Some singular limits for evolutionary Ginzburg-Landau equations.

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Abstract: We give some partial results regarding to evolutionary dispersive Ginzburg-Landau equations associated to the static case considered in Béthuel, Brezis and Hélein [1]. We treat the partially dissipative case. Moreover we perform an exact "semi-classical" limit. We obtain alternatively the Laplace, heat and wave equations as limit equation, depending on the scaling that we consider.

1 Introduction and statement of the results.

In [1], Béthuel, Brezis and Hélein study the limit behavior as $\varepsilon \to 0$ of the minimizers for the problem

$$\min_{u \in H^1_g} \left\{ \frac{1}{2} \int_\Omega |\nabla u|^2 + \frac{1}{4\varepsilon^2} \int_\Omega (1-|u|^2)^2 \right\}, \tag{1}$$

where Ω is a simply connected domain in $\mathbf{R}^2$, $g : \partial\Omega \to \mathbf{C}$ is a given boundary value such that $|g| = 1$ a.e. on $\partial\Omega$ and $H^1_g = \{u \in H^1(\Omega, \mathbf{C}), u = g \text{ on } \partial\Omega\}$. Note that the minimizers of (1) satisfy

$$\Delta u_\varepsilon = -u_\varepsilon \frac{1-|u_\varepsilon|^2}{\varepsilon^2}. \tag{2}$$

They give a complete description of the behavior of the minimizers. The result depends on the degree of g: $d = deg(g, \partial\Omega)$. If $d = 0$ then $u_\varepsilon \to u_0$ in $\mathcal{C}^{1,\alpha}(\bar{\Omega})$ for every $\alpha < 1$ where u_0 is the unique harmonic map from Ω into S^1 such that $u_0 = g$ on $\partial\Omega$. More precisely, $u_0 = e^{i\phi_0}$ where

$$\left\{ \begin{array}{rcl} \Delta\phi_0 & = & 0 \text{ in } \Omega, \\ e^{i\phi_0} & = & g \text{ on } \partial\Omega. \end{array} \right. \tag{3}$$

In the case where $d \neq 0$, then the limit develop singularities of infinite energy which are called Ginzburg-Landau vortices.

The heat flow associated to (2), i.e.

$$u^\varepsilon_t = \Delta u^\varepsilon + \frac{1}{\varepsilon^2}(1-|u^\varepsilon|^2)u^\varepsilon, \tag{4}$$

has been studied by Lin in [3]. More precisely, the following results are proved under some natural conditions:

Theorem *[3] If the initial datum* u_0^ε *converges to* $\prod_{j=1}^d \frac{x-b_j}{|x-b_j|} e^{ih_0(x)}$*, then the solution* $u^\varepsilon(x,t)$ *to (4) converges to* $\prod_{j=1}^d \frac{x-b_j}{|x-b_j|} e^{ih_0(x,t)}$ *in* $L^2_{loc}(\bar\Omega \times \mathbf{R}^+)$ *strongly and in* $H^1_{loc}(\bar\Omega - \{b_1, ...b_d\} \times \mathbf{R}^+)$ *weakly, where* $h_0(x,t)$ *satisfies*

$$\frac{\partial h_0}{\partial t} = \Delta h_0, \ \textit{in } \Omega - \{b_1, ...b_d\} \times \mathbf{R}^+.$$

On the other hand, a result is also given for a rescalled version of (4):

$$\frac{1}{\lambda_\varepsilon} u_t^\varepsilon = \Delta u^\varepsilon + \frac{1}{\varepsilon^2}(1 - |u^\varepsilon|^2)u^\varepsilon, \tag{5}$$

and it reads:

Theorem *[3] Suppose* $\lim_{\varepsilon \to 0} \lambda_\varepsilon / \log(\frac{1}{\varepsilon}) = 0$ *and* $\lambda_\varepsilon \to \infty$ *as* $\varepsilon \to 0$*. Then the solution* u_ε *to (5) converges to* $\prod_{j=1}^d \frac{x-b_j}{|x-b_j|} e^{ih(x)}$ *in* $L^2_{loc}(\bar\Omega \times \mathbf{R}^+)$*, where* $\Delta h(x) = 0$ *in* Ω *and* $h(x) = h_0(x)$ *on* $\partial\Omega$*.*

The aim of this work is to study the limit $\varepsilon \to 0$ for the flow:

$$\left\{ \begin{array}{rcl} iu_{\varepsilon t} + \Delta u_\varepsilon & = & -u_\varepsilon \frac{1-|u_\varepsilon|^2}{\varepsilon^2} \text{ in } \mathbf{R}^+ \times \Omega, \\ u_\varepsilon(x,t) & = & g(x) \text{ on } \partial\Omega, \\ u_\varepsilon(x,0) & = & \psi(x) \text{ in } \Omega, \end{array} \right. \tag{6}$$

which is the Schrödinger equation associated to (2). We only consider the case when $|g| = 1$ a.e. on $\partial\Omega$, $|\psi_0| = 1$ a.e. on Ω and we assume the compatibility condition $\psi_0 = g$ on $\partial\Omega$. Of course all these conditions imply

$$deg(g, \partial\Omega) = 0.$$

This assumption will allways be done in this paper. For the physical meaning of the static model (2) see the references in [1]. Equation (6) is presented by J.Neu [5] as an universal model in mathematical physics and he studies the case of equation (8) below when $deg(g, \partial\Omega) \neq 0$. This case arises in quantum mecanics see [4].

The different limit equations that we obtain from variations of (6) are:
i) The Laplace equation:

$$\begin{array}{rcl} -\Delta u & = & u|\nabla u|^2, \\ |u| & = & 1 \text{ a.e.} \\ u|_{\partial\Omega} & = & g. \end{array}$$

ii) The heat equation:

$$\begin{array}{rcl} \frac{\delta}{1+\delta^2}u_t - \Delta u & = & u|\nabla u|^2, \\ |u| & = & 1 \text{ a.e.} \\ u|_{\partial\Omega} & = & g, \\ u(x,0) & = & \psi(x). \end{array}$$

iii) The wave equation:

$$\begin{array}{rcl} u_{tt} - 2\Delta u & = & -u(|u_t|^2 - 2|\nabla u|^2), \\ |u| & = & 1 \text{ a.e.} \\ u|_{\partial\Omega} & = & g, \\ u(x,0) & = & \psi(x), \\ u_t(x,0) & = & 0. \end{array}$$

More precisely, this paper is organized as follows:
In section 2, we show the following result (see Proposition 2 and 3).

Theorem *If the solution u_ε to (6) converges to some limit $u(x,t)$ in $L^2((0,T)\times\Omega)$ for some $T>0$, then $u(x,t)$ does not depend on t and $u(x,t)\equiv u_0$, where u_0 denotes the unique harmonic map from Ω into S^1 defined by (3) i.e.*

$$\begin{array}{rcl} -\Delta u_0 & = & u_0|\nabla u_0|^2, \\ |u_0| & = & 1 \textit{ a.e.}, \\ u_0|_{\partial\Omega} & = & g. \end{array}$$

Moreover, if the initial value ψ of u_ε is equal to u_0 then $u_\varepsilon(x,t) \to u_0$ in $L^p(0,T,H^1(\Omega))$ for all $p<\infty$ and $T<\infty$.

In section 3, we introduce some dissipation in the problem, namely:

$$u_{\varepsilon t} = (i+\delta(\varepsilon))(\Delta u_\varepsilon + u_\varepsilon \frac{1-|u_\varepsilon|^2}{\varepsilon^2}). \tag{7}$$

We prove (see proposition 4 and theorem 1)

Theorem *If $\delta>0$ is constant, then $u_\varepsilon \to u$ in $\mathcal{C}(0,T,L^2)$ strongly and in $L^\infty(0,T,H^1)$ weakly, where u satisfies the heat flow equation for harmonic maps:*

$$\begin{array}{rcl} \frac{\delta}{1+\delta^2}u_t - \Delta u & = & u|\nabla u|^2, \\ |u| & = & 1 \textit{ a.e.}, \\ u & = & g \textit{ on } \partial\Omega, \\ u(t=0) & = & \psi. \end{array}$$

On the other hand, if $\delta(\varepsilon)\to 0$ but $\frac{\epsilon}{\delta(\varepsilon)} \to 0$ then, $u_\varepsilon \to u_0$ in $L^p(0,T,H^1)$ strongly for all $p<\infty$ and $T<\infty$.

We can compare this result with that of lin [3] that we have recalled above.

Finally in the fourth section, we rescale u_ε in time:

$$v_\varepsilon := u_\varepsilon(x, \varepsilon t),$$

so that v_ε satisfies:

$$i\varepsilon v_{\varepsilon t} + \varepsilon^2 \Delta v_\varepsilon = -v_\varepsilon(1 - |v_\varepsilon|^2), \tag{8}$$

and we get (theorem 2)

Theorem *The fonction v_ε converges to v in $\mathcal{C}(0, T, L^2)$ strongly and in $L^\infty(0, T, H^1)$ weakly, where v is the solution of the wave equation for harmonic maps:*

$$\begin{cases} v_{tt} - 2\Delta v = -v(|v_t|^2 - 2|\nabla v|^2), \ |v| = 1 \ a.e., \\ \qquad v = g \ on \ \partial\Omega, \\ \qquad v(t = 0) = \psi, \\ \qquad v_t(t = 0) = 0. \end{cases}$$

We also give a formal computation of the next term of the asymptotic expansion in this case.

2 Some partial results on the dispersive case.

In this section, we consider

$$\begin{cases} iu_{\varepsilon t} + \Delta u_\varepsilon &= -u_\varepsilon \frac{1-|u_\varepsilon|^2}{\varepsilon^2} \text{ in } \mathbf{R}^+ \times \Omega, \\ u_\varepsilon(x, t) &= g(x) \text{ on } \partial\Omega, \\ u_\varepsilon(x, 0) &= \psi(x) \text{ in } \Omega, \end{cases} \tag{9}$$

where $deg(g, \partial\Omega) = 0$ and $\psi \in H^2(\Omega)$. Any solution in $\mathcal{C}([0, T], H^2)$ of (9) satisfies

$$\int_\Omega \frac{1}{2}|\nabla u_\varepsilon|^2 + \frac{(1 - |u_\varepsilon|^2)^2}{4\varepsilon^2} = \int_\Omega |\nabla\psi|^2. \tag{10}$$

We first have the

Proposition 1 *For any $\psi \in H^2$, there exists a unique global solution u_ε to (9) such that*

$$u_\varepsilon \in \mathcal{C}(\mathbf{R}^+, H^2) \cap \mathcal{C}^1(\mathbf{R}^+, L^2).$$

Moreover, u_ε satisfies (10).

Proof: This result is obtained using Brezis-Gallouet's technics [2]. ■

Note that the estimate (10) is strongly needed to prove the existence result. On the other hand, (10) is the only estimate that we know on this problem. This is why we restrict ourself to the case when $deg(g, \partial\Omega) = 0$.

Concerning the limit behavior, we have:

Proposition 2 *Suppose that $u_\varepsilon \to u$ as $\varepsilon \to 0$ in $L^2((0, T) \times \Omega)$ strongly, then $u(x, t) = u_0(x)$, the harmonic map defined by (3).*

Proof: Let us multiply (9) by $\bar{u}_\varepsilon$ and let us then take the imaginary part, one obtains:

$$\frac{1}{2}\frac{\partial}{\partial t}|u_\varepsilon|^2 + Im(\Delta u_\varepsilon \bar{u}_\varepsilon) = 0,$$

or equivalently

$$\frac{1}{2}\frac{\partial}{\partial t}|u_\varepsilon|^2 + Im\frac{\partial}{\partial x_j}(\frac{\partial u_\varepsilon}{\partial x_j}\bar{u}_\varepsilon) = 0. \tag{11}$$

Then, since u_ε is bounded in $L^\infty(\mathbf{R}^+, H^1)$, we can extract a subsequence that converges weakly in this space. We get

$$Im\frac{\partial}{\partial x_j}(\frac{\partial u_\varepsilon}{\partial x_j}\bar{u}_\varepsilon) \rightharpoonup Im\frac{\partial}{\partial x_j}(\frac{\partial u}{\partial x_j}\bar{u}) \text{ in } \mathcal{D}',$$

since $u_\varepsilon \rightarrow u$ in $L^2((0,T)\times\Omega)$ strongly. On the other hand, $|u_\varepsilon|^2 \rightarrow 1$ in $L^\infty(\mathbf{R}^+, L^2)$, so that

$$\frac{1}{2}\frac{\partial}{\partial t}|u_\varepsilon|^2 \rightharpoonup 0 \text{ in } \mathcal{D}'.$$

Hence (11) implies

$$Im\frac{\partial}{\partial x_j}(\frac{\partial u}{\partial x_j}\bar{u}) = 0,\ |u|^2 = 1 \text{ a.e. and } u \in L^\infty(0,T,H^1). \tag{12}$$

The results (12) classically implies that for each t, $u(.,t)$ is equal to the harmonic map. ■

So we are only left with proving that u_ε converges. In the most general setting, we are unable to prove or disprove this statement (see the remark at the end of the paper). However we have the

Proposition 3 *If the initial data* ψ *is identically equal to the harmonic map* u_0 *(defined by (3)), then* $u_\varepsilon(x,t) \rightarrow u_0(x)$ *in* $L^p(0,T,H^1)$ *for all* $p < \infty$ *and* $T < \infty$.

Proof: Let us first recall that the harmonic map is the unique minimizer to:

$$\min_{\begin{cases} u \in H^1(\Omega), \\ u_{|\partial\Omega} = g, \\ |u| = 1 \text{ a.e.} \end{cases}} \int_\Omega |\nabla u|^2. \tag{13}$$

We fix the time t. Since by (10), $u_\varepsilon(t)$ is bounded in $H^1(\Omega)$, we can extract a subsequence $u_{\varepsilon'}(t)$ that converges in $H^1(\Omega)$ weakly and in L^2 strongly: $u_{\varepsilon'}(t) \rightarrow v$. Moreover, $|v| = 1$ a.e. It follows by (10) that $\int_\Omega |\nabla v|^2 \leq \int_\Omega |\nabla u_0|^2$. The caracterization (13) then implies that $v = u_0$ and $\int_\Omega |\nabla v|^2 = \int_\Omega |\nabla u_0|^2$. Since for any subsequence, the limit is the same, the whole sequence u_ε converges. Hence, $u_\varepsilon(t) \rightarrow u_0$ in H^1 strongly and this is true for all $t \geq 0$. Finally, we remark that since u_ε is bounded in $L^\infty(\mathbf{R}, H^1)$, Lebesgue's theorem implies that $u_\varepsilon \rightarrow u_0$ in $L^p(0,T,H^1)$ for all $p < \infty$ and $T < \infty$. ■

These results are the only ones that we get on the original system (6). The remaining of this paper deals with modifications of this original system.

3 The dissipative case.

We introduce some viscosity in system (6) in the following way:

$$\left\{\begin{array}{c} u_{\varepsilon t} = (i+\delta(\varepsilon))(\Delta u_\varepsilon + u_\varepsilon \frac{1-|u_\varepsilon|^2}{\varepsilon^2}), \\ u_\varepsilon(x,0) = \psi(x), \\ u_\varepsilon = \psi = g(x), \text{ on } \partial\Omega, \\ |\psi| = 1 \text{ a.e., } \psi \in H^1(\Omega),\ deg(\psi, \partial\Omega) = 0. \end{array}\right. \tag{14}$$

The first result we get is the

Proposition 4 *We suppose that δ does not depend on ε. The solution u_ε to (14) satisfies:*
$u_\varepsilon \longrightarrow u$ in $\mathcal{C}(0,T,L^2)$ strongly and $L^\infty(0,T,H^1)$ weakly, where the limit u is the solution to

$$\frac{\delta}{1+\delta^2} u_t - \Delta u = u|\nabla u|^2,\ |u| = 1 \ a.e.\ ,\ u(0) = \psi.$$

Proof: We first compute some energy estimates. We first multiply (14) by $\bar{u}_{\varepsilon t}$ and integrate:

$$\int_\Omega |u_{\varepsilon t}|^2 = (i+\delta)(\int_\Omega \Delta u_\varepsilon \bar{u}_{\varepsilon t} + u_\varepsilon \frac{1-|u_\varepsilon|^2}{\varepsilon^2} \bar{u}_{\varepsilon t}). \tag{15}$$

Taking the real part of (15), we obtain

$$\int_\Omega |u_{\varepsilon t}|^2 = -Im(\int_\Omega \Delta u_\varepsilon \bar{u}_{\varepsilon t} + u_\varepsilon \frac{1-|u_\varepsilon|^2}{\varepsilon^2} \bar{u}_{\varepsilon t}) - \delta \frac{d}{dt}(\int_\Omega \frac{|\nabla u_\varepsilon|^2}{2} + \frac{(1-|u_\varepsilon|^2)^2}{4\varepsilon^2}). \tag{16}$$

We now take the imaginary part of (15):

$$0 = -\frac{d}{dt}(\int_\Omega \frac{|\nabla u_\varepsilon|^2}{2} + \frac{(1-|u_\varepsilon|^2)^2}{4\varepsilon^2}) + \delta Im(\int_\Omega \Delta u_\varepsilon \bar{u}_{\varepsilon t} + u_\varepsilon \frac{1-|u_\varepsilon|^2}{\varepsilon^2} \bar{u}_{\varepsilon t}). \tag{17}$$

A linear combination of (16) and (17) yields

$$\delta \int_\Omega |u_{\varepsilon t}|^2 + (1+\delta^2) \frac{d}{dt}(\int_\Omega \frac{|\nabla u_\varepsilon|^2}{2} + \frac{(1-|u_\varepsilon|^2)^2}{4\varepsilon^2}) = 0.$$

After integration in time, this leads to

$$\frac{\delta}{1+\delta^2} \int_0^T \int_\Omega |u_{\varepsilon t}|^2 dx\, dt + \int_\Omega \frac{|\nabla u_\varepsilon|^2}{2} + \frac{(1-|u_\varepsilon|^2)^2}{4\varepsilon^2} = \frac{1}{2} \int_\Omega |\nabla \psi|^2. \tag{18}$$

We now compute the equivalent of (11) for system (14). We first multiply (14) by $\bar{u}_\varepsilon$, this gives

$$\bar{u}_\varepsilon u_{\varepsilon t} = (i+\delta)(\Delta u_\varepsilon \bar{u}_\varepsilon + \frac{1-|u_\varepsilon|^2}{\varepsilon^2} |u_\varepsilon|^2). \tag{19}$$

Taking the imaginary part of (19) leads to

$$Im(\bar{u}_\varepsilon u_{\varepsilon t}) = Re(\Delta u_\varepsilon \bar{u}_\varepsilon + \frac{1-|u_\varepsilon|^2}{\varepsilon^2}|u_\varepsilon|^2) + \delta Im(\Delta u_\varepsilon \bar{u}_\varepsilon). \tag{20}$$

Taking the real part of (19) gives

$$\frac{\partial}{\partial t}\frac{|u_\varepsilon|^2}{2} = -Im(\Delta u_\varepsilon \bar{u}_\varepsilon) + \delta Re(\Delta u_\varepsilon \bar{u}_\varepsilon + \frac{1-|u_\varepsilon|^2}{\varepsilon^2}|u_\varepsilon|^2). \tag{21}$$

A linear combination of (20) and (21) leads to

$$\frac{1}{1+\delta^2}\frac{\partial}{\partial t}\frac{|u_\varepsilon|^2}{2} = Im(\frac{\delta}{1+\delta^2}\bar{u}_\varepsilon u_{\varepsilon t}) - (\bar{u}_\varepsilon u_{\varepsilon x_j})_{x_j}). \tag{22}$$

Now, by (18), u_ε is bounded in $L^\infty(\mathbf{R}^+, H^1)$ while $u_{\varepsilon t}$ is bounded in $L^2((0,T)\times\Omega)$. The classical compactness result implies that $u_\varepsilon \longrightarrow u$ in $\mathcal{C}(0,T,L^2)$ strongly and in $L^\infty(0,T,H^1)$ weakly. Moreover, by (18) we know that $|u| = 1$ a.e. We can now pass to the limit in (22) and find:

$$Im(\frac{\delta}{1+\delta^2}\bar{u}u_t - (\bar{u}u_{x_j})_{x_j}) = 0,$$

which is equivalent to

$$\frac{\delta}{1+\delta^2}u_t - \Delta u = u|\nabla u|^2, \; |u| = 1 \text{ a.e. } .$$

■

Remark: If we write $u = e^{i\theta}$, then θ satisfies:

$$\frac{\delta}{1+\delta^2}\frac{\partial\theta}{\partial t} - \Delta\theta = 0.$$

Hence, when $\delta \longrightarrow 0$, $\theta_\delta \longrightarrow \theta_0$ with $\Delta\theta_0 = 0$. Therefore, $u \longrightarrow u_0 \equiv$ the harmonic map. This is a motivation for the:

Theorem 1 *Suppose that* $\delta(\varepsilon) \longrightarrow 0$ *but that* $\frac{\varepsilon}{\delta(\varepsilon)} \longrightarrow 0$. *Then*

$$u_\varepsilon(x,t) \longrightarrow u_0(x),$$

in $L^p(0,T,H^1)$ *for all* $p < \infty$, $T < \infty$.

Proof: We first rescale u_ε as follows:

$$v_\varepsilon(x,t) := u_\varepsilon(x, \frac{\delta(\varepsilon)}{1+\delta(\varepsilon)^2}t).$$

Therefore

$$\int_0^T\int_\Omega |v_{\varepsilon t}|^2 dxdt = \frac{\delta(\varepsilon)}{1+\delta(\varepsilon)^2}\int_0^{\frac{\delta(\varepsilon)T}{1+\delta(\varepsilon)^2}}\int_\Omega |u_{\varepsilon t}(x,t)|^2 dxdt.$$

Equality (18) then becomes

$$\int_0^T \int_\Omega |v_{\varepsilon t}|^2 dxdt + \int_\Omega \frac{1}{2}|\nabla v_\varepsilon|^2 + \frac{(1-|v_\varepsilon|^2)^2}{4\varepsilon^2} = \int_\Omega \frac{1}{2}|\nabla \psi|^2.$$

Hence v_ϵ is bounded in $L^\infty(\mathbf{R}^+, H^1)$ and $v_{\varepsilon t}$ is bounded in $L^2((0,T)\times\Omega)$. Therefore (up to subsequences) the following convergences hold:

$$\begin{array}{rcl} v_\varepsilon \rightharpoonup v & \text{in} & L^\infty(\mathbf{R}^+, H^1) \text{ weakly,} \\ v_{\varepsilon t} \rightharpoonup v_t & \text{in} & L^\infty(\mathbf{R}^+, L^2) \text{ weakly,} \\ v_\varepsilon \to v & \text{in} & \mathcal{C}((0,T), L^2) \text{ strongly,} \end{array}$$

and $|v| = 1$ a.e.
Rewriting (22), we get

$$\frac{1}{\delta}\frac{\partial}{\partial t}\frac{|v_\varepsilon|^2}{2} = Im(\bar{v}_\varepsilon v_{\varepsilon t} - (\bar{v}_\varepsilon v_{\varepsilon x_j})_{x_j}).$$

On the other hand

$$\frac{1}{\delta}\frac{\partial}{\partial t}\frac{|v_\varepsilon|^2}{2} = \frac{\varepsilon}{2\delta}\frac{\partial}{\partial t}\frac{|v_\varepsilon|^2-1}{\varepsilon},$$

and since $\frac{|v_\varepsilon|^2-1}{\varepsilon}$ is bounded in $L^2((0,T)\times\Omega)$,

$$\frac{\varepsilon}{2\delta}\frac{\partial}{\partial t}\frac{|v_\varepsilon|^2-1}{\varepsilon} \to 0 \text{ in } H^{-1}((0,T)\times\Omega),$$

so that v satisfies:

$$Im(\bar{v}v_t - (\bar{v}v_{x_j})_{x_j}) = 0, \text{ and } |v| = 1 \text{ a.e.} \tag{23}$$

Hence v is a solution to

$$v_t - \Delta v = v|\nabla v|^2,\ |v| = 1.$$

Moreover,

$$v(t) \to u_0 \text{ in } H^1 \text{ strongly as } t \to \infty.$$

Coming back to u_ε, we get

$$\frac{\delta}{1+\delta^2}\int_0^T \int_\Omega |u_{\varepsilon t}|^2 dxdt + (\int_\Omega \frac{|\nabla u_\varepsilon|^2}{2} + \frac{(1-|u_\varepsilon|^2)^2}{4\varepsilon^2})(T) = \int_\Omega \frac{|\nabla\psi|^2}{2}. \tag{24}$$

On the other hand, the energy estimate on (23) yields

$$\int_0^T \int_\Omega |v_t|^2 + \int_\Omega \frac{|\nabla v|^2}{2}(T) = \int_\Omega \frac{|\nabla\psi|^2}{2}. \tag{25}$$

Replacing u_ε by v_ε in the first term of the left-hand side of (24) and using (25), we obtain

$$\begin{array}{c} \int_0^{\frac{T(1+\delta^2)}{\delta}} \int_\Omega |v_{\varepsilon t}|^2 dxdt + E_\varepsilon(u_\varepsilon)(T) \\ = \int_0^{\frac{T(1+\delta^2)}{\delta}} \int_\Omega |v_t|^2 + \int_\Omega \frac{|\nabla v|^2}{2}(T) - \int_T^{\frac{T(1+\delta^2)}{\delta}} \int_\Omega |v_t|^2, \end{array} \tag{26}$$

where $E_\varepsilon(u_\varepsilon) = \int \frac{1}{2}|\nabla u_\varepsilon|^2 + \frac{(1-|u_\varepsilon|^2)^2}{4\varepsilon^2}$. Using the energy estimate for v between times T and $\frac{T(1+\delta^2)}{\delta}$, equation (26) leads to

$$\int_0^{\frac{T(1+\delta^2)}{\delta}} \int_\Omega |v_{\varepsilon t}|^2 dxdt + E_\varepsilon(u_\varepsilon)(T) = \int_0^{\frac{T(1+\delta^2)}{\delta}} \int_\Omega |v_t|^2 + \int_\Omega \frac{|\nabla v|^2}{2}(\frac{T(1+\delta^2)}{\delta}). \tag{27}$$

Let us now fix a real number A and ε such that $\frac{T(1+\delta^2)}{\delta} \geq A$. Equality (27) gives

$$\int_0^A \int_\Omega |v_{\varepsilon t}|^2 dxdt + E_\varepsilon(u_\varepsilon)(T) \leq \int_0^\infty \int_\Omega |v_t|^2 + \int_\Omega \frac{|\nabla v|^2}{2}(\frac{T(1+\delta^2)}{\delta}).$$

Letting ε tends to zero in this last inequality, we get:

$$\int_0^A \int_\Omega |v_t|^2 dxdt + \limsup_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon)(T) \leq \int_0^\infty \int_\Omega |v_t|^2 + \int_\Omega \frac{|\nabla u_0|^2}{2},$$

since $v_{\varepsilon t} \rightharpoonup v_t$ in $L^2(0, A, L^2)$ weakly as $\varepsilon \to 0$ and since $v(x,t) \to u_0(x)$ in H^1 strongly as $t \to \infty$. Letting A tends to infinity yields

$$\limsup_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon)(T) \leq \int_\Omega \frac{|\nabla u_0|^2}{2}.$$

It follows that for fixed T, $u_\varepsilon(T) \to u_0$ in H^1 strongly, and therefore the whole sequence converges. The result follows by Lebesgue's theorem. ■

4 Semi-classical limit.

In this section, we come back to the original system (9) and rescale u_ε in time as follows:

$$v_\varepsilon(x,t) = u_\varepsilon(x, \epsilon t).$$

The function v_ε satisfies

$$i\varepsilon v_{\varepsilon t} + \varepsilon^2 \Delta v_\varepsilon + v_\varepsilon(1 - |v_\varepsilon|^2) = 0. \tag{28}$$

The result is

Theorem 2 *We have $v_\varepsilon \to v$ in $L^\infty(\mathbf{R}^+, H^1)$ weakly and in $\mathcal{C}(0, T, L^2)$ strongly, where v satisfies*

$$v_{tt} - 2\Delta v = -v(|v_t|^2 - 2|\nabla v|^2),\ |v| = 1 \ a.e.,$$

$$v(t=0) = \psi,$$

$$v_t(t=0) = 0,$$

or equivalently $v = e^{i\theta}$ and

$$\theta_{tt} - 2\Delta\theta = 0.$$

Proof: Let us multiply (28) by $\bar{v}_\varepsilon$ and take the imaginary part. One gets

$$-\frac{\partial}{\partial t}(\frac{1-|v_\varepsilon|^2}{\varepsilon}) + 2Im(\bar{v}_\varepsilon v_{\varepsilon x_j})_{x_j} = 0. \tag{29}$$

On the other hand,

$$\int_\Omega |\nabla v_\varepsilon|^2 + \frac{(1-|v_\varepsilon|^2)^2}{2\varepsilon^2} = \int_\Omega |\nabla \psi|^2,$$

so that v_ε is bounded in $L^\infty(\mathbf{R}^+, H^1)$ and $v_{\varepsilon t}$ is bounded in $L^\infty(\mathbf{R}^+, H^{-1})$ and we can extract subsequences such that $v_\varepsilon \longrightarrow v$ in $\mathcal{C}(0,T,L^2)$ strongly and $v_\varepsilon \rightharpoonup v$ in $L^\infty(0,T,H^1)$ weakly. On the other hand, (29) gives

$$\frac{1-|v_\varepsilon|^2}{\varepsilon} = 2\int_0^t Im(\bar{v}_\varepsilon v_{\varepsilon x_j})_{x_j}(\tau)d\tau. \tag{30}$$

The quantity $\frac{1-|v_\varepsilon|^2}{\varepsilon}$ is bounded in $L^\infty(0,T,L^2)$ hence it converges weakly to some function w and by (30),

$$\frac{1-|v_\varepsilon|^2}{\varepsilon} \rightharpoonup 2\int_0^t Im(\bar{v} v_{x_j})_{x_j}(\tau)d\tau \text{ in } \mathcal{D}',$$

hence $w = 2\int_0^t Im(\bar{v}v_{x_j})_{x_j}(\tau)d\tau$. Therefore, as ε goes to zero in (28) we get:

$$iv_t + 2v\int_0^t Im(\bar{v}v_{x_j})_{x_j}(\tau)d\tau = 0,$$

with $v(t=0) = \psi$, $v_t(t=0) = 0$ and $|v| = 1$. Writing $v = e^{i\theta}$ we obtain

$$-\theta_t + 2\int_0^t Im(i\Delta\theta)d\tau = 0,$$

or

$$\theta_{tt} - 2\Delta\theta = 0$$

and the result is proved. ■

We can obtain formally the next term of the asymptotic expansion of v_ε. Namely:

Proposition 5 *Let $v_\varepsilon = v + \varepsilon w + \varepsilon^2 ...$, then w formally satisfies:*

$$Re(w\bar{v}) = -\frac{1}{2}\theta_t,$$

and $u = Im(w\bar{v})$ satisfies

$$u_{tt} - 2\Delta u = -\theta_{tt}\theta_t,\ u(t=0) = 0,\ u_t(t=0) = Re(\Delta\psi\bar{\psi}) = -|\nabla\psi|^2.$$

Proof: We use the formal expansion of v_ε in (28), using (30), the terms of order 1 in ε are

$$iw_t+\Delta v+2v\int_0^t Im(v_{x_j}\bar{w})_{x_j}d\tau+2v\int_0^t Im(w_{x_j}\bar{v})_{x_j}d\tau+2w\int_0^t Im(v_{x_j}\bar{v})_{x_j}d\tau=0. \tag{31}$$

We remark that $2\int_0^t Im(v_{x_j}\bar{v})_{x_j}d\tau=\theta_t$. Multiplying (31) by $\bar{v}$ gives

$$iw_t\bar{v}+\Delta v\bar{v}+2\int_0^t Im(v_{x_j}\bar{w})_{x_j}d\tau+2\int_0^t Im(w_{x_j}\bar{v})_{x_j}d\tau+2w\bar{v}\theta_t=0. \tag{32}$$

On the other hand $\bar{v}_t=-i\theta_t\bar{v}$, so that (32) leads to

$$i(w\bar{v})_t+\Delta v\bar{v}+2\int_0^t Im(v_{x_j}\bar{w})_{x_j}d\tau+2\int_0^t Im(w_{x_j}\bar{v})_{x_j}d\tau=0. \tag{33}$$

We first take the imaginary part of (33):

$$Re(w\bar{v})_t=-Im(\Delta v\bar{v})=-\Delta\theta=-\frac{1}{2}\theta_{tt}.$$

Hence

$$Re(w\bar{v})=-\frac{1}{2}\theta_t, \tag{34}$$

since $w(t=0)=0$.

We take now the real part of (33):

$$-Im(w\bar{v})_t+Re(\Delta v\bar{v})+2\int_0^t Im(v_{x_j}\bar{w})_{x_j}d\tau+2\int_0^t Im(w_{x_j}\bar{v})_{x_j}d\tau=0. \tag{35}$$

Since $0=\Delta|v|^2=2Re(\Delta v\bar{v})+2|\nabla v|^2$, (35) becomes

$$Im(w\bar{v})_t-|\nabla v|^2-2\int_0^t Im(v_{x_j}\bar{w})_{x_j}d\tau-2\int_0^t Im(w_{x_j}\bar{v})_{x_j}d\tau=0.$$

We now introduce $u=Im(w\bar{v})$ and we differentiate the previous equation with respect to t:

$$u_{tt}+(|\nabla v|^2)_t-2Im(v_{x_j}\bar{w})_{x_j}-2Im(w_{x_j}\bar{v})_{x_j}=0.$$

The next step is then to compute the different terms:

$$\begin{aligned}-2Im(v_{x_j}\bar{w})_{x_j}-2Im(w_{x_j}\bar{v})_{x_j}&=-2\Delta Im(w\bar{v})\\+4Im(w_{x_j}\bar{v}_{x_j})+2Im(w\bar{v}_{x_jx_j})-2Im(v_{x_jx_j}\bar{w}),\end{aligned}$$

$$=-2\Delta u+4Im(w_{x_j}\bar{v}_{x_j})+4Im(w\bar{v}_{x_jx_j}).$$

The result now follows from a straightforward computation using the equation on v. ■

Remark: We would like to be able to say that the solution of the initial problem behaves like $v(x, t/\varepsilon)$, but we are not able to prove that result. However, this function could be a correct Ansatz. Indeed if we denote by $u_0 = e^{i\phi_0}$ (where u_0 is the harmonic map) and $v = e^{i\theta}$. Introducing $\theta_1 = \theta - \phi_0$, θ_1 satisfies:

$$\begin{cases} \theta_{1tt} - 2\Delta\theta_1 = 0, \\ \theta_1(t=0) = \phi_0 - \theta(t=0), \\ \theta_{1t}(t=0) = 0, \\ \theta_1 = 0 \text{ on } \partial\Omega. \end{cases}$$

Denoting by w_j and λ_j respectively the eigenfunctions and eigenvalues of -2Δ on Ω, we get

$$v(x, t/\varepsilon) = e^{i\phi_0} e^{i\sum_{j\geq 1} a_j w_j(x)\cos(\lambda_j t/\varepsilon)},$$

where a_j are the Fourier coefficients of $\theta_1(t=0)$.

Now, if $\theta(t=0) = \phi_0$, then all the a_j's are equal to 0 and $v(x, t/\varepsilon) = e^{i\phi_0}$ which is in agreement with the results of the first section.

If $\theta(t=0) \neq \phi_0$, let us take the explicit case (we do not know how to treat the others) $\theta(t=0) = \phi_0 + w_1(x)$. The function $e^{iw_1(x)\cos(\lambda_1 t/\varepsilon)}$ is bounded in $L^\infty(0, T, H^1)$ and converges weakly to

$$e^{iw_1(x)\cos(\lambda_1 t/\varepsilon)} \rightharpoonup \frac{\lambda_1}{2\pi}\int_0^{\frac{2\pi}{\lambda_1}} e^{i\cos(\lambda_1 t)w_1(x)}dt,$$

which is a non zero function, independent of the time t and whose modulus is a.e. strictly less than 1.

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