

# On the standing waves solutions to a nonlocal, nonlinear Schrödinger equation occuring in plasma Physics

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## Abstract

In this work, our goal is to study the standing waves solutions to some generalization of the following system.

$$\begin{cases} i\phi_t + \Delta\phi = -div(|\nabla\psi|^2\nabla\psi), \\ \Delta\psi = \phi. \end{cases}$$

We shall prove the existence of standing waves, solution to this equations and provide some stability results.

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## 1 Introduction and statement of the results.

### 1.1 Introduction.

In a previous work ([4]), we study the Cauchy problem for the following family of system:

$$\begin{cases} i\phi_t + \sum_{k,l=1}^3 a_{kl} \frac{\partial^2 \phi}{\partial x_k \partial x_l} = -\operatorname{div}(|\nabla \psi|^\sigma \nabla \psi) \\ \Delta \psi = \phi \\ \phi(x, 0) = \phi_0(x), \end{cases} \quad (1)$$

where  $a_{kl} = a_{lk}$  are real constants, the matrix  $(a_{kl})$  being nonsingular and  $\sigma > 0$ .

These systems are generalizations of the following ones which occurs in Plasma Physics (see [6]).

$$\begin{cases} i\phi_t + \Delta\phi = -\operatorname{div}(|\nabla\psi|^2\nabla\psi), \\ \Delta\psi = \phi. \end{cases} \quad (2)$$

If  $\phi$  is a solution of (1), one can show that the following quantities are invariants of the motion:

$$m(t) = \int_{R^3} |\nabla\psi(t)|^2 dx = m(0), \quad (3)$$

$$E(t) = \int_{R^3} \left( \frac{1}{2} \sum_{i=1}^3 q\left(\nabla \frac{\partial\psi}{\partial x_i}\right) - \frac{1}{\sigma+2} |\nabla\psi|^{\sigma+2} \right) dx = E(0), \quad (4)$$

where  $q$  is the following hermitian form:

$$q(\mathbf{u}) = \sum_{k,l=1}^3 a_{kl} u_k \bar{u}_l.$$

Indeed, multiplying the first equation of (1) by  $\bar{\psi}$  leads, after integration, to:

$$-i \int \nabla\psi_t \cdot \nabla\bar{\psi} + \int a_{kl} \frac{\partial^2 \bar{\psi}}{\partial x_k \partial x_l} \frac{\partial^2 \psi}{\partial x_i \partial x_i} = \int |\nabla\psi|^{\sigma+2}. \quad (5)$$

But  $\int a_{kl} \frac{\partial^2 \psi}{\partial x_i \partial x_i} \frac{\partial^2 \bar{\psi}}{\partial x_k \partial x_l}$  is real, indeed an integration by parts gives:

$$\int a_{kl} \frac{\partial^2 \psi}{\partial x_i \partial x_i} \frac{\partial^2 \bar{\psi}}{\partial x_k \partial x_l} = \int a_{kl} \frac{\partial^2 \psi}{\partial x_i \partial x_l} \frac{\partial^2 \bar{\psi}}{\partial x_k \partial x_i}.$$

Hence, taking the imaginary part of (5) leads to:

$$\operatorname{Re} \int \nabla\psi_t \cdot \nabla\bar{\psi} = 0$$

and we obtain (3). On the other hand, multiplying the first equation of (1) by  $\bar{\psi}_t$  and using the same method, we arrive at (with the summation convention):

$$-i \int \nabla\psi_t \cdot \nabla\bar{\psi}_t + \int a_{kl} \frac{\partial^2 \bar{\psi}_t}{\partial x_k \partial x_l} \frac{\partial^2 \psi}{\partial x_i \partial x_i} = \int |\nabla\psi|^\sigma \nabla\psi \cdot \nabla\bar{\psi}_t. \quad (6)$$

But:

$$\int a_{kl} \frac{\partial^2 \psi}{\partial x_i \partial x_i} \frac{\partial^2 \bar{\psi}_t}{\partial x_k \partial x_l} = \int a_{kl} \frac{\partial^2 \psi}{\partial x_i \partial x_l} \frac{\partial^2 \bar{\psi}_t}{\partial x_k \partial x_i},$$

therefore,

$$\begin{aligned} \operatorname{Re} \left( \int a_{kl} \frac{\partial^2 \psi}{\partial x_i \partial x_i} \frac{\partial^2 \bar{\psi}_t}{\partial x_k \partial x_l} \right) &= \frac{1}{2} \int \frac{d}{dt} a_{kl} \frac{\partial^2 \psi}{\partial x_i \partial x_i} \frac{\partial^2 \bar{\psi}}{\partial x_k \partial x_l} \\ &= \frac{1}{2} \frac{d}{dt} \int \sum_i q \left( \nabla \frac{\partial \psi}{\partial x_i} \right). \end{aligned}$$

Since

$$\operatorname{Re} \left( \int |\nabla \psi|^\sigma \nabla \psi \cdot \nabla \bar{\psi}_t \right) = (\sigma + 2) \frac{d}{dt} \left( \int |\nabla \psi|^{\sigma+2} \right),$$

we now obtain (4) using (6).

Let us recall the results that we have obtained concerning the Cauchy problem in [4].

We introduce

$$H = \{ \psi \in L^6 \cap C_0(R^3), \nabla \psi \in H^1 \},$$

endowed with the norm  $\| |\nabla \psi| \|_{H^1}$ .

**Theorem 1.** *Let  $0 < \sigma < 4$ .*

*\*Let  $\psi_0 \in H$ , then there exists an unique maximal solution on  $[0, T(\psi_0)[$   $\psi \in C([0, T(\psi_0)[, H)$  to:*

$$\begin{cases} i(\nabla \psi)_t + L(\nabla \psi) = \nabla(-\Delta)^{-1} \operatorname{div}(|\nabla \psi|^\sigma \nabla \psi) \\ \psi(x, 0) = \psi_0(x) \end{cases}$$

*\*Moreover,  $\phi = \Delta \psi \in L^r(0, t, L^{\sigma+2})$  with  $\frac{2}{r} + \frac{3}{\sigma+2} = \frac{3}{2}$  for all  $t < T$ .*

*\*The function  $\psi$  is a solution to:*

$$\begin{cases} i\phi_t + L\phi = -\operatorname{div}(|\nabla \psi|^\sigma \nabla \psi) \\ \nabla \psi = \phi \\ \phi(x, 0) = \phi_0(x) \end{cases}$$

*and if  $0 < \sigma < 3$ , it is the only solution in  $C([0, T[, H)$ .*

*\*The solution  $\psi$  depends continuously on  $\psi_0$  in  $C([0, T[, H)$  in the following sense: if  $\psi_0^n \rightarrow \psi_0$  in  $H$  then for all  $T < T(\psi_0)$ , if  $n$  is sufficiently large, the corresponding solutions exist on a common interval  $[0, T]$  and  $\psi_n \rightarrow \psi$  in  $C([0, T], H)$ .*

We have a result of regularity:

**Theorem 2.** *If  $\psi_0 \in H$  with  $\nabla\psi_0 \in H^2$ , then the solution given by Theorem 1 satisfies*

$$\nabla\psi \in C([0, T(\psi_0)[, H^2).$$

In some cases, we are able to prove that the solution exists globally for all  $t > 0$ .

**Theorem 3.** *a) Let  $\psi_0 \in H$ , then the solution of (1) satisfies:*

$$m(t) = \int_{R^3} |\nabla\psi(t)|^2 dx = m(0),$$

$$E(t) = \int_{R^3} \left( \frac{1}{2} \sum_{i=1}^3 q \left( \nabla \frac{\partial\psi}{\partial x_i} \right) - \frac{1}{\sigma+2} |\nabla\psi|^{\sigma+2} \right) dx = E(0),$$

where  $q$  is the following hermitian form:

$$q(u) = \sum_{k,l=1}^3 a_{kl} u_k \bar{u}_l.$$

*b) If the matrix  $(a_{kl})$  is negative, then  $\|\psi\|_H$  remains bounded and the solution is global in time.*

*c) If the matrix  $(a_{kl})$  is positive, then if  $\sigma < \frac{4}{3}$ ,  $\|\psi\|_H$  remains bounded and the solution is global in time. If  $\sigma \geq \frac{4}{3}$ , then if  $\|\psi_0\|_H$  is sufficiently small,  $\|\psi\|_H$  remains bounded and the solution is global in time.*

*d) If  $\psi$  is a solution of (2), if  $\nabla\psi_0 \in H^{m+1} \cap W^{m+1, \frac{6}{5}}$  for  $m \geq 4$ , then there exists  $\delta > 0$  such that, if*

$$\|\nabla\bar{\psi}_0\|_{H^{m+1}} + \|\nabla\psi_0\|_{W^{m+1, \frac{6}{5}}} < \delta,$$

then the solution is global in time and there exists  $\bar{\phi}$  satisfying

$$\begin{cases} i\bar{\phi}_t + \Delta\bar{\phi} = 0 \\ \Delta\bar{\psi} = \bar{\phi}, \end{cases}$$

such that:

$$\begin{cases} \|\nabla\psi\|_{W^{m-2,6}} \leq \frac{C}{1+t} \\ \|\nabla\psi\|_{H^{m+1}} \leq C \\ \|\nabla\psi - \nabla\bar{\psi}\|_{H^{m+1}} \leq \frac{C}{1+t}. \end{cases}$$

We have the following finite-time blow-up result:

**Theorem 4.** *Let  $\psi_0 \in H$  be such that  $\int |x|^2 |\nabla \psi|^2 < \infty$ , then the solution of (1) with  $L = \Delta$  satisfies*

$$|x| |\nabla \psi| \in L^\infty(0, t, L^2) \cap L^r(0, t, L^{\sigma+2}),$$

for all  $t < T(\psi_0)$ .

*There exists some radial initial values such that the corresponding solutions blow up in finite time.*

## 1.2 Statement of the results.

The main results of this work are:

**Theorem 5.** *Let us suppose that the matrix  $a_{ij}$  is positive, and let  $0 < \sigma < 4$ . For every  $\omega > 0$ , there exists a function  $\psi_\omega \in H$  satisfying:*

$$\begin{cases} -\omega \phi_\omega + L\phi_\omega = -\operatorname{div}(|\nabla \psi_\omega|^\sigma \nabla \psi_\omega) \\ \Delta \psi_\omega = \phi_\omega \end{cases}$$

*i.e.  $e^{i\omega t} \psi_\omega(x)$  satisfies (1).*

*Moreover,  $\psi_\omega$  is a solution to the following minimization problem:*

$$(P1) \inf \left\{ - \int |\nabla \psi|^{\sigma+2}, \psi \in H, \int |\nabla \psi|^2 = \lambda, \int \sum_{i=1}^3 q \left( \nabla \frac{\partial \psi}{\partial x_i} \right) = \mu \right\}$$

*for convenient  $\lambda, \mu$  and  $\nabla \psi \in \mathcal{C}^2(\mathbb{R}^3)$ .*

*Moreover*

$$\int |\nabla \psi_\omega|^{\sigma+2}, \int \sum_{i=1}^3 q \left( \nabla \frac{\partial \psi_\omega}{\partial x_i} \right) \text{ and } \int |\nabla \psi_\omega|^2$$

*are independent of the solution  $\psi_\omega$  of (P1) that we consider.*

We will prove this theorem in next section, adapting to our functional setting the concentration-compactness arguments of P.L. Lions [5]. About standing waves, let us mention the following result.

**Theorem 6.** For all  $\omega > 0$  and  $0 < \sigma < 4$ , there exists radial functions  $\psi_\omega$  such that  $\frac{\partial}{\partial r}\psi_\omega \in \mathcal{C}^2(\mathbb{R}^3)$  satisfying:

$$\begin{cases} -\omega\phi_\omega + \Delta\phi_\omega = -\operatorname{div}(|\nabla\psi_\omega|^\sigma\nabla\psi_\omega) \\ \Delta\psi_\omega = \phi_\omega. \end{cases}$$

We prove this theorem using compactness properties of spaces of radial functions (see Strauss [8]). Moreover, these radial standing waves are related to the finite time blow up by the following proposition.

**Proposition 1** Let  $\sigma = 4/3$ , then for all  $t_0 > 0$ , there exists a radial function  $\psi_0$  so that the corresponding solution to:

$$\begin{cases} i\phi_t + \Delta\phi = -\operatorname{div}(|\nabla\psi|^{4/3}\nabla\psi) \\ \Delta\psi = \phi \end{cases}$$

blows up exactly at  $t = t_0$ .

We can now state some stability results for these standing waves:

**Theorem 7.** \*If  $0 < \sigma < \frac{4}{3}$ , we consider

$$(P2) \inf\left\{\frac{1}{2}\int\sum_{i=1}^3q\left(\nabla\frac{\partial\psi}{\partial x_i}\right)-\frac{1}{\sigma+2}\int|\nabla\psi|^{\sigma+2}+\frac{\omega}{2}\int|\nabla\psi|^2,\int|\nabla\psi|^2=\int|\nabla\psi_\omega|^2\right\}$$

where  $\psi_\omega$  is a solution of (P1). Then (P1) and (P2) are equivalent.

\*The set  $S_\omega$  of standing waves (solutions of (P1) or (P2)) is stable, ie for all  $\epsilon > 0$  there exists  $\delta > 0$  such that

$$\forall\psi_0 \in H, \inf_{\psi_\omega \in S_\omega} \|\psi_0 - \psi_\omega\|_H < \delta \rightarrow \inf_{\psi_\omega \in S_\omega} \|\psi(\cdot, t) - \psi_\omega\|_H \leq \epsilon.$$

The first part of this theorem will be proved using concentration-compactness methods, the second part adapting the method of Cazenave-Lions [2].

**Theorem 8.** \*For  $0 < \sigma < 4$  we consider:

$$(P3) \inf\left\{\frac{1}{2}\int\sum_{i=1}^3q\left(\nabla\frac{\partial\psi}{\partial x_i}\right)-\frac{1}{\sigma+2}\int|\nabla\psi|^{\sigma+2}+\frac{\omega}{2}\int|\nabla\psi|^2,\right.$$

$$\int \sum_{i=1}^3 q(\nabla \frac{\partial \psi}{\partial x_i}) = \int \sum_{i=1}^3 q(\nabla \frac{\partial \psi_\omega}{\partial x_i})$$

where  $\psi_\omega$  is a solution of (P1). Then (P1) and (P3) are equivalent.

\*If  $\sigma > \frac{4}{3}$ , the orbite  $e^{i\theta}\psi_\omega$  is unstable, i.e. there exists  $\epsilon_0 > 0$  and  $\psi_n^0 \rightarrow \psi_\omega$  in  $H$  such that:

$$\sup_{t>0} \inf_{\theta \in \mathbb{R}} \|\psi_n(\cdot, t) - e^{i\theta}\psi_\omega\|_H \geq \epsilon_0.$$

The first part of this theorem will be proved using concentration-compactness methods. For the second part, we will prove that our equations enter in the framework of Shatah-Strauss [7].

The results of this paper were announced in [3].

## 2 Standing waves solutions

We want to find solutions to

$$\begin{cases} i\phi_t + L\phi = -\operatorname{div}(|\nabla\psi|^\sigma \nabla\psi) \\ \Delta\psi = \phi \end{cases}$$

of the form  $e^{i\omega t}u(x)$ , where  $L$  is elliptic. Therefore if we make a scaling, we can restrict ourself to the case where  $L = \Delta$ ; i.e. we want to solve:

$$\begin{cases} -\omega u + \Delta u = -\operatorname{div}(|\nabla v|^\sigma \nabla v) \\ \Delta v = u \end{cases} \quad (7)$$

In order to solve (7), we introduce the following minimization problem:

$$\inf \left\{ -\int |\nabla\psi|^{\sigma+2}, \phi \in H, \int |\nabla\psi|^2 = 1, \int |\Delta\psi|^2 = 1 \right\},$$

where  $H = \{\psi \in L^6, \nabla\psi \in H^1\}$ .

If  $\psi$  is a solution of the minimization problem, then there exists two numbers  $\alpha$  and  $\beta$  (Lagrange mulipliers) such that:

$$\beta\Delta^2\psi - \alpha\Delta\psi = -\operatorname{div}(|\nabla\psi|^\sigma \nabla\psi).$$

A suitable scaling will give a solution of (7).

## 2.1 Concentration-compactness lemma

A few notations are in order.

We set

$$I(\lambda, \mu) = \inf(- \int |\nabla \psi|^{\sigma+2}, \psi \in S_{\lambda, \mu}),$$

for  $\lambda, \mu > 0$ , and

$$S_{\lambda, \mu} = \{\psi \in H, \int |\nabla \psi|^2 = \lambda, \int |\Delta \psi|^2 = \mu\}.$$

We now have:

**Lemma 1.**

$$I(\lambda, \mu) = I(1, 1)\lambda^{1-\sigma/4}\mu^{3\sigma/4}$$

and  $-\infty < I(1, 1) < 0$ .

**Proof:** The Lemma follows obviously from the Gagliardo-Nirenberg's inequality:

$$|\nabla \psi|_{L^{\sigma+2}}^{\sigma+2} \leq C|\Delta \psi|_{L^2}^{3\sigma/2}|\nabla \psi|_{L^2}^{2-\sigma/2}.$$

■

We will use the concentration-compactness lemma of P.L. Lions (see [5]). The aim of this section is to prove the following version of this lemma:

**Lemma 2.** 1) Let  $\rho_n$  a sequence in  $L^1(\mathbb{R}^3)$  satisfying  $\rho_n \geq 0$  in  $\mathbb{R}^3$  and  $\int \rho_n dx = \lambda$  for a  $\lambda > 0$  fixed. Then there exists a subsequence  $\rho_{n_k}$  satisfying one of the three following alternatives:

i) (Compactness) : there exists  $y_{n_k} \in \mathbb{R}^3$  such that  $\forall \epsilon > 0, \exists R < \infty$  such that:

$$\int_{y_{n_k} + B_R} \rho_{n_k}(x) dx \geq \lambda - \epsilon.$$

ii) (Vanishing) :  $\forall R < \infty$

$$\sup_{y \in \mathbb{R}^3} \int_{y + B_R} \rho_{n_k} \rightarrow 0$$

when  $k \rightarrow \infty$ .

iii) (Dichotomy) : there exists  $\alpha \in ]0, \lambda[$  such that  $\forall \epsilon > 0$ , there exists  $\rho_{n_k}^1, \rho_{n_k}^2 \in L^1_+(R^3)$  such that if  $k \geq k_0$  :

$$\left| \int \rho_{n_k}^1 dx - \alpha \right| < \epsilon, \quad |\rho_{n_k} - (\rho_{n_k}^1 + \rho_{n_k}^2)|_{L^1} \leq \epsilon,$$

$\rho_{n_k}^1$  has compact support and  $\text{dist}(\text{supp} \rho_{n_k}^1, \text{supp} \rho_{n_k}^2) \rightarrow \infty$  as  $k \rightarrow \infty$ .

2) Moreover, if  $\rho_n = |\nabla u_n|^2$ , with  $u_n \in H$ , if  $u_n$  is bounded in  $H$  and if iii) occurs, then one can take  $\rho_{n_k}^i = (|\nabla u_{n_k}^i|^2)^2$  for  $i = 1, 2$  and  $u_{n_k}^i$  are bounded in  $H$  and satisfy

$$\int |\Delta u_{n_k}|^2 \geq \int (|\Delta u_{n_k}^1|^2 + |\Delta u_{n_k}^2|^2) - \epsilon$$

and

$$|\nabla u_{n_k} - (\nabla u_{n_k}^1 + \nabla u_{n_k}^2)|_{L^p} \leq \delta_p(\epsilon),$$

with  $\delta_p(\epsilon) \rightarrow 0$  when  $\epsilon \rightarrow 0$  for  $2 \leq p < 6$ .

**Proof:** The 1) is lemma I1 of P.L. Lions [5] part I. The 2) is an adaptation of lemma III1 in [5] part I.

1) Let us recall briefly how to prove the first part, (see [5] for details). We introduce:

$$Q_n(t) = \sup_{y \in R^3} \int_{y+B_t} \rho_n(x) dx$$

The  $Q_n$  are nondecreasing, nonnegative functions, uniformly bounded on  $R^+$ . We can extract a subsequence  $Q_{n_k}(t) \rightarrow Q(t)$  for every  $t \geq 0$ , where  $Q$  is a nondecreasing, nonnegative function. Let  $\alpha = \lim_{t \rightarrow \infty} Q(t) \in [0, \lambda]$ .

If  $\alpha = 0$  then  $Q \equiv 0$  and vanishing occurs.

If  $\alpha = \lambda$ , no mass disappears : compactness occurs ( see [5] for details).

If  $0 < \alpha < \lambda$ : some mass disappears, and dichotomy occurs. Let us explain why: let  $\epsilon > 0$ , choose  $R$  such that  $Q(R) > \alpha - \epsilon$ . If  $k$  is sufficiently large we have:  $\alpha - \epsilon < Q_{n_k}(R) < \alpha + \epsilon$ . Moreover, one can find  $R_k \rightarrow \infty$  such that  $Q_{n_k}(R_k) \leq \alpha + \epsilon$ . Finally, there exists  $y_k \in R^3$  such that  $\int_{y_k+B_R} \rho_{n_k}(x) dx \in ]\alpha - \epsilon, \alpha + \epsilon[$ . Let  $\rho_{n_k}^1 = \rho_{n_k} 1_{y_k+B_R}$  and  $\rho_{n_k}^2 = \rho_{n_k} 1_{R^3 - (y_k+B_{R_k})}$  then

$$\int \{\rho_{n_k} - \rho_{n_k}^1 - \rho_{n_k}^2\} dx = \int_{R \leq |x-y_k| \leq R_k} \rho_{n_k} dx \leq (Q_{n_k}(R_k) + \epsilon) - (Q_{n_k}(R) - \epsilon) \leq 4\epsilon$$

The condition on the supports is satisfied and this implies that  $\rho_{n_k}^1 \rho_{n_k}^2 = 0$  a.e.

2) To prove 2), we make a truncation of  $\rho_{n_k} = |\nabla u_{n_k}|^2$  with regular functions: let  $\epsilon > 0$ ,  $\xi \in \mathcal{D}(R^3)$ ,  $0 \leq \xi \leq 1$ , with  $\xi \equiv 1$  if  $|x| \leq 1$  and  $\xi \equiv 0$  if  $|x| \geq 2$ . Let  $\phi = 1 - \xi$ ,  $\xi_\mu = \xi(\frac{\cdot}{\mu})$  and  $\phi_\mu = \phi(\frac{\cdot}{\mu})$ . We take  $R_0$  such that  $Q(R) \leq \alpha - \epsilon$  for  $R \geq R_0$ . Let  $v \in H$  such that  $|v|_H \leq M$  where  $M \geq \sup_n |u_n|_H$ .

We then have:

**Lemma 3.** *There exists a constant  $C(M)$  such that:*

$$\left| \int |\Delta(\xi_R v)|^2 dx - \int \xi_R^2 |\Delta v|^2 dx \right| \leq \frac{C(M)}{R} \quad (8)$$

and

$$\left| \int |\nabla(\xi_R v)|^2 dx - \int \xi_R^2 |\nabla v|^2 dx \right| \leq \frac{C(M)}{R} \quad (9)$$

**Proof:** Let us prove the first one for example. We have:

$$\begin{aligned} \left| \int |\Delta(\xi_R v)|^2 dx - \int \xi_R^2 |\Delta v|^2 dx \right| &= \int \Delta \xi_R^2 v^2 + 4 \int (\nabla \xi_R \cdot \nabla v)^2 \\ &+ 4 \int \Delta \xi_R v \nabla \xi_R \cdot \nabla v + 4 \int \nabla \xi_R \cdot \nabla v \xi_R \Delta v + 2 \int \Delta \xi_R v \xi_R \Delta v, \end{aligned}$$

using Hölder's and Cauchy-Schwartz inequalities we obtain that this last quantity is lower or equal to:

$$\begin{aligned} &\left( \int (\Delta \xi_R)^3 \right)^{1/3} \left( \int v^6 \right)^{1/6} + 4 \int \nabla \xi_R^2 \nabla v^2 + 4 \left( \int \Delta \xi_R^3 \right)^{1/3} \left( \int v^6 \right)^{1/6} \left( \int \nabla \xi_R^2 \nabla v \right)^{1/2} \\ &+ 4 \left( \int \nabla \xi_R^2 \nabla v^2 \right)^{1/2} \left( \int \xi_R^2 \Delta v^2 \right)^{1/2} + 2 \left( \int \Delta \xi_R^3 \right)^{1/3} \left( \int v^6 \right)^{1/6} \left( \int \xi_R^2 \Delta v^2 \right)^{1/2}. \end{aligned}$$

But  $\nabla \xi_R(x) = \frac{1}{R} \nabla \xi(\frac{x}{R})$  and  $\Delta \xi_R(x) = \frac{1}{R^2} \Delta \xi(\frac{x}{R})$ . Moreover  $\xi, \nabla \xi, \Delta \xi$  are bounded and  $\int |\nabla v|^2, \int |\Delta v|^2 \leq M^2$ . Since  $(\int v^6)^{1/6} \leq C(\int |\nabla v|^2)^{1/2}$  for all  $v \in H$ , the above expression is bounded by  $\frac{C(M)}{R}$  as claimed.  $\blacksquare$

Choose  $R_1 \geq R_0$  such that  $\frac{C(M)}{R_1} \leq \epsilon$ , thus

$$Q(R_1) \geq \alpha - \epsilon. \quad (10)$$

If  $k$  is sufficiently large, then

$$Q_{n_k}(R_1) \leq \int_{y_k + B_{R_1}} |\nabla u_{n_k}|^2 + \epsilon. \quad (11)$$

We define  $u_{n_k}^1 = \xi_{R_1}(\cdot - y_k)u_{n_k} \in H$  and:

$$\begin{aligned} \left| \int |\nabla u_{n_k}^1|^2 dx - \alpha \right| &\leq \left| \int |\nabla u_{n_k}^1|^2 dx - \int \xi_{R_1}^2(\cdot - y_k) |\nabla u_{n_k}|^2 dx \right| \\ &\quad + \left| \int \xi_{R_1}^2(\cdot - y_k) |\nabla u_{n_k}|^2 dx - \alpha \right|. \end{aligned}$$

(9) implies that this last quantity is smaller than:

$$\epsilon + \left| \int_{y_k + B_{R_1}} |\nabla u_{n_k}|^2 - \alpha \right| \leq 3\epsilon,$$

by (10) and (11).

We may find  $R_k \rightarrow \infty$  such that

$$Q_{n_k}(2R_k) \leq \alpha + 2\epsilon, \quad (12)$$

take  $\phi_k = \phi_{R_k}(\cdot - y_k)$  and  $u_{n_k}^2 = \phi_k u_{n_k} \in H$ . We have the condition on the supports and

$$\begin{aligned} \int |\nabla u_{n_k} - (\nabla u_{n_k}^1 + \nabla u_{n_k}^2)|^2 dx &= \int |\nabla(\{1 - \xi_{R_1}(x - y_k) - \phi_k\}u_{n_k})|^2 dx \\ &\leq \frac{2C(M)}{R_1} + \int \{1 - \xi_{R_1}(\cdot - y_k) - \phi_k\}^2 |\nabla u_{n_k}|^2 dx \\ &\leq 2\epsilon + \int_{R_1 \leq |x - y_k| \leq 2R_k} |\nabla u_{n_k}|^2 dx, \end{aligned}$$

using (11):

$$\leq 2\epsilon + Q_{n_k}(2R_k) - Q_{n_k}(R_1) + \epsilon,$$

by (12):

$$\leq 3\epsilon + \alpha + 2\epsilon - Q_{n_k}(R_1).$$

But if  $k \geq k_0$ ,  $|Q_{n_k}(R_1) - \alpha| \leq \epsilon$ , this implies that

$$\int |\nabla u_{n_k} - (\nabla u_{n_k}^1 + \nabla u_{n_k}^2)|^2 dx \leq 6\epsilon$$

Now interpolating  $L^p$  between  $L^2$  and  $L^6$  for  $2 \leq p < 6$  leads to

$$|\nabla u_{n_k} - (\nabla u_{n_k}^1 + \nabla u_{n_k}^2)|_{L^p} \leq \delta_p(\epsilon).$$

Moreover

$$\left| \int |\Delta u_{n_k}^1|^2 dx - \int \xi_{R_1}^2(x - y_k) |\Delta u_{n_k}|^2 dx \right| \leq \epsilon,$$

and lemma 2 is proved. ■

## 2.2 Application of the concentration-compactness lemma.

The aim of this section is to prove:

**Proposition 2.** *For all  $\lambda, \mu > 0$ , the following minimization problem:*

$$\inf\left\{-\int |\nabla\psi|^{\sigma+2}, \psi \in H, \int |\nabla\psi|^2 = \lambda, \int |\Delta\psi|^2 = \mu\right\}$$

has a solution.

Moreover, every minimizing sequence is relatively compact in  $H$  up to translations.

**Proof:** We follow [5]. Take  $\psi_n \in H$  satisfying  $-\int |\nabla\psi_n|^{\sigma+2} \rightarrow I(\lambda, \mu)$  with  $\int |\nabla\psi_n|^2 = \lambda$  and  $\int |\Delta\psi_n|^2 = \mu$ . Let us apply lemma 2 with  $\rho_n = |\nabla\psi_n|^2$ .

Suppose that *iii*) occurs: We apply 2) of lemma 2: there exists  $\psi_{n_k}^1, \psi_{n_k}^2$  bounded in  $H$  such that for  $k \geq k_0$

$$|\nabla\psi_{n_k} - (\nabla\psi_{n_k}^1 + \nabla\psi_{n_k}^2)|_{L^{\sigma+2}} \leq \delta_{\sigma+2} \rightarrow 0.$$

On the other hand, since  $\nabla\psi_{n_k}^i$  is bounded in  $H^1$  and hence in  $L^{\sigma+2}$ , we see that

$$\int |\nabla\psi_{n_k}|^{\sigma+2} \leq K\delta_{\sigma+2}(\epsilon) + \int |\nabla\psi_{n_k}^1 + \nabla\psi_{n_k}^2|^{\sigma+2}.$$

Moreover  $\text{dist}(\text{supp}\nabla\psi_{n_k}^1, \text{supp}\nabla\psi_{n_k}^2) \rightarrow \infty$  so that  $(\nabla\psi_{n_k}^1)^2(\nabla\psi_{n_k}^2)^2 = 0$  a.e. This leads to

$$\int (\nabla\psi_{n_k}^1 + \nabla\psi_{n_k}^2)^{\sigma+2} = \int |\nabla\psi_{n_k}^1|^{\sigma+2} + \int |\nabla\psi_{n_k}^2|^{\sigma+2}.$$

Finally we obtain

$$-\int |\nabla\psi_{n_k}|^{\sigma+2} \geq -K\delta_{\sigma+2}(\epsilon) - \int |\nabla\psi_{n_k}^1|^{\sigma+2} - \int |\nabla\psi_{n_k}^2|^{\sigma+2}.$$

Then:

$$\begin{aligned} -\int |\nabla\psi_{n_k}|^{\sigma+2} &\geq -K\delta_{\sigma+2}(\epsilon) + I(\int |\nabla\psi_{n_k}^1|^2, \int |\Delta\psi_{n_k}^1|^2) \\ &\quad + I(\int |\nabla\psi_{n_k}^2|^2, \int |\Delta\psi_{n_k}^2|^2), \end{aligned} \tag{13}$$

with

$$\left| \int |\nabla\psi_{n_k}^1|^2 - \alpha \right| \leq \epsilon,$$

$$|\int |\nabla \psi_{n_k}^2|^2 - (\lambda - \alpha)| \leq \epsilon$$

and

$$\mu = \int |\Delta \psi_{n_k}|^2 \geq \int |\Delta \psi_{n_k}^1|^2 + \int |\Delta \psi_{n_k}^2|^2 - \epsilon.$$

Hence possibly extracting subsequences:

$$\begin{cases} \int |\nabla \psi_{n_k}^1|^2 \rightarrow_{k \rightarrow \infty} \bar{\alpha}_\epsilon \\ \int |\nabla \psi_{n_k}^2|^2 \rightarrow_{k \rightarrow \infty} \bar{\beta}_\epsilon \\ \int |\Delta \psi_{n_k}^1|^2 \rightarrow_{k \rightarrow \infty} \mu_1^\epsilon \\ \int |\Delta \psi_{n_k}^2|^2 \rightarrow_{k \rightarrow \infty} \mu_2^\epsilon, \end{cases}$$

with

$$\begin{aligned} |\bar{\alpha}_\epsilon - \alpha| &\leq \epsilon \\ |\bar{\beta}_\epsilon - (\beta - \alpha)| &\leq \epsilon \\ \mu &\geq \mu_1^\epsilon + \mu_2^\epsilon - \epsilon. \end{aligned}$$

We let  $k \rightarrow \infty$  in (13) and we obtain:

$$I(\lambda, \mu) \geq -K\delta_{\sigma+2}(\epsilon) + I(\bar{\alpha}_\epsilon, \mu_1^\epsilon) + I(\bar{\beta}_\epsilon, \mu_2^\epsilon).$$

Since  $\mu_1^\epsilon + \mu_2^\epsilon - \epsilon \leq \mu$ , possibly extracting subsequences:  $\mu_1^\epsilon \rightarrow \mu_1$ ,  $\mu_2^\epsilon \rightarrow \mu_2$ , with  $0 \leq \mu_1 + \mu_2 \leq \mu$ . This leads to

$$I(\lambda, \mu) \geq I(\alpha, \mu_1) + I(\lambda - \alpha, \mu_2).$$

Using the explicit value of  $I(\lambda, \mu)$  given by lemma 1, we get that

$$\mu^{3\sigma/4} \leq \left(\frac{\alpha}{\lambda}\right)^{1-\sigma/4} \mu_1^{3\sigma/4} + \left(1 - \frac{\alpha}{\lambda}\right)^{1-\sigma/4} \mu_2^{3\sigma/4} < \mu_1^{3\sigma/4} + \mu_2^{3\sigma/4}$$

which is a contradiction since  $\mu \geq \mu_1 + \mu_2$ . So dichotomy does not occur.

*Suppose that vanishing occurs:* We use lemma II part II of P.L. Lions [5]:

**Lemma 4.** *Let  $1 < p \leq \infty$ ,  $1 \leq q < \infty$  with  $q \neq \frac{Np}{N-p}$  if  $p < N$ . Suppose that  $u_n$  is bounded in  $L^q(\mathbb{R}^N)$ , and  $\nabla u_n$  is bounded in  $L^p(\mathbb{R}^n)$  and*

$$\sup_{y \in \mathbb{R}^N} \int_{y+B_R} |u_n|^q dx \rightarrow_{n \rightarrow \infty} 0$$

for one  $R > 0$ .

*Then  $u_n \rightarrow 0$  as  $n \rightarrow \infty$  in  $L^\alpha(\mathbb{R}^N)$  for all  $\alpha \in ]q, \frac{Np}{N-p}[$*

For the proof see [5].

In our case  $p = 2$ ,  $N = 3$ ,  $q = 2$ . Then  $\nabla u_n \rightarrow 0$  in  $L^{\sigma+2}$ , this is a contradiction since  $I(\lambda, \mu) < 0$ , and vanishing does not occur.

Hence i) occurs:

\*We define  $\tilde{\psi}_n = \psi_n(\cdot + y_n)$ , possibly extracting a subsequence we have  $\nabla \tilde{\psi}_n \rightharpoonup \nabla \tilde{\psi}$  in  $H^1$  and  $L^{\sigma+2}$  weak. Moreover  $\lambda \geq \int_{B_R} |\nabla \tilde{\psi}_n|^2 dx \geq \lambda - \epsilon$ ; then since  $\nabla \tilde{\psi}_n$  is bounded in  $H^1_{loc}$ ,  $\nabla \tilde{\psi}_n \rightarrow \nabla \tilde{\psi}$  in  $L^2_{loc}$  strong and this leads to:

$$\lambda \geq \int_{B_R} |\nabla \tilde{\psi}|^2 \geq \lambda - \epsilon.$$

Letting  $R \rightarrow \infty$  and  $\epsilon \rightarrow 0$  gives:

$$\int |\nabla \tilde{\psi}|^2 = \lambda = \lim_n \int |\nabla \tilde{\psi}_n|^2.$$

This implies that  $\nabla \tilde{\psi}_n \rightarrow \nabla \tilde{\psi}$  in  $L^2$  strong.

\*But  $\nabla \tilde{\psi}_n$  is bounded in  $H^1$  and hence in  $L^6$ , interpolation implies that:  $\nabla \tilde{\psi}_n \rightarrow \nabla \tilde{\psi}$  in  $L^p$  strong for  $2 \leq p < 6$  in particular for  $p = \sigma + 2$ , it follows that

$$I(\lambda, \mu) = - \int |\nabla \tilde{\psi}|^{\sigma+2}.$$

\*In order to conclude, we still have to show that  $\int |\Delta \tilde{\psi}|^2 = \mu$ . We note:  $\int |\Delta \tilde{\psi}|^2 = \nu$ .

If  $\nu = 0$ , then  $\tilde{\psi} = 0$ , and this is a contradiction.

If  $0 < \nu < \mu$ , Gagliardo-Nirenberg's inequality implies:

$$-I(\lambda, \mu) \leq C\lambda^{1-\sigma/4}\nu^{3\sigma/4},$$

and

$$I(\lambda, \mu) \geq -C\lambda^{1-\sigma/4}\nu^{3\sigma/4} > -C\lambda^{1-\sigma/4}\nu^{3\sigma/4} = I(\lambda, \mu),$$

which is a contradiction, so that  $\nu = \mu$ , and  $\Delta \tilde{\psi} \rightarrow \Delta \tilde{\psi}$  in  $L^2$  strong. ■

We still have to prove the second part of Theorem 5. Indeed, we found  $\phi$  satisfying

$$\begin{aligned} -div(|\nabla \psi|^\sigma \nabla \psi) &= \alpha \Delta \phi + \beta \phi \\ \Delta \psi &= \phi, \end{aligned}$$

$\alpha$  and  $\beta$  being Lagrange multipliers. We introduce  $\phi(x) = a\tilde{\phi}(bx)$  for  $a, b \neq 0$ . One obtains:

$$a(\alpha b^2 \Delta \tilde{\psi} + \beta \tilde{\phi}) = -a \left( \frac{|a|}{|b|} \right)^\sigma \operatorname{div}(|\nabla \tilde{\psi}|^\sigma \nabla \tilde{\psi}).$$

Let us take  $a = b$  and  $b^2 = \frac{1}{|\alpha|}$ , we find:

$$\beta \tilde{\phi} + \nu \Delta \tilde{\phi} = -\operatorname{div}(|\nabla \tilde{\psi}|^\sigma \nabla \tilde{\psi}),$$

where  $\nu = \pm 1$ .

We first prove a result of regularity:

**Lemma 5.** *If  $\nabla \psi$  is a solution of:*

$$\begin{cases} \beta \tilde{\phi} + \nu \Delta \tilde{\phi} = -\operatorname{div}(|\nabla \tilde{\psi}|^\sigma \nabla \tilde{\psi}) \\ \Delta \psi = \phi, \end{cases}$$

with  $\psi \in H$ , then  $\nabla \psi \in \mathcal{C}^2(R^3)$ .

**Proof:** Let us take  $\nabla \psi \in H^s$  with  $s \geq 1$ , then  $\nabla \psi \in L^z$  with  $\frac{1}{z} = \frac{1}{2} - \frac{s}{3}$  so that

$$|\nabla \psi|^{\sigma+1} \in L^{\frac{6}{(3-2s)(\sigma+1)}} \subset H^{-s_0}$$

with  $s_0 = \frac{(3-2s)(\sigma+1)-3}{2}$ . Now  $\nabla \psi$  satisfies:

$$\xi \cdot \mathcal{F}(|\nabla \psi|^\sigma \nabla \psi) = \beta \mathcal{F}\phi + \nu |\xi|^2 \mathcal{F}\phi$$

and

$$\nu |\xi|^{1-s_0} |\mathcal{F}\phi| \leq |\beta| \frac{|\mathcal{F}\phi|}{|\xi|^{1+s_0}} + |\xi|^{-s_0} |\mathcal{F}(|\nabla \psi|^\sigma \nabla \psi)|$$

and all these terms are in  $L^2(|\xi| \geq 1)$ , hence  $\phi \in H^{1-s_0}$  and  $\nabla \psi \in H^{2-s_0} = H^{s_1}$ . If now we define  $s_j = \frac{4-3\sigma}{2} + s_{j-1}(\sigma+1)$  and  $s_0 = 1$ , then  $s_j \rightarrow \infty$  therefore if  $j$  is sufficiently large, then  $\nabla \psi \in H^{3/2+\epsilon}$  which is an algebra, we conclude that  $\operatorname{div}(|\nabla \psi|^\sigma \nabla \psi) \in H^{1/2+\epsilon}$  and  $\phi \in H^{5/2+\epsilon}$ ; this leads to the result. ■

**Lemma 6.** *We have:*

$$-\beta \int |\nabla\psi|^2 + \nu \int |\phi|^2 = \int |\nabla\psi|^{\sigma+2}.$$

**Proof:** Let us multiply

$$\beta\phi + \nu\Delta\phi = -\operatorname{div}(|\nabla\psi|^\sigma\nabla\psi)$$

by  $v \in \mathcal{D}(R^3)$ , after integration, one obtains:

$$-\beta \int \nabla\psi \cdot \nabla v + \nu \int \phi \Delta v = \int |\nabla\psi|^\sigma \nabla\psi \cdot \nabla v.$$

Let  $v_n \in \mathcal{D}(R^3)$  such that  $v_n \rightarrow \psi$  in  $H$ , passing to the limit leads to the lemma. ■

**Proposition 3.**  *$\psi$  satisfies*

$$\beta/2 \int |\nabla\psi|^2 + \nu/2 \int |\phi|^2 = \left(1 - \frac{3}{\sigma+2}\right) \int |\nabla\psi|^{\sigma+2}$$

**Proof:** This identity is called Pohozaev Identity, and we proceed as follows: we multiply the equation by  $(x \cdot \nabla)\psi$  and then integrate. However, this is not directly possible since we do not know the exact behavior of  $\psi$  at infinity. We will use  $(x \cdot \nabla\psi)e^{-\epsilon x^2}$  and then we make  $\epsilon \rightarrow 0$ . We leave the detail of the proof in the Appendix. ■

The relationships given by lemma 6 and lemma 3 lead to:

$$\beta \int |\nabla\psi|^2 = -\frac{4-\sigma}{2(\sigma+2)} \int |\nabla\psi|^{\sigma+2}$$

and  $\beta < 0$ . On the other hand:

$$\nu \int |\phi|^2 = 3\left(\frac{1}{2} - \frac{1}{\sigma+2}\right) \int |\nabla\psi|^{\sigma+2} > 0$$

and  $\nu = +1$ . Let  $\omega = -\beta$ , we have found  $\phi$  satisfying

$$\begin{cases} -\omega\phi + \Delta\phi = -\operatorname{div}(|\nabla\psi|^\sigma\nabla\psi) \\ \Delta\psi = \phi \end{cases}$$

and the equalities

$$\begin{cases} \omega f |\nabla\psi|^2 + f |\phi|^2 = f |\nabla\psi|^{\sigma+2} \\ -\omega f |\nabla\psi|^2 + f |\phi|^2 = (2 - \frac{6}{\sigma+2}) f |\nabla\psi|^{\sigma+2}. \end{cases} \quad (14)$$

So we have proved Theorem 5 for **one**  $\omega$ , by scaling, using the homogeneous properties of the equation, we have a solution for every  $\omega > 0$ . Moreover, thanks to the relationships (14), the values of  $\int |\nabla\psi|^{\sigma+2}$ ,  $\int |\nabla\psi|^2$ ,  $\int |\Delta\psi|^2$  are the same for all the solutions of

$$\inf\{-\int |\nabla\psi|^{\sigma+2}, \int |\nabla\psi|^2 = \lambda(\omega), \int |\Delta\psi|^2 = \mu(\omega)\}.$$

■

### 2.3 Radial standing waves.

The aim of this section is to prove that there exists some radial standing waves for the equation:

$$\begin{cases} i\phi_t + \Delta\phi = -\operatorname{div}(|\nabla\psi|^\sigma \nabla\psi) \\ \nabla\psi = \phi, \end{cases} \quad (15)$$

i.e. solution of the form  $e^{i\omega t}\phi(x)$  with  $\phi$  depending only on  $r$ . Then (15) becomes

$$\begin{cases} -\omega\phi + \frac{1}{r^2} \frac{d}{dr}(r^2 \frac{d\phi}{dr}) = -\frac{1}{r^2} \frac{d}{dr}(r^2 |\frac{d\psi}{dr}|^\sigma \frac{d\psi}{dr}) \\ \phi = \frac{1}{r^2} \frac{d}{dr}(r^2 \frac{d\psi}{dr}) \end{cases} \quad (16)$$

Multiplying by  $r^2$  and then integrating leads to

$$-\omega\psi_r + \frac{d}{dr}\left(\frac{1}{r^2} \frac{d}{dr}(r^2 \frac{d\psi}{dr})\right) = -|\psi_r|^\sigma \psi_r$$

Now let  $f = \psi_r$ ,  $f$  satisfies:

$$\frac{d^2 f}{dr^2} + \frac{2}{r} \frac{df}{dr} - \left(\omega + \frac{2}{r^2}\right)f = -|f|^\sigma f. \quad (17)$$

In order to solve (17), we consider the following minimization problem:

$$I(\lambda, \mu) = \inf\{-\int_0^\infty f^{\sigma+2} r^2 dr, \int_0^\infty f'^2 r^2 dr + 2 \int_0^\infty f^2 dr = \mu, \int_0^\infty f^2 r^2 dr = \lambda\} \quad (18)$$

About (18), we first prove:

**Lemma 7.** \*For  $f \in H_r^1(\mathbb{R}^3)$ ,  $\int_0^\infty |f|^2 dr$  makes sense and

$$\int_0^\infty |f|^2 dr \leq 4 \int_0^\infty |\nabla f|^2 r^2 dr.$$

$$* I(\lambda, \mu) > -\infty.$$

**Proof:** Since  $f \in H_r^1(\mathbb{R}^3)$ ,  $f$  as a function of one variable is in  $H^1([\alpha, +\infty[)$  for all  $\alpha > 0$ , and

$$f^2(y) - f^2(x) = \int_x^y 2f(t)f'(t)dt \quad \forall x, y.$$

Integrating for  $x \in [\epsilon, y]$  leads to

$$f^2(y)(y - \epsilon) - \int_\epsilon^y f^2(x)dx = \int_\epsilon^y \left( \int_x^y 2f(t)f'(t)dt \right) dx.$$

We use Fubini's theorem in order to obtain

$$\int_\epsilon^y f^2(x)dx = f^2(y)(y - \epsilon) - \int_\epsilon^y 2f(t)f'(t)(t - \epsilon)dt.$$

Moreover,

$$\int_\epsilon^y |f^2(x)|dx \leq |f^2(y)(y - \epsilon)| + 2 \left( \int_\epsilon^y |f^2(t)|dt \right)^{1/2} \left( \int_\epsilon^y (t - \epsilon)^2 |f'(t)|^2 dt \right)^{1/2}.$$

This implies that

$$\begin{aligned} \left( \int_\epsilon^y |f^2(x)|dx \right)^{1/2} &\leq \left( \int_\epsilon^y (t - \epsilon)^2 |f'(t)|^2 dt \right)^{1/2} \\ &+ \left( \int_\epsilon^y (t - \epsilon)^2 |f'(t)|^2 dt + |f^2(y)(y - \epsilon)| \right)^{1/2}. \end{aligned} \quad (19)$$

Now since  $f \in H_r^1(\mathbb{R}^3)$ , we have  $|f(y)| \leq C|y|^{-1}|f|_{H^1}$  (see Berestycki-Lions [1]). Letting  $\epsilon \rightarrow 0$  and  $y \rightarrow \infty$  in (19), we get

$$\int_0^\infty f(x)^2 dx \leq 4 \int_0^\infty |f'(t)|^2 t^2 dt$$

as claimed.

A Gagliardo-Nirenberg's inequality gives:

$$\int_0^\infty f^{\sigma+2} r^2 dr \leq C \left( \int_0^\infty f^2 r^2 dr \right)^{1-\sigma/4} \left( \int_0^\infty f'^2 r^2 dr \right)^{3\sigma/4},$$

therefore

$$I(\lambda, \mu) \geq -C\lambda^{1-\sigma/4}\mu^{3\sigma/4}$$

and  $I(\lambda, \mu) > -\infty$ . ■

We now solve problem (18).

**Proposition 4.** *The minimum:*

$$I(\lambda, \mu) = \inf \left\{ - \int f^{\sigma+2} r^2 dr, \int f'^2 r^2 dr + 2 \int f^2 dr = \mu, \int f^2 dr = \lambda \right\}$$

is attained.

**Proof:** Let us take a minimizing sequence  $f_n$  such that

$$- \int_0^\infty f_n^{\sigma+2} r^2 dr \rightarrow I(\lambda, \mu)$$

and

$$\int_0^\infty f_n'^2 r^2 dr + 2 \int_0^\infty f_n^2 dr = \mu, \int_0^\infty f_n^2 r^2 dr = \lambda.$$

Then we can suppose that  $f_n \rightharpoonup f$  in  $H_r^1(R^3)$  weakly and since the injection of  $H_r^1(R^3)$  in  $L^{\sigma+2}(R^3)$  is compact (see Strauss [8]), then  $f_n \rightarrow f$  in  $L^{\sigma+2}$  strong, so that

$$- \int_0^\infty f_n^{\sigma+2} r^2 dr \rightarrow - \int_0^\infty f^{\sigma+2} r^2 dr.$$

Moreover  $f \mapsto \int_0^\infty f^2 dr$  is convex and continuous on  $H_r^1(R^3)$ , thus it is weakly lower semi-continuous and

$$b \equiv \int_0^\infty f'^2 r^2 dr + 2 \int_0^\infty f^2 dr \leq \mu,$$

$$a \equiv \int_0^\infty f^2 r^2 dr \leq \lambda.$$

We define  $\tilde{f}(x) = \alpha f(\beta x)$ , with  $\alpha, \beta$  satisfying

$$\lambda = \int \tilde{f}^2 r^2 dr = \frac{\alpha^2}{\beta^3} a,$$

$$\mu = \int \tilde{f}'^2 r^2 dr + 2 \int \tilde{f}^2 dr = \frac{\alpha^2}{\beta} b.$$

We obtain

$$- \int \tilde{f}^{\sigma+2} r^2 dr = - \frac{\alpha^{\sigma+2}}{\beta^3} I(\lambda, \mu).$$

By definition of  $I(\lambda, \mu)$  we get

$$-\int \tilde{f}^{\sigma+2} r^2 dr \geq I(\lambda, \mu).$$

This implies that

$$\frac{\alpha^{\sigma+2}}{\beta^3} \leq 1, \quad (20)$$

since  $I(\lambda, \mu) < 0$ . Now  $\frac{\alpha^4}{\beta^4} = \frac{\lambda\mu}{ab}$  and  $\beta = (\frac{a\mu}{b\lambda})^{1/2}$  and  $\frac{\alpha^{\sigma+2}}{\beta^3} = (\frac{\mu}{b})^{3\sigma/4} (\frac{\lambda}{a})^{1-\sigma/4} \geq 1$ . Together with (20), this inequality implies that  $\lambda = a$  and  $\mu = b$  and the minimum is attained.  $\blacksquare$

For all  $v \in \mathcal{D}(R^+)$

$$-\int |f|^\sigma f v r^2 dr = 2\alpha \int f v r^2 dr + 2\beta \int \frac{df}{dr} r^2 v' dr + 4\beta \int f v dr,$$

so that we found a solution of (16).

We still have to prove Proposition 1.

**Proposition 1.** *Let  $\sigma = 4/3$ , then for all  $t_0 > 0$ , there exists a radial function  $\psi_0$  so that the corresponding solution to:*

$$\begin{cases} i\phi_t + \Delta\phi = -\operatorname{div}(|\nabla\psi|^{4/3}\nabla\psi) \\ \Delta\psi = \phi \end{cases}$$

*blows up exactly at  $t = t_0$ .*

**Proof:** If we restrict (1) to the radial functions, we obtain

$$iu_t + \Delta u - \frac{2}{|x|^2}u = -|u|^\sigma u, \quad (21)$$

where  $u = \psi_r$ . This last equation satisfies the pseudo-conformal transformation laws, i.e if  $u(x, t)$  is a solution of (21), then the function

$$v(x, t) = \frac{1}{t^{3/2}} e^{\frac{i|x|^2}{4t}} \bar{u}\left(\frac{x}{t}, \frac{1}{t}\right)$$

is a solution to (21) too. If we apply this transformation to any radial standing waves that we found, we obtain the Proposition.  $\blacksquare$

### 3 Stability of standing waves

In the previous section, we found some functions  $\phi_\omega$  satisfying

$$\begin{cases} \Delta\phi_\omega - \omega\phi_\omega = -\operatorname{div}(|\nabla\psi_\omega|^\sigma \nabla\psi_\omega) \\ \Delta\psi_\omega = \phi_\omega. \end{cases} \quad (22)$$

$\nabla\psi_\omega$  are solutions of the following minimization problem

$$(P1) \inf\left\{-\int |\nabla\psi|^{\sigma+2}, \psi \in H \mid \int |\nabla\psi|^2 = \lambda, \int |\Delta\psi|^2 = \mu\right\}.$$

Each solution of (P1) satisfies:

$$\begin{cases} \omega \int |\nabla\psi_\omega|^2 = \left(\frac{3}{\sigma+2} - \frac{1}{2}\right) \int |\nabla\psi_\omega|^{\sigma+2}, \\ \int |\phi_\omega|^2 = 3\left(\frac{1}{2} - \frac{1}{\sigma+2}\right) \int |\nabla\psi_\omega|^{\sigma+2} \end{cases} \quad (23)$$

and since  $\nabla\psi_\omega$  is a solution of (P1), the quantities  $\int |\nabla\psi_\omega|^2$ ,  $\int |\nabla\psi_\omega|^{\sigma+2}$  and  $\int |\phi_\omega|^2$  do not depend on the solution of (P1) that we consider, but only on  $\omega$ . The aim of this section is to investigate the stability of these standing waves. A few notations are in order:

$$\begin{cases} E(\psi) = \int \left(\frac{1}{2}|\Delta\psi|^2 - \frac{1}{\sigma+2}|\nabla\psi|^{\sigma+2}\right) dx \\ Q(\psi) = \frac{1}{2} \int |\nabla\psi|^2. \end{cases} \quad (24)$$

So that (22) becomes  $E'(\psi) + \omega Q'(\psi) = 0$ . In order to prove the stability and instability properties, we introduce new minimization problems and we prove:

**Proposition 5.** *If  $\sigma < 4/3$ , then the following minimization problem has a solution*

$$(\tilde{P}2) \min\{E(\psi), \psi \in H, \int |\nabla\psi|^2 = \int |\nabla\psi_\omega|^2\} = -a$$

with  $a > 0$ .

Moreover, let  $\psi_n \in H$  such that  $E(\psi_n) \rightarrow -a$  and  $\int |\nabla\psi_n|^2 \rightarrow \int |\nabla\psi_\omega|^2$ , then there exists  $y_n \in \mathbb{R}^3$  such that  $\psi_n(\cdot - y_n)$  is compact in  $H$ .

**Proposition 6.** *If  $0 < \sigma < 4$ , then the following minimization problem has a solution:*

$$(\tilde{P}3) \min\left\{-\frac{1}{\sigma+2} \int |\nabla\psi|^{\sigma+2} + \frac{\omega}{2} \int |\nabla\psi|^2, \psi \in H, \int |\Delta\psi|^2 = \int |\Delta\psi_\omega|^2\right\} = -b$$

with  $b > 0$ .

### 3.1 Proof of the propositions.

#### 3.1.1 First proposition.

We fix  $0 < \sigma < 4/3$ . We first prove that the minimum is finite. Indeed, Gagliardo-Nirenberg's inequality gives:

$$E(\psi) \geq -C\left(\int |\Delta\psi|^2\right)^{3\sigma/4} + \frac{1}{2} \int |\Delta\psi|^2,$$

for  $\psi \in H$  such that  $\int |\nabla\psi|^2 = \int |\nabla\psi_\omega|^2$ . Since  $\sigma < 4/3$ ,

$$-C\left(\int |\Delta\psi|^2\right)^{3\sigma/4} + \frac{1}{2} \int |\Delta\psi|^2$$

is bounded from below and  $-a > -\infty$ .

Let us now show that  $-a < 0$ . Indeed, consider  $\psi \in H$  such that  $\int |\nabla\psi|^2 = \int |\nabla\psi_\omega|^2$  and  $\tilde{\psi}(x) = \alpha\psi(\beta x)$  for  $\alpha, \beta > 0$ , then if  $\alpha^2 = \beta$ ,  $\int |\nabla\tilde{\psi}|^2 = \int |\nabla\psi|^2$  and

$$E(\tilde{\psi}) = -\beta^{3\sigma/2} \int \frac{|\nabla\psi|^{\sigma+2}}{\sigma+2} + \frac{\beta^2}{2} \int |\Delta\psi|^2.$$

Since  $3\sigma/2 < 2$ , for  $\beta$  small,  $E(\tilde{\psi}) < 0$  and  $-a < 0$ .

Using the same technique as for the proof of Theorem 5, in particular the concentration-compactness lemma, one can prove that  $(P2)$  has a solution. ■

#### 3.1.2 Second proposition.

Gagliardo-Nirenberg's inequality yields:

$$-\frac{1}{\sigma+2} \int |\nabla\psi|^{\sigma+2} + \frac{\omega}{2} \int |\nabla\psi|^2 \geq -C\left(\int |\nabla\psi|^2\right)^{1-\sigma/4} + \frac{\omega}{2} \int |\nabla\psi|^2,$$

for all  $\psi \in H$  such that

$$\int |\Delta\psi|^2 = \int |\Delta\psi_\omega|^2.$$

Since  $1 - \sigma/4 < 1$ , we obtain that  $-b > -\infty$ .

On the other hand, for  $\psi \in H$  with  $\int |\Delta\psi|^2 = \int |\Delta\psi_\omega|^2$ , we define:  $\tilde{\psi}(x) = \alpha\psi(\frac{x}{\alpha^2})$ . We have  $\int |\Delta\tilde{\psi}|^2 = \int |\Delta\psi_\omega|^2$ , and

$$-\frac{1}{\sigma+2} \int |\Delta\tilde{\psi}|^{\sigma+2} + \frac{\omega}{2} \int |\nabla\tilde{\psi}|^2 = -\frac{1}{\sigma+2} \alpha^{4-\sigma} \int |\nabla\psi|^{\sigma+2} + \frac{\omega}{2} \alpha^4 \int |\nabla\psi|^2.$$

Thus if  $\alpha$  is small enough, this last quantity is negative.

To prove that problem  $(\tilde{P}2)$  has a solution, we proceed as for the proof of theorem 5. ■

### 3.2 Equivalent problems

Problem  $(\tilde{P}2)$  is equivalent to:

$$(P2) \inf \left\{ \frac{1}{2} \int |\Delta\psi|^2 - \frac{1}{\sigma+2} \int |\nabla\psi|^{\sigma+2} + \frac{\omega}{2} \int |\nabla\psi|^2, \int |\nabla\psi|^2 = \int |\nabla\psi_\omega|^2 \right\}$$

for  $\sigma < \frac{4}{3}$ .

Problem  $(\tilde{P}3)$  is equivalent to:

$$(P3) \inf \left\{ \frac{1}{2} \int |\Delta\psi|^2 - \frac{1}{\sigma+2} \int |\nabla\psi|^{\sigma+2} + \frac{\omega}{2} \int |\nabla\psi|^2, \int |\Delta\psi|^2 = \int |\Delta\psi_\omega|^2 \right\}$$

for  $\sigma < 4$ .

We now investigate the relationships between  $(P1)$ ,  $(P2)$  and  $(P3)$ .

**Proposition 7.** *Let  $0 < \sigma < \frac{4}{3}$ , then  $(P1)$  and  $(P2)$  are equivalent.*

**Proof:** Let us write the Euler-Lagrange equation corresponding to problem  $(P2)$ : there exists a Lagrange multiplier  $\gamma$  so that:

$$\Delta^2\psi + \operatorname{div}(|\nabla\psi|^\sigma \nabla\psi) - \omega\Delta\psi = \gamma\Delta\psi. \quad (25)$$

It is roughly the same equation as in the proof of Theorem 5, so that the corresponding result of regularity apply and if we multiply (25) by  $\bar{\psi}$  and then integrate, we obtain

$$\int |\Delta\psi|^2 - \int |\nabla\psi|^{\sigma+2} + (\omega + \gamma) \int |\nabla\psi|^2 = 0. \quad (26)$$

Multiplying (25) by  $(x \cdot \nabla)\psi$  and then integrating leads to:

$$\frac{1}{2} \int |\Delta\psi|^2 + \left(\frac{3}{\sigma+2} - 1\right) \int |\nabla\psi|^{\sigma+2} - \frac{\omega + \gamma}{2} \int |\nabla\psi|^2 = 0. \quad (27)$$

We make a combination of (26) and (27):

$$2 \int |\Delta\psi|^2 = 3\left(1 - \frac{2}{\sigma+2}\right) \int |\nabla\psi|^{\sigma+2}, \quad (28)$$

$$\left(1 - \frac{6}{\sigma + 2}\right) \int |\nabla \psi|^{\sigma+2} = -2(\omega + \gamma) \int |\nabla \psi|^2. \quad (29)$$

We note  $\lambda = \int |\nabla \psi_\omega|^2 = \int |\nabla \psi|^2$ , we have, using (28) and (29):

$$\int |\nabla \psi|^{\sigma+2} = \frac{2(\sigma + 2)(\omega + \gamma)}{4 - \sigma} \lambda \quad (30)$$

and

$$\int |\Delta \psi|^2 = \frac{3\sigma(\omega + \gamma)}{4 - \sigma} \lambda. \quad (31)$$

An analogue calculation, using the corresponding formulas for  $\nabla \psi_\omega$  leads to

$$\int |\nabla \psi_\omega|^{\sigma+2} = \frac{2\omega(\sigma + 2)}{4 - \sigma} \lambda \quad (32)$$

and

$$\int |\Delta \psi_\omega|^2 = \frac{3\sigma}{4 - \sigma} \lambda. \quad (33)$$

Hence

$$-\frac{1}{\sigma + 2} \int |\nabla \psi_\omega|^{\sigma+2} + \frac{1}{2} \int |\Delta \psi_\omega|^2 = \frac{\omega\lambda(3\sigma - 4)}{2(4 - \sigma)} \quad (34)$$

while

$$-\frac{1}{\sigma + 2} \int |\nabla \psi|^{\sigma+2} + \frac{1}{2} \int |\Delta \psi|^2 = \frac{(\omega + \gamma)\lambda(3\sigma - 4)}{2(4 - \sigma)}. \quad (35)$$

Since  $3\sigma - 4 < 0$  and  $E(\psi_\omega) \geq E(\psi)$ , (34) and (35) imply

$$\gamma \geq 0. \quad (36)$$

Set  $\tilde{\psi}(x) = \alpha\psi(\alpha^2 x)$ , then  $\int |\nabla \tilde{\psi}|^2 = \int |\nabla \psi|^2$ , and we choose  $\alpha$  such that  $\int |\Delta \tilde{\psi}|^2 = \int |\Delta \psi|^2$ , this gives:

$$\alpha^4 = \frac{\omega}{\omega + \gamma} \quad (37)$$

Using (37), we obtain that

$$-\int |\nabla \tilde{\psi}|^{\sigma+2} = -\alpha^{3\sigma-4} \int |\nabla \psi_\omega|^{\sigma+2}.$$

If we use the definition of (P1), we find

$$-\int |\nabla\psi_\omega|^{\sigma+2} \leq -\alpha^{3\sigma-4} \int |\nabla\psi_\omega|^{\sigma+2}$$

or equivalently

$$1 \geq \alpha,$$

and since  $3\sigma - 4 < 0$ ,  $\alpha \geq 1$  and  $\gamma \leq 0$ . Together with (36), this inequality implies

$$\gamma = 0.$$

In fact, we have proved that if  $\psi$  is a solution of (P2), and  $\psi_\omega$  is a solution of (P1), then

$$\begin{aligned} \int |\nabla\psi|^2 &= \int |\nabla\psi_\omega|^2, \\ \int |\Delta\psi|^2 &= \int |\Delta\psi_\omega|^2, \\ \int |\nabla\psi|^{\sigma+2} &= \int |\nabla\psi_\omega|^{\sigma+2}, \end{aligned}$$

so that (P1) and (P2) are equivalent, and the proposition is proved.  $\blacksquare$

**Proposition 8.** *Let  $0 < \sigma < 4$ , then (P1) and (P2) are equivalent.*

**Proof:** Let us write the Euler-Lagrange equation corresponding to problem (P3)

$$\operatorname{div}(|\nabla\psi|^\sigma \nabla\psi) - \omega\Delta\psi + \Delta^2\psi = \delta\Delta^2\psi, \quad (38)$$

where  $\delta$  is a Lagrange multiplier. Multiplying (38) by  $\bar{\psi}$  and then integrating leads to:

$$(1 - \delta) \int |\Delta\psi|^2 + \omega \int |\nabla\psi|^2 = \int |\nabla\psi|^{\sigma+2}. \quad (39)$$

Multiplying (38) by  $(x \cdot \nabla)\psi$  and integrating gives

$$\frac{1 - \delta}{2} \int |\Delta\psi|^2 - \frac{\omega}{2} \int |\nabla\psi|^2 = \left(1 - \frac{3}{\sigma + 2}\right) \int |\nabla\psi|^{\sigma+2}. \quad (40)$$

Making linear combination of (39) and (40), we obtain

$$\omega \int |\nabla\psi|^2 = \left(\frac{3}{\sigma + 2} - \frac{1}{2}\right) \int |\nabla\psi|^{\sigma+2} \quad (41)$$

and

$$(1 - \delta) \int |\phi|^2 = 3\left(\frac{1}{2} - \frac{1}{\sigma + 2}\right) \int |\nabla\psi|^{\sigma+2}. \quad (42)$$

We recall that for  $\psi_\omega$  the solution of (P1) we have

$$\omega \int |\nabla\psi_\omega|^2 = \left(\frac{3}{\sigma + 2} - \frac{1}{2}\right) \int |\nabla\psi|^{\sigma+2} \quad (43)$$

and

$$\int |\phi_\omega|^2 = 3\left(\frac{1}{2} - \frac{1}{\sigma + 2}\right) \int |\nabla\psi_\omega|^{\sigma+2}, \quad (44)$$

so that, using (43) and (44), we get that

$$-\frac{1}{\sigma + 2} \int |\nabla\psi_\omega|^{\sigma+2} + \frac{\omega}{2} \int |\nabla\psi_\omega|^2 = -\frac{1}{6} \int |\Delta\psi_\omega|^2.$$

On the other hand, using (41) and (42)

$$-\frac{1}{\sigma + 2} \int |\nabla\psi|^{\sigma+2} + \frac{\omega}{2} \int |\nabla\psi|^2 = -\frac{1 - \delta}{6} \int |\Delta\psi|^2 = -\frac{1 - \delta}{6} \int |\Delta\psi_\omega|^2$$

and necessarily

$$\delta \leq 0. \quad (45)$$

If we let  $\mu = \int |\Delta\psi|^2 = \int |\Delta\psi_\omega|^2$ , we get:

$$\int |\nabla\psi|^{\sigma+2} = \frac{2(1 - \delta)(\sigma + 2)}{3\sigma} \mu, \quad \int |\nabla\psi_\omega|^{\sigma+2} = \frac{2(\sigma + 2)}{3\sigma} \mu \quad (46)$$

and

$$\int |\nabla\psi|^2 = \frac{(1 - \delta)(4 - \sigma)}{3\omega\sigma}, \quad \int |\nabla\psi_\omega|^2 = \frac{4 - \sigma}{3\omega\sigma} \mu. \quad (47)$$

Set  $\tilde{\psi}(x) = \alpha\psi\left(\frac{x}{\alpha^2}\right)$ , then  $\int |\Delta\tilde{\psi}|^2 = \int |\Delta\psi|^2$ , and we choose  $\alpha$  such that  $\int |\nabla\tilde{\psi}|^2 = \int |\nabla\psi_\omega|^2$ . Using (46) and (47), we obtain  $\alpha^4 = \frac{1}{1 - \delta}$ . Then

$$-\int |\nabla\tilde{\psi}|^{\sigma+2} = -\alpha^{-\sigma} \int |\nabla\psi_\omega|^{\sigma+2},$$

so that

$$-\int |\nabla\psi_\omega|^{\sigma+2} \leq -\alpha^{-\sigma} \int |\nabla\psi_\omega|^{\sigma+2}$$

and  $\alpha \geq 1$  therefore

$$\delta \geq 0.$$

This inequality with (45) implies that  $\delta = 0$ , and we concluded as for the previous proposition. ■

### 3.3 Stability for $\sigma < 4/3$ .

Adapting the method of Cazenave-Lions [2], we may prove the second part of Theorem 7. Let  $S_\omega$  the set of standing waves, solutions of (P1) or (P2). Let us suppose that this set is unstable under the flow of

$$i(\nabla\psi)_t + \Delta(\nabla\psi) = \nabla(-\Delta)^{-1} \operatorname{div}(|\nabla\psi|^\sigma \nabla\psi). \quad (48)$$

There exists  $\psi_n^0 \in H$ ,  $t_n \in \mathbb{R}^+$  and  $\epsilon > 0$  such that

$$\inf_{\psi_\omega \in S_\omega} |\psi_n^0 - \psi_\omega|_H \rightarrow 0$$

and the solution  $\psi_n(t)$  of (48) with  $\psi_n(0) = \psi_n^0$  satisfies

$$\inf_{\psi_\omega \in S_\omega} |\psi_n(t_n, \cdot) - \psi_\omega|_H \geq \epsilon.$$

We have that

$$\begin{cases} \int |\nabla\psi_n^0|^2 \rightarrow \int |\nabla\psi_\omega|^2, \\ \int |\nabla\psi_n^0|^{\sigma+2} \rightarrow \int |\nabla\psi_\omega|^{\sigma+2}, \\ \int |\Delta\psi_n^0|^2 \rightarrow \int |\Delta\psi_\omega|^2. \end{cases}$$

Since  $E$  and  $Q$  are conserved by the flow of (48), we know that

$$Q(\psi_n(t_n)) \rightarrow Q(\psi_\omega)$$

and

$$E(\psi_n(t_n)) \rightarrow E(\psi_\omega) = -a.$$

By proposition 5,  $\psi_n(t_n)$  is compact up to translation and

$$\inf_{\psi_\omega \in S_\omega} |\psi_n(t_n, \cdot) - \psi_\omega|_H \rightarrow 0$$

which is a contradiction. ■

### 3.4 Instability for $\sigma > 4/3$ .

The aim of this section is to show how it is possible to adapt the proof of J.Shatah and W.Strauss [7] (which works for the Klein-Gordon equation), to our problem. We introduce

$$d(\omega) = E(\psi_\omega) + \omega Q(\psi_\omega).$$

Then, since (P1) and (P3) are equivalent

$$d(\omega) = \inf\{E(u) + \omega Q(u), u \in H, \int |\Delta u|^2 = \int |\Delta \psi_\omega|^2\}.$$

Using the relationships (32) and (33), we obtain that

$$d(\omega) = \frac{1}{3} \int |\Delta \psi_\omega|^2.$$

Let us now find the dependence of  $d(\omega)$  in  $\omega$ . Let  $\psi_\omega(x) = \alpha \tilde{\psi}(\beta x)$ ,  $\tilde{\psi}$  satisfies:

$$-\frac{\omega}{\alpha^\sigma \beta^\sigma} \Delta \tilde{\psi} + \frac{1}{\alpha^\sigma \beta^{\sigma-2}} \Delta^2 \tilde{\psi} = -\operatorname{div}(|\nabla \tilde{\psi}|^\sigma \nabla \tilde{\psi})$$

we take  $\alpha$  and  $\beta$  such that  $\alpha^\sigma \beta^{\sigma-2} = 1$  and  $\alpha^\sigma \beta^\sigma = \omega$ , this leads to:

$$\alpha = \omega^{\frac{1}{\sigma}-\frac{1}{2}}, \beta = \omega^{1/2}.$$

Then  $\tilde{\psi} = \psi_1$  and

$$d(\omega) = \frac{\omega^{\frac{2}{\sigma}-\frac{1}{2}}}{3} \int |\Delta \psi_1|^2.$$

This function is strictly concave if and only if  $\frac{4}{3} < \sigma < 4$ , and from now on, we fix such a  $\sigma$ .

**Lemma 8.**

$$d'(\omega) = Q(\psi_\omega).$$

**Proof:** Indeed,  $d(\omega) = E(\psi_\omega) + \omega Q(\psi_\omega)$  and

$$d'(\omega) = \langle E'(\psi_\omega) + \omega Q'(\psi_\omega), \frac{\partial \psi_\omega}{\partial \omega} \rangle + Q(\psi_\omega).$$

But

$$E'(\psi_\omega) + \omega Q'(\psi_\omega) = 0,$$

thereby proving the lemma. ■

**Lemma 9.** Fix  $\omega = \omega_0$  and  $\psi_0 = \psi_{\omega_0}$ . The for all  $C^2$  curve  $u(\lambda)$  such that  $u(0) = \psi_0$  and  $Q(u(\lambda)) = Q(\psi_0)$  then

$$\frac{d^2}{dt^2}E(u(\lambda))|_{\lambda=0} = \langle (E''(\psi_0) + \omega_0 Q''(\psi_0))y_0, y_0 \rangle,$$

where  $y_0 = u'(0)$ .

**Proof:**

$$\frac{d^2}{d\lambda^2}E(u(\lambda)) = \langle E''(u(\lambda))u', u' \rangle + \langle E'(u), u'' \rangle. \quad (49)$$

On the other hand, since  $Q$  is constant along the curve:

$$\langle Q''(u)u', u' \rangle + \langle Q'(u), u'' \rangle = 0 \quad (50)$$

and (49)+ $\omega_0$  (50)| $_{\lambda=0}$  gives the result, since  $E'(\psi_0) + \omega_0 Q'(\psi_0) = 0$ .  $\blacksquare$

We introduce:

$$\nabla\chi(\omega, x) = (\nabla\psi_\omega)\left(\frac{x}{\lambda(\omega)}\right),$$

with

$$\lambda^3(\omega) = \frac{Q(\psi_0)}{Q(\psi_\omega)}. \quad (51)$$

Then we have  $Q(\chi(\omega, x)) = Q(\psi_0)$ .

**Lemma 10.**

$$\frac{d^2}{d\omega^2}E(\chi(\omega))|_{\omega=\omega_0} \leq d''(\omega_0).$$

**Proof:** we define  $\alpha(\omega) = E(\chi(\omega)) - d(\omega) + \omega d'(\omega_0)$ , then  $\alpha(\omega_0) = 0$  by definition of  $E$ ,  $Q$  and  $d$  and lemma 8. Therefore by the definition of  $\chi$  and since  $Q(\chi(\omega)) = Q(\psi_0)$ , we have:

$$E(\chi(\omega)) = -\omega Q(\psi_0) + \frac{\lambda}{2} \int |\Delta\psi_\omega|^2 - \frac{\lambda^3}{\sigma+2} \int |\nabla\psi_\omega|^{\sigma+2} + \frac{\lambda^3\omega}{2} \int |\nabla\psi_\omega|^2.$$

Using the relationships (32) and (33), one obtain:

$$\omega \int |\nabla\psi_\omega|^2 = \frac{4-\sigma}{3\sigma} \int |\phi_\omega|^2, \quad (52)$$

$$\int |\nabla \psi_\omega|^{\sigma+2} = \frac{2(\sigma+2)}{3\sigma} \int |\phi_\omega|^2, \quad (53)$$

whence

$$E(\chi(\omega)) = -\omega Q(\psi_0) + \left(\frac{\lambda}{2} - \frac{\lambda^3}{6}\right) \int |\phi_0|^2.$$

Since  $\frac{\lambda}{2} - \frac{\lambda^3}{6} \leq \frac{1}{3}$  for  $\lambda \geq 0$  we have:

$$E(\chi(\omega)) \leq -\omega d'(\omega_0) + d(\omega_0)$$

and  $\alpha(\omega) \leq 0$ , therefore  $\alpha(\omega)$  is maximum for  $\omega = \omega_0$  and  $\frac{\partial^2 \alpha}{\partial \omega^2}(\omega_0) \leq 0$ . The lemma follows.  $\blacksquare$

**Lemma 11.** *We have:*

$$E(\chi(\omega)) < E(\psi_0),$$

for  $\omega \neq \omega_0$ ,  $\omega$  near  $\omega_0$ .

**Proof:** First we note that

$$\frac{d}{d\omega} E(\chi(\omega))|_{\omega=\omega_0} = \langle E'(\chi(\omega_0)), \frac{d\chi}{d\omega}(\omega_0) \rangle. \quad (54)$$

Since  $Q(\chi(\omega)) = Cte$ ,

$$\langle Q'(\chi(\omega_0)), \frac{d\chi}{d\omega}(\omega_0) \rangle = 0. \quad (55)$$

(54)+(55) leads to

$$\frac{d}{d\omega} E(\chi(\omega)) = \langle E'(\psi_0) + \omega_0 Q'(\psi_0), \frac{d\psi}{d\omega}(\omega_0) \rangle = 0.$$

Moreover  $\frac{d^2}{d\omega^2} E(\chi(\omega))|_{\omega=\omega_0} \leq d''(\omega_0) < 0$  so that  $E$  is locally maximal in  $\psi_0$  and the lemma is proved.  $\blacksquare$

We define:

$$y_0 = \frac{\partial \nabla \chi(\omega)}{\partial \omega} |_{\omega=\omega_0}. \quad (56)$$

About  $y_0$ , we prove:

**Lemma 12.** (a)  $\langle (E''(\psi_0) + \omega_0 Q''(\psi_0))y_0, y_0 \rangle \leq d''(\omega_0)$ .  
(b)  $\langle Q'(\psi_0), y_0 \rangle = - \int \nabla \psi_0 \cdot \bar{y}_0 = 0$ .  
(c)  $\int \Delta \psi_0 \operatorname{div}(y_0) > 0$ .

**Proof:** (a) follows from lemmas 9 and 10.

(b) is trivial.

(c) By definition of  $\chi$ ,  $\int |\Delta \chi(\omega)|^2 dx = \lambda \int |\Delta \psi_\omega|^2$ , hence by differentiation in  $\omega = \omega_0$ :

$$2 \int \Delta \psi_0 \operatorname{div}(y_0) = \lambda' \int |\Delta \psi_0|^2 + 3d'(\omega_0), \quad (57)$$

Since  $d(\omega) = \frac{1}{3} \int |\Delta \psi_\omega|^2$ . But  $\lambda^3 = \frac{d'(\omega_0)}{d'(\omega)}$ , therefore differentiation in  $\omega = \omega_0$  leads to

$$3\lambda'(\omega_0) = -\frac{d''(\omega_0)}{d'(\omega_0)}.$$

But  $d''(\omega_0) < 0$ , so that  $\lambda'(\omega_0) > 0$  and the lemma follows. ■

Set

$$G = \{f \in (H^1(\mathbb{R}^3))^3, \exists \psi \in \mathcal{D}' \text{ such that } f = \nabla \psi\}.$$

We know that (see [4])

$$G = \{\nabla \psi, \psi \in L^6 \cap C_0, \nabla \psi \in H^1\}.$$

Let  $I$  the following injection of  $G$  in  $G'$ :

$$\begin{aligned} G &\rightarrow G' \\ \nabla u &\rightarrow I(\nabla u) \end{aligned}$$

defined by:

$$\langle I(\nabla u), \nabla v \rangle = \operatorname{Re} \int \nabla u \cdot \nabla \bar{v}.$$

With this preliminaries, we are in the mathematical setting of Shatah-Strauss [7] and we have the result of Theorem 8.

# Appendix.

## A Proof of proposition 2.

**Proposition 3.**  $\psi$  satisfies

$$\beta/2 \int |\nabla\psi|^2 + \nu/2 \int |\phi|^2 = (1 - \frac{3}{\sigma+2}) \int |\nabla\psi|^{\sigma+2}.$$

**Proof:** We multiply the equation:

$$\beta\phi + \tilde{\epsilon}\Delta\phi = -\operatorname{div}(|\nabla\psi|^\sigma \nabla\psi)$$

by  $(x \cdot \nabla\psi)e^{-\epsilon x^2}$ , and we compute every terms:

i)  $\int \phi(x \cdot \nabla\psi)\psi e^{-\epsilon x^2}$ .

By numerous integrations by parts we get that

$$\int \phi(x \cdot \nabla\psi)e^{-\epsilon x^2} = \frac{1}{2} \int |\nabla\psi|^2 e^{-\epsilon x^2} + 2\epsilon \int e^{-\epsilon x^2} (x \cdot \nabla\psi)^2 - \epsilon \int |\nabla\psi|^2 x^2 e^{-\epsilon x^2}. \quad (58)$$

When we make  $\epsilon \rightarrow 0$ ,  $\int |\nabla\psi|^2 e^{-\epsilon x^2} \rightarrow \int |\nabla\psi|^2$  by Lebesgue's theorem.

The other terms will be treated by:

**Lemma 13.** Let  $f \in L^1_+(R^3)$  then

$$\epsilon \int f x^2 e^{-\epsilon x^2} dx \rightarrow 0,$$

as  $\epsilon \rightarrow 0$ .

**Proof:** Consider  $g(x) = x^2 e^{-\epsilon x^2}$ , then  $|g(x)| \leq \frac{1}{\epsilon} e^{-1}$ . Define  $r_\epsilon = \frac{1}{\epsilon^{1/4}}$ , then  $g(r_\epsilon) = \frac{1}{\epsilon^{1/2}} e^{-\epsilon^{1/2}} \leq \frac{1}{\epsilon^{1/2}}$ . We have

$$\epsilon \int f x^2 e^{-\epsilon x^2} dx = \epsilon \int_{|x| < r_\epsilon} f x^2 e^{-\epsilon x^2} dx + \epsilon \int_{|x| > r_\epsilon} f x^2 e^{-\epsilon x^2} dx.$$

Hence

$$\begin{aligned} \epsilon \int f x^2 e^{-\epsilon x^2} dx &\leq \epsilon^{1/2} \int_{|x| < r_\epsilon} f dx + \int_{|x| > r_\epsilon} f(x) e^{-1} dx \\ &\leq \epsilon^{1/2} \int_{\mathbb{R}^3} f dx + \int_{|x| > r_\epsilon} f(x) e^{-1} dx \rightarrow 0, \end{aligned}$$

as  $\epsilon \rightarrow 0$  since  $r_\epsilon \rightarrow \infty$ . And the lemma is proved. ■

Using lemma 13, and passing to the limit in (58) leads to:

$$\lim_{\epsilon \rightarrow 0} \int \phi(x \cdot \nabla \psi) e^{-\epsilon x^2} = \frac{1}{2} \int |\nabla \psi|^2. \quad (59)$$

ii)  $\int \operatorname{div}(|\nabla \psi|^\sigma \nabla \psi)(x \cdot \nabla \psi) e^{-\epsilon x^2}$ .

By numerous integrations by parts, one obtain:

$$\begin{aligned} \int \operatorname{div}(|\nabla \psi|^\sigma \nabla \psi)(x \cdot \nabla \psi) e^{-\epsilon x^2} &= \left(\frac{3}{\sigma+2} - 1\right) \int |\nabla \psi|^{\sigma+2} \\ &+ 2\epsilon \int (x \cdot \nabla \psi)^2 |\nabla \psi|^\sigma e^{-\epsilon x^2} - \frac{1}{2} \epsilon \int x^2 |\nabla \psi|^{\sigma+2} e^{-\epsilon x^2}. \end{aligned} \quad (60)$$

Using lemma 13 and letting  $\epsilon \rightarrow 0$  in (60) leads to:

$$\lim_{\epsilon \rightarrow 0} \int \operatorname{div}(|\nabla \psi|^\sigma \nabla \psi)(x \cdot \nabla \psi) e^{-\epsilon x^2} = \left(\frac{3}{\sigma+2} - 1\right) \int |\nabla \psi|^{\sigma+2}. \quad (61)$$

iii)  $\int \Delta \phi(x \cdot \nabla \psi) e^{-\epsilon x^2}$ .

Numerous integrations by parts and the use of lemma 13 leads to:

$$\lim_{\epsilon \rightarrow 0} \int \Delta \phi(x \cdot \nabla \psi) e^{-\epsilon x^2} = \frac{1}{2} \int |\phi|^2. \quad (62)$$

From (59), (61) and (62) we deduce

$$\frac{\beta}{2} \int |\nabla \psi|^2 + \frac{\tilde{\epsilon}}{2} \int |\phi|^2 = \left(1 - \frac{3}{\sigma+2}\right) \int |\nabla \psi|^{\sigma+2},$$

as claimed in the proposition. ■

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