WAVES IN FERROMAGNETIC MEDIA

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Abstract. It is shown that small perturbations of equilibrium states in ferromagnetic media give rise to standing and traveling waves that are are stable for long times. The evolution of the wave profiles is governed by semilinear heat equations. The mathematical model underlying these results consists of the Landau–Lifshitz equation for the magnetization vector and Maxwell's equations for the electromagnetic field variables. The model belongs to a general class of hyperbolic equations for vector-valued functions, whose asymptotic properties are analyzed rigorously. The results are illustrated with numerical examples.

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1 Introduction

In this article, we show that small perturbations of equilibrium states in ferromagnetic media give rise to standing and traveling waves that are stable for long times. The evolution of the wave profiles is governed by semilinear heat equations.

These results are obtained in the framework of micromagnetics—a continuum approximation that allows the calculation of magnetization phenomena on a length scale intermediate between the size of a magnetic domain and the mean distance between crystal lattice sites. The basic variable is the magnetization vector, which is assumed to vary continuously with position and describes the magnetization structures and reversal mechanisms in the medium. Its dynamics are those of a spinning top driven by the local effective magnetic field, subject to damping. The dynamic equation was first formulated by Landau and Lifshitz [1] and later given in an equivalent form by Gilbert [2]. The equation is complemented by Maxwell's equations for the electromagnetic field variables [3]. The Landau–Lifshitz–Maxwell equations admit an equilibrium solution, where the magnetization is uniform and everywhere parallel to the effective magnetic field. In this article, we are interested in small perturbations of such equilibrium states. The size of the perturbations is measured by a small parameter ε. The complete mathematical model is given in Section 2.

The spatio-temporal evolution of long-wave perturbations is governed by a system of partial differential equations. The system is a special case of a hyperbolic equation for vector-valued functions, which we analyze in detail in Section 3. Using formal expansion techniques, we show that the equation admits an asymptotic solution that exhibits standing and traveling waves. The wave profiles evolve on a slow-time scale according to system of semilinear heat equations. Using analytical techniques inspired by nonlinear optics, we then show that the asymptotic solution approaches the exact solution of the hyperbolic equation in the limit as $\varepsilon \downarrow 0$. The main result is stated in Theorem 3.1 (Section 3.3).

The derivation of the asymptotic solution and the proof of the convergence theorem require several hypotheses, which are satisfied in the case of the Landau–Lifshitz–Maxwell model. The application is discussed in Section 4. We find that the magnetization as well as the electromagnetic field variables develop standing waves and up to four traveling waves, whose speed of propagation depends on the equilibrium state. The asymptotic solution generalizes the expansion developed in [4].

The numerical results of Section 5 show standing and traveling waves that are stable for long times, as predicted by the analysis.

2 Mathematical Model

The state of a ferromagnet is described by the magnetization vector M. The evolution of M with time (t) is governed by the Landau–Lifshitz (LL) equation [1],

$$\partial_t M = -(M \times H) - \frac{g}{|M|}\left(M \times (M \times H)\right). \tag{2.1}$$

This is the equation of a spinning top driven by the magnetic field H and subject to damping; the constant g is the (dimensionless) damping coefficient. Note that the magnitude $|M|$ of M is an invariant of the motion. The electromagnetic field variables obey Maxwell's equations [3],

$$\partial_t H - \nabla \times E = -\partial_t M, \tag{2.2}$$

$$\partial_t E + \nabla \times H = 0. \tag{2.3}$$

The equations are in dimensionless form, and the coefficients have been set equal to one. The spatial domain is all of $\mathbf{R}^3$.

2.1 Basic Solution

The system of Eqs. (2.1)–(2.3) admits a family of constant solutions,

$$(M, H, E)_\alpha = (M_0, \alpha^{-1} M_0, 0), \quad \alpha > 0. \tag{2.4}$$

Here, M_0 is an arbitrary vector in $\mathbf{R}^3$; without loss of generality, we may assume that $|M_0| = 1$. We are interested in the spatio-temporal evolution of long-wave perturbations of such solutions. The perturbations are measured in terms of an arbitrarily small positive parameter ε and have the form

$$M(x,t) = M_0 + \varepsilon \tilde{M}(\tilde{x}, \tilde{t}, \tau), \tag{2.5}$$

$$H(x,t) = \alpha^{-1} M_0 + \varepsilon \tilde{H}(\tilde{x}, \tilde{t}, \tau), \tag{2.6}$$

$$E(x,t) = \varepsilon \tilde{E}(\tilde{x}, \tilde{t}, \tau), \tag{2.7}$$

where $\tilde{M}$, $\tilde{H}$, and $\tilde{E}$ are $O(1)$ as $\varepsilon \downarrow 0$, and

$$\tilde{x} = \varepsilon x, \quad \tilde{t} = \varepsilon t, \quad \tau = \varepsilon^2 t. \tag{2.8}$$

If the triple (M, H, E) is a solution of Eqs. (2.1)–(2.3), then $\tilde{M}$, $\tilde{H}$, and $\tilde{E}$ must satisfy the system of equations

$$\begin{aligned} \varepsilon \partial_{\tilde{t}} \tilde{M} + \varepsilon^2 \partial_\tau \tilde{M} &= -(M_0 \times \tilde{H}) + \alpha^{-1}(M_0 \times \tilde{M}) - \varepsilon(\tilde{M} \times \tilde{H}) \\ &- \frac{g}{|M|} \Big[M_0 \times (M_0 \times \tilde{H}) - \alpha^{-1} M_0 \times (M_0 \times \tilde{M}) + \varepsilon (M_0 \times (\tilde{M} \times \tilde{H}) \\ &+ \varepsilon \tilde{M} \times (M_0 \times \tilde{H}) - \varepsilon \alpha^{-1} \tilde{M} \times (M_0 \times \tilde{M}) + \varepsilon^2 \tilde{M} \times (\tilde{M} \times \tilde{H}) \Big], \end{aligned} \tag{2.9}$$

$$\varepsilon \partial_{\tilde{t}} \tilde{H} + \varepsilon^2 \partial_\tau \tilde{H} - \varepsilon (\tilde{\nabla} \times \tilde{E}) = -\varepsilon \partial_{\tilde{t}} \tilde{M} - \varepsilon^2 \partial_\tau \tilde{M}, \tag{2.10}$$

$$\varepsilon \partial_{\tilde{t}} \tilde{E} + \varepsilon^2 \partial_\tau \tilde{E} + \varepsilon (\tilde{\nabla} \times \tilde{H}) = 0. \tag{2.11}$$

Here, we have reduced the powers of ε by one everywhere.

Remark 2.1 *In Eq. (2.9), we have left the term $|M|$ in the denominator without expanding it. This term introduces complications that are merely technical and nonessential for the arguments to be presented. To avoid these complications entirely, we will change the model slightly: we replace the term $|M|$ by $|M_0|$, which is 1, and thus reduce the factor multiplying the damping term to g. This modification does not affect the leading-order asymptotics. We describe the changes that need to be made if the term $|M|$ is retained in several remarks below.*

We are interested in solutions of Eqs. (2.9)–(2.11) that describe plane waves propagating in the direction of $\tilde{k}$, a fixed unit vector in $\mathbf{R}^3$ that is not parallel or antiparallel to M_0. Since variations occur only in the direction of $\tilde{k}$, we may make the substitution

$$\tilde{\nabla} = \tilde{k}\partial_{\tilde{x}}, \tag{2.12}$$

if $\tilde{x}$ is the coordinate in the direction of $\tilde{k}$. Henceforth, we omit the tilde, so the equations to be considered are

$$\begin{aligned} \varepsilon\partial_t M + \varepsilon^2\partial_\tau M = &-(M_0 \times H) + \alpha^{-1}(M_0 \times M) - \varepsilon(M \times H) \\ &- g\left[M_0 \times (M_0 \times H) - \alpha^{-1}M_0 \times (M_0 \times M) + \varepsilon M_0 \times (M \times H)\right. \\ &\left.+ \varepsilon M \times (M_0 \times H) - \varepsilon\alpha^{-1}M \times (M_0 \times M) + \varepsilon^2 M \times (M \times H)\right], \end{aligned} \tag{2.13}$$

$$\varepsilon\partial_t H + \varepsilon^2\partial_\tau H - \varepsilon k \times \partial_x E = -\varepsilon\partial_t M - \varepsilon^2\partial_\tau M, \tag{2.14}$$

$$\varepsilon\partial_t E + \varepsilon^2\partial_\tau E + \varepsilon k \times \partial_x H = 0. \tag{2.15}$$

2.2 Vector Formulation

The system of Eqs. (2.13)–(2.15) can be written as a single equation for a function $U : \mathbf{R} \times [0, \infty) \times [0, T] \to \mathbf{R}^9 = \mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{R}^3$,

$$U(x,t,\tau) = \begin{pmatrix} \alpha^{-1/2} M(x,t,\tau) \\ H(x,t,\tau) \\ E(x,t,\tau) \end{pmatrix}, \quad x \in \mathbf{R},\ t \geq 0,\ \tau \in [0,T]. \tag{2.16}$$

The factor $\alpha^{-1/2}$ is introduced for convenience, so the problem has certain symmetry properties (see Section 2.3). After dividing once more by ε, we obtain the following equation for U:

$$\partial_t U + \varepsilon \partial_\tau U + A \partial_x U + \varepsilon^{-1}(L_0 + L_1) U = B(U,U) + \varepsilon T(U,U,U), \tag{2.17}$$

where A, L_0, and L_1 are linear operators in $\mathbf{R}^9$,

$$Au = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -k \times \cdot \\ 0 & k \times \cdot & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, \tag{2.18}$$

$$L_0 u = \begin{pmatrix} -\alpha^{-1}(M_0 \times \cdot) & \alpha^{-1/2}(M_0 \times \cdot) & 0 \\ \alpha^{-1/2}(M_0 \times \cdot) & -(M_0 \times \cdot) & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, \tag{2.19}$$

$$L_1 u = g \begin{pmatrix} -\alpha^{-1} M_0 \times (M_0 \times \cdot) & \alpha^{-1/2} M_0 \times (M_0 \times \cdot) & 0 \\ \alpha^{-1/2} M_0 \times (M_0 \times \cdot) & -M_0 \times (M_0 \times \cdot) & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}; \tag{2.20}$$

B is a bilinear map on $\mathbf{R}^9 \times \mathbf{R}^9$,

$$B(u,v) = \begin{pmatrix} B_1(u,v) \\ -\alpha^{1/2} B_1(u,v) \\ 0 \end{pmatrix}, \tag{2.21}$$

with

$$B_1(u,v) = -\tfrac{1}{2}(u_1 \times v_2 + v_1 \times u_2) - \tfrac{1}{2} g M_0 \times (u_1 \times v_2 + v_1 \times u_2))$$
$$- \tfrac{1}{2} g[(u_1 \times (M_0 \times (v_2 - \alpha^{-1/2} v_1))) + (v_1 \times (M_0 \times (u_2 - \alpha^{-1/2} u_1)))];$$

and T is a trilinear map on $\mathbf{R}^9 \times \mathbf{R}^9 \times \mathbf{R}^9$,

$$T(u,v,w) = \alpha^{1/2} g \begin{pmatrix} T_1(u,v,w) \\ -\alpha^{1/2} T_1(u,v,w) \\ 0 \end{pmatrix}, \tag{2.22}$$

with

$$T_1(u,v,w) = \tfrac{1}{6}[u_1 \times (v_2 \times w_1) + u_1 \times (w_2 \times v_1) + w_1 \times (v_2 \times u_1)$$
$$+ v_1 \times (u_2 \times w_1) + v_1 \times (w_2 \times u_1) + w_1 \times (u_2 \times v_1)].$$

Here, u, v, and w are arbitrary vectors in $\mathbf{R}^9$, $u = (u_1, u_2, u_3)^{\mathrm{t}}$, $v = (v_1, v_2, v_3)^{\mathrm{t}}$, and $w = (w_1, w_2, w_3)^{\mathrm{t}}$ with $u_i, v_i, w_i \in \mathbf{R}^3$, $i = 1, 2, 3$.

Equation (2.17) must be satisfied for all $x \in \mathbf{R}$, $t > 0$, and $\tau \in [0, T]$.

Remark 2.2 *When $|M|$ is retained in Eq. (2.9) (cf. Remark 2.1), L_1 remains unchanged, but a new term, $\phi(U) L_1 U$ with $\phi(U)$ a linear form on $\mathbf{R}^9$, is added to the bilinear form B. The trilinear map T is also modified. The additional terms do not affect the first term of the asymptotic expansion.*

2.3 Auxiliary Properties

Since the vector product is antisymmetric, the operator A is symmetric with respect to the usual scalar product in $\mathbf{R}^9$,

$$Au \cdot v = u \cdot Av, \quad u, v \in \mathbf{R}^9. \tag{2.23}$$

The operators L_0 and L_1 are antisymmetric and symmetric, respectively, with respect to the scalar product in $\mathbf{R}^9$,

$$(L_0 u) \cdot v = -u \cdot (L_0 v),\ (L_1 u) \cdot v = u \cdot (L_1 v), \quad u, v \in \mathbf{R}^9. \tag{2.24}$$

The bilinear map B is symmetric, $B(u, v) = B(v, u)$ for all $u, v \in \mathbf{R}^9$, and the trilinear map T is symmetric in the sense that $T(u, v, w) = T(\pi(u, v, w))$ for all $u, v, w \in \mathbf{R}^9$ and any permutation π.

Lemma 2.1 *The operator $L_0 + L_1$ induces an orthogonal decomposition,*

$$\mathbf{R}^9 = \ker(L_0 + L_1) \oplus \operatorname{im}(L_0 + L_1). \tag{2.25}$$

Proof. The symmetry properties of L_0 and L_1 imply that $((L_0 + L_1)u) \cdot v = -u \cdot (L_0 v) + u \cdot (L_1 v)$ for any $u, v \in \mathbf{R}^9$. A straightforward computation shows that $\ker(L_0) = \ker(L_1) = \ker(L_0 + L_1)$, so $((L_0 + L_1)u) \cdot v = 0$ for any $u \in \mathbf{R}^9$, $v \in \ker(L_0 + L_1)$. ∎

The kernel and image of $L_0 + L_1$ are given explicitly by

$$\ker(L_0 + L_1) = \{v = (v_1, v_2, v_3)^{\mathrm{t}} \in \mathbf{R}^9 : (\alpha^{-1/2} v_1 - v_2) \times M_0 = 0\}, \tag{2.26}$$

$$\operatorname{im}(L_0 + L_1) = \{v = (v_1, v_2, v_3)^{\mathrm{t}} \in \mathbf{R}^9 : v_1 \cdot M_0 = 0, v_2 = -\alpha^{1/2} v_1, v_3 = 0\}. \tag{2.27}$$

Let P and Q be the orthogonal projections on $\ker(L_0 + L_1)$ and $\operatorname{im}(L_0 + L_1)$, respectively, and let R be the inverse of $L_0 + L_1$ on $\operatorname{im}(L_0 + L_1)$, trivially extended to $\mathbf{R}^9$. Then

$$R(L_0 + L_1) = (L_0 + L_1)R = I - P = Q. \tag{2.28}$$

Furthermore, if $(L_0 + L_1)u = v$ for some $u, v \in \mathbf{R}^9$, then $Pv = 0$ and $Qu = Rv$.

The following lemma is verified by direct computation.

Lemma 2.2 *(i) The operator L_1 is coercive on* $\mathrm{im}(L_0 + L_1)$,

$$(L_1 Qv) \cdot (Qv) = g(1 + \alpha^{-1})(Qv) \cdot (Qv), \quad v \in \mathbf{R}^9. \tag{2.29}$$

(ii) The maps B and T are transparent on $\ker(L_0 + L_1)$,

$$PB(Pu, Pv) = 0, \quad PT(Pu, Pv, Pw) = 0, \quad u, v, w \in \mathbf{R}^9. \tag{2.30}$$

Remark 2.3 *If $|M|$ is retained in Eq. (2.9) (cf. Remarks 2.1 and 2.2), the transparency properties given in Eq. (2.30) remain valid, since $L_1 P = 0$.*

3 A General Hyperbolic Equation

Equation (2.17) is a special case of the general differential equation

$$\partial_t U + \varepsilon \partial_\tau U + A \partial_x U + \varepsilon^{-1} LU = B(U, U) + \varepsilon T(U, U, U) \tag{3.1}$$

in $\mathbf{R}^n$ ($n \geq 1$), where A is a symmetric linear operator, L a linear operator, B a symmetric bilinear map, and T a symmetric trilinear map. In this section we consider Eq. (3.1); the application to the special case of Eq. (2.17) follows in Section 4. Our procedure is as follows. First, we construct an asymptotic solution of Eq. (3.1) using formal power series expansions in the small parameter ε (Section 3.1). Then we give precise asymptotic estimates of the various terms in the asymptotic solution (Section 3.2). Finally, we show that the asymptotic solution actually converges to the solution of Eq. (3.1) on the slow-time scale as $\varepsilon \downarrow 0$ (Section 3.3).

3.1 Formal Expansion

We first take an asymptotic approach to Eq. (3.1) and look for a solution $U \equiv U(x, t, \tau)$ of the form

$$U \equiv U_1 + \varepsilon U_2 + \varepsilon^2 U_3 + \cdots , \tag{3.2}$$

proceeding formally by substituting, expanding, and equating the coefficients of like powers of ε. The underlying assumptions are $U_1 = O(1)$, $\varepsilon U_2 = o(1)$, and $\varepsilon U_3 = o(1)$ as $\varepsilon \downarrow 0$. Note that the second assumption differs from the usual assumption, $U_2 = O(1)$. It is indeed a key point in the asymptotic analysis [5, 6], as U_2 is unbounded on the large-time (τ) scale. The construction requires three hypotheses.

Hypothesis 1 $\mathbf{R}^n = \ker(L) \oplus \operatorname{im}(L)$.

Hypothesis 2. $(Lu) \cdot u \geq C\|Qu\|^2$ for all $u \in \mathbf{R}^n$, for some $C > 0$.

Hypothesis 3 $PB(Pu, Pv) = 0$ and $PT(Pu, Pv, Pw) = 0$ for all $u, v, w \in \mathbf{R}^n$.

Here, P and Q are the orthogonal projections on $\ker(L)$ and $\operatorname{im}(L)$, respectively. The hypotheses are satisfied in the case of Eq. (2.17). Hypothesis 3 is commonly referred to as the *transparency* property, a term borrowed from nonlinear optics [7].

Let R be the partial inverse of L on $\operatorname{im}(L)$, trivially extended to all of $\mathbf{R}^n$. Then $RL = LR = Q$. The proof of the following lemma is trivial.

Lemma 3.1 *If $Lu = v$ for some $u, v \in \mathbf{R}^n$, then $Pv = 0$ (solvability condition) and $Qu = Rv$.*

The equation of order $\mathbf{O}(\varepsilon^{-1})$. The equation is

$$LU_1 = 0, \tag{3.3}$$

so $QU_1 = 0$, and, therefore,

$$U_1 = PU_1. \tag{3.4}$$

The equation of order $\mathbf{O}(1)$. The equation is

$$LU_2 = V_2(U_1) = B(U_1, U_1) - (\partial_t + A\partial_x)U_1. \tag{3.5}$$

Because $U_1 = PU_1$ and B is transparent on $\ker(L)$ (Hypothesis 3), the solvability condition $PV_2 = 0$ reduces to

$$(\partial_t + PAP\partial_x)U_1 = 0, \tag{3.6}$$

with $U_1(t = 0) = PU(t = 0)$.

The operator PAP is symmetric, so there exist k projections P_j and k numbers v_j ($j = 1, \dots, k$, $k \le n$) such that

$$P = \sum_{j=1}^{k} P_j; \quad PAPP_j = v_j P_j,\ j = 1, \dots, k. \tag{3.7}$$

Hence, the solvability condition is met if

$$(\partial_t + v_j\partial_x)P_jU_1 = 0, \quad j = 1, \dots, k. \tag{3.8}$$

Because $U_1 = PU_1$ and $R = 0$ on $\ker(L)$, the equation $QU_2 = RV_2$ reduces to

$$QU_2 = RB(U_1, U_1) - RA\partial_x U_1. \tag{3.9}$$

Remark 3.1 *The numbers v_j can be characterized in terms of the characteristic variety $X = \{(\omega, \xi) \in \mathbf{C} \times \mathbf{R} : \det(-i\omega + iA\xi + L) = 0\}$ of the operator $\partial_t + A\partial_x + L$. Since L is not invertible, $(0, 0) \in X$. Suppose that $(0, 0)$ is an isolated singular point of X. Then there exist k functions ω_j ($j = 1, \dots, k$, $k \le n$) satisfying $\omega_j(0) = 0$ that describe X in the neighborhood of $(0, 0)$, and $v_j = \omega_j'(0)$ [8].*

The equation of order $\mathbf{O}(\varepsilon)$. The equation is

$$LU_3 = V_3(U_1, U_2) = 2B(U_1, U_2) + T(U_1, U_1, U_1) - \partial_\tau U_1 - (\partial_t + A\partial_x)U_2. \quad (3.10)$$

Because $U_1 = PU_1$ and T is transparent on $\ker(L)$ (Hypothesis 3), the solvability condition $PV_3 = 0$ reduces to

$$\partial_\tau U_1 + \partial_t PU_2 + PA\partial_x U_2 = 2PB(U_1, U_2). \quad (3.11)$$

We rewrite this condition, using Eq. (3.9) and the transparency of B on $\ker(L)$,

$$\begin{aligned}\partial_\tau U_1 + (\partial_t + PAP\partial_x)PU_2 - PARAP\partial_x^2 U_1 &= 2PB(U_1, RB(U_1, U_1)) \\ &- PAR\partial_x B(U_1, U_1) - 2PB(U_1, RA\partial_x U_1). \quad (3.12)\end{aligned}$$

This equation represents a system of k equations,

$$\begin{aligned}\partial_\tau P_j U_1 + (\partial_t + v_j\partial_x)P_j U_2 - P_j ARA\sum_{i=1}^{k} \partial_x^2 P_i U_1 &= 2P_j B(U_1, RB(U_1, U_1)) \\ &- P_j AR\partial_x B(U_1, U_1) - 2P_j B(U_1, RA\partial_x U_1), \quad j = 1, \ldots, k. \quad (3.13)\end{aligned}$$

The jth equation involves the rate of change of P_jU_1 on the slow (τ) time scale, as well as the rate of change of P_jU_2 along the characteristic determined by v_j on the regular (t) time scale. We can separate these two effects if U_2 satisfies a sublinear growth condition,

$$\lim_{t\to\infty} \frac{1}{t}\|U_2(\cdot\,, t, \tau)\|_{H^s} = 0, \quad (3.14)$$

uniformly on $[0, T]$, for some sufficiently large s. (H^s is the usual Sobolev space of order s.) The condition (3.14) implies, in particular, that $\varepsilon\|U_2\|_{H^s} = o(1)$ as $\varepsilon \downarrow 0$. The separation is accomplished by averaging over t along characteristics. Formally,

$$G_v u(x, t) = \lim_{T\to\infty} \frac{1}{T}\int_0^T u(x + vs, t + s)\,\mathrm{d}s, \quad v \in \mathbf{R}, \quad (3.15)$$

whenever the limit exists. The following lemma is taken from [6, Lemmas 3–6].

Lemma 3.2 *(i) If $(\partial_t + v\partial_x)u = 0$, then $G_{v'}u$ exists for all v'; $G_{v'}u = u$ if $v' = v$, and $G_{v'}u = 0$ otherwise.*

(ii) If $(\partial_t + v\partial_x)u = 0$ and $(\partial_t + v'\partial_x)u' = 0$, then $G_{v''}(uu') = uu'$ if $v'' = v' = v$, and $G_{v''}(uu') = 0$ otherwise.

(iii) If u satisfies a sublinear growth condition, $\lim_{t\to\infty} t^{-1}\|u(\cdot\,,t)\|_{L^\infty} = 0$, then $G_v(\partial_t + v\partial_x)u$ exists, and $G_v(\partial_t + v\partial_x)u = 0$.

The application of G_{v_j} to both sides of Eq. (3.13) eliminates the transport term and reduces the equation to

$$\begin{aligned}\partial_\tau P_jU_1 - P_jARAP_j\partial_x^2 P_jU_1 &= 2P_jB(P_jU_1, RB(P_jU_1, P_jU_1)) \\ &- P_jAR\partial_x B(P_jU_1, P_jU_1) - 2P_jB(P_jU_1, RA\partial_x P_jU_1). \quad j = 1, \ldots, k. \end{aligned} \tag{3.16}$$

The operator P_jARAP_j is nonnegative, because of Hypothesis 2, and proportional to P_j,

$$P_jARAP_j = D_jP_j, \tag{3.17}$$

where D_j is a scalar, $D_j \geq 0$. (In fact, $D_j = \frac{1}{2}\omega_j''(0)$ [8].) The solvability condition $PV_3 = 0$ thus yields a system of k diffusion equations on the slow (τ) time scale,

$$(\partial_\tau - D_j\partial_x^2)P_jU_1 = F_j(P_jU_1), \quad j = 1, \ldots, k, \tag{3.18}$$

where

$$\begin{aligned}F_j(P_jU_1) &= 2P_jB(P_jU_1, RB(P_jU_1, P_jU_1)) \\ &- P_jAR\partial_x B(P_jU_1, P_jU_1) - 2P_jB(P_jU_1, RA\partial_x P_jU_1).\end{aligned}$$

Furthermore, if we use Eq. (3.16) to eliminate the τ derivative in Eq. (3.13), we find that the solvability condition $PV_3 = 0$ also yields a system of k transport equations for P_jU_2 on the regular (t) time scale,

$$(\partial_t + v_j\partial_x)P_jU_2 = S_j(U_1), \quad j = 1, \ldots, k, \tag{3.19}$$

subject to the initial conditions $P_jU_2(t=0)=0$,

where

$$\begin{aligned}
S_j(U_1) &= P_jARA \sum_{i=1, i\neq j}^{k} \partial_x^2 P_i U_1 \\
&\quad + 2P_j[B(U_1, RB(U_1,U_1)) - B(P_jU_1, RB(P_jU_1, P_jU_1))] \\
&\quad - P_jAR\partial_x[B(U_1,U_1) - B(P_jU_1,P_jU_1)] \\
&\quad - 2P_j[B(U_1, RA\partial_x U_1) - B(P_jU_1, RA\partial_x P_jU_1)].
\end{aligned}$$

If Eqs. (3.18)–(3.19) are satisfied, then $QU_3 = RV_3$. The equation reduces to

$$QU_3 = 2RB(U_1,U_2) - R\partial_t U_2 - RA\partial_x U_2. \tag{3.20}$$

This is as far as we go with formal asymptotic analysis. We summarize the results of the analysis in a lemma.

Lemma 3.3 *If U_2 satisfies the sublinear growth condition (3.14), then $U_1 = \sum_{j=1}^{k} P_jU_1$. Each P_jU_1 satisfies a homogeneous transport equation, Eq. (3.8), on the regular (t) time scale and an inhomogeneous heat equation, Eq. (3.18), on the slow (τ) time scale.*

It is interesting to compare the present results with those obtained for the case where the operator L is real antisymmetric, which has been studied extensively in nonlinear optics [5]–[13]. The asymptotics of Eq. (3.1) are intimately connected with the characteristic variety $X = \{(\omega,\xi) \in \mathbf{C}\times\mathbf{R} : \det(-i\omega + iA\xi + L) = 0\}$ of the operator $\partial_t + A\partial_x + L$. For example, plane-wave initial data lead to superpositions of modulated plane waves $\exp(i\varepsilon^{-1}(k_j \cdot x - \omega_j t))$, provided (ω_j, k_j) is a regular point of X [9, 10]. The asymptotic solutions are valid on time intervals of the order $O(1)$ as $\varepsilon \downarrow 0$ (geometrical optics). On the slow (τ) time scale, the dispersive effects of diffraction come into play. Generically, if $(0,0) \in X$, a mean field is created (rectification effect) that evolves according to a nonlinear Schrödinger equation [5, 6]. If $(0,0)$ is a singular point of X, a long-wave asymptotic analysis yields Korteweg–de Vries equations, where the dispersive phenomena are described by third-order differential expressions [11]. A similar situation arises in the water-wave problem, where the long-wave limit yields two counterpropagating waves, each described by a Korteweg–de Vries equation [14, 15].

In the case considered here, L has a symmetric component, and the eigenvalues ω are generally complex. Waves described by an expression of the form $\exp(i\varepsilon^{-1}(k\cdot x - \omega t))$ decay or grow exponentially as $t \to \infty$, so the proofs given, for example, in [6] no longer apply, nor can they be adapted. As in [7, 12, 13], stability on the slow-time scale results from the transparency of B. Nevertheless, it is remarkable that the asymptotic behavior of a *reversible* system is described by a system of *irreversible* equations.

3.2 Asymptotic Estimates

For the convergence proof in the next section, we need asymptotic estimates of the coefficients U_1, U_2, and U_3. The estimates require an additional hypothesis.

Hypothesis 4. Either $D_j > 0$, or, if $D_j = 0$, both terms involving x derivatives in F_j (Eq. (3.18)) are zero, $j = 1, \ldots, k$.

Our first concern is the existence and uniqueness of U_1.

Lemma 3.4 *For any $U_1^0 \in H^s(\mathbf{R})$ $(s \geq 1)$ satisfying the condition $U_1^0 = PU_1^0$, there exists a $T > 0$ and a unique function $U_1 \in C^l([0,\infty) \times [0,T]; H^{s-2l}(\mathbf{R}))$ such that $U_1 = \sum_{j=1}^{k} P_j U_1$, where the functions $P_j U_1$ satisfy Eqs. (3.8) and (3.18). Furthermore, $U_1(\cdot\,,0,0) = U_1^0$.*

Proof. Let $u_j^0 = P_j U_1^0$. Because of Hypothesis 4, there exists a $T_j > 0$ and a unique solution $u_j \in C^l([0, T_j], H^{s-2l}(\mathbf{R}))$ of Eq. (3.18) such that $u_j(0) = u_j^0$. Take $T = \min\{T_j : j = 1, \ldots, k\}$. Then the function U_1 defined by the expression

$$U_1(x, t, \tau) = \sum_{j=1}^{k} u_j(x - v_j t, \tau), \quad x \in \mathbf{R}, t \geq 0, \tau \geq 0,$$

satisfies the conditions of the lemma. ∎

Next, we address the asymptotic estimates of U_1, U_2, and U_3. We introduce the spaces $X_{s,T}$ and $Y_{s,T}$ $(s > 0, T > 0)$ of real-valued functions u defined on $\mathbf{R} \times [0, \infty) \times [0, T]$,

$$X_{s,T} = \{u : \sup\{\|\partial_\tau^l \partial_t^\alpha \partial_x^\beta u(\cdot\,, t, \tau)\|_{L^2(\mathbf{R})} : t \in [0, \infty), \tau \in [0, T]\} < \infty\}, \quad (3.21)$$

$$Y_{s,T} = \{u : \lim_{t \to \infty} t^{-1} \sup\{\|\partial_\tau^l \partial_t^\alpha \partial_x^\beta u(\cdot\,, t, \tau)\|_{L^2(\mathbf{R})} : \tau \in [0, T]\} = 0\}, \quad (3.22)$$

for all α, β, and l such that $\alpha + \beta + 2l = s$ and $\frac{1}{2}s \geq l \geq 0$. The following lemma is taken from [6, Proposition 5].

Lemma 3.5 *If $(\partial_t + v\partial_x)u = f \in X_{s,T}$ and $G_v f = 0$, then $u \in Y_{s,T}$.*

Lemma 3.5 enables us to establish the desired asymptotic estimates.

Lemma 3.6 *If $U_1^0 \in H^s(\mathbf{R})$, then $U_1 \in X_{s,T}$, $QU_2 \in X_{s-1,T}$, $PU_2 \in Y_{s-2,T}$, and $QU_3 \in Y_{s-3,T}$.*

Proof. The first assertion is an immediate consequence of the construction of U_1 (Lemma 3.4) and the definition of the space $X_{s,T}$. Equation (3.9) defines $QU_2 \in X_{s-1,T}$. Then Eq. (3.19) defines $P_j U_2$ for $j = 1, \ldots, k$. The inhomogenous term S_j in Eq. (3.19) averages to zero along the characteristic determined by v_j, $G_{v_j} S_j = 0$, so $P_j U_2 \in Y_{s-2,T}$ for each j. Hence, $PU_2 \in Y_{s-2,T}$. Equation (3.20) defines $QU_3 \in Y_{s-3,T}$. ∎

3.3 Convergence Proof

Given any $U^0 \in H^5(\mathbf{R})$, we define $U_1^0 = PU^0$ and construct $U_1 \equiv U_1(x,t,\tau)$ in accordance with Lemma 3.4 and $U_2 \equiv U_2(x,t,\tau)$ and $QU_3 = QU_3(x,t,\tau)$ in accordance with Lemma 3.6. Our goal in this section is to prove that there exists a solution $U \equiv U(x,t)$ of Eq. (3.1) satisfying $U(\cdot\,,0) = U^0$ such that $U(\cdot\,,t) - U_1(\cdot\,,t,\varepsilon t) \to 0$ on $[0,T/\varepsilon]$ in a suitable norm.

Theorem 3.1 *Let Hypotheses 1–4 be satisfied. For any $U^0 \in H^5(\mathbf{R})$, there exists a $T > 0$, which does not depend on ε, such that Eq. (3.1) has a unique solution $U \in C([0,T/\varepsilon]; H^1(\mathbf{R}))$ satisfying $U(\cdot\,,0) = U^0$. Furthermore,*

$$\sup\{\|PU(\cdot\,,t) - U_1(\cdot\,,t,\varepsilon t)\|_{H^1} : t \in [0,T/\varepsilon]\} = o(1) \ \textit{as}\ \varepsilon \downarrow 0, \tag{3.23}$$

$$\sup\{\|QU(\cdot\,,t)\|_{H^2} : t \in [t_0,T/\varepsilon]\} = o(1) \ \textit{as}\ \varepsilon \downarrow 0,\ \textit{for any}\ t_0 > 0, \tag{3.24}$$

$$\frac{1}{\varepsilon}\int_0^{T/\varepsilon} \|QU(\cdot\,,t)\|_{H^2}\,\mathrm{d}t = o(1). \tag{3.25}$$

Proof. We introduce the function $U_a \equiv U_a(x,t,\tau)$ on $\mathbf{R} \times [0,T/\varepsilon] \times [0,T]$ by the definition

$$U_a = U_1 + \varepsilon U_2 + \varepsilon^2 QU_3. \tag{3.26}$$

Because the variable τ is restricted to the compact interval $[0,T]$, it does not play a critical role. Without loss of generality we may make the identification $\tau = \varepsilon t$ and consider U_1, U_2 and QU_3, as well as U_a, as functions of x and t on $\mathbf{R} \times [0,T/\varepsilon]$.

By assumption, $U_1 \in H^5(\mathrm{R})$, so Lemma 3.6 gives the asymptotic estimates

$$\|U_1\|_{H^1} = O(1),\ \|QU_2\|_{H^1} = O(1),\ \varepsilon\|PU_2\|_{H^1} = o(1),\ \varepsilon\|QU_3\|_{H^1} = o(1). \tag{3.27}$$

It follows that $\|PU_a\|_{H^1} = O(1)$ and $\|QU_a\|_{H^1} = O(\varepsilon)$. The estimates hold uniformly on $[0,T/\varepsilon]$.

The proof of Theorem 3.1 consists of several steps; each step is summarized in a lemma.

Lemma 3.7 *The function* U_a *satisfies a differential equation,*

$$\partial_t U_a + A\partial_x U_a + \varepsilon^{-1} L U_a = B(U_a, U_a) + \varepsilon T(U_a, U_a, U_a) + \varepsilon r_1 + Q r_2, \quad (3.28)$$

where $\|r_1\|_X,\ \|r_2\|_X = o(1),\ X = L^\infty([0, T/\varepsilon]; H^1(\mathbf{R})),$ *as* $\varepsilon \downarrow 0$.

Proof. The function U_a satisfies the following differential equation:

$$\begin{aligned}
&\partial_t U_a + A\partial_x U_a + \varepsilon^{-1} L U_a = B(U_a, U_a) - \varepsilon^2 B(U_2, U_2) - 2\varepsilon^2 B(U_1, QU_3) \\
&- 2\varepsilon^3 B(U_2, QU_3) - \varepsilon^4 B(QU_3, QU_3) + \varepsilon T(U_a, U_a, U_a) - 3\varepsilon^2 T(U_1, U_1, U_2) \\
&- 3\varepsilon^3 T(U_1, U_2, U_2) - 3\varepsilon^3 T(U_1, U_1, QU_3) - \varepsilon^4 T(U_2, U_2, U_2) \\
&- 6\varepsilon^4 T(U_1, U_2, QU_3) - 3\varepsilon^5 T(U_1, QU_3, QU_3) - 3\varepsilon^5 T(U_2, U_2, QU_3) \\
&- 3\varepsilon^6 T(U_2, QU_3, QU_3) - \varepsilon^7 T(QU_3, QU_3, QU_3) \\
&+ \varepsilon^2 [\partial_t QU_3 + A\partial_x QU_3]. \qquad (3.29)
\end{aligned}$$

Using Eq. (3.27), we estimate each term that does not involve U_a. The term $B(U_2, U_2)$ is special because of Hypothesis 3,

$$B(U_2, U_2) = 2PB(PU_2, QU_2) + PB(QU_2, QU_2) + QB(U_2, U_2). \qquad (3.30)$$

Hence,

$$\varepsilon^2 B(U_2, U_2) = \varepsilon p_1 + Q p_2, \qquad (3.31)$$

where $\|p_1\|_X, \|p_2\|_X = o(1)$ as $\varepsilon \downarrow 0$. The remaining terms are easy to estimate; they are all at least $o(\varepsilon)$. The assertion of the lemma follows. ∎

Lemma 3.8 *For any* $U^0 \in H^4(\mathbf{R})$, *there exist a* $T > 0$ *and a unique function* $U \in C([0, T/\varepsilon]; H^1(\mathbf{R}))$ *that satisfies Eq. (3.1) on* $[0, T/\varepsilon]$ *and the initial* *condition* $U(\cdot\,,0) = U^0$*. The difference* $V = U - U_a$ *satisfies a differential inequality*

$$\frac{\mathrm{d}}{\mathrm{d}t}\|V(t)\|_{H^1}^2 + \varepsilon^{-1}\|QV\|_{H^1}^2$$
$$\leq \varepsilon C\left[\|V\|_{H^1}^2 + \|V\|_{H^1}^4 + ||U_a||_{H^1}\|V\|_{H^1}^3 + o(1)\right], \quad t \in (0, T/\varepsilon], \quad (3.32)$$

for some positive constant C *that does not depend on* ε*.*

Proof. If U satisfies Eq. (3.1) and U_a satisfies Eq. (3.28), then V satisfies the equation

$$\partial_t V + A\partial_x V + \varepsilon^{-1}LV = B(V,V) + 2B(U_a,V) + \varepsilon T(V,V,V)$$
$$+ 3\varepsilon T(U_a,V,V) + 3\varepsilon T(U_a,U_a,V) - (\varepsilon r_1 + Qr_2). \quad (3.33)$$

We take the scalar product of both sides of this equation with V and $-\partial_x^2 V$, add the two equations, and integrate the resulting equation over $\mathbf{R}$,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|V(t)\|_{H^1}^2 + \varepsilon^{-1}((LV,V)) = ((B(V,V),V)) + 2((B(U_a,V),V))$$
$$+ \varepsilon((T(V,V,V),V)) + 3\varepsilon((T(U_a,V,V),V)) + 3\varepsilon((T(U_a,U_a,V),V))$$
$$- ((\varepsilon r_1 + Qr_2, V)). \quad (3.34)$$

Here, $((\cdot\,,\cdot))$ denotes the H^1-inner product, $((u,v)) = \int_{\mathbf{R}}((u\cdot v) + (\partial_x u \cdot \partial_x v))(x)\,\mathrm{d}x$ for $u,v \in [H^1(\mathbf{R})]^n$.

We estimate each term in the right member. Again, the transparency of B on $\ker(L)$ plays a critical role: any product $((B(u,v),w))$ is estimated by a sum of terms, each of which contains at least one of Qu, Qv, and Qw,

$$|((B(u,v),w))| \leq C(\|Qu\|_{H^1}\|v\|_{H^1}\|w\|_{H^1} + \|u\|_{H^1}\|Qv\|_{H^1}\|w\|_{H^1}$$
$$+ \|u\|_{H^1}\|v\|_{H^1}\|Qw\|_{H^1}), \quad u,v,w \in [H^1(\mathbf{R})]^n.$$

Thus we find that there exists a positive constant C that does not depend on ε such that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|V(t)\|_{H^1}^2 + \varepsilon^{-1}((LV,V))$$

$$\leq C\,[\|V\|_{H^1}^2\|QV\|_{H^1} + \|QU_a\|_{H^1}\|V\|_{H^1}^2 + \|U_a\|_{H^1}\|V\|_{H^1}\|QV\|_{H^1}$$
$$+\,\varepsilon\,(\|V\|_{H^1}^4 + \|U_a\|_{H^1}\|V\|_{H^1}^3 + \|U_a\|_{H^1}^2\|V\|_{H^1}^2)$$
$$+\varepsilon\|r_1\|_{H^1}\|V\|_{H^1} + \|r_2\|_{H^1}\|QV\|_{H^1}]\,. \tag{3.35}$$

Hypothesis 2 enables us to estimate the left member from below, replacing the term $((LV,V))$ by $C\|QV\|_{H^1}^2$. We estimate the right member from above by means of Young's inequality, absorbing every term involving $\|QV\|_{H^1}$ in the left member. (At this step we make use of the fact that $\|QU_a\|_{H^1} = O(\varepsilon)$.) The differential inequality (3.32) follows. ∎

The convergence does not follow from Lemma 3.8, since $V(\cdot\,,0)$ does not tend to zero as $\varepsilon \downarrow 0$. As a matter of fact, $PU(\cdot\,,0) = PU_1(\cdot\,,0) + o(1) = U_1(\cdot\,,0) + o(1)$, so $PV(\cdot\,,0) = o(1)$ as $\varepsilon \downarrow 0$. But $QU(\cdot\,,0)$ is not necessarily zero, so we can conclude only that $QV(\cdot\,,0) = O(1)$.

Lemma 3.9 *There exists a positive constant C that does not depend on ε such that*

$$\sup\{\|V(t)\|_{H^i} : t \in [0, T/\varepsilon]\} \leq C,\ \frac{1}{\varepsilon}\int_0^{T/\varepsilon} \|QV(t)\|_{H^i}^2(t)\,\mathrm{d}t \leq C, \quad i = 1, 2. \tag{3.36}$$

Proof. The estimates in H^1 follow from Lemma 3.8; those in H^2 follow similarly, because $U_1(\cdot\,,0) \in H^5(\mathbf{R})$. ∎

Lemma 3.10 *For any $t_0 > 0$ that does not depend on ε,*

$$\sup\{\|PV(t)\|_{H^1} : t \in [0, t_0]\} = o(1). \tag{3.37}$$

Proof. We apply P to Eq. (3.33),

$$\partial_t PV + PA\partial_x(PV + QV) = PB(V,V) + 2PB(U_a,V)$$
$$+\,\varepsilon PT(V,V,V) + 3\varepsilon PT(U_a,V,V) + 3\varepsilon PT(U_a,U_a,V) - \varepsilon Pr_1, \tag{3.38}$$

and take the H^1 inner product with PV,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|PV(t)\|_{H^1}^2 + ((PA\partial_x QV, PV)) = ((PB(V,V), PV))$$
$$+ 2((PB(U_a,V), PV)) + \varepsilon((PT(V,V,V), PV)) + 3\varepsilon((PT(U_a,V,V), PV))$$
$$+ 3\varepsilon((PT(U_a,U_a,V), PV)) - \varepsilon((Pr_1, PV)). \qquad (3.39)$$

Again, because of the transparency of B on $\ker(L)$,

$$|((PB(V,V), PV))| \le C(\|PV\|_{H^1}\|QV\|_{H^1} + \|QV\|_{H^1}^2)\|PV\|_{H^1},$$
$$|((PB(U_a,V), PV))| \le C(\|U_a\|_{H^1}\|QV\|_{H^1} + \|QU_a\|_{H^1}\|V\|_{H^1})\|PV\|_{H^1},$$

so

$$\frac{\mathrm{d}}{\mathrm{d}t}\|PV(t)\|_{H^1}^2 \le C\Big[\|QV\|_{H^2}\|PV\|_{H^1} + \|PV\|_{H^1}^2\|QV\|_{H^1}$$
$$+ \|QV\|_{H^1}^2\|PV\|_{H^1} + \|U_a\|_{H^1}\|QV\|_{H^1}\|PV\|_{H^1} + \|QU_a\|_{H^1}\|V\|_{H^1}\|PV\|_{H^1}$$
$$+\varepsilon\|V\|_{H^1}^4 + \varepsilon\|U_a\|_{H^1}\|V\|_{H^1}^3 + \varepsilon\|U_a\|_{H^1}^2\|V\|_{H^1}^2 + \varepsilon\|r_1\|_{H^1}\|PV\|_{H^1}\Big]. \quad (3.40)$$

It follows from Lemma 3.9 that $\|V(t)\|_{H^1}$ is bounded on $[0, T/\varepsilon]$. We already know that $\|U_a\|_{H^1}$ is bounded and $\|QU_a\|_{H^1} = O(\varepsilon)$, so

$$\frac{\mathrm{d}}{\mathrm{d}t}\|PV(t)\|_{H^1}^2 \le C\left[(\|QV\|_{H^2} + \|QV\|_{H^1})\|PV\|_{H^1} + \varepsilon\right]. \qquad (3.41)$$

Applying Young's inequality, we obtain the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\|PV(t)\|_{H^1}^2 \le \|PV\|_{H^1}^2 + C\Big[(\|QV\|_{H^2}^2 + \|QV\|_{H^1}^2) + \varepsilon\Big]. \qquad (3.42)$$

According to Lemma 3.9,

$$\int_0^{T/\varepsilon}\Big[\|QV(t)\|_{H^1}^2 + \|QV(t)\|_{H^2}^2\Big]\,\mathrm{d}t \le C\varepsilon. \qquad (3.43)$$

Applying Gronwall's lemma to Eq. (3.42), we obtain the estimate

$$\|PV(t)\|_{H^1}^2 \le \|PV(0)\|_{H^1}^2 \mathrm{e}^{t_0} + C\varepsilon, \quad t \in [0, t_0], \qquad (3.44)$$

for any $t_0 > 0$ that does not depend on ε. The lemma follows, because $\|PV(0)\|_{H^1} = \varepsilon\|PU_2(0)\|_{H^1} = o(1)$. ∎

We now complete the proof of Theorem 3.1.

According to Lemma 3.9, $\int_0^{t_0} \|QV(t)\|_{H^1}^2 \,\mathrm{d}t \le C\varepsilon$, so $\|QV(t_1)\|_{H^1}^2 \le 2C\varepsilon$ for some $t_1 \in (0, t_0)$. Using this estimate and Lemma 3.10, we conclude that $\|V(t_1)\|_{H^1} = o(1)$. On $[t_1, T/\varepsilon]$, V satisfies the asymptotic differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\|V(t)\|_{H^1}^2 + \varepsilon^{-1}\|QV\|_{H^1}^2 = o(\varepsilon). \tag{3.45}$$

Hence, $\|V(t)\|_{H^1} = o(1)$ on $[t_1, T/\varepsilon]$. The proof of the theorem is complete. ∎

4 The Landau–Lifshitz–Maxwell Equations

We now return to Eq. (2.17) and the system of partial differential equations of micromagnetics, Eqs. (2.13)–(2.15).

As we observed in Section 2.2, Eq. (2.17) is a special case of the general equation (3.1). (In fact, Eq. (2.17) provided the motivation for the analysis of Section 3.) Hypotheses 1–3 are satisfied (see Section 2.3); we will verify the remaining Hypothesis 4 once we have found the coefficients D_j. The asymptotic approximation is therefore unique and valid on the slow-time scale. How the asymptotic approximation is actually constructed is irrelevant. This observation is important because it allows us to use the Landau–Lifshitz equation in the form given by Gilbert [2],

$$\partial_t M = -(M \times H) + \frac{g}{|M|}\,(M \times \partial_t M)\,. \tag{4.1}$$

This equation, which is known as the Landau–Lifshitz–Gilbert (LLG) equation, is equivalent with the LL equation (2.1), except for a rescaling of time by a factor $1 + g^2$. As it turns out, the LLG equation is more convenient for constructing the asymptotic expansions.

We need to make one more change. In Section 2.1, we introduced a simplification of the mathematical model, replacing the term $|M|$ in the denominator of the damping term by $|M_0|$; see Remark 2.1. We make the same simplification in Eq. (4.1) and take the factor multiplying the damping term to be g. Thus, we start from the following system of equations:

$$\varepsilon\partial_t M + \varepsilon^2\partial_\tau M = -(M_0 \times H) + \alpha^{-1}(M_0 \times M) - \varepsilon(M \times H)$$
$$+ g[\varepsilon(M_0 \times \partial_t M) + \varepsilon^2(M_0 \times \partial_\tau M) + \varepsilon^2(M \times \partial_t M) + \varepsilon^3(M \times \partial_\tau M)], \quad (4.2)$$

$$\varepsilon\partial_t H + \varepsilon^2\partial_\tau H - \varepsilon k \times \partial_x E = -\varepsilon\partial_t M - \varepsilon^2\partial_\tau M, \quad (4.3)$$

$$\varepsilon\partial_t E + \varepsilon^2\partial_\tau E + \varepsilon k \times \partial_x H = 0. \quad (4.4)$$

Note that the exponent of ε in this system is 1 more than in Eq. (3.1). There is no need to symmetrize the equations.

We construct an asymptotic solution of Eqs. (2.13)–(2.15) along the lines of Section 3.1,

$$M = M_1 + \varepsilon M_2 + \varepsilon^2 M_3 + \cdots, \quad (4.5)$$

$$H = H_1 + \varepsilon H_2 + \varepsilon^2 H_3 + \cdots, \quad (4.6)$$

$$E = E_1 + \varepsilon E_2 + \varepsilon^2 E_3 + \cdots. \quad (4.7)$$

4.1 The Equations of Order $O(1)$

To leading order, Eqs. (4.2)–(4.4) reduce to a single equation,

$$-M_0 \times (H_1 - \alpha^{-1} M_1) = 0. \quad (4.8)$$

The equation gives an expression for M_1 in terms of $M_1 \cdot M_0$ and H_1,

$$M_1 = (M_1 \cdot M_0)M_0 - \alpha M_0 \times (M_0 \times H_1). \quad (4.9)$$

4.2 The Equations of Order $O(\varepsilon)$

To first order, Eqs. (4.2)–(4.4) yield a set of differential equations,

$$\partial_t M_1 = -M_0 \times (H_2 - \alpha^{-1} M_2 - g\partial_t M_1) - M_1 \times H_1, \tag{4.10}$$

$$\partial_t H_1 - k \times \partial_x E_1 = -\partial_t M_1, \tag{4.11}$$

$$\partial_t E_1 + k \times \partial_x H_1 = 0. \tag{4.12}$$

Taking the scalar product of Eq. (4.10) with M_0 and adding to it the scalar product of Eq. (4.8) with M_1, we find that $\partial_t(M_1 \cdot M_0) = 0$, so

$$M_1 \cdot M_0 = f_0, \quad f_0 \equiv f_0(x, \tau). \tag{4.13}$$

(Note that $M_1 \cdot M_0$ is the $O(\varepsilon)$ term in the expansion of $|M|^2$, which is constant.) If, instead of the scalar product, we take the vector product, we obtain an expression for M_2 in terms of $M_2 \cdot M_0$ and H_2,

$$M_2 = (M_2 \cdot M_0)M_0 - \alpha M_0 \times (M_0 \times H_2) + \alpha M_0 \times q, \tag{4.14}$$

where the vector q is given in terms of M_1 and H_1,

$$q = -\partial_t M_1 + g M_0 \times \partial_t M_1 - M_1 \times H_1. \tag{4.15}$$

We substitute the expression (4.9) in the right member of Eq. (4.11), use the fact that $\partial_t(M_1 \cdot M_0) = 0$, and solve the resulting equation for $\partial_t H_1$ to obtain a system of equations for H_1 and E_1,

$$\partial_t H_1 + \frac{\alpha}{1+\alpha}(k \cdot (M_0 \times \partial_x E_1))M_0 - \frac{1}{1+\alpha} k \times \partial_x E_1 = 0, \tag{4.16}$$

$$\partial_t E_1 + k \times \partial_x H_1 = 0. \tag{4.17}$$

4.2.1 Choice of Coordinates

The system of Eqs. (4.16)–(4.17) is most easily solved if we adopt a coordinate system in $\mathbf{R}^3$ that is spanned by k, $k \times M_0$, and M_0. (Here, we rely on the assumption that k and M_0 are not parallel or antiparallel.) Given any vector $v \in \mathbf{R}^3$, we define

$$v_a = v \cdot M_0, \quad v_b = v \cdot (k \times M_0), \quad v_c = v \cdot k, \quad v \in \mathbf{R}^3. \tag{4.18}$$

Then

$$v = \frac{1}{1-k_a^2}\left[(v_a - k_a v_c)M_0 + v_b(k \times M_0) + (v_c - k_a v_a)k\right], \quad v \in \mathbf{R}^3, \tag{4.19}$$

where

$$k_a = M_0 \cdot k. \tag{4.20}$$

An easy computation shows that

$$u \cdot v = \frac{1}{1-k_a^2}[u_a v_a + u_b v_b + u_c v_c - k_a(u_a v_c + u_c v_a)], \quad u, v \in \mathbf{R}^3, \tag{4.21}$$

$$u \times v = \frac{1}{1-k_a^2}\begin{vmatrix} M_0 & k \times M_0 & k \\ u_a & u_b & u_c \\ v_a & v_b & v_c \end{vmatrix}, \quad u, v \in \mathbf{R}^3. \tag{4.22}$$

The system of Eqs. (4.16)–(4.17) becomes

$$\partial_t u_1 + K \partial_x u_1 = 0, \tag{4.23}$$

where $u_1 = (H_{1a}, H_{1b}, H_{1c}, E_{1a}, E_{1b}, E_{1c})^{\mathrm{t}}$ and

$$K = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -(1+\alpha)^{-1} & 0 & k_a(1+\alpha)^{-1} \\ 0 & 0 & 0 & 0 & k_a\alpha(1+\alpha)^{-1} & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -k_a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.24}$$

The derivatives in Eq. (4.23) are taken componentwise.

4.2.2 Solution of Equation (4.23)

The characteristic determinant of K is

$$\det(\lambda I - K) = (\lambda^2 - v_0^2)(\lambda^2 - v_1^2)(\lambda^2 - v_2^2), \tag{4.25}$$

where

$$v_0 = 0, \quad v_1 = \left(\frac{1}{1+\alpha}\right)^{1/2}, \quad v_2 = \left(\frac{1+(1-k_a^2)\alpha}{1+\alpha}\right)^{1/2}, \tag{4.26}$$

so the eigenvalues of K are $v_0 = 0$ (algebraic multiplicity 2), $\pm v_1$, and $\pm v_2$. Note that $v_0 < v_1 < v_2$; furthermore,

$$1 - v_2^2 = k_a^2(1 - v_1^2). \tag{4.27}$$

In terms of v_1 and v_2, we have

$$K = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -v_1^2 & 0 & k_a v_1^2 \\ 0 & 0 & 0 & 0 & k_a(1-v_1^2) & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -k_a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.28}$$

The matrix K is diagonalized by the linear transformation F,

$$K = F^{-1}VF, \quad V = \operatorname{diag}(v_0, v_0, v_1, -v_1, v_2, -v_2), \tag{4.29}$$

where

$$F = \begin{pmatrix} k_a(1-v_1^{-2}) & 0 & v_1^{-2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -v_1 & 0 & k_a v_1 \\ 0 & 1 & 0 & v_1 & 0 & -k_a v_1 \\ v_2^{-1} & 0 & -k_a v_2^{-1} & 0 & 1 & 0 \\ -v_2^{-1} & 0 & k_a v_2^{-1} & 0 & 1 & 0 \end{pmatrix}, \tag{4.30}$$

$$
F^{-1} = \begin{pmatrix} k_a v_1^2 v_2^{-2} & 0 & 0 & 0 & \frac{1}{2} v_2^{-1} & -\frac{1}{2} v_2^{-1} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ v_1^2 v_2^{-2} & 0 & 0 & 0 & \frac{1}{2} k_a (1 - v_1^2) v_2^{-1} & -\frac{1}{2} k_a (1 - v_1^2) v_2^{-1} \\ 0 & k_a & -\frac{1}{2} v_1^{-1} & \frac{1}{2} v_1^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.31}
$$

Applying F to both members of Eq. (4.23), we obtain a diagonal system,

$$
(\partial_t + V \partial_x) F u_1 = 0. \tag{4.32}
$$

(This system corresponds to Eq. (3.8).) The equations are decoupled, and each equation can be integrated along its characteristics. Upon application of the inverse transformation F^{-1} we find

$$
u_1 = F^{-1} f, \tag{4.33}
$$

where

$$
u_1 = \begin{pmatrix} H_{1a} \\ H_{1b} \\ H_{1c} \\ E_{1a} \\ E_{1b} \\ E_{1c} \end{pmatrix}, \quad f = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \end{pmatrix}, \quad \begin{aligned} f_1 &\equiv f_1(x, \tau), \\ f_2 &\equiv f_2(x, \tau), \\ f_3 &\equiv f_3(x - v_1 t, \tau), \\ f_4 &\equiv f_4(x + v_1 t, \tau), \\ f_5 &\equiv f_5(x - v_2 t, \tau), \\ f_6 &\equiv f_6(x + v_2 t, \tau). \end{aligned} \tag{4.34}
$$

The functions f_1 and f_2 represent standing waves; f_3 and f_4 are traveling waves propagating with the velocity v_1 and $-v_1$, respectively; and f_5 and f_6 are traveling waves propagating with the velocity v_2 and $-v_2$, respectively.

The components of M_1 are found from Eqs. (4.9), (4.13), and (4.33),

$$
M_{1a} = f_0, \; M_{1b} = \frac{1 - v_1^2}{2 v_1^2} (f_3 + f_4), \; M_{1c} = k_a f_0 + \frac{v_2^2 - v_1^2}{v_2^2} f_1 - k_a \frac{1 - v_1^2}{2 v_2} (f_5 - f_6). \tag{4.35}
$$

This completes the analysis of the first-order approximation. We now know that the coefficients of order 1 in the expansions (4.5), (4.6), and (4.7) are linear combinations of standing ($v_0 = 0$) and traveling waves ($\pm v_1$, $\pm v_2$). In the next section we will see how the profile functions $f_0, \dots, f_6$ evolve on the slow-time scale (that is, as a function of τ).

4.3 The Equations of Order $O(\varepsilon^2)$

To second order, Eqs. (4.2)–(4.4) yield the differential equations

$$\partial_t M_2 + \partial_\tau M_1 = -M_0 \times \left(H_3 - \alpha^{-1} M_3 - g\partial_t M_2 - g\partial_\tau M_1\right) - M_1 \times (H_2 - g\partial_t M_1) - (M_2 \times H_1), \tag{4.36}$$

$$\partial_t H_2 + \partial_\tau H_1 - k \times \partial_x E_2 = -\left(\partial_t M_2 + \partial_\tau M_1\right), \tag{4.37}$$

$$\partial_t E_2 + \partial_\tau E_1 + k \times \partial_x H_2 = 0. \tag{4.38}$$

We follow the same procedure as in the preceding section. First, we take the scalar product of Eq. (4.36) with M_0 and add to it the scalar product of Eq. (4.10) with M_1 and the scalar product of Eq. (4.8) with M_2. The result is

$$\partial_t (M_2 \cdot M_0 + \tfrac{1}{2}|M_1|^2) + \partial_\tau (M_1 \cdot M_0) = 0. \tag{4.39}$$

Recall Eq. (4.13), $M_1 \cdot M_0 = f_0$, where f_0 does not depend on t. Hence, Eq. (4.39) implies that $M_2 \cdot M_0 + \frac{1}{2}|M_1|^2$ grows linearly with t as $t \to \infty$, unless f_0 is independent not only of t but also of τ. We avoid this type of secular behavior by imposing the condition $f_0 \equiv f_0(x)$.

The scalar product of Eq. (4.36) with M_0 thus yields the relation

$$2M_2 \cdot M_0 + |M_1|^2 = f_{20}, \quad f_{20} \equiv f_{20}(x, \tau). \tag{4.40}$$

(Note that $2M_2 \cdot M_0 + |M_1|^2$ is the $O(\varepsilon^2)$ term in the expansion of $|M|^2$, which is constant.) If, instead of the scalar product, we take the vector product of

Eq. (4.36) with M_0, we obtain an expression for M_3 in terms of H_3 (as well as H_2 and H_1),

$$M_3 = (M_3 \cdot M_0)M_0 + (1 - v_1^{-2})M_0 \times (M_0 \times H_3) - (1 - v_1^{-2})M_0 \times q_2, \quad (4.41)$$

where

$$q_2 = -\partial_t M_2 - \partial_\tau M_1 + gM_0 \times \partial_t M_2 + gM_0 \times \partial_\tau M_1 + gM_1 \times \partial_t M_1 - M_1 \times H_2 - M_2 \times H_1. \quad (4.42)$$

We substitute the expression (4.14) in the right member of Eq. (4.37) and solve the resulting equation for $\partial_t H_2$ to obtain a system of equations for H_2 and E_2,

$$\partial_t H_2 + (1 - v_1^2)(k \cdot (M_0 \times \partial_x E_2))M_0 - v_1^2 k \times \partial_x E_2 = (1 - v_1^2)(M_0 \cdot v_H)M_0 + v_1^2 v_H, \quad (4.43)$$

$$\partial_t E_2 + k \times \partial_x H_2 = v_E. \quad (4.44)$$

The vectors v_H and v_E are known,

$$v_H = -\partial_\tau(H_1 + M_1) + \partial_t[\tfrac{1}{2}|M_1|^2 M_0 - \alpha M_0 \times q], \quad (4.45)$$

$$v_E = -\partial_\tau E_1. \quad (4.46)$$

Here, q is the vector defined in Eq. (4.15).

Remark 4.1 *If the term $|M|$ is retained in Eq. (2.9) (cf. Remarks 2.1, 2.2, and 2.3), one must add a term $g(M_0 \cdot M_1)M_0 \times \partial_t M_1$ to the right-hand side of Eq. (4.36). But Eq. (4.39) remains unchanged, and the modification affects only the expression for M_3 in Eq. (4.41). Therefore, the derivation of the equations for f in the next paragraph remains unchanged.*

4.3.1 Coordinate Representation

We use the coordinate system introduced in Section 4.2.1 and the abbreviations defined in Eq. (4.18). Equations (4.43) and (4.44) correspond to the equation

$$\partial_t u_2 + K\partial_x u_2 = -\partial_\tau F^{-1} f + \partial_t r, \tag{4.47}$$

where K is the matrix defined in Eq. (4.24), f is the vector $f = (f_1, \ldots, f_6)^{\mathrm{t}}$, and u_2 and r stand for the vectors

$$u_2 = \begin{pmatrix} H_{2a} \\ H_{2b} \\ H_{2c} \\ E_{2a} \\ E_{2b} \\ E_{2c} \end{pmatrix}, \quad r = \begin{pmatrix} \frac{1}{2}|M_1|^2 \\ (1-v_1^2)q_c \\ \frac{1}{2}k_a|M_1|^2 - (1-v_1^2)q_b \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{4.48}$$

The elements of r are known (in terms of f_1 and f_3 through f_6; f_2 does not enter). Notice, however, that f_1 does not depend on t and that the derivatives of f_3 through f_6 with respect to t can be expressed in terms of their derivatives with respect to x; see Eq. (4.33). Thus,

$$\begin{aligned} |M_1|^2 &= \frac{1}{1-k_a^2}\left[\frac{(v_2^2-v_1^2)^2}{v_2^4}f_1^2 - k_a\frac{(v_2^2-v_1^2)(1-v_1^2)}{v_2^3}f_1(f_5-f_6)\right. \\ &\quad \left. + \frac{(1-v_1^2)^2}{4v_1^4}(f_3+f_4)^2 + \frac{(1-v_1^2)(1-v_2^2)}{4v_2^2}(f_5-f_6)^2\right] + f_0^2, \end{aligned} \tag{4.49}$$

$$\begin{aligned} q_b &= -k_a\frac{v_1^2(v_2^2-v_1^2)}{v_2^4}f_1^2 + \frac{v_1^2(1-v_2^2)-(v_2^2-v_1^2)}{2v_2^3}f_1(f_5-f_6) \\ &\quad + k_a\frac{1-v_1^2}{4v_2^2}(f_5-f_6)^2 + \frac{1-v_1^2}{2v_1}\partial_x(f_3-f_4) - \tfrac{1}{2}gk_a(1-v_1^2)\partial_x(f_5+f_6) \\ &\quad + (1-k_a^2)\frac{v_1^2}{v_2^2}f_0f_1 - \frac{k_a v_1^2}{2v_2}f_0(f_5-f_6), \end{aligned} \tag{4.50}$$

$$\begin{aligned} q_c &= k_a\frac{1-v_1^2}{2v_2^2}f_1(f_3+f_4) + \frac{1-v_1^2}{4v_1^2v_2}(f_3+f_4)(f_5-f_6) \\ &\quad - g\frac{1-v_1^2}{2v_1}\partial_x(f_3-f_4) - \tfrac{1}{2}k_a(1-v_1^2)\partial_x(f_5+f_6) - k_af_0. \end{aligned} \tag{4.51}$$

4.3.2 Solution of Equation (4.47)

We apply the transformation F defined in Eq. (4.30) to both sides of Eq. (4.47) and absorb the t-derivative term in the left member, compensating with an x derivative in the right member,

$$(\partial_t + V\partial_x)F(u_2 - r) = -\partial_\tau f - \partial_x VFr. \tag{4.52}$$

Because V is diagonal, Eq. (4.52) decouples into six first-order hyperbolic equations with constant coefficients, which can be integrated along their characteristics. If the solution is to remain bounded, the right member must be such that it does not lead to secular behavior. This condition imposes constraints, which we can find by following the averaging strategy of Section 3.1, Lemma 3.2.

We decompose VFr, separating the terms that are constant along the characteristics from those that are not,

$$VFr = -D_1\partial_x f + D_2 f^2 + w. \tag{4.53}$$

The first two terms are constant along the characteristics; D_1 and D_2 are diagonal matrices with nonnegative entries that are readily found from Eqs. (4.49), (4.50), and (4.51),

$$D_1 = \tfrac{1}{2}g(1-v_1^2)^2 \operatorname{diag}(0,0,1,1,1,1), \tag{4.54}$$
$$D_2 = \frac{3(1-v_1^2)(1-v_2^2)}{8v_2^2} \operatorname{diag}(0,0,0,0,1,1); \tag{4.55}$$

f^2 is the vector whose entries are the squares of the entries of f,

$$f^2 = (f_1^2, f_2^2, f_3^2, f_4^2, f_5^2, f_6^2)^{\mathrm{t}}. \tag{4.56}$$

The remainder w consists exclusively of terms that vary along the characteristics: its first and second components involve at least one of f_3 through f_6, its third component at least one of f_1 and f_4 through f_6, and so on. Thus, Eq. (4.52) becomes

$$(\partial_t + V\partial_x)F(u_2 - r) = -[\partial_\tau f - D_1\partial_x^2 f + D_2\partial_x f^2] + w. \tag{4.57}$$

Application of the averaging operator to each component yields the equation

$$\partial_\tau f - D_1\partial_x^2 f + D_2\partial_x f^2 = 0. \tag{4.58}$$

Thus, a necessary condition for the solution of Eq. (4.52) to remain bounded for long times as $\varepsilon \downarrow 0$ is that the first-order profile functions f_1 through f_6 satisfy a heat equation on the (slow) time scale of τ. The equations for f_1 and f_2 are particularly simple: $\partial_\tau f_1 = 0$, $\partial_\tau f_2 = 0$, so f_1 and f_2 must be constant on the slow time scale, and we have $f_1 \equiv f_1(x)$ and $f_2 \equiv f_2(x)$. The equations for f_3 and f_4 are linear, those for f_5 and f_6 nonlinear with a quadratic nonlinearity.

Remark 4.2 *Equation (4.58) corresponds to Eq. (3.18). The nonzero entries of D_1 are positive, and the equations for f_1 and f_2, which involve the zero entries of D_1, are trivial. This observation validates Hypothesis 4.*

If the condition (4.58) is satisfied, Eq. (4.57) reduces to

$$(\partial_t + V\partial_x)F(u_2 - r) = w, \tag{4.59}$$

from which we obtain the solution u_2 of Eq. (4.47),

$$u_2 = r + F^{-1}(f_2 + (\partial_t + V\partial_x)^{-1}w). \tag{4.60}$$

Here, $(\partial_t + V\partial_x)^{-1}$ denotes the integral along characteristics, and

$$u_2 = \begin{pmatrix} H_{2a} \\ H_{2b} \\ H_{2c} \\ E_{2a} \\ E_{2b} \\ E_{2c} \end{pmatrix}, \quad f_2 = \begin{pmatrix} f_{21} \\ f_{22} \\ f_{23} \\ f_{24} \\ f_{25} \\ f_{26} \end{pmatrix}, \quad \begin{array}{l} f_{21} \equiv f_{21}(x,\tau), \\ f_{22} \equiv f_{22}(x,\tau), \\ f_{23} \equiv f_{23}(x - v_1 t,\tau), \\ f_{24} \equiv f_{24}(x + v_1 t,\tau), \\ f_{25} \equiv f_{25}(x - v_2 t,\tau), \\ f_{26} \equiv f_{26}(x + v_2 t,\tau). \end{array} \tag{4.61}$$

In addition, we have the expression $M_{2a} = \frac{1}{2}(f_{20} - |M_1|^2)$, where $f_{20} \equiv f_{20}(x,\tau)$; see Eq. (4.40). The remaining components of M_2 follow from Eq. (4.14),

$$M_{2b} = \alpha(H_{2b} - q_c), \quad M_{2c} = k_a M_{2a} + \alpha(H_{2c} - k_a H_{2a} + q_b). \tag{4.62}$$

This completes the construction of the asymptotic approximation.

5 Numerical Results

In this section we illustrate the analytical results of the preceding section with the results of some numerical computations. The computations are done in a Cartesian (x, y, z) coordinate system. The (x, y, z) coordinates are obtained from the (a, b, c) coordinates (Eq. (4.18)) by applying the matrix

$$T = \frac{1}{\sin\phi} \begin{pmatrix} 0 & 0 & \sin\phi \\ 1 & 0 & -\cos\phi \\ 0 & 1 & 0 \end{pmatrix}. \tag{5.1}$$

The basic solution is given by

$$M_0 = \begin{pmatrix} \cos\phi \\ \sin\phi \\ 0 \end{pmatrix}, \quad H_0 = \begin{pmatrix} \cos\phi \\ \sin\phi \\ 0 \end{pmatrix}, \quad E_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \tag{5.2}$$

for some $\phi \in (0, \pi)$. At $t = 0$, we perturb this basic solution near the origin. The perturbation is uniform in y and z, sharply peaked near the origin in x,

$$M(x,0) = H(x,0) = E(x,0) = \mathrm{e}^{-20x^2} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}. \tag{5.3}$$

With $k = (1,0,0)^{\mathrm{t}}$, we have $k_a = \cos\phi$, while $k \times M_0$ falls along the z axis. When we apply the transformation T to Eqs. (4.33), (4.34), and (4.35), the asymptotic analysis gives the following expressions for M, H, and E:

$$M \approx \begin{pmatrix} (\cos\phi) f_0 + \frac{v_2^2 - v_1^2}{v_2^2} f_1 - \frac{(1-v_1^2)\cos\phi}{2v_2}(f_5 - f_6) \\ (\sin\phi) f_0 - \frac{(v_2^2-v_1^2)\cos\phi}{v_2^2 \sin\phi} f_1 + \frac{1-v_2^2}{2v_2\sin\phi}(f_5 - f_6) \\ \frac{1-v_1^2}{2v_1^2\sin\phi}(f_3 + f_4) \end{pmatrix}, \tag{5.4}$$

$$H \approx \begin{pmatrix} \frac{v_1^2}{v_2^2} f_1 + \frac{(1-v_1^2)\cos\phi}{2v_2}(f_5 - f_6) \\ \frac{v_2}{2\sin\phi}(f_5 - f_6) \\ \frac{1}{2\sin\phi}(f_3 + f_4) \end{pmatrix}, \quad E \approx \begin{pmatrix} f_2 \\ -\frac{1}{2v_1\sin\phi}(f_3 - f_4) \\ \frac{1}{2\sin\phi}(f_5 + f_6) \end{pmatrix}. \tag{5.5}$$

Here,

$$v_1 = \left(\frac{1}{1+\alpha}\right)^{1/2}, \quad v_2 = \left(\frac{1+\alpha\sin^2\phi}{1+\alpha}\right)^{1/2}. \tag{5.6}$$

The functions f_0, f_1, and f_2 are independent of time; f_3, f_4, f_5, and f_6 represent propagating waves traveling with the velocities v_1, $-v_1$, v_2, and $-v_2$, respectively. Thus, leading-order asymptotics predict that E_x is constant in time; M_z, H_z, and E_y split into waves traveling at the velocities $\pm v_1$; H_y and E_z split into waves traveling at the velocities $\pm v_2$; and M_x, M_y, and H_x combine a standing wave with waves traveling at the velocities $\pm v_2$.

5.1 Numerical Results

All computations reported in this section refer to the case $\varepsilon = 0.01$ and $g = 1$. We use a finite-difference approximation on a uniform mesh on an interval

$-L \leq x \leq L$ with $2N+1$ mesh points. With an implicit treatment of the linear terms and an explicit treatment of the nonlinear terms, the computation requires the factorization of a (sparse) matrix of dimension $9(2N+1)$.

Figure 1: Solution of Eqs. (2.13)–(2.15); $\alpha = 1, \phi = \frac{1}{3}\pi$.

Figure 2: Solution of Eqs. (2.13)–(2.15); $\alpha = 1$, $\phi = \frac{1}{2}\pi$.

Figure 1 shows the x, y, and z components (top to bottom) of M, H, and E (left to right) vs. x (measured along the front) and t (increasing toward the back), for $\alpha = 1$ and $\phi = \frac{1}{3}\pi$. They display the features predicted by the asymptotic theory. We see standing waves and waves traveling with the velocities $v_1 = 0.69$ and $v_2 = 0.91$. The specific wave configuration depends on the initial data. In fact, by changing the initial data for the individual components we can change a positive wave into a negative wave or vice versa.

Figure 3: Variation of the wave speeds with α.

A variation of the angle ϕ, changing the direction of the basic solution (5.2) in the (x, y) plane, does not affect the velocity of the slower waves (v_1); on the other hand, the velocity of the faster waves (v_2) increases with ϕ until it is close to 1 when $\phi = \frac{1}{2}\pi$.

At $\phi = \frac{1}{2}\pi$, some waves disappear, in accordance with the asymptotic theory; see Fig. 2.

As α increases, both v_1 and v_2 decrease; the former approaches 0, the latter $\sin\phi$ as $\alpha \to \infty$. Figure 3 shows v_1 and v_2 as a function of α, the latter for three different values of ϕ. The continuous curves were obtained from the asymptotic expressions, Eq. (4.27), the discrete marks from the numerical solution. The asymptotic expressions appear to slightly overestimate the computed values.

Figure 4: Evolution of H_z on the slow-time scale; $\alpha = 2$, $\phi = \frac{1}{6}\pi$.

Figure 5: Evolution of E_z on the slow-time scale; $\alpha = 2$, $\phi = \frac{1}{6}\pi$.

Both Figs. 1 and 2 show results for times of order $O(1)$. When we integrate over longer time intervals, we begin to see the evolution of the wave profiles on the slow-time scale. To show the different types of evolution, we computed the solution $(M^{(1)},\ H^{(1)},\ E^{(1)})$ from the initial data (5.3) and the solution $(M^{(2)},\ H^{(2)},\ E^{(2)})$ from initial data that had the same shape but twice the magnitude, using the vector $(2,4,6)^{\mathrm{t}}$ instead of $(1,2,3)^{\mathrm{t}}$. Figures 4 and 5 show the quantities

$$\Delta H_z(t) = \frac{H_z^{(1)}(\cdot\,,t) - \frac{1}{2}H_z^{(2)}(\cdot\,,t)}{\max\{H_z^{(1)}(x,t) : x \in [-L,L]\}}, \tag{5.7}$$

$$\Delta E_z(t) = \frac{E_z^{(1)}(\cdot\,,t) - \frac{1}{2}E_z^{(2)}(\cdot\,,t)}{\max\{E_z^{(1)}(x,t) : x \in [-L,L]\}}, \tag{5.8}$$

on the interval $[-L, L]$ with $L = 60$, at $t = 0$, 20, 40, and 60. Note that, at the last time frame ($t = 60$), t is of the order of ε^{-1}. We observe that ΔH_z, which depends only on f_3 and f_4, scales with the initial data; ΔH_z is of the same order as ε ($\Delta H_z \approx 0.01$). On the other hand, E_z depends on f_5 and f_6, which evolve nonlinearly and do not scale with the initial data. The effect of the nonlinearity is evident in the numerical results; ΔE_z is an order of magnitude larger than ΔH_z ($\Delta E_z \approx 0.1$).

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