 ds + \int_0^t v_x^2(0, s)ds + 2 \int_0^t |g_t(1+g)|(s)ds\sqrt{L} \left(|v_0|_2 + 2\sqrt{L}\alpha_1\right). \end{aligned} \quad (2.40)$$

Now, since  $\int_0^L ((1+g)v^2 + \frac{v^3}{3})dx \leq |v|_2^2 \left(\alpha_2 + \frac{1}{3} \frac{3\alpha_0}{4}\right)$ , we deduce from (2.40) that  $|v_x(\cdot, t)|_2^2 \leq M_1$  and (2.35) is proved.

Once (2.28) is satisfied, (2.32) is indeed achieved by making the data  $v_0$  and  $g$  small.

### 2.3 A regularized problem

Let  $\varepsilon > 0$  be given. Our aim in this section is to construct a solution to the following problem. Find  $u^\varepsilon = u^\varepsilon(x, t)$ ,  $x \in [0, L]$  and  $t \geq 0$  solution to

$$\frac{\partial u^\varepsilon}{\partial t} + \frac{\partial u^\varepsilon}{\partial x} + u^\varepsilon \frac{\partial u^\varepsilon}{\partial x} + \frac{\partial^3 u^\varepsilon}{\partial x^3} - \varepsilon \frac{\partial^3 u^\varepsilon}{\partial x^2 \partial t} = 0, \quad 0 < x < L, \quad t > 0 \quad (2.41)$$

together with the boundary and initial conditions (2.2), (2.3) and (2.4).

Let us make the following change of independent variables by setting (the dependence of  $w$  below with respect to  $\varepsilon$  is dropped for simplification in the notations) :

$$w(x, t) = \varepsilon u^\varepsilon(\varepsilon^{1/2}(x-t), \varepsilon^{3/2}t) \quad (2.42)$$

so that (2.41) - (2.2) - (2.3) - (2.4) is transformed into

$$\frac{\partial w}{\partial t} + (1 + \varepsilon) \frac{\partial w}{\partial x} + w \frac{\partial w}{\partial x} - \frac{\partial^3 w}{\partial x^2 \partial t} = 0, \quad t < x < t + \varepsilon^{-1/2} L, \quad t > 0, \quad (2.43)$$

$$w(t, t) = \varepsilon g(\varepsilon^{3/2} t), \quad t \geq 0 \quad (2.44)$$

$$\frac{\partial w}{\partial x}(t + \varepsilon^{-1/2} L, t) = 0, \quad \frac{\partial^2 w}{\partial x^2}(L + \varepsilon^{-1/2} L, t) = 0, \quad t \geq 0 \quad (2.45)$$

$$w(x, 0) = w_0(x) \equiv \varepsilon u_0(\varepsilon^{1/2} x), \quad 0 < x < \varepsilon^{-1/2} L. \quad (2.46)$$

In order to solve (2.43) - (2.46), which is posed on a tilded domain, we notice that (2.43) can also be written as

$$\left(1 - \frac{\partial^2}{\partial x^2}\right) \frac{\partial w}{\partial t} = -\frac{\partial}{\partial x} \left\{ (1 + \varepsilon)w + \frac{w^2}{2} \right\}. \quad (2.47)$$

Inverting formally the operator  $1 - \frac{\partial^2}{\partial x^2}$ , this equation can be seen as an o.d.e. since  $\left(1 - \frac{\partial^2}{\partial x^2}\right)^{-1} \frac{\partial}{\partial x}$  is a bounded operator. This o.d.e. is then solved via a fixed point procedure. This is the matter of the next section.

### 2.3.1 An integral formulation of the regularized problem

Since  $L$  and  $\varepsilon$  are fixed, we change a little bit the notations and study

$$w_t + cw_x + ww_x - w_{xxt} = 0, \quad t < x < t + \lambda, \quad t > 0, \quad (2.48)$$

$$w(t, t) = \gamma(t), \quad t \geq 0, \quad (2.49)$$

$$w_x(t + \lambda, t) = 0, \quad w_{xx}(t + \lambda, t) = 0, \quad t \geq 0, \quad (2.50)$$

$$w(x, 0) = \varphi(x), \quad 0 < x < \lambda. \quad (2.51)$$

Next we set  $v(x, t) = w(x, t) - \gamma(t)$  and obtain

$$v_t - v_{xxt} = -\gamma_t - V_x, \quad t < x < t + \lambda, \quad t > 0, \quad (2.52)$$

$$v(t, t) = 0, \quad t \geq 0, \quad (2.53)$$

$$v_x(t + \lambda, t) = 0, \quad v_{xx}(t + \lambda, t) = 0, \quad t \geq 0, \quad (2.54)$$

$$v(x, 0) = \varphi(x) - \gamma(0), \quad 0 < x < L \quad (2.55)$$

where

$$V(x, t) \equiv (C + \gamma(t))v(x, t) + \frac{v^2(x, t)}{2}. \quad (2.56)$$

As previously said, one of the key point in the resolution of (2.52)-(2.55) is the study of the inverse of the operator  $1 - \frac{\partial^2}{\partial x^2}$ .

Of course we can only retain two boundary conditions on  $]t, t + \lambda[$  and we will take the homogeneous Dirichlet boundary condition at  $x = t$  and the homogeneous Neumann condition at  $x = t + \lambda$ .

It is then straightforward to check the following Proposition.

**Proposition 2.1** *Let  $t \geq 0$  and  $\lambda > 0$  be given. For every  $h \in L^2(t, t + \lambda)$ , there exists a unique solution  $\psi \in H^2(t, t + \lambda)$  to the following boundary value problem :*

$$\left(1 - \frac{\partial^2}{\partial x^2}\right) \psi(x) = h(x), \quad t < x < t + \lambda, \quad (2.57)$$

$$\psi(t) = 0, \quad \frac{\partial \psi}{\partial x}(t + \lambda) = 0. \quad (2.58)$$

Moreover this function is given by the formula

$$\psi(x) = \int_t^x h(z) \sinh(z - x) dz + \frac{\sinh(x - t)}{\cosh \lambda} \int_t^{t + \lambda} h(z) \cosh(t + \lambda - z) dz. \quad (2.59)$$

Let us now transform (2.52) - (2.55) into an integral equation thanks to this Proposition. First we integrate (2.52) with respect to time. Two cases are to be considered (i)  $x \leq \lambda$  and (ii)  $x \geq \lambda$ .

Case (i) :  $x \leq \lambda$ . We obtain integrating between 0 and  $t$

$$\left(1 - \frac{\partial^2}{\partial x^2}\right) v(x, t) = \left(1 - \frac{\partial^2}{\partial x^2}\right) (\varphi(x) - \gamma(0)) - \int_0^t [\gamma_t(s) + V_x(x, s)] ds.$$

Case (ii) :  $x \geq \lambda$ . We obtain integrating between  $x - \lambda$  and  $t$  :

$$\begin{aligned} \left(1 - \frac{\partial^2}{\partial x^2}\right) v(x, t) &= \left(1 - \frac{\partial^2}{\partial x^2}\right) v(x, x - \lambda) - \int_{x - \lambda}^t [\gamma_t(s) + V_x(x, s)] ds \\ &= v(x, x - \lambda) - \int_{x - \lambda}^t [\gamma_t(s) + V_x(x, s)] ds \end{aligned}$$

by using the second condition in (2.50).

It follows that if we set

$$W(x, t) = \begin{cases} \left(1 - \frac{\partial^2}{\partial x^2}\right) \varphi(x) - \gamma(t) - \int_0^t V_x(x, s) ds, & x < \lambda, \\ v(x, x - \lambda) - \gamma(t) + \gamma(x - \lambda) - \int_{x - \lambda}^t V_x(x, s) ds, & x \geq \lambda, \end{cases} \quad (2.60)$$

we have obtained that

$$\left(1 - \frac{\partial^2}{\partial x^2}\right) v(x, t) = W(x, t), \quad t < x < t + \lambda. \quad (2.61)$$

The next step is to use the Proposition 2.1 in order to obtain  $v$  explicitly as a function of  $W$ . This is done by setting (for a fixed  $t$ ),  $h(x) = W(x, t)$  in the formula (2.59). After a straightforward but long and technical computation (see the Appendix of [13]) one finds :

$$v(x, t) = (\mathcal{T}v)(x, t), \quad t < x < t + \lambda, \quad 0 \leq t \leq \lambda, \quad (2.62)$$

where  $\mathcal{T}$  is an integral operator defined hereafter.

Let us denote by  $S_\lambda^\infty$  the tilded strip

$$S_\lambda^\infty = \{(x, t) \in \mathbb{R}^2, \quad t \geq 0 \text{ and } t \leq x \leq t + \lambda\}$$

and also for  $T > 0$

$$S_\lambda^T = \{(x, t) \in \mathbb{R}^2, 0 \leq t \leq T \text{ and } t \leq x \leq t + \lambda\}.$$

Given a function  $v : S_\lambda^T$  into  $\mathbb{R}$ , we first construct  $V : S_\lambda^T$  into  $\mathbb{R}$  by setting

$$V(x, t) = (c + \gamma(t))v(x, t) + \frac{1}{2}v^2(x, t). \quad (2.63)$$

Then we set

$$\begin{aligned} (\mathcal{T}v)(x, t) &= \frac{\sinh(x-t)}{\cosh \lambda} [\varphi(t) \sinh \lambda - \varphi(\lambda) \sinh t - \gamma(t) \sinh \lambda \\ &+ \int_\lambda^{t+\lambda} (v(z, z-\lambda) + \gamma(z-\lambda)) \cosh(t+\lambda-z) dz - \int_0^t V(s+\lambda, s) \cosh(t-s) ds \\ &- \int_0^t \int_t^{s+\lambda} V(z, s) \sinh(t+\lambda-z) dz ds] - \varphi(t) \cosh(t-x) \\ &+ \gamma(t) [\cosh(t-x) - 1] + \chi_\lambda(x) \left( \varphi(x) + \int_t^x \int_0^t V(z, s) \cosh(z-x) ds dz \right) \\ &+ (1 - \chi_\lambda(x)) \left( \varphi(\lambda) \cosh(\lambda-x) + \int_\lambda^x \int_{\lambda-z-\lambda}^+ V(z, s) \cosh(z-x) ds dz \right. \\ &+ \int_t^\lambda \int_0^t V(z, s) \cosh(z-x) ds dz - \int_0^{x-\lambda} V(s+\lambda, s) \sinh(s+\lambda-x) ds \\ &\left. + \int_\lambda^x \{V(z, z-\lambda) + \gamma(z-\lambda)\} \sinh(z-x) dz \right), \end{aligned} \quad (2.64)$$

where  $\chi_\lambda$  is the characteristic function of the set  $\{x \in \mathbb{R}, x \leq \lambda\}$ :  $\chi_\lambda(x) = 1$  for  $x \leq \lambda$  and  $\chi_\lambda(x) = 0$  for  $x > \lambda$ .

**Remark 2.2** The discontinuity of  $\chi_\lambda$  at  $x = \lambda$  do not introduce a discontinuity of  $\mathcal{T}$  at this point since the coefficients of  $\chi_\lambda$  and  $1 - \chi_\lambda$  are equal at  $x = \lambda$ .

As a result, we have *formally* transformed the original p.d.e. problem (2.48)-(2.51) into a fixed point problem (2.62). Our strategy now is very classical: (i) to prove that this fixed point problem has indeed a solution on a suitable space of functions; (ii) to establish in what extent a solution to (2.62) solves (2.48)-(2.51).

### 2.3.2 Existence of solutions to the integral equation

Let us assume that  $\gamma \in \mathcal{C}^0(\mathbb{R}_+)$  the space of continuous function on  $\mathbb{R}_+$  and that  $\varphi \in H^1(0, \lambda)$ . Since  $H^1(0, \lambda) \subset \mathcal{C}^0([0, \lambda])$ ,  $\varphi(0)$  makes senses and the only compatibility condition which is needed in order to solve (2.62) will be

$$\gamma(0) = \varphi(0). \quad (2.65)$$

Let us then denote by  $\mathcal{E}_T$  the Banach space of those (class of) functions which are defined on  $S_T$  and such that for almost every  $t \in ]0, T[$ ,  $w(\cdot, t)$  belongs to  $H^1(t; t+\lambda)$  and satisfy

$$\|w\|_{\mathcal{E}_t} = \text{ess sup}_{0 < t < T} \|w(\cdot, t)\|_{H^1(t; t+\lambda)} < \infty. \quad (2.66)$$

It is easy to check that for every  $0 < T < \infty$ ,  $\mathcal{T}$  maps  $\mathcal{E}_T$  into itself. This follows mainly from the fact that the mapping  $v \mapsto V$  (see (2.63)) is locally lipschitzian on  $H^1$ . Hence the existence of a solution to the fixed point problem

$$w = \mathcal{T}w, \quad w \in \mathcal{E}_T \quad (2.67)$$

will follow if we can exhibit  $R$  such that

$$\|w\|_{\mathcal{E}_T} \leq R \implies \|\mathcal{T}w\|_{\mathcal{E}_T} \leq R, \quad (2.68)$$

$$\|\mathcal{T}w_1 - \mathcal{T}w_2\|_{\mathcal{E}_T} \leq \|w_1 - w_2\|_{\mathcal{E}_T}/2, \quad \forall w_i \quad (2.69)$$

such that  $\|w_i\|_{\mathcal{E}_T} \leq R$ .

We are now in a position to state and prove the following result.

**Proposition 2.3** *Let  $\gamma \in \mathcal{C}^0(\mathbb{R}_+)$  and  $\varphi \in H^1(0, \lambda)$  be given. There exists  $T > 0$  such that (2.67) has a unique solution  $w \in \mathcal{E}_T$ . Moreover the function  $w$  is continuous on  $S_\lambda^T$  and satisfies (provided the compatibility condition  $\gamma(0) = \varphi(0)$  holds true)*

$$w(x, 0) = \varphi(x) - \gamma(0), \quad w(t, t) = 0. \quad (2.70)$$

*Proof :* Since it is only a local in time result (see however below), it is sufficient to assume a priori that  $T \leq \lambda$ . We write

$$\mathcal{T}w = \mathcal{T}0 + \mathcal{T}w - \mathcal{T}0$$

and observe that there exists  $R_0 > 0$  independent of  $T$  such that  $\|\mathcal{T}0\|_{\mathcal{E}_T} \leq R_0$ . Moreover since  $H^1$  is an algebra we have  $\|\mathcal{T}w - \mathcal{T}0\|_{\mathcal{E}_T} \leq T^\alpha C_1 (C_2 + \|w\|_{\mathcal{E}_T})^2$  for some  $\alpha > 0$ , where  $C_1$  and  $C_2$  are constants which are independent of  $T$ ,  $0 < T \leq \lambda$ . Hence

$$\|\mathcal{T}w\|_{\mathcal{E}_T} \leq R_0 + T^\alpha C_1 (C_2 + \|w\|_{\mathcal{E}_T})^2,$$

and choosing  $R = 2R_0$ , we know that (2.68) will hold true for sufficiently small  $T$ . Then in order to obtain (2.69) we observe that for  $\|w\|_{\mathcal{E}_T} \leq R = 2R_0$ , there exists  $\beta > 0$  and  $C_3$ , independent of  $t$ , such that ( $C_3$  depends on  $R$ )

$$\|\mathcal{T}w_1 - \mathcal{T}w_2\|_{\mathcal{E}_T} \leq C_3 T^\beta \|w_1 - w_2\|_{\mathcal{E}_T}$$

and by making  $T$  even smaller, we can ensure  $C_3 T^\beta \leq 1/2$  and (2.69) is proved.

Now by the contraction principle we have shown that for  $T$  sufficiently small, (2.67) has a unique solution  $w \in \mathcal{E}_T$ . By Remark 2.1,  $\mathcal{T}w$  is a continuous function of  $(x, t) \in S_\lambda^T$  and therefore  $w = \mathcal{T}w$  is continuous on  $S_\lambda^T$ .

Then  $w(t, t) = (\mathcal{T}w)(t, t) = 0$ ; and  $w(x, 0) = (\mathcal{T}w)(x, 0) = (\text{by (2.64)}) = \varphi(x) - \varphi(0) + \frac{\cosh(x - \lambda)}{\cosh \lambda} (\gamma(0) - \varphi(0))$ , which proves (2.70).

**Remark 2.4** *Classically, (2.67) satisfies a continuation principle, however the situation is slightly different from the usual Cauchy-Lipschitz setting for o.d.e's for two reasons. The first one is that the problem is not posed on a cylindrical domain but on a tilded strip but this do not introduce major changes. The second reason is related to the time-delay  $\lambda$ . This also do not introduce a difficulty. By the continuation principle, we have the following alternative. Either  $T = \lambda$  or  $T < \lambda$  and in this last case, either the solution can be continued on  $[T, T + \tau]$ ,  $\tau > 0$ ,  $T + \tau \leq \lambda$  or  $\lim_{t \rightarrow T} \|w\|_{\mathcal{E}_T} = +\infty$ . This allows us to construct a maximal solution on  $[0, T_{\max}[$  with  $T_{\max} \leq \lambda$ . At this point we see that the restriction  $T \leq \lambda$  which appears in the course of the proof of Proposition 2.3 can be removed because if  $T_{\max} = \lambda$ , we can start again at time  $\lambda$  and Proposition 2.2 allows us to continue the solution beyond this point provided we ensures the ad hoc compatibility condition at the point  $x = t = \lambda$ . It is indeed the case since  $\gamma$  and the solution  $w(., .)$  are continuous. ■*

Proposition 2.3 and the previous Remark allows us to state the following result.

**Theorem 2.5** *Let  $\gamma \in \mathcal{C}^0(\mathbb{R}_+)$  and  $\varphi \in H^1(0, \lambda)$  be such that  $\gamma(0) = \varphi(0)$ . There exists  $T \in ]0, +\infty]$  such that the integral equation (2.67) has a unique solution  $w \in \mathcal{E}_T$  and either  $T = +\infty$  or  $0 < T < \infty$  and then  $\lim_{t \rightarrow T} \|w\|_{\mathcal{E}_t} = +\infty$ .*

## 2.4 About smoothness of the solution to the integral equation

Since we want to satisfy (2.54), it is clear that we need more smoothness on the solution to (2.67). It is straightforward to show the following result.

**Theorem 2.6** *Let  $\gamma \in \mathcal{C}^0(\mathbb{R}_+)$  and  $\varphi \in H^1(0, \lambda)$  be such that  $\gamma(0) = \varphi(0)$  and let  $w$  denote the maximal solution  $w \in \mathcal{E}_T$  obtained in Theorem 2.1.*

(i) *If  $\varphi \in H^2(0, \lambda)$  satisfies  $\varphi_x(\lambda) = 0$  then*

$$\sup_{0 \leq t \leq \tau} \|w(\cdot, t)\|_{H^2(t, t+\lambda)} < \infty, \quad \forall \tau < T. \quad (2.71)$$

Moreover  $w_x \in \mathcal{C}^0(S_\lambda^T)$  and

$$w_x(t + \lambda, t) = 0. \quad (2.72)$$

(ii) *If  $\varphi \in H^3(0, \lambda)$  satisfies  $\varphi_x(\lambda) = 0$ ,  $\varphi_{xx}(\lambda) = 0$  then*

$$\sup_{0 \leq t \leq \tau} \|w(\cdot, t)\|_{H^3(t, t+\lambda)} < \infty, \quad \forall \tau < T. \quad (2.73)$$

Moreover  $w_{xx} \in \mathcal{C}^0(S_\lambda^T)$  and

$$w_{xx}(t + \lambda, t) = 0. \quad (2.74)$$

(iii) *If  $\varphi \in H^3(0, \lambda)$  satisfies  $\varphi_x(\lambda) = 0$ ,  $\varphi_{xx}(\lambda) = 0$  and  $\gamma \in \mathcal{C}^1(\mathbb{R}_+)$  then*

$$\sup_{0 \leq t \leq \tau} \left\| \frac{\partial w}{\partial t} \right\|_{H^4(t, t+\lambda)} < \infty. \quad (2.75)$$

The points (i) and (ii) are obvious consequences of the fact that  $w = \mathcal{T}w$ ; while (iii) follows from the expression of  $\frac{\partial}{\partial t}(\mathcal{T}w)$ .  $\blacksquare$

**Corollary 2.7** *Under the hypotheses of Theorem 2.6 point (iii),  $w$  solves (2.52)-(2.55) in the classical sense.*

### 2.4.1 Estimates on the solutions to the regularized problem

Thanks to Theorem 2.6 (Corollary 2.7) we have constructed solutions to (2.52)-(2.55) which satisfy (2.72) and (2.74) on a time interval  $0 < t < T_\varepsilon$  with a continuation principle in  $\mathcal{E}_t$  (Theorem 2.5). Performing the inverse change of variables in (2.42) i.e. setting

$$u^\varepsilon(x, t) = \varepsilon^{-1} w(\varepsilon^{-1/2}x + \varepsilon^{-3/2}t, \varepsilon^{-3/2}t), \quad (2.76)$$

we have shown the existence of classical solutions to (2.41) satisfying the boundary and initial conditions (2.2), (2.3) and (2.4). At this point, we use (2.5) and then obtain for  $v^\varepsilon$  :

$$v^\varepsilon(x, t) \equiv u^\varepsilon(x, t) - g(t), \quad (2.77)$$

and for  $0 \leq t < T_\varepsilon$  :

$$\frac{\partial v^\varepsilon}{\partial t} + (1 + g) \frac{\partial v^\varepsilon}{\partial x} + v^\varepsilon \frac{\partial v^\varepsilon}{\partial x} + \frac{\partial^3 v^\varepsilon}{\partial x^3} - \varepsilon \frac{\partial^3 v^\varepsilon}{\partial x^2 \partial t} = -\frac{dg}{dt}, \quad (2.78)$$

$$v^\varepsilon(0, t) = 0, \quad (2.79)$$

$$\frac{\partial v^\varepsilon}{\partial x}(L, t) = 0, \quad \frac{\partial^2 v^\varepsilon}{\partial x^2}(L, t) = 0. \quad (2.80)$$

**Proposition 2.8** Let  $v^\varepsilon \in \mathcal{C}([0, T_\varepsilon]; H^3(0, L))$ , satisfy  $\frac{\partial v^\varepsilon}{\partial t} \in \mathcal{C}([0, T_\varepsilon]; H^4(0, L))$  and (2.78) - (2.80). then one has the two following “energy” equations :

$$\begin{aligned} & \frac{d}{dt} \int_0^L ((v^\varepsilon)^2 + \varepsilon(v_x^\varepsilon)^2) dx + (1+g)(v^\varepsilon)^2(L) + \frac{2(v^\varepsilon)^3(L)}{3} \\ & + (v_x^\varepsilon)^2(0) = -2g_t \int_0^L v^\varepsilon ds; \end{aligned} \quad (2.81)$$

$$\begin{aligned} & \frac{d}{dt} \int_0^L \left( (v_x^\varepsilon)^2 - (1+g)v^\varepsilon)^2 - \frac{(v^\varepsilon)^3}{3} \right) dx + \varepsilon(v_t^\varepsilon)^2(L) \\ & + (v_{xx}^\varepsilon - \varepsilon v_{xt})^2(0) = g_t \int_0^L (v^\varepsilon)^2 dx + \left( (1+g)v^\varepsilon(L) + \frac{(v^\varepsilon)^2(L)}{2} \right)^2 \\ & + 2g_t(1+g) \int_0^L v^\varepsilon dx - 2g_t v(0) - 2\varepsilon v_t^\varepsilon(L)g_t. \end{aligned} \quad (2.82)$$

**Proof :** Due to the smoothness properties of  $v^\varepsilon$ , it is allowed to take the scalar product in  $L^2(0, L)$  of (2.78) with  $v^\varepsilon$  and with  $(1+g)v^\varepsilon + \frac{(v^\varepsilon)^2}{2} + \frac{\partial v^\varepsilon}{\partial x^2} - \varepsilon \frac{\partial v^\varepsilon}{\partial x \partial t}$ . Then (2.81) and (2.82) follow after several integrations by parts and use of (2.79) and (2.80).  $\blacksquare$

We observe now that we can mimic what we did in section 2.11 i.e. we denote by  $X(\cdot)$  and  $Y(\cdot)$  the functions defined in (2.12)-(2.13) with this time  $v^\varepsilon$  instead of  $v$  and observe that (2.15) and (2.22) are again satisfied. Hence combining these estimates with Proposition 2.8, we obtain the following result.

**Proposition 2.9** Let  $u_0 \in H^3([0, L])$  and  $g \in \mathcal{C}^1(\mathbb{R}^+)$  be such that  $u_0(0) = g(0)$ ,  $\frac{\partial}{\partial x} u_0(L) = \frac{\partial^2}{\partial x^2} u_0(L) = 0$ . Then there exists  $T(u_0, g)$  depending only on  $|u_0|_{H^1}$  and  $|g|_{\mathcal{C}^1}$  such that there exists an unique solution  $u^\varepsilon$  to (2.41) satisfying  $u^\varepsilon(x, 0) = u_0(x)$  and the boundary conditions (2.2)-(2.3). Moreover  $u^\varepsilon \in \mathcal{C}([0, T(u_0, g)]; H^3) \cap \mathcal{C}^1([0, T(u_0, g)]; H^4)$  and  $|u^\varepsilon|_{L^\infty(0, T(u_0, g); H^1)}$  is bounded independently of  $\varepsilon$  and this bound depends only on  $|u_0|_{H^1([0, L])}$  and  $|g|_{\mathcal{C}^1(\mathbb{R}^+)}$ .

**Proposition 2.10** If moreover  $g \in L^\infty(\mathbb{R}^+)$ . Suppose that  $\inf_{t \geq 0} (1+g(t)) > 0$ . There exists  $\delta > 0, \delta$  depending only on  $|g|_{L^\infty(\mathbb{R}^+)}$ , such that if  $\|u_0(\cdot) - g(0)\|_{H^1([0, L])} \leq \delta$ ,  $|g_t|_{L^1(\mathbb{R}^+)} \leq \delta$  and  $|g_t|_{L^2(\mathbb{R}^+)} \leq \delta$ , then one can take  $T(u_0, g) = +\infty$  in proposition 2.9.

## 2.5 Proof of Theorem 1.1 and 1.2 in the homogeneous case

Let  $u_0 \in H^1([0, L])$  and  $g \in \mathcal{C}^1(\mathbb{R}^+)$  be such that  $u_0(0) = g(0)$ . Take a sequence  $u_0^\varepsilon \in H^3([0, L])$  such that  $u_0^\varepsilon(0) = g(0)$ ,  $\frac{\partial}{\partial x} u_0^\varepsilon(L) = \frac{\partial^2}{\partial x^2} u_0^\varepsilon(L) = 0$  and  $u_0^\varepsilon \rightarrow u_0$  in  $H^1([0, L])$ . We still call  $u^\varepsilon$  the solution to (2.41) given by proposition (2.9). Since  $u_0^\varepsilon \rightarrow u_0$  in  $H^1([0, L])$  and since  $T(u_0^\varepsilon)$  depends only on  $|u_0^\varepsilon|_{H^1}$ , there exists  $T_0 > 0$  such that  $T(u_0^\varepsilon) > T_0$  for  $\varepsilon$  sufficiently small.

Moreover the sequence  $u^\varepsilon(x, t)$  is bounded in  $L^\infty(0, T_0; H^1([0, L]))$ . Now, since  $u^\varepsilon$  satisfies (2.41),  $\partial_t u^\varepsilon$  is bounded in  $L^\infty(0, T_0; H^{-2}([0, L]))$ , and therefore up to a subsequence,  $u^\varepsilon \rightarrow u$  in  $\mathcal{C}([0, T_0]; L^2([0, L]))$  strongly and  $u^\varepsilon \rightharpoonup u$  in  $L^\infty(0, T_0; H^1([0, L]))$  weakly. It is then easy to see that  $u$  satisfies (2.1), (2.2) and (2.4). Moreover, if (2.28) is satisfied and if  $g \in L^\infty(\mathbb{R}^+)$ ,  $\partial_t g \in L^1 \cap L^2(\mathbb{R}^+)$  and if  $v_0(x) \equiv u_0(x) - g(0)$  the constants

$\alpha_1, \alpha_2, \alpha_3$  in (2.29) are small enough (in the sense of (2.32), then  $T_\varepsilon = +\infty \forall \varepsilon > 0$  and  $u \in \mathcal{C}(\mathbb{R}^+; L^2([0, L])) \cap L^\infty(\mathbb{R}^+; H^1([0, L]))$ , and  $u$  satisfies (2.1), (2.2) and (2.4).

We still have to show that (2.3) is satisfied in a suitable sense. To this aim, we prove the following result.

**Proposition 2.11** *Let  $T > 0$  and  $A$  be the following space :*

$A = \{(v, f) \in \mathcal{C}([0, T]; H^3) \times \mathcal{C}([0, T]; H^1)\}$  such that  $v_t + v_x + v_{xxx} = f_x, 0 < x < L, 0 < t < T\}$  and  $\mathcal{T}_1, \mathcal{T}_2$  the following linear maps defined on  $A$  by

$$\mathcal{T}_1(v, f) = v_x(L, t),$$

$$\mathcal{T}_2(v, f) = v_{xx}(L, t).$$

One has

$$|\mathcal{T}_1(v, f)|_{H^{-1}([0, T])} \leq C (|v|_{L^\infty(0, T; L^2)} + |f|_{L^\infty(0, T; L^2)}) \quad (2.83)$$

and

$$\begin{aligned} |\mathcal{T}_2(v, f)|_{H^{-1}([0, T])} &\leq C \left( |v|_{L^\infty(0, T; L^2)}^{1/2} |v|_{L^\infty(0, T; H^1)}^{1/2} \right. \\ &\quad \left. + |f|_{L^\infty(0, T; L^2)}^{1/2} |f|_{L^\infty(0, T; H^1)}^{1/2} \right). \end{aligned} \quad (2.84)$$

**Proof :** In order to establish (2.83), one multiplies (2.83) by a function  $\varphi(x) \in \mathcal{C}^3([0, L])$  that will be chosen later one and one integrates over  $x \in [0, L]$  :

$$\frac{d}{dt} \int_0^L v \varphi + \int_0^L v_x \varphi + \int_0^L v_{xxx} \varphi = \int_0^L f_x \varphi.$$

Integration by parts leads to

$$\begin{aligned} \frac{d}{dt} \int_0^L v \varphi - \int_0^L v \varphi_x + [v \varphi]_0^L - \int_0^L v \varphi_{xxx} \\ + [v_{xx} \varphi]_0^L - [v_x \varphi_x]_0^L + [v \varphi_{xx}]_0^L = - \int_0^L f \varphi_x + [f \varphi]_0^L. \end{aligned} \quad (2.85)$$

Choose now  $\varphi$  such that  $\varphi(0) = \varphi(L) = \varphi_x(0) = \varphi_{xx}(0) = \varphi_{xx}(L) = 0$  and  $\varphi_x(L) = 1$ . Equation (2.85) becomes

$$\frac{d}{dt} \int_0^L v \varphi - \int_0^L v \varphi_{xxx} - \int_0^L v \varphi_x + v_{xx}(L, t) = - \int_0^L f \varphi_x,$$

which implies (2.83).

For (2.84), one takes  $\varphi$  satisfying  $\varphi(0) = \varphi_x(0) = \varphi_x(L) = 0$  and  $\varphi(L) = 1$  and (2.85) gives

$$\begin{aligned} \frac{d}{dt} \int_0^L v \varphi - \int_0^L v \varphi_x + v(L, t) - \int_0^L v \varphi_{xxx} + v_{xx}(L, t) \\ + v(L, t) \varphi_{xx}(L) - v(0, t) \varphi_{xx}(0) = - \int_0^L f \varphi_x + f(L, t). \end{aligned} \quad (2.86)$$

Using the fact that  $\forall \psi \in H^1([0, L])$ ,  $|\psi|_{L^\infty} \leq C |\psi|_{L^2}^{1/2} |\psi|_{H^1}^{1/2}$  in (2.86), one gets (2.84).

Applying this proposition to equation (2.1), i.e. with  $f = \frac{u^2}{2}$ , one gets that for any  $u \in L^\infty(0, T; H^1([0, L]))$  solution to (2.1)  $u_x(L, t)$  and  $u_{xx}(L, t)$  exist in  $H^{-1}([0, T])$  and that the application  $u \mapsto (u_x(L, t), u_{xx}(L, t))$  is continuous on bounded sets of  $L^\infty(0, T; H^1([0, L]))$  for the norm of  $L^\infty(0, T; L^2([0, T]))$  with values in  $(H^{-1}([0, T]))^2$ . ■

For the regularized problem (2.41) one has :

**Proposition 2.12** *Let  $T > 0$  and  $B^\varepsilon$  the following space :*

$B^\varepsilon = \{(v^\varepsilon, f^\varepsilon) \in \mathcal{C}([0, T]; 3) \times \mathcal{C}([0, T]; H^1)\}$  such that

$$v_t^\varepsilon + v_x^\varepsilon + v_{xxx}^\varepsilon - \varepsilon v_{xxt}^\varepsilon = f_x^\varepsilon, \quad 0 < x < L, \quad 0 < t < T$$

We consider the maps  $\mathcal{T}_1, \mathcal{T}_2$  as in proposition 2.9.

One has

$$\begin{aligned} |\mathcal{T}_1(v^\varepsilon, f^\varepsilon)|_{H^{-1}([0, T])} &\leq C (|v^\varepsilon|_{L^\infty(0, T; L^2)} + |f|_{L^\infty(0, T; L^2)}) \\ &+ \varepsilon |v^\varepsilon|_{L^\infty(0, T; H^1)} \end{aligned} \quad (2.87)$$

and

$$\begin{aligned} |\mathcal{T}_2(v^\varepsilon, f^\varepsilon)|_{H^{-2}([0, T])} &\leq C \left( |v^\varepsilon|_{L^\infty(0, T; L^2)}^{1/2} |v^\varepsilon|_{L^\infty(0, T; H^1)}^{1/2} \right. \\ &\left. + |f^\varepsilon|_{L^\infty(0, T; L^2)}^{1/2} |f^\varepsilon|_{L^\infty(0, T; H^1)}^{1/2} + \varepsilon |v^\varepsilon|_{L^\infty(0, T; H^1)} \right). \end{aligned} \quad (2.88)$$

**Proof :** We still have to deal with the term  $-\varepsilon v_{xxt}^\varepsilon$ .

For (2.87), one obtains

$$\begin{aligned} -\varepsilon \int_0^L v_{xxt}^\varepsilon \varphi(x) &= \varepsilon \int_0^L v_{xt}^\varepsilon \varphi - \varepsilon [v_{xt} \varphi]_0^L \\ &= \varepsilon \frac{d}{dt} \int_0^L v_x^\varepsilon \varphi - 0, \end{aligned}$$

thereby proving (2.87).

For (2.88), one gets

$$\begin{aligned} -\varepsilon \int_0^L v_{xxt}^\varepsilon \varphi(x) &= \varepsilon \int_0^L v_{xt}^\varepsilon \varphi - \varepsilon [v_{xt} \varphi]_0^L \\ &= \varepsilon \frac{d}{dt} \int_0^L v_x^\varepsilon \varphi - \varepsilon \frac{d}{dt} v_x(L), \end{aligned}$$

Using (2.87), we get (2.88). ■

**Remark 2.13** *If  $v_x^\varepsilon(L) = 0$ , then (2.88) holds with  $|\mathcal{T}_2(v^\varepsilon, f^\varepsilon)|_{H^{-1}([0, T])}$*

Applying these results to  $v^\varepsilon(x, t) = u^\varepsilon(x, t)$  the solution to the regularized problem and  $f^\varepsilon = \frac{(u^\varepsilon)^2}{2}$  finishes the proofs of Theorem 1.1 and 1.2. ■

### 3 General Case

In this section, we address the non-homogeneous case. Namely, find  $u = u(x, t), x \in [0, L]$  and  $t \geq 0$ , solution to

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} + u \frac{\partial u}{\partial x} + u \frac{\partial^3 u}{\partial x^3} = 0, \quad 0 < x < L, \quad t > 0, \quad (3.1)$$

$$u(0, t) = g(t), \quad t \geq 0, \quad (3.2)$$

$$\frac{\partial u}{\partial x}(L, t) = h(t), \quad \frac{\partial^2 u}{\partial x^2}(L, t) = k(t), \quad t \geq 0, \quad (3.3)$$

$$u(x, 0) = u_0(x), \quad 0 < x < L. \quad (3.4)$$

We will extend the proofs of the previous section.

### 3.1 Extension of the a priori estimates

As usual, we perform formal computations on the solutions to (3.1) - (3.4). Let  $v$  be defined by

$$\begin{aligned} v(x, t) &= u(x, t) - [g(t) + (h(t) - k(t)L)x + \frac{k(t)}{2}x^2] \\ &\equiv u(x, t) - f(x, t). \end{aligned} \quad (3.5)$$

the (3.1) - (3.4) reads :

$$\frac{\partial v}{\partial t} + \frac{\partial}{\partial x}((1+f)v) + v\frac{\partial v}{\partial x} + \frac{\partial^3 v}{\partial x^3} = -\frac{\partial f}{\partial t} - f\frac{\partial f}{\partial x}, \quad (3.6)$$

$$v(0, t) = 0, \quad t \geq 0, \quad (3.7)$$

$$\frac{\partial v}{\partial x}(L, t) = 0, \quad \frac{\partial^2 v}{\partial x^2}(L, t) = 0, \quad t \geq 0, \quad (3.8)$$

$$v(x, 0) = v_0(x) \equiv u_0(x) - f(x, 0). \quad (3.9)$$

Multiplying (3.6) by  $v$  leads to

$$\begin{aligned} &\frac{d}{dt} \int_0^L v^2(x, t) dx + \int_0^L v^2(x, t) \frac{\partial f}{\partial x}(x, t) dx + (1+f(L, t))v^2(L, t) \\ &+ \frac{2v^3}{3}(L, t) + v_x^2(0, t) = -2 \int_0^L \left( \frac{\partial f}{\partial t}(x, t) + \frac{\partial}{\partial x} \frac{f^2}{2}(x, t) \right) v(x, t) dx. \end{aligned} \quad (3.10)$$

Since the left-hand side of (3.6) can be written  $\frac{\partial v}{\partial t} + \frac{\partial}{\partial x} \left( (1+f)v + \frac{v^2}{2} + \frac{\partial^2 v}{\partial x^2} \right)$ , we deduce, after multiplication by  $(1+f)v + \frac{v^2}{2} + v_{xx}$  that

$$\begin{aligned} &\frac{d}{dt} \left( \int_0^L v_x^2 - (1+f)v^2 - \frac{v^3}{3} dx \right) + v_{xx}^2(0, t) - \int_0^L \frac{\partial f^2}{\partial x^2} v^2 dx \\ &= \left( (1+f(L, t))v(L, t) + \frac{v^2}{2}(L, t) \right)^2 - 2 \left( \frac{\partial f}{\partial t} + \frac{\partial f^2}{\partial x} \right)(0, t)v_x(0, t) \\ &+ 2 \int_0^L \left( \frac{\partial f}{\partial t} + \frac{\partial}{\partial x} \frac{f^2}{2} \right) (1+f)v dx + 2 \int_0^L \left( \frac{\partial^3}{\partial t \partial x^2} f + \frac{\partial^3}{\partial x^3} \frac{f^2}{2} \right) v dx. \end{aligned} \quad (3.11)$$

It is clear that the same kind of proof than that of section 2.1.1. yields a local in time estimate. The new terms do not imply serious problem. For the global in time estimate, the situation is more complicated. The two basic assumptions are :

$$\inf_{t \geq 0} (1+f(L, t)) = \alpha_0 > 0 \quad (3.12)$$

and

$$\frac{\partial f}{\partial x}(x, t) \geq 0, \quad 0 \leq x \leq L, \quad t \geq 0, \quad (3.13)$$

which is equivalent to

$$h(t) - k(t)(L-x) \geq 0, \quad 0 \leq x \leq L, \quad t \geq 0. \quad (3.14)$$

**Remark 3.1** The condition (3.13) can be replaced by a weaker condition. Let  $m$  denote the function defined by  $m(t) = h(t) - L(k(t) + |k(t)|)/2$ . It is easy to check that  $\forall t \geq 0, \forall x \in [0, L], \frac{\partial f}{\partial x}(x, t) \geq m(t) = \min_{0 \leq y \leq L} \frac{\partial f}{\partial x}(y, t)$ . Hence (3.13) reads  $m \geq 0$ . Actually all the estimates can be done assuming that

$$\sup_{t \geq 0} \left( \exp - \int_0^t m(s) ds \right) < \infty \quad (3.15)$$

(although they become even much more technical).

Now the proof follows along the same lines as in section 2.1.2. assuming that the following quantities are small enough:

$$\begin{aligned} \mu_1 &= \int_0^{+\infty} \left| \frac{\partial f}{\partial t} + \frac{\partial f^2}{\partial x} \right|_2(s) ds, \quad \mu_2 = \sup_{t \geq 0} (1 + f(L, t)), \\ \mu_3 &= \int_0^{+\infty} \left| \frac{\partial f}{\partial t} + \frac{\partial f^2}{\partial x} \right|_2^2(0, s) ds, \quad \mu_4 = \int_0^t \left| \frac{\partial^3}{\partial t \partial x^2} f + \frac{\partial^3}{\partial x^3} \frac{f^2}{2} \right|_2(s) ds, \\ \mu_5 &= \int_0^{+\infty} \left| \frac{\partial f^2}{\partial x} \right|_\infty(s) ds, \end{aligned} \quad (3.16)$$

and we refer to [13] for details.

### 3.1.1 Solutions to a regularized problem

Let  $\varepsilon > 0$  be given. We want to construct a solution to the following problem. Find  $u^\varepsilon = u^\varepsilon(x, t)$ ,  $x \in [0, L]$  and  $t \geq 0$  solution to

$$\frac{\partial u^\varepsilon}{\partial t} + \frac{\partial u^\varepsilon}{\partial x} + u^\varepsilon \frac{\partial u^\varepsilon}{\partial x} + \frac{\partial^3 u^\varepsilon}{\partial x^3} - \varepsilon \frac{\partial^3 u^\varepsilon}{\partial x^2 \partial t} = 0, \quad 0 < x < L, \quad t \geq 0, \quad (3.17)$$

together with the boundary and initial conditions (3.2), (3.3) and (3.4). The same change of variables as in section 2.2 (formula (2.42))  $w(x, t) = \varepsilon u^\varepsilon(\varepsilon^{1/2}(x - t), \varepsilon^{3/2}t)$  yields

$$\frac{\partial w}{\partial t} + (1 + \varepsilon) \frac{\partial w}{\partial x} + w \frac{\partial w}{\partial x} - \frac{\partial^3 w}{\partial x^2 \partial t} = 0, \quad t < x < t + \varepsilon^{-1/2}L, \quad t > 0, \quad (3.18)$$

$$w(t, t) = \varepsilon g(\varepsilon^{3/2}t) \quad t \geq 0, \quad (3.19)$$

$$\partial_x w(t + \varepsilon^{-1/2}L, t) = \varepsilon^{3/2} h(\varepsilon^{3/2}t), \quad (3.20)$$

$$\partial_{xx}^2 w(t + \varepsilon^{-1/2}L, t) = \varepsilon^2 k(\varepsilon^{3/2}t) \quad t \geq 0.$$

Since  $L$  and  $\varepsilon$  are fixed, as in section 2.2.1, we have to study

$$w_t + cw_x + ww_x - w_{xxt} = 0, \quad t < x < t + \lambda, \quad t > 0, \quad (3.21)$$

$$w(t, t) = \gamma(t), \quad t \geq 0, \quad (3.22)$$

$$w_x(t + \lambda, t) = \eta(t), \quad w_{xx}(t + \lambda, t) = K(t), \quad t \geq 0 \quad (3.23)$$

$$w(x, 0) = \varphi(x), \quad 0 < x < \lambda. \quad (3.24)$$

We introduce  $f_\lambda(x, t) = \gamma(t) + (\eta(t) - K(t)\lambda)(x - t) + \frac{K(t)^2}{2}(x - t)^2$ , and  $v(x, t) = w(x, t) - f_\lambda(x, t)$  and obtain

$$\frac{\partial v}{\partial t} - v_{xxt} = -f_{\lambda t} - v_x, \quad t < x < t + \lambda, t > 0, \quad (3.25)$$

$$v(t, t) = 0, \quad t \geq 0, \quad (3.26)$$

$$v_x(t + \lambda, \lambda, t) = 0, \quad v_{xx}(t + \lambda, t) = 0, \quad t \geq 0, \quad (3.27)$$

$$v(x, 0) = \varphi(x) - (\gamma(0) + (\eta(0) - K(0)\lambda)x + \frac{K(0)^2}{2}x^2), \quad (3.28)$$

where

$$V(x, t) = (1 + f_\lambda)v + \frac{v^2}{2} + \frac{f_\lambda^2}{2}. \quad (3.29)$$

It is clear that one can solve (3.25) - (3.29) as in section 2.2 using an integral formulation. One then obtain the same kind of local existence result than Theorem 2.5 and 2.6 for the integral equation; we omit the details.

Concerning the regularized equation (3.17), one has the analogues of Proposition (2.9) and (2.10) :

**Proposition 3.2** *Let  $u_0 \in H^3([0, L])$ ,  $g, h, k, \in C^1(\mathbb{R}^+)$  satisfying  $u_0(0) = g(0)$ ,  $u_x(L) = h(0)$ ,  $u_{xx}(L) = k(0)$ . Then there exists  $T_0$  depending only on  $|u_0|_{H^1}$ ,  $|g|_{C^1}$ ,  $|h|_{C^1}$  and  $|k|_{C^1}$  such that there exists a unique solution  $u^\varepsilon$  to (3.17) satisfying  $u^\varepsilon(x, 0) = u_0(x)$  and the boundary conditions (3.2) - (3.3). Moreover  $u^\varepsilon \in C([0, T_0]; H^3) \cap C^1([0, T_0]; H^4)$  and  $|u^\varepsilon|_{L^\infty(0, T_0; H^1)}$  is bounded independently of  $\varepsilon$  and this bound depends only on  $|u_0|_{H^1([0, L])}$ ,  $|g|_{C^1}$ ,  $|h|_{C^1}$ ,  $|k|_{C^1}$ .*

**Proposition 3.3** *Let  $f(x, t) = g(t) + (h(t) - k(t)L)x + \frac{k^2(t)}{2}x^2$ . Suppose that  $f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial t} \in L^\infty([0, L] \times \mathbb{R}^+)$  and that*

$$\inf(1 + f(L, t)) = \alpha_0 > 0 \text{ and } h(t) - k(t)(L, x) \geq 0, 0 \leq x \leq L, t \geq 0.$$

*Then there exists  $\delta > 0$ , such that if  $|u_0(\cdot) - f(\cdot, 0)|_{H^1} \leq \delta$  and if the quantities  $(\mu_i)_{i=1}^5$  to 5 defined in (3.16) satisfy  $\mu_i \leq \delta$ ,  $i = 1$  to 5, the one can take  $T_0 = +\infty$  in Proposition 3.1.*

The only thing that one has to prove is that the ‘‘energy’’ estimates proved in section 3.1 are still valid for the regularized problem (3.17). It is the same as in section 2.2.  $\blacksquare$

### 3.2 Proof of Theorem 1.1 and Theorem 1.2 in the general case

Let  $u_0 \in H^1([0, L])$ ,  $(g, h, k) \in (C^1(\mathbb{R}^+))^3$  be such that  $u_0(0) = g(0)$ . Take a sequence  $u_0^\varepsilon \in H^3([0, L])$  such that  $u_0^\varepsilon(0) = g(0)$ ,  $\frac{\partial u_0^\varepsilon}{\partial x}(L) = h(0)$ ,  $\frac{\partial^2 u_0^\varepsilon}{\partial x^2}(L) = k(0)$ , and  $u_0^\varepsilon \rightarrow u_0$  in  $H^1([0, L])$ . We still call  $u^\varepsilon$  the solution to (3.17) satisfying  $u^\varepsilon(x, 0) = u_0^\varepsilon$  and the boundary conditions (3.2) and (3.3), given by Proposition 3.1.

Since  $u_0^\varepsilon \rightarrow u_0$  in  $H^1([0, L])$  and since  $T_0$  depends only on  $|u_0^\varepsilon|_{H^1}$ , one can take  $T_0$  depending only on  $|u_0|_{H^1}$ . One concludes as in section 2.3.

Of course the boundary conditions  $u_x(L, t) = h(t)$  and  $u_{xx}(L, t) = k(t)$  are satisfied respectively in  $H^{-1}([0, T])$  and  $H^{-2}([0, T])$  in the sense of Proposition 2.11.  $\blacksquare$

## 4 Local weak solutions in $L^2([0, L])$

Let us consider the homogeneous problem

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} + u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0, \quad 0 < x < L, \quad t > 0 \quad (4.1)$$

$$u(0, t) = \frac{\partial u}{\partial x}(L, t) = \frac{\partial^2 u}{\partial x^2}(L, t) = 0, \quad t \geq 0 \quad (4.2)$$

$$u(0, t) = u_0(x), \quad 0 < x < L. \quad (4.3)$$

The aim of this section is to construct local weak solutions to (4.1) - (4.3).

### 4.1 Construction of the linear semi-group

We first construct a solution  $u$  to

$$\frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} = 0, \quad 0 < x < L, \quad t > 0, \quad (4.4)$$

satisfying (4.2) and (4.3). One has

**Theorem 4.1** *i) Let  $u_0 \in H^3(0, L)$  such that  $u_0(0) = \frac{\partial u_0}{\partial x}(L) = \frac{\partial^2 u_0}{\partial x^2}(L) = 0$ . There exists a unique solution denoted by  $S(\cdot)u_0$  to (4.4) - (4.2) - (4.3) satisfying  $S(\cdot)u_0 \in L^\infty(\mathbb{R}^+; H^3(0; L))$ .*

*ii) There exists a constant  $C(L)$  such that*

$$|S(\cdot)u_0|_{L^\infty(\mathbb{R}^+; L^2)} + \left| \frac{\partial}{\partial x} S(\cdot)u_0 \right|_{L^2(\mathbb{R}^+; L^2)} + \left| \sqrt{t} \frac{\partial^2}{\partial x^2} S(\cdot)u_0 \right|_{L^2_{loc}(\mathbb{R}^+; L^2)} \leq C(L) |u_0|_{L^2}.$$

Let

$$E = \left\{ \{V \in C \cap L^\infty(\mathbb{R}^+; L^2) \cap L^2(\mathbb{R}^+; H^1) \text{ such that } \sqrt{t} \frac{\partial^2 V}{\partial x^2} \in L^2_{loc}(\mathbb{R}^+; L^2)\} \right\}.$$

we therefore obtain :

**Corollary 4.2** *The semi-group  $(S(t))_{t \geq 0}$  extends continuously from  $L^2(0, L)$  into  $E$ .*

Proof of Theorem 4.1 : In order to prove i), one first construct regularized solutions as in Section 2. Let  $u^\varepsilon(x, t)$  be the solution to

$$\frac{\partial u^\varepsilon}{\partial t} + \frac{\partial^3 u^\varepsilon}{\partial x^3} - \varepsilon \frac{\partial^3 u^\varepsilon}{\partial x^2 \partial t} = 0, \quad 0 < x < L, \quad t \geq 0 \quad (4.5)$$

and  $u^\varepsilon$  satisfies (4.2) and (4.3).

Local in time solutions to this problem can be obtained as in Section 2. In order to show that the solutions are global, it is enough to obtain global bounds in  $H^3(0, L)$ .

Multiplying (4.5) by  $u^\varepsilon$  leads to

$$\frac{d}{dt} \int_0^L (u^\varepsilon)^2 + \varepsilon \left( \frac{\partial u^\varepsilon}{\partial x} \right)^2 dx + \left( \frac{\partial u^\varepsilon}{\partial x} \right)^2 (0, t) = 0. \quad (4.6)$$

Multiplying  $\frac{\partial}{\partial t}$  (4.5) by  $\frac{\partial u^\varepsilon}{\partial t}$  leads to :

$$\frac{d}{dt} \int_0^L \left( \frac{\partial u^\varepsilon}{\partial t} \right)^2 + \varepsilon \left( \frac{\partial^2}{\partial x \partial t} u^\varepsilon \right)^2 dx + \left( \frac{\partial^2 u^\varepsilon}{\partial x \partial t} \right)^2 (0, t) = 0. \quad (4.7)$$

Using (4.6) and (4.7), one gets that  $u^\varepsilon \rightharpoonup u$  in  $L^\infty(\mathbb{R}^+, H^3)$   $w_\star$  and  $\frac{\partial u^\varepsilon}{\partial t} \rightharpoonup \frac{\partial u}{\partial t}$  in  $L^\infty(\mathbb{R}^+; L^2)$   $w_\star$  with  $\frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} = 0$ , thereby proving that  $u \in L^\infty(\mathbb{R}^+; H^3)$ . Hence i) of theorem 4.2 follows (of course  $u \in C(\mathbb{R}^+; H^2)$ ).

In order to prove ii), first multiply (4.4) successively by  $u(x, t)$  and  $xu(x, t)$ , one obtains :

$$\frac{d}{dt} \int_0^L u^2 dx + \left( \frac{\partial u}{\partial x} \right)^2 (0, t) = 0, \quad (4.8)$$

$$\frac{d}{dt} \int_0^L xu^2 dx + 3 \int_0^L \left( \frac{\partial u}{\partial x} \right)^2 dx = 0. \quad (4.9)$$

Now multiplying (4.4) by  $\frac{\partial^2 u}{\partial x^2}$  and  $x \frac{\partial^2 u}{\partial x^2}$  yields

$$\frac{d}{dt} \int_0^L \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial^2 u}{\partial x^2} \right)^2 (0, t) = 0, \quad (4.10)$$

$$\frac{d}{dt} \int_0^L x \left( \frac{\partial u}{\partial x} \right)^2 + 3 \int_0^L \left( \frac{\partial^2 u}{\partial x^2} \right)^2 + 2 \frac{\partial^2 u}{\partial x^2} (0, t) \frac{\partial u}{\partial x} (0, t) = 0. \quad (4.11)$$

Equations (4.8) and (4.9) give readily that

$$|S(t)u_0|_{L^\infty(\mathbb{R}^+; L^2)} + \left| \frac{\partial}{\partial x} S(t)u_0 \right|_{L^2(\mathbb{R}^+; L^2)} \leq C(L)|u_0|_{L^2}.$$

Note that (4.8) implies

$$\left| \frac{\partial u}{\partial x} (0, t) \right|_{L^2(\mathbb{R}^+)} \leq C(L)|u_0|_{L^2}. \quad (4.12)$$

Let us multiply (4.10) by  $t$  and integrate the result with respect to  $t$ , one gets :

$$t \int_0^L \left( \frac{\partial u}{\partial x} \right)^2 (x, t) dx + \int_0^t s \left( \frac{\partial^2 u}{\partial x^2} \right)^2 (0, s) ds = \int_0^t \int_0^L \left( \frac{\partial u}{\partial x} \right)^2 dx ds. \quad (4.13)$$

Multiplying (4.11) by  $t$  integrating in time leads to

$$\begin{aligned} & - \int_0^t \int_0^L x \left( \frac{\partial u}{\partial x} \right)^2 dx ds + t \int_0^L x \left( \frac{\partial u}{\partial x} \right)^2 dx + 3 \int_0^t s \int_0^L \left( \frac{\partial^2 u}{\partial x^2} \right)^2 dx ds \\ & + 2 \int_0^t s u_{xx}(0, s) u_x(0, s) ds = 0. \end{aligned}$$

So that thanks to (4.13) :

$$\begin{aligned} & \left| \sqrt{t} \frac{\partial^2 u}{\partial x^2} \right|_{L^2(0, T; L^2)} \leq C(L) \left[ \left| \frac{\partial u}{\partial x} \right|_{L^2(\mathbb{R}^+; L^2)} \right. \\ & \left. + \left| \frac{\partial u}{\partial x} \right|_{L^2(\mathbb{R}^+; L^2)}^{1/2} 2 \left( \int_0^T s u_x^2(0, s) ds \right)^{1/2} \right] \\ & \leq C(L) \left[ 1 + T^{1/2} \right] |u_0|_{L^2}, \end{aligned}$$

thereby proving ii) of Theorem 4.1. ■

**Definition 4.3** A weak solution to (4.1) - (4.3) is a function  $u \in \mathcal{C}([0, T]; L^2) \cap L^2_{loc}([0, T]; H^1)$  satisfying

$$u = S(t)u_0 - \int_0^t S(t-s) \left[ \frac{\partial u}{\partial x} + u \frac{\partial u}{\partial x} \right](s) ds.$$

## 4.2 Existence and uniqueness of weak solutions

We will prove :

**Theorem 4.4** Let  $u_0 \in L^2(0, L)$ . There exists a unique maximal weak solution to (4.1)-(4.3)  $u \in \mathcal{C}([0, T_{max}[, L^2) \cap L^2_{loc}([0, T_{max}[, H^1(0, L))$ .

Moreover  $\sqrt{t} \frac{\partial^2 u}{\partial x^2} \in L^2_{loc}([0, T_{max}[, L^2)$  and if  $T_{max} < +\infty$ , then  $\lim_{t \rightarrow T_{max}} \|u(t)\|_{L^2} = +\infty$ .

**Proof :** We first introduce for any  $f \in L^1(0, T; L^2)$

$$\Lambda f(t) = \int_0^t S(t-s) f(s) ds.$$

One has the following dual estimates on  $\Lambda$  :

**Proposition 4.5** There exists  $C > 0$  such that for any  $f \in L^1(0, T; L^2)$

$$\|\Lambda f(t)\|_{L^\infty(0, T; L^2)} \leq C \|f\|_{L^1(0, T; L^2)}, \quad (4.14)$$

$$\left\| \frac{\partial}{\partial x} \Lambda f(t) \right\|_{L^2(0, T; L^2)} \leq C \|f\|_{L^1(0, T; L^2)}. \quad (4.15)$$

*Proof of the Proposition :* One has  $\|\Lambda f(t)\|_{L^2(0, L)} \leq \int_0^t \|f\|_{L^2(s)} ds$  and (4.14) follows.

On the other hand  $\left\| \frac{\partial}{\partial x} \Lambda f(t) \right\|_{L^2} \leq \int_0^t \left\| \frac{\partial}{\partial x} S(t-s) f(s) \right\|_{L^2} ds$ . So that for all  $\varphi \geq 0$

$$\begin{aligned} \int_0^T \left\| \frac{\partial \Lambda}{\partial x} f(t) \right\| \varphi(t) dt &\leq \int_0^T \int_0^t \left\| \frac{\partial}{\partial x} S(t-s) f(s) \right\|_{L^2} ds \varphi(t) dt \\ &= \int_0^T \int_s^T \left\| \frac{\partial}{\partial x} S(t-s) f(s) \right\|_{L^2} \varphi(t) dt ds \\ &\leq \int_0^T \int_s^T \left\| \frac{\partial}{\partial x} S(t-s) f(s) \right\|_{L^2}^2 dt \left( \int_s^T \varphi(t)^2 dt \right)^{1/2} ds. \end{aligned}$$

Now thanks to Theorem 4.1,

$$\left( \int_0^T \left\| \frac{\partial}{\partial x} S(t-s) f(s) \right\|_{L^2}^2 dt \right)^{1/2} \leq C \|f(s)\|_{L^2(0, L)}$$

so that

$$\int_0^T \left\| \frac{\partial}{\partial x} \Lambda f(t) \right\| \varphi(t) dt \leq C \int_0^T \|f(s)\|_{L^2(0, L)} ds \left( \int_0^T \varphi(t)^2 dt \right)^{1/2}$$

from which (4.15) follows. ■

In order to prove Theorem 4.4, we introduce the following functional  $\mathcal{T}$  defined on  $L^\infty(0, T; L^2) \cap L^2(0, T; H^1)$  by

$$\mathcal{T}(u) = S(t)u_0 - \int_0^t S(t-s) \left[ \frac{\partial u}{\partial x} + u \frac{\partial u}{\partial x} \right](s) ds.$$

One has

$$\left| \frac{\partial u}{\partial x} + u \frac{\partial u}{\partial x} \right|_{L^1(0, T; L^2)} \leq |u|_{L^1(0, T; H^1)} + \left| u \frac{\partial u}{\partial x} \right|_{L^1(0, T; L^2)}.$$

Now

$$\left| u \frac{\partial u}{\partial x} \right|_{L^2(0, L)} \leq |u|_{L^\infty} \left| \frac{\partial u}{\partial x} \right|_{L^2} \leq C |u|_{L^2}^{1/2} \left| \frac{\partial u}{\partial x} \right|_{L^2}^{3/2},$$

so that

$$\left| u \frac{\partial u}{\partial x} \right|_{L^1(0, T; L^2)} \leq C |u|_{L^\infty(0, T; L^2)}^{1/2} \left| \frac{\partial u}{\partial x} \right|_{L^{3/2}(0, T; L^2)}^{3/2}.$$

It follows

$$\begin{aligned} \left| \frac{\partial u}{\partial x} + u \frac{\partial u}{\partial x} \right|_{L^1(0, T; L^2)} &\leq T^{1/2} |u|_{L^2(0, T; H^1)} \\ &\quad + C |u|_{L^\infty(0, T; L^2)}^{1/2} T^{1/4} \left| \frac{\partial u}{\partial x} \right|_{L^2(0, T; H^1)}^{3/2}. \end{aligned}$$

Applying Proposition 4.5 and Theorem 4.1, one gets

$$\begin{aligned} |\mathcal{T}(u)|_{L^\infty(0, T; L^2) \cap L^2(0, T; H^1)} &\leq C_1 |u_0|_{L^2} \\ &\quad + C_2 \left( T^{1/2} |u|_{L^2(0, T; H^1)} + T^{1/4} |u|_{L^\infty(0, T; L^2)}^{1/2} \left| \frac{\partial u}{\partial x} \right|_{L^2(0, T; H^1)}^{3/2} \right) \end{aligned} \quad (4.16)$$

Let  $R = 2C_1 |u_0|_{L^2}$  and  $B_R$  denotes the ball of radius  $R$  dans  $L^\infty(0, T; L^2) \cap L^2(0, T; H^1)$ , we have proved thanks to (4.16):

**Lemma 4.6** *It  $T$  is sufficiently small,  $\mathcal{T}$  maps  $B_R$  into itself.*

In the same way, one can show,

**Lemma 4.7** *It  $T$  is small enough,  $\mathcal{T}$  is a contraction on  $B_R$ .*

Applying Banach's fixed point theorem for  $\mathcal{T}$  on  $B_R$  (which is a complete metric space, yields Theorem 4.4.  $\blacksquare$

## 5 Open questions and further investigations

5.1. In view of Theorems 1.1 and 1.2 a natural question arises : is it possible to prove global existence of solutions for (1.7) - (1.11) for e.g. smooth solutions (as it is the case for both the quarter plane and the whole line cases) ? For these problems, uniqueness relies on a priori estimates in  $H^2$  that we are not able to extend here and therefore establish the existence of more regular solutions.

R. Temam [21] (see also Bubnov [6]) has proposed a different set of boundary conditions at  $x = L$  which leads to global estimates with respect to time in  $H^1(0, L)$ . These conditions are nonlinear and reads

$$u_x(L, t) = 0, \quad u_{xx}(L, t) + \frac{u^2}{2}(L, t) = 0. \quad (5.17)$$

A natural question which arises then is the study of the long time behavior of solutions to this problem. One could also consider the dissipative case : (KdV-Burgers)

$$u_t + u_x + uu_x + u_{xxx} - \gamma u_{xx} = 0 \quad (5.18)$$

with time periodic forcing of the form  $u(0, t) = g(t)$  and (4.1). In this case it would be interesting to obtain the existence of global attractors like in the space-periodic case (see also Ghidaglia [15] and [16]).

5.2. Another question is the study of the solutions constructed in theorem 1.1 and 1.2 when  $L \rightarrow \infty$ , see [14].

5.3. We have not addressed the study of the mapping  $(g, h, k) \rightarrow u(L, \cdot)$ . This question is inspired from scattering theory.

5.4 Since the linear equation

$$\begin{aligned} u_t + u_{xxx} &= 0, \\ u(0, t) = 0, u_x(L, t) = 0, u_{xx}(L, t) &= 0 \end{aligned}$$

has regularizing effects, one can expect, as in the case of the whole line, existence results to the IBVP (1.7) - (1.11) for initial values in  $H^s(0, L)$ ,  $s < 0$ . Let us also mention the paper of Cattabriga [8] for a complete study of Airy's equation  $u_t + u_{xxx} = 0$  on a finite interval.

5.5. Finally we refer to Bubnov [6] and [7] for various *a priori* estimates on a similar equation.

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