Introduction to Homological Algebra

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CHAPTER 1

Categories

1. Categories

1.1. Basic ideas:

- Study classes of objects (as sets, modules, topological spaces,...);
- Avoid the set-theoretic language (do not suppose that our objects are sets).

Definition. *A category* $\mathcal{A}$ *consists of*

– *a class* $\mathbf{Obj}(\mathcal{A})$ *of objects;*
– *for all* $X, Y \in \mathbf{Obj}(\mathcal{A})$*, a set* $\mathrm{Mor}_{\mathcal{A}}(X, Y)$ *called the set of morphisms from* X *to* Y*. We write* $f : X \to Y$ *to say that* $f \in \mathrm{Mor}_{\mathcal{A}}(X, Y)$*.*
– *For each* $X \in \mathbf{Obj}(\mathcal{A})$*, an identity morphism* $\mathrm{id}_X \in \mathrm{Mor}_{\mathcal{A}}(X, X)$:
– *For every ordered triple of objects* $X, Y, Z \in \mathbf{Obj}(\mathcal{A})$*, a map of sets*

$$\mathrm{Mor}_{\mathcal{A}}(X, Y) \times \mathrm{Mor}_{\mathcal{A}}(Y, Z) \to \mathrm{Mor}_{\mathcal{A}}(X, Z),$$

called a composition function. This map associates to each $f \in \mathrm{Mor}_{\mathcal{A}}(X, Y)$ *and* $g \in \mathrm{Mor}_{\mathcal{A}}(X, Y)$ *a morphism* $g \circ f : X \to Z$ *(or just simply* gf*) called the composition of* f *and* g*.*

These data should satisfy the following axioms:

Cat1) *(Associativity axiom) For all* $f \in \mathrm{Mor}_{\mathcal{A}}(X, Y)$*,* $g \in \mathrm{Mor}_{\mathcal{A}}(Y, Z)$ *and* $h \in \mathrm{Mor}_{\mathcal{A}}(Z, U)$*,*

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

Cat2) *(Unit axiom) For all* $X, Y \in \mathbf{Obj}(\mathcal{A})$ *and* $f \in \mathrm{Mor}_{\mathcal{A}}(X, Y)$*,*

$$\mathrm{id}_Y \circ f = f = f \circ \mathrm{id}_X.$$

Examples. 1) The category **Sets** of sets. The objects are sets and the morphisms are set functions (maps):

$$\mathbf{Obj}(\mathbf{Sets}) = \{\text{sets}\}, \qquad \mathrm{Mor}_{\mathbf{Sets}}(X, Y) := \{\text{maps } f : X \to Y\}.$$

2) The category $A-\mathbf{Mod}$ of left modules over a fixed ring A. The morphisms are homomorphisms of modules:

$$\mathrm{Mor}_{A-\mathbf{Mod}}(X, Y) := \mathrm{Hom}_A(X, Y).$$

3) The category **Groups** of groups. The morphisms are morphisms of groups.
4) The category **Rings** of rings. The morphisms are morphisms of rings.
5) The category **TSpaces**. The objects are topological spaces and

$$\mathrm{Mor}_{\mathbf{TSpaces}}(X, Y) := \{\text{continuous } f : X \to Y\}.$$

Definition. *Let $\mathcal{A}$ be a category. We define the dual (or opposite) category $\mathcal{A}^\circ$ of $\mathcal{A}$ setting:*

a) $\mathbf{Obj}(\mathcal{A}^\circ) := \mathbf{Obj}(\mathcal{A})$. *For each $X \in \mathbf{Obj}(\mathcal{A})$, we write X° for X viewed as an object of $\mathcal{A}^\circ$.*

b) $\mathrm{Mor}_{\mathcal{A}^\circ}(X^\circ, Y^\circ) := \mathrm{Mor}_{\mathcal{A}}(Y, X)$.

In general, $\mathbf{Obj}(\mathcal{A})$ is not a set.

Definition. *A category $\mathcal{A}$ is small if $\mathbf{Obj}(\mathcal{A})$ is a set.*

1.2. We want to define the notions of isomorphism, monomorphism and epimorphism in a completely general setting.

Definition. *Let $f : X \to Y$.*

i) *f is an isomorphism if there exists $g : Y \to X$ such that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$.*

ii) *f is monic (or a monomorphism) if for all $Z \in \mathbf{Obj}(\mathcal{A})$, the map*

$$\mathrm{Mor}_{\mathcal{A}}(Z, X) \to \mathrm{Mor}_{\mathcal{A}}(Z, Y),$$
$$g \mapsto f \circ g$$

is injective.

iii) *f is epi (or an epimorphism) if for all $Z \in \mathbf{Obj}(\mathcal{A})$, the map*

$$\mathrm{Mor}_{\mathcal{A}}(Y, Z) \to \mathrm{Mor}_{\mathcal{A}}(X, Z),$$
$$g \mapsto g \circ f$$

is injective.

Exercise 1. *1) f is an isomorphism $\Rightarrow$ f is monic and epi.*

2) Show that in the category **Rings**, *the natural inclusion $f : \mathbf{Z} \to \mathbf{Q}$ is monic, epi, but not an isomorphism.*

2. Functors

2.1. Let $\mathcal{A}$ and $\mathcal{B}$ be two categories.

Definition. *A covariant functor from $\mathcal{A}$ to $\mathcal{B}$ is a rule $\mathscr{F} : \mathcal{A} \to \mathcal{B}$ that associates to each $X \in \mathbf{Obj}(\mathcal{A})$ an object $\mathscr{F}(X) \in \mathbf{Obj}(\mathcal{B})$ and to each morphism $f : X \to Y$ in $\mathcal{A}$ a morphism $\mathscr{F}(f) : \mathscr{F}(X) \to \mathscr{F}(Y)$ in $\mathcal{B}$ and such that:*

Fun1) *For all $f \in \mathrm{Mor}_{\mathcal{A}}(X, Y)$ and $g \in \mathrm{Mor}_{\mathcal{A}}(Y, Z)$,*

$$\mathscr{F}(g \circ f) = \mathscr{F}(g) \circ \mathscr{F}(f).$$

Fun2) $\mathscr{F}(\mathrm{id}_X) = \mathrm{id}_{\mathscr{F}(X)}$ *for all $X \in \mathbf{Obj}(\mathcal{A})$.*

Therefore, we have a map

$$\mathrm{Mor}_{\mathcal{A}}(X, Y) \to \mathrm{Mor}_{\mathcal{B}}(\mathscr{F}(X), \mathscr{F}(Y)), \qquad f \mapsto \mathscr{F}(f).$$

Definition. *A contravariant functor from $\mathcal{A}$ to $\mathcal{B}$ is a rule $\mathscr{F} : \mathcal{A} \to \mathcal{B}$ that associates to each $X \in \mathbf{Obj}(\mathcal{A})$ an object $\mathscr{F}(X) \in \mathbf{Obj}(\mathcal{B})$ and to each morphism $f : X \to Y$ in $\mathcal{A}$ a morphism $\mathscr{F}(f) : \mathscr{F}(Y) \to \mathscr{F}(X)$ in $\mathcal{B}$ and such that:*

Fun°1) *For all* $f \in \mathrm{Mor}_{\mathcal{A}}(X, Y)$ *and* $g \in \mathrm{Mor}_{\mathcal{A}}(Y, Z)$,

$$\mathscr{F}(g \circ f) = \mathscr{F}(f) \circ \mathscr{F}(g) \qquad (sic!).$$

Fun°2) $\mathscr{F}(\mathrm{id}_X) = \mathrm{id}_{\mathscr{F}(X)}$ *for all* $X \in \mathbf{Obj}(\mathcal{A})$.

Therefore, we have a map

$$\mathrm{Mor}_{\mathcal{A}}(X, Y) \to \mathrm{Mor}_{\mathcal{B}}(\mathscr{F}(Y), \mathscr{F}(X)), \qquad f \mapsto \mathscr{F}(f).$$

A contravariant functor $\mathscr{F}$ defines a covariant functor on the dual category:

$$\mathscr{F}^\circ : \mathcal{A}^\circ \to \mathcal{B}, \qquad \mathscr{F}^\circ(X^\circ) := \mathscr{F}(X).$$

If $\mathscr{F} : \mathcal{A} \to \mathcal{B}$ and $\mathscr{G} : \mathcal{B} \to \mathcal{C}$ are two functors, then

$$\mathscr{G} \circ \mathscr{F} : \mathcal{A} \to \mathcal{C}, \qquad \mathscr{G} \circ \mathscr{F}(X) = \mathscr{G}(\mathscr{F}(X))$$

is a functor.

Exercise 2. If $\mathscr{F}$ and $\mathscr{G}$ are both covariant or contravariant, then $\mathscr{G} \circ \mathscr{F}$ is covariant. If one of functors is covariant and the other is contravariant, then $\mathscr{G} \circ \mathscr{F}$ is contravariant.

2.2. We define functors in several variables. If $\mathcal{A}_1$ and $\mathcal{A}_2$ are two categories, we define the product category $\mathcal{A}_1 \times \mathcal{A}_2$ by:

$\mathbf{Obj}(\mathcal{A}_1 \times \mathcal{A}_2) = \{$ordered pairs (X_1, X_2), where $X_1 \in \mathcal{A}_1$, and $X_2 \in \mathcal{A}_2\}$,

$\mathrm{Mor}_{\mathcal{A}_1 \times \mathcal{A}_2}((X_1, X_2), (Y_1, Y_2)) = \mathrm{Mor}_{\mathcal{A}_1}(X_1, Y_1) \times \mathrm{Mor}_{\mathcal{A}_2}(X_2, Y_2)$.

Let

$$\mathscr{F} : \mathcal{A}_1 \times \mathcal{A}_2 \to \mathcal{B}$$

be a rule which assignes to each $(X_1, X_2) \in \mathcal{A}_1 \times \mathcal{A}_2$ an object $\mathscr{F}(X_1, X_2) \in \mathcal{B}$. Fixing $X_2 \in \mathcal{A}_2$, we can consider the assignement

$$\mathscr{F}(-, X_2) : \mathcal{A}_1 \to \mathcal{B}, \qquad Z \mapsto \mathscr{F}(Z, X_2).$$

Analogously, fixing $X_1 \in \mathcal{A}_1$, we can consider the assignment

$$\mathscr{F}(X_1, -) : \mathcal{A}_2 \to \mathcal{B}, \qquad Z \mapsto \mathscr{F}(X_2, Z).$$

Definition. $\mathscr{F} : \mathcal{A}_1 \times \mathcal{A}_2 \to \mathcal{B}$ *is a functor in two variables if the rules* $\mathscr{F}(X_1, -)$ *and* $\mathscr{F}(-, X_2)$ *are functors for all* $X_1 \in \mathcal{A}_1$ *and* $X_2 \in \mathcal{A}_2$. *The functor* $\mathscr{F}$ *is covariant (resp. contravariant) in the first variable if* $\mathscr{F}(-, X_2)$ *is covariant (resp. contravariant) for all* X_2. *The functor* $\mathscr{F}$ *is covariant (resp. contravariant) in the second variable if* $\mathscr{F}(X_1, -)$ *is covariant (resp. contravariant) for all* X_1.

Examples. 1) Let $\mathcal{A}$ be an arbitrary category. Fix $A \in \mathcal{A}$. Then

a) $h_A : \mathcal{A} \to \mathbf{Sets}$ defined by $h_A(X) = \mathrm{Mor}_{\mathcal{A}}(A, X)$ is a covariant functor.

b) a) $h^A : \mathcal{A} \to \mathbf{Sets}$ defined by $h^A(X) = \mathrm{Mor}_{\mathcal{A}}(X, A)$ is a contravariant functor.

c) $\mathrm{Mor}_{\mathcal{A}}(-, -) : \mathcal{A} \times \mathcal{A} \to \mathbf{Sets}$ given by

$$(X, Y) \mapsto \mathrm{Mor}_{\mathcal{A}}(X, Y)$$

is a functor which is contravariant in the first variable and covariant in the second variable.

2) In particular, if $\mathcal{A} = A - \mathbf{Mod}$ is the category of modules over A, when $\mathrm{Mor}_{\mathcal{A}}(X, Y) = \mathrm{Hom}_A(X, Y)$ is an abelian group. Therefore

$$\mathrm{Hom}_A(-, -) \,:\, \mathcal{A} \times \mathcal{A} \to \mathbf{Ab},$$

where **Ab** denotes the category of abelian groups.

3) Let A be a ring. We already introduced the category $A - \mathbf{Mod}$ of left A-modules. We can also consider the category $\mathbf{Mod} - A$ of right A-modules. For any $X \in \mathbf{Mod} - A$ and $Y \in A - \mathbf{Mod}$ the tensor product $X \otimes_A Y$ is well defined. This gives us a two-variable functor $(X, Y) \mapsto X \otimes_A Y$. It is covariant in the both variables.

2.3. We define the notion of natural transformation of functors.

Definition. *Let $\mathscr{F} : \mathcal{A} \to \mathcal{B}$ and $\mathscr{F} : \mathscr{G} \to \mathcal{B}$ be two covariant functors. A natural transformation $\alpha : \mathscr{F} \to \mathscr{G}$ is a rule that to each $X \in$ **Obj**$(\mathcal{A})$ associates a morphism in $\mathcal{B}$*

$$\alpha_X \,:\, \mathscr{F}(X) \to \mathscr{G}(X),$$

such that for any $f : X \to Y$ the following diagram is commutative:

$$\begin{array}{ccc} \mathscr{F}(X) & \xrightarrow{\alpha_X} & \mathscr{G}(X) \\ \downarrow{\scriptstyle \mathscr{F}(f)} & & \downarrow{\scriptstyle \mathscr{G}(f)} \\ \mathscr{F}(Y) & \xrightarrow{\alpha_Y} & \mathscr{G}(Y). \end{array}$$

A natural transformation of contrvariant functors can be defined similarly.

Examples. 1) For each category $\mathcal{A}$, the identity functor is the functor $\mathrm{id}_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}$ defined by $\mathrm{id}_{\mathcal{A}}(X) = X$ and $\mathrm{id}_{\mathcal{A}}(f) = f$.

2) Let $A, B \in \mathcal{A}$ and $g : A \to B$. For each $X \in \mathcal{A}$ we have the map

$$\alpha_X \,:\, h_B(X) := \mathrm{Mor}_{\mathcal{A}}(B, X) \to h_A(X) := \mathrm{Mor}_{\mathcal{A}}(A, X),$$
$$\alpha_X(f) := f \circ g.$$

Then α is a natural transformation $\alpha : h_B \to h_A$ of covariant functors (exercise). Similarly, the maps

$$\beta_X \,:\, h^A(X) := \mathrm{Mor}_{\mathcal{A}}(X, A) \to h^B(X) := \mathrm{Mor}_{\mathcal{A}}(X, B),$$
$$\beta_X(f) := g \circ f$$

define a natural transformation $\beta : h^A \to h^B$ of contravariant functors.

Definition. *A natural transformation $\alpha : \mathscr{F} \to \mathscr{G}$ is a natural isomorphism of functors if α_X is an isomorphism in $\mathcal{B}$ for all $X \in \mathcal{A}$. Equivalently, α is a natural isomorphism if there exists a natural transformation $\beta : \mathscr{G} \to \mathscr{F}$ such that $\beta \circ \alpha = \mathrm{id}_{\mathcal{A}}$ and $\alpha \circ \beta = \mathrm{id}_{\mathcal{B}}$.*

The usual notation for a naturally isomorphic functors is $\mathscr{F} \simeq \mathscr{G}$.

2.4. We can define the notion of isomorphism for categories:

Definition. *An isomorphism of categories is a functor $\mathscr{F} : \mathcal{A} \to \mathcal{B}$ which is bijection both on objects and morphisms. Equivalently, $\mathscr{F} : \mathcal{A} \to \mathcal{B}$ is an isomorphism if there exists a functor $\mathscr{G} : \mathcal{B} \to \mathcal{A}$ such that*

$$\mathscr{G} \circ \mathscr{F} = \mathrm{id}_{\mathcal{A}}, \qquad \mathscr{F} \circ \mathscr{G} = \mathrm{id}_{\mathcal{B}}.$$

This notion is not very useful (too restrictive!). In particular, if $\mathcal{A}$ and $\mathcal{B}$ are isomorphic, there exists a one-to-one correspondence between $\mathbf{Obj}(\mathcal{A})$ and $\mathbf{Obj}(\mathcal{B})$. The following notion is more natural:

Definition. *An equivalence between two categories $\mathcal{A}$ and $\mathcal{B}$ is a pair of functors $\mathscr{F} : \mathcal{A} \to \mathcal{B}$ and $\mathscr{G} : \mathcal{B} \to \mathcal{A}$ such that*

$$\mathscr{G} \circ \mathscr{F} \simeq \mathrm{id}_{\mathcal{A}}, \qquad \mathscr{F} \circ \mathscr{G} \simeq \mathrm{id}_{\mathcal{B}}.$$

2.5. We define some important classes of functors:

Definition. *A functor $\mathscr{F} : \mathcal{A} \to \mathcal{B}$ is*

i) *faithful, if for all $X, Y \in \mathcal{A}$ the map*

$$\mathrm{Mor}_{\mathcal{A}}(X, Y) \to \mathrm{Mor}_{\mathcal{B}}(\mathscr{F}(X), \mathscr{F}(Y))$$

is injective;

ii) *full, if for all $X, Y \in \mathcal{A}$ the map*

$$\mathrm{Mor}_{\mathcal{A}}(X, Y) \to \mathrm{Mor}_{\mathcal{B}}(\mathscr{F}(X), \mathscr{F}(Y))$$

is surjective.

iii) *fully faithful, if it is full and faithful.*

Example. A forgetful functor is a functor that forgets some structures. For example, the functor $\mathscr{F} : A - \mathbf{Mod} \to \mathbf{Ab}$ which associates to each A-module X the same set X equipped only with its abelian group structure, is a forgetful functor. It is fully faithful. In general, it is not full because for a general ring $\mathrm{Hom}_A(X, Y)$ is smaller that $\mathrm{Hom}_{\mathbf{Ab}}(X, Y)$.

Theorem 2.6. *A functor $\mathscr{F} : \mathcal{A} \to \mathcal{B}$ is an equivalence of categories if and only if it is fully faithful and for any $Y \in \mathbf{Obj}(\mathcal{B})$, there exists $X \in \mathbf{Obj}(\mathcal{A})$ such that $Y \simeq \mathscr{F}(X)$ (Y is isomorphic to $\mathscr{F}(X)$).*

3. Products

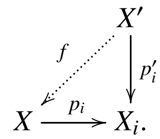
In ths section, we define the notions of direct product and direct sum (or direct coproduct) in general categories.

Let I be a set and let $(X_i)_{i \in I}$ a family of objects $X_i \in \mathbf{Obj}(\mathcal{A})$ indexed by I.

Definition. *i) An object $X \in \mathbf{Obj}(\mathcal{A})$ is a product of $(X_i)_{i \in I}$ if it is equipped with morphisms $p_i : X \to X_i$ ($i \in I$) such that the following universal property holds:*

For any $X' \in \mathbf{Obj}(\mathcal{A})$ equipped with morphisms $p'_i : X' \to X_i$ ($i \in I$) there exists a unique morphism $f : X' \to X$ such that

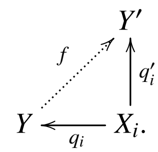
$$p'_i = p_i \circ f, \qquad \forall i \in I :$$



*ii) An object $Y \in$ **Obj**$(\mathcal{A})$ is a coproduct of $(X_i)_{i\in I}$ if it is equipped with morphisms $q_i : X_i \to Y$ $(i \in I)$ such that the following universal property holds:*

*For any $Y' \in$ **Obj**$(\mathcal{A})$ equipped with morphisms $q_i' : X_i \to Y$ $(i \in I)$ there exists a unique morphism $f : Y \to Y'$ such that*

$$q_i' = f \circ q_i, \qquad \forall i \in I:$$



From the universal property it follows that the direct product and the direct coproduct (if exist!) are unique up to an isomorphism. The usual notations for the product and coproduct are

$$\prod_{i\in I} X_i, \qquad \coprod_{i\in I} X_i$$

Examples. 1) In the category $A-$**Mod**, the products and coproducts exist. The product of modules X_i is the usual cartesian (direct) product

$$\prod_{i\in I} X_i = \{(x_i)_{i\in I} \mid x_i \in X_i\}$$

equipped with the componentwise addition and scalar multiplication:

$$(x_i)_{i\in I} + (y_i)_{i\in I} = (x_i + y_i)_{i\in I}, \qquad a \cdot (x_i)_{i\in I} = (ax_i)_{i\in I}.$$

The coproduct of X_i can be constructed as follows:

$$\coprod_{i\in I} X_i = \{(x_i)_{i\in I} \mid x_i \in X_i \text{ and } x_i = 0 \text{ for almost all } i\}.$$

2) In **Sets**, the product is the cartesian product of sets. The coproduct $\coprod_{i\in I} X_i$ is the disjoint union of X_i.

3) In **Groups**, the product is the cartesian (direct) product of groups with the componentwise multiplication. One can show that coproducts exist (free product of groups).

Exercise 3. X is a direct product of $(X_i)_{i\in I}$ in $\mathcal{A}$ if and only if X° is a direct sum of $(X_i^\circ)_{i\in I}$ in $\mathcal{A}^\circ$.

4. Additive categories

4.1. Initial, final and zero objects. Let $\mathcal{A}$ be a category.

Definition. *An object $X \in$ **Obj**$(\mathcal{A})$ is*

– *initial, if for any $Y \in \mathbf{Obj}(\mathcal{A})$ there exists exactly one morphism $f : X \to Y$.*
– *final (or terminal) , if for any $Y \in \mathbf{Obj}(\mathcal{A})$ there exists exactly one morphism $f : Y \to X$.*
– *zero if it is initial and final.*

Properties 4.2. *1) Initial, terminal and zero objects (if exist) are unique up to isomorphism.*

Proof. We will prove only the uniqueness of the initial object. Assume that X_1 and X_2 be two initial objects. Then we have unique morphisms $f_1 : X_1 \to X_2$ and $f_2 : X_2 \to X_1$. The composition $f_2 \circ f_1 : X_1 \to X_1$ coincides with the unique morphism $X_1 \to X_1$. Therefore $f_2 \circ f_1 = \mathrm{id}_{X_1}$. The same argument shows that $f_1 \circ f_2 = \mathrm{id}_{X_2}$. Therefore $X_1 \simeq X_2$. □

2) X is initial (resp. final) in $\mathcal{A} \Leftrightarrow X$ is final (resp. initial) in $\mathcal{A}^\circ$.
*3) In $A -$ **Mod**, the module $\{0\}$ is a zero object.*
4) In **Sets**, *the $\emptyset$ is an initial object. Any one-point set is a final object.*

Proposition 4.3. *Assume that $\mathcal{A}$ has a zero object $0_{\mathcal{A}}$. Then:*

i) *For each $X \in \mathbf{Obj}(\mathcal{A})$, the sets $\mathrm{Mor}_{\mathcal{A}}(X, O_{\mathcal{A}})$ and $\mathrm{Mor}_{\mathcal{A}}(0_{\mathcal{A}}, X)$ consist of one element, which we denote by 0.*

ii) *For all $X, Y \in \mathbf{Obj}(\mathcal{A})$, let O_{XY} denote the composition*

$$X \to 0_{\mathcal{A}} \to Y.$$

Then the morphism $0_{X,Y}$ does not depend on the choice of $0_{\mathcal{A}}$. For any morphism $f : Y \to Z$ one has $f \circ 0_{X,Y} = 0_{X,Z}$:

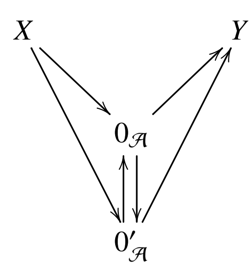
$$X \xrightarrow{0_{X,Y}} Y \xrightarrow{f} Z, \qquad X \xrightarrow{0_{X,Z}} Z.$$

For any morphism $f : Z \to X$ one has $0_{X,Y} \circ f = 0_{Z,X}$:

$$Z \xrightarrow{f} X \xrightarrow{0_{X,Y}} Y, \qquad Z \xrightarrow{0_{Z,Y}} Y.$$

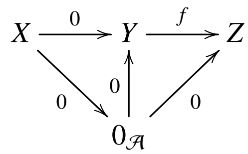
Proof. i) is clear.

iia) Assume that $0'_{\mathcal{A}}$ is another zero element. Then there exist unique morphisms $0_{\mathcal{A}} \to 0'_{\mathcal{A}}$ and $0'_{\mathcal{A}} \to 0_{\mathcal{A}}$. We have a commutative diagram



which shows that the compositions $X \to 0_{\mathcal{A}} \to Y$ and $X \to 0'_{\mathcal{A}} \to Y$ coincide.

iib) We have a commutative diagram



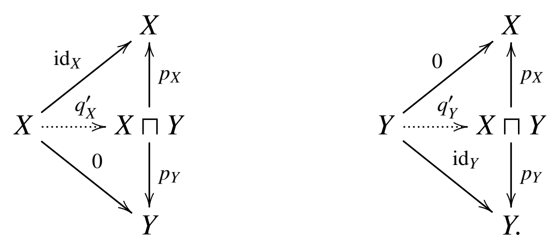
which shows that $f \circ 0_{X,Y} = 0_{X,Z}$. The proof of the second formula is analogous. □

4.4. Assume that $\mathcal{A}$ satisfies the following axioms:

Ad1) $\mathcal{A}$ has a zero object.
Ad2) $\mathcal{A}$ has finite products and coproducts:

$$\text{for all } X, Y \in \mathbf{Obj}(\mathcal{A}), \text{ there exist } X \sqcap Y \text{ and } X \sqcup Y \text{ in } \mathcal{A}.$$

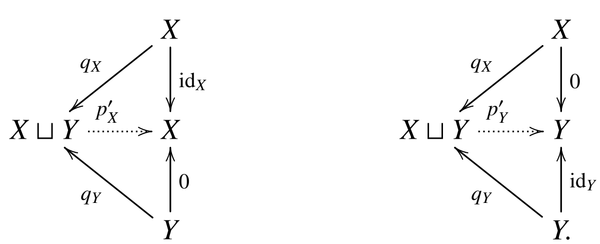
Consider the diagrams



By the universal property of products, there exist unique morphisms $q'_X : X \to X \sqcap Y$ and $q'_Y : Y \to X \sqcap Y$ such that

$$p_X \circ q'_X = \mathrm{id}_X, \quad p_Y \circ q'_Y = \mathrm{id}_Y$$
$$p_Y \circ q'_X = 0_{X,Y}, \quad p_X \circ q'_Y = 0_{Y,X}.$$

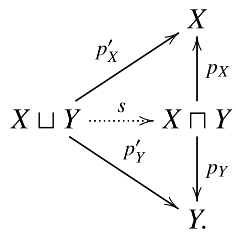
Dually, consider the diagrams



By the universal property of coproducts, there exist unique morphisms $p'_X : X \sqcup Y \to X$ and $p'_Y : X \sqcup Y \to Y$ such that

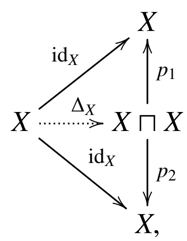
$$p'_X \circ q_X = \mathrm{id}_X, \quad p'_Y \circ q_Y = \mathrm{id}_Y$$
$$p'_X \circ q_Y = 0_{Y,X}, \quad p'_Y \circ q_X = 0_{X,Y}.$$

Therefore we can consider the diagram

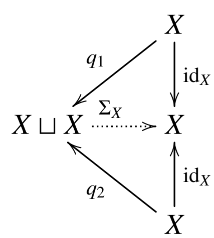


By the universal property of direct products, there exists a unique morphism $s : X \sqcup Y \to X \sqcap Y$ which makes this diagram commute.

4.5. Diagonal and codiagonal. Consider the diagram



where p_1 (resp. p_2) denotes the projection on the first (resp. second) copy of X. By the universal property of direct products, there exists a unique morphism $\Delta_X : X \to X \sqcap X$ which makes this diagram commute. Dually the diagram



defines a morphism $\sigma_X : X \sqcup X \to X$.

Definition. *The morphism Δ_X is called the diagonal morphism. The morphism Σ_X is called the codiagonal morphism or the sum.*

Example. *In A – **Mod**, we have $X \sqcup Y \simeq X \sqcap Y \simeq X \times Y$. It is easy to see that*

$$\Delta_X(x) = (x, x),$$
$$\Sigma_X(x_1, x_2) = x_1 + x_2.$$

Remark 4.6. *The concepts of the diagonal and codiagonal morphisms are dual to each other. Namely in the dual category, $\Delta_X^\circ = \Sigma_{X^\circ}$ and $\Sigma_X^\circ = \Delta_{X^\circ}$.*

4.7. Additive categories. Assume that, in addition, $\mathcal{A}$ satisfies the following axiom:

Ad3) For all objects $X, Y \in \mathbf{Obj}(\mathcal{A})$, the morphism s is an isomorphism:

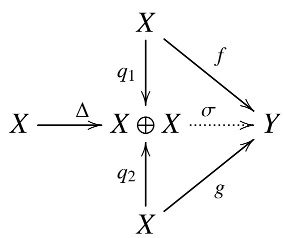
$$s : X \sqcup Y \xrightarrow{\sim} X \sqcap Y.$$

We set

$$X \oplus Y := X \sqcup Y \simeq X \sqcap Y$$

and call it the direct sum of X and Y.

We define a law of composition on $\mathrm{Mor}_{\mathcal{A}}(X, Y)$. For any $f, \gamma \in \mathrm{Mor}_{\mathcal{A}}(X, Y)$, consider the commutative diagram



Here the map σ exists by the universal property of coproducts. Set:

$$f + g := \Delta \circ \sigma.$$

Proposition 4.8. i) *For all* $X, Y \in \mathbf{Obj}(\mathcal{A})$, $(\mathrm{Mor}_{\mathcal{A}}(X, Y), +)$ *is an abelian semigroup. Namely:*

a) *The law of composition + is associative;*

b) $0_{X,Y} + f = f + 0_{X,Y} = f$ *for all* $f \in \mathrm{Mor}_{\mathcal{A}}(X, Y)$;

c) *The law of composition + is commutative;*

ii) *The composition of morphisms is bilinear with respect to +. Namely,*

$$(f + g) \circ h, \quad \forall f, g \in \mathrm{Mor}_{\mathcal{A}}(X, Y) \text{ and } h \in \mathrm{Mor}_{\mathcal{A}}(Z, X),$$
$$h \circ (f + g), \quad \forall f, g \in \mathrm{Mor}_{\mathcal{A}}(X, Y) \text{ and } h \in \mathrm{Mor}_{\mathcal{A}}(Y, Z).$$

Proof. Admitted. □

Definition. *A category* $\mathcal{A}$ *is additive if, in addition to axioms* **Ad1-3)***, it satisfies the following axiom:*

Ad4) *For all* $X, Y \in \mathrm{Mor}_{\mathcal{A}}(X, Y)$, *the semigroup* $(\mathrm{Mor}_{\mathcal{A}}(X, Y), +)$ *is an abelian group, i.e. each element has an inverse:*

$$\forall f \in \mathrm{Mor}_{\mathcal{A}}(X, Y),\ \exists -f \in \mathrm{Mor}_{\mathcal{A}}(X, Y) \text{ such that } f + (-f) = 0_{X,Y}.$$

Example. *The categories* $A-$**Mod** *and* **Mod** $-A$ *are additive.*

Definition. *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be two additive categories. A covariant functor* $\mathscr{F} : \mathcal{A} \to \mathcal{B}$ *is additive if for all* $X, Y \in \mathbf{Obj}(\mathcal{A})$, *the map*

$$\mathrm{Mor}_{\mathcal{A}}(X, Y) \to \mathrm{Mor}_{\mathcal{B}}(\mathscr{F}(X), \mathscr{F}(Y))$$

is a morphism of groups.

A contravariant functor $\mathscr{F} : \mathcal{A} \to \mathcal{B}$ *is additive if for all* $X, Y \in \mathbf{Obj}(\mathcal{A})$, *the map*

$$\mathrm{Mor}_{\mathcal{A}}(X, Y) \to \mathrm{Mor}_{\mathcal{B}}(\mathscr{F}(Y), \mathscr{F}(X))$$

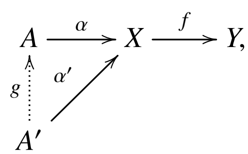
is a morphism of groups.

5. Abelian categories

5.1. Kernels and cokernels. In this section, we assume that $\mathcal{A}$ is a category which has a zero object. To simplify notation, we will often write 0 instead $0_{X,Y}$.

Definition. *Let $f : X \to Y$. A morphism $\alpha : A \to X$ represents the kernel of f if*

i) $f \circ \alpha = 0$;
ii) *The following universal property holds: for any $\alpha' : A' \to X$ such that $f \circ \alpha' = 0$, there exists a unique $g : A' \to A$ such that $\alpha' = \alpha \circ g$:*

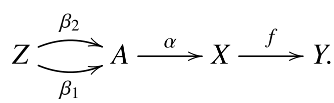


Properties 5.2. *1) If it exists, the kernel (A, α) is unique up to isomorphism.*

Proof. This follows immediately from the universal property. □

2) If (A, α) represents the kernel of a morphism $f : X \to Y$, then α is monic.

Proof. Consider the diagram



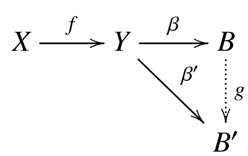
Assume that $\alpha \circ \beta_1 = \alpha \circ \beta_2$. Then

$$f \circ (\alpha \circ \beta_1) = f \circ (\alpha \circ \beta_2).$$

By the universal property of (A, α), for any $\gamma : Z \to X$ such that $f \circ \gamma = 0$, there exists a unique $\beta : Z \to A$ such that $\gamma = \alpha \circ \beta$. Take $\gamma = \alpha \circ \beta_1 = \alpha \circ \beta_2$. Then the above property implies that $\beta_1 = \beta_2$. □

Definition. *Let $f : X \to Y$. A morphism $\beta : Y \to B$ represents the cokernel of f if*

i) $\beta \circ f = 0$;
ii) *The following universal property holds: for any $\beta' : Y \to B'$ such that $\beta' \circ f = 0$, there exists a unique $g : B \to B'$ such that $\beta' = g \circ \beta$:*



The universal property shows that if it exists, the cokernel is unique up to isomorphism. We adopt the following notation:

$$(\ker(f) \xrightarrow{\alpha} X) := \text{kernel of } f,$$

$$(Y \xrightarrow{\beta} \operatorname{coker}(f)) := \text{cokernel of } f.$$

We will often write $\ker(f)$ and $\operatorname{coker}(f)$ instead $\ker(f) \xrightarrow{\alpha} X$ and $Y \xrightarrow{\beta} \operatorname{coker}(f)$.

It is easy to see that the notions of kerner and cokernel are dual to each other. If $f \in \operatorname{Mor}_{\mathcal{A}}(X, Y)$ and $f^\circ \in \operatorname{Mor}_{\mathcal{A}}(X^\circ, Y^\circ)$ is the corresponding morphism in the dual category, then $\ker(f)^\circ \simeq \operatorname{coker}(f^\circ)$ and $\operatorname{coker}(f)^\circ \simeq \ker(f^\circ)$. Dualizing property 5.2, 2) above, we obtain that the morphism $(Y \xrightarrow{\beta} \operatorname{coker}(f)$ is epi.

Definition. *Let $f : X \to Y$. We define the image* $\operatorname{Im}(f)$ *and the coimage* $\operatorname{Coim}(f)$ *of f as:*

$$\operatorname{Im}(f) := \ker(Y \xrightarrow{\beta} \operatorname{coker}(f)),$$

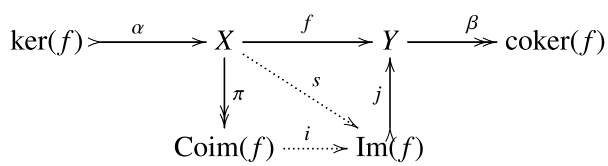
$$\operatorname{Coim}(f) := \operatorname{coker}(\ker(f) \xrightarrow{\alpha} X).$$

Remark 5.3. *The notions of image and coimage are dual to each other:*

$$\operatorname{Coim}(f) \simeq \operatorname{Im}(f^\circ).$$

5.4. Definition of abelian categories. To say that $f : X \to Y$ is a monic (resp. epi) we will often write $f : X \rightarrowtail Y$ (resp. $f : X \twoheadrightarrow Y$).

Let $f : X \to Y$ be an arbitrary morphism. Assume that f has kernel, cokernel, image and coimage. These data can be represented by the diagram



where α, j are monic and β, π are epi.

We analyze this diagram. Since $\beta \circ f = 0$, by the definition of the kernel, there exists a unique map $s : X \to \operatorname{Im}(f)$ such that $f = j \circ s$. We remark that $j \circ s \circ \alpha = f \circ \alpha = 0$. Since j is monic, this implies that $s \circ \alpha = 0$. By the universal property of the cokernel, we deduce that there exists a unique morphism

$$i : \operatorname{Coim}(f) \to \operatorname{Im}(f)$$

such that $s = i \circ \pi$.

Definition. *A category $\mathcal{A}$ is abelian if it is additive and, in addition, satisfies the following axioms:*

Ab1) *Each morphism has a kernel and a cokernel.*

Ab2) *For any morphism f, the morphism $i : \operatorname{Coim}(f) \to \operatorname{Im}(f)$ is an isomorphism.*

Example. The categories $A -$ **Mod** and **Mod** $- A$ are abelian. The following exercise gives an example of an additive category which satisfies **Ab1**), but is not abelian:

Exercise 4. Let K be a field. A filtered finite-dimensional vector space $X = (V, (V_i)_{i \in \mathbf{Z}})$ over K is a finite dimensional K-vector space K equipped with an increasing filtration by K-subspaces:

$$\ldots \subseteq V_{i-1} \subseteq V_i \subseteq V_{i+1} \subseteq \ldots$$

Let $Y = (W, (W_i)_{i\in\mathbf{Z}})$. A morphism $f : X \to Y$ is a linear map $f : V \to W$ such that $f(V_i) \subseteq W_i$ for all $i \in \mathbf{Z}$. Let $\mathbf{FVect}_K$ denote the category of filtered finite-dimensional vector spaces over K.

1) Show that $\mathbf{FVect}_K$ is additive.
2) Show that each morphism in $\mathbf{FVect}_K$ has a kernel and a cokernel.
3) Let V be a nonzero vector space and let $X = (V, (V_i)_{i\in\mathbf{Z}})$ and $Y = (V, (V'_i)_{i\in\mathbf{Z}})$ be the objects defined as:

$$V_i = \begin{cases} 0, & \text{if } i \leqslant 0, \\ V, & \text{if } i \geqslant 1, \end{cases} \qquad V'_i = \begin{cases} 0, & \text{if } i \leqslant -1, \\ V, & \text{if } i \geqslant 0, \end{cases}$$

Show that the identity map on V induces a morphism $f : X \to Y$ which is monic and epi, but is not an isomorphism. Deduce that $\mathbf{FVect}_K$ is not abelian.

5.5. Basic properties of abelian categories. Let $\mathcal{A}$ be an abelian category.

Conventions. i) We write $X \oplus Y := X \sqcup Y \simeq X \sqcap Y$ and call it the direct sum or biproduct of X and Y.

ii) If $\alpha : X \to Y$ is monic, we will write Y/X for $\operatorname{coker}(\alpha)$:

$$Y/X := \operatorname{coker}(\alpha).$$

iii) We will write $\operatorname{Hom}_{\mathcal{A}}(X, Y)$ instead $\operatorname{Mor}_{\mathcal{A}}(X, Y)$.

Properties 5.6. *1) $\mathcal{A}$ is abelian if and only if $\mathcal{A}^\circ$ is abelian.*

Proof. This follows from the observation that the dual of $i : \operatorname{Coim}(f) \to \operatorname{Im}(f)$ is $i^\circ : \operatorname{Coim}(f^\circ) \to \operatorname{Im}(f^\circ)$. □

2) $f : X \to Y$ is monic if and only if $\ker(f) = 0_{\mathcal{A}}$.

Proof. a) Assume that $\ker(f) = 0_{\mathcal{A}}$. Let $Z \in \mathbf{Obj}(\mathcal{A})$. We want to show that the map

$$f_Z^* : \operatorname{Hom}_{\mathcal{A}}(Z, X) \to \operatorname{Hom}_{\mathcal{A}}(Z, Y), \qquad g \mapsto f \circ g$$

is injective. Since $\mathcal{A}$ is additive, this map is a morphism of abelian groups, and it is sufficient to show that $\ker(f_Z^*) = 0$. Let $g \in \ker(f_Z^*)$. Then $f \circ g = 0$. By the universal property of kernels, there exists a map $Z \to \ker(f)$ such that g coincides with the composition $Z \to \ker(f) \to X$. But $\ker(f) = 0_{\mathcal{A}}$, and therefore $g = 0$. This shows that $\ker(f_Z^*) = 0$ and f is monic.

b) Conversely, assume that f is monic. Let $\alpha : A \to X$ be such that $f \circ \alpha = 0$. Then $f \circ \alpha = f \circ 0_{A,X}$ and therefore $\alpha = 0_{A,X}$. This shows that α is the composition of morphisms $A \to 0_{\mathcal{A}} \to X$. Therefore $0_{\mathcal{A}}$ satisfies the universal property of $\ker(f)$. □

3) $f : X \to Y$ is epi if and only if $\operatorname{coker}(f) = 0_{\mathcal{A}}$.

Proof. Apply property 2) to the morphism $f^\circ : Y^\circ \to X^\circ$. □

4) If $f : X \to Y$ is monic, then $X \simeq$ $\operatorname{Im}(f)$.

Proof. We have $\ker(f) = 0_{\mathcal{A}}$. It is not difficult to see that $\operatorname{coker}(0_{\mathcal{A}} \to X) \simeq X$ (exercise). Therefore $\operatorname{Coim} f \simeq \operatorname{coker}(0_{\mathcal{A}} \to X) \simeq X$, and the isomorphism i reads $X \simeq \operatorname{Im}(f)$. □

5) If $f : X \to Y$ is epi, then $Y \simeq \operatorname{Im}(f)$ and $Y \simeq X/\ker(f)$.

Proof. a) One has:

$$f \text{ is epi} \Rightarrow f^\circ \text{ is mono} \Rightarrow \operatorname{coker}(f) \simeq \ker(f^\circ) = 0.$$

Therefore $Y \simeq \operatorname{Im}(f)$.

b) By property 3), we have $Y^\circ \simeq \operatorname{Im}(f^\circ)$. Therefore

$$Y \simeq \operatorname{Coim}(f) \simeq X/\ker(f).$$

□

6) If f is monic and epi, then f is an isomorphism.

Proof. By properties 3) and 5), we have $X \simeq \operatorname{Im}(f) \simeq Y$. □

Exercise 5. *Show that in an additive category the following statements hold true:*

a) $\ker(X \to 0) = X$;
b) $\operatorname{coker}(0 \to X) = X$;
c) $\operatorname{Im}(0 \to X) = 0$;
d) $\operatorname{Coim}(X \to 0) = 0$;

Exercise 6. *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$. Show that if f and g are monic (resp. epi) then $g \circ f$ is monic (resp. epi).*

Exercise 7. *In a an additive category, the zero map $X \xrightarrow{0} Y$ is monic (resp. epi) if and only if $X = 0$ (resp. $Y = 0$).*

5.7. Exact sequences.

Definition.

1) *A sequence of morphisms*

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

is exact if $\operatorname{Im}(f) \simeq \ker(g)$.

2) *A sequence*

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} \cdots X_{n-1} \xrightarrow{f_{n-1}} X_n$$

is exact if it is exact in each term:

$$\ker(f_{i+1}) = \operatorname{Im}(f_i), \qquad \textit{for all } 1 \leqslant i \leqslant n-2.$$

3) *A short exact sequence is an exact sequence of the form*

$$0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0.$$

Consider a short exact sequence

$$0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0.$$

Then:

- $\ker(f) = \mathrm{Im}(0 \to X) = 0$ and therefore f is monic;
- $\mathrm{Im}(g) = \ker(Z \to 0) = Z$. The composition map

$$\mathrm{Im}(g) \simeq Z \to \mathrm{coker}(g)$$

  is zero and epi. Therefore $\mathrm{coker}(g) = 0$ (cf. Exercise 7). and g is epi.
- $Z \simeq Y/X$.

  Proof. We have:

$$Z = \mathrm{Im}(g) \simeq \mathrm{Coim}(g) \simeq \mathrm{coker}(\ker(g) \to Y) \simeq$$
$$\simeq \mathrm{coker}(\mathrm{Im}(f) \to Y) \simeq \mathrm{coker}(X \to Y) =: Y/X.$$

□

6. Exact functors

6.1. Exact functors.

Definition. *i) Let $\mathcal{A}$ and $\mathcal{B}$ be two additive categories. A covariant functor $\mathscr{F} : \mathcal{A} \to \mathcal{B}$ is additive if for all $X, Y \in$* **Obj**$(\mathcal{A})$, *the map*

$$\mathrm{Mor}_{\mathcal{A}}(X, Y) \to \mathrm{Mor}_{\mathcal{B}}(\mathscr{F}(X), \mathscr{F}(Y))$$

is a morphism of groups.

ii) A contravariant functor $\mathscr{F} : \mathcal{A} \to \mathcal{B}$ is additive if the covariant functor $\mathscr{F} : \mathcal{A}^\circ \to \mathcal{B}$ is additive. Explicitly, $\mathscr{F} : \mathcal{A} \to \mathcal{B}$ is additive if for all $X, Y \in$ **Obj**$(\mathcal{A})$, *the map*

$$\mathrm{Mor}_{\mathcal{A}}(X, Y) \to \mathrm{Mor}_{\mathcal{B}}(\mathscr{F}(Y), \mathscr{F}(X))$$

is a morphism of groups.

In this section, we will always assume that the categories $\mathcal{A}$ and $\mathcal{B}$ are abelian and write $\mathrm{Hom}_{\mathcal{A}}$ instead $\mathrm{Mor}_{\mathcal{A}}$.

Definition. *An additive functor $\mathscr{F} : \mathcal{A} \to \mathcal{B}$ is exact if for each exact sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ the induced sequence*

$$\mathscr{F}(X) \xrightarrow{\mathscr{F}(f)} \mathscr{F}(Y) \xrightarrow{\mathscr{F}(g)} \mathscr{F}(Z)$$

is exact.

Proposition 6.2. *i) $\mathscr{F}$ is exact if and only if for each short exact sequence*

$$0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$$

the induced sequence

$$0 \to \mathscr{F}(X) \xrightarrow{\mathscr{F}(f)} \mathscr{F}(Y) \xrightarrow{\mathscr{F}(g)} \mathscr{F}(Z) \to 0$$

is exact.

ii) Assume that $\mathscr{F}$ is exact. Then f respects kernels, cokernels, images and coimages. Namely, for any morphism $X \xrightarrow{f} Y$, one has:

$$\ker(\mathscr{F}(f)) \simeq \mathscr{F}(\ker(f)), \qquad \operatorname{Im}(\mathscr{F}(f)) \simeq \mathscr{F}(\operatorname{Im}(f)),$$
$$\operatorname{coker}(\mathscr{F}(f)) \simeq \mathscr{F}(\operatorname{coker}(f)), \qquad \operatorname{Coim}(\mathscr{F}(f)) \simeq \mathscr{F}(\operatorname{Coim}(f)).$$

Proof. a) Assume that $\mathscr{F}$ is exact. Then for any exact sequence

$$X_1 \to X_2 \to \cdots \to X_n$$

the induced sequence

$$\mathscr{F}(X_1) \to \mathscr{F}(X_2) \to \cdots \to \mathscr{F}(X_n)$$

is exact. Applying this remark to a short exact sequence $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ we see that the induced sequence $0 \to \mathscr{F}(X) \xrightarrow{\mathscr{F}(f)} \mathscr{F}(Y) \xrightarrow{\mathscr{F}(g)} \mathscr{F}(Z) \to 0$ is exact.

b) We prove ii). Let $X \xrightarrow{f} Y$. Applying $\mathscr{F}$ to the exact sequence

$$0 \to \ker(f) \to X \xrightarrow{f} Y \to \operatorname{coker}(f) \to 0,$$

we obtain that the sequence

$$0 \to \mathscr{F}(\ker(f)) \to \mathscr{F}(X) \xrightarrow{f} \mathscr{F}(Y) \to \mathscr{F}(\operatorname{coker}(f)) \to 0$$

is exact. Comparing this sequence with the tautological exact sequence

$$0 \to \ker(\mathscr{F}(f)) \to \mathscr{F}(X) \xrightarrow{f} \mathscr{F}(Y) \to \operatorname{coker}(\mathscr{F}(f)) \to 0$$

we obtain that $\ker(\mathscr{F}(f)) \simeq \mathscr{F}(\ker(f))$ and $\operatorname{coker}(\mathscr{F}(f)) \simeq \mathscr{F}(\operatorname{coker}(f))$.

Applying $\mathscr{F}$ to the tautological exact sequence

$$0 \to \operatorname{Im}(f) \to Y \to \operatorname{coker}(f) \to 0$$

we obtain an exact sequence

$$0 \to \mathscr{F}(\operatorname{Im}(f)) \to \mathscr{F}(Y) \to \mathscr{F}(\operatorname{coker}(f)) \to 0.$$

Since $\mathscr{F}(\operatorname{coker}(f)) \simeq \operatorname{coker}(\mathscr{F}(f))$, we obtain that

$$\mathscr{F}(\operatorname{Im}(f)) \simeq \ker(\mathscr{F}(Y) \to \operatorname{coker}(\mathscr{F}(f))) =: \operatorname{Im}(\mathscr{F}(f)).$$

An analogous argument shows that $\operatorname{Coim}(\mathscr{F}(f)) \simeq \mathscr{F}(\operatorname{Coim}(f))$.

c) It remains to prove that if $\mathscr{F}$ preserves short exact sequences, then it is exact. Consider an exact sequence of the form $X \xrightarrow{f} Y \xrightarrow{g} Z$ and the induced sequence

$$\mathscr{F}(X) \xrightarrow{\mathscr{F}(f)} \mathscr{F}(Y) \xrightarrow{\mathscr{F}(g)} \mathscr{F}(Z).$$

Since $\mathscr{F}$ respects kernels and images, we have

$$\operatorname{Im}\mathscr{F}(f) \simeq \mathscr{F}(\operatorname{Im}(f)) \simeq \mathscr{F}(\ker(g)) \simeq \ker(\mathscr{F}(g)).$$

Therefore $\mathscr{F}$ is exact, and the proposition is proved. □

6.3. Left and right exact functors. In some sense, the most interesting functors are not exact, but satisfy some weaker properties, which we introduce in this section.

Definition. *i) Let* $\mathscr{F} : \mathcal{A} \to \mathcal{B}$ *be a covariant additive functor. Then* $\mathscr{F}$ *is said to be left exact if for any exact sequence*

$$0 \to X \xrightarrow{f} Y \xrightarrow{g} Z$$

the induced sequence

$$0 \to \mathscr{F}(X) \xrightarrow{\mathscr{F}(f)} \mathscr{F}(Y) \xrightarrow{\mathscr{F}(g)} \mathscr{F}(Z)$$

is exact. It is said to be right exact if for any exact sequence

$$X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$$

the induced sequence

$$\mathscr{F}(X) \xrightarrow{\mathscr{F}(f)} \mathscr{F}(Y) \xrightarrow{\mathscr{F}(g)} \mathscr{F}(Z) \to 0$$

is exact.

ii) A contravariant additive functor $\mathscr{F} : \mathcal{A} \to \mathcal{B}$ *is left (respectively right) exact if the covariant functor* $\mathscr{F}^\circ : \mathcal{A}^\circ \to Y$ *is left (respectively right) exact. Namely,* $\mathscr{F}$ *is left exact if for any exact sequence*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$$

the sequence

$$0 \to \mathscr{F}(Z) \xrightarrow{\mathscr{F}(g)} \mathscr{F}(Y) \xrightarrow{\mathscr{F}(f)} \mathscr{F}(X)$$

is exact. It is right exact if for any exact sequence

$$0 \to X \xrightarrow{f} Y \xrightarrow{g} Z$$

the sequence

$$\mathscr{F}(Z) \xrightarrow{\mathscr{F}(g)} \mathscr{F}(Y) \xrightarrow{\mathscr{F}(f)} \mathscr{F}(X) \to 0$$

is exact.

Proposition 6.4. *Let* $\mathscr{F} : \mathcal{A} \to \mathcal{B}$ *be a covariant additive functor. Then the following assertions hold true:*

i) $\mathscr{F}$ *is left exact if and only if for any short exact sequence* $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ *the sequence*

$$0 \to \mathscr{F}(X) \xrightarrow{\mathscr{F}(f)} \mathscr{F}(Y) \xrightarrow{\mathscr{F}(g)} \mathscr{F}(Z)$$

is exact.

ii) It is right exact if and only if for any short exact sequence $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ *the induced sequence*

$$\mathscr{F}(X) \xrightarrow{\mathscr{F}(f)} \mathscr{F}(Y) \xrightarrow{\mathscr{F}(g)} \mathscr{F}(Z) \to 0$$

is exact.

Proof. The proof is purely technical and is omitted here. □

Consider the rule

$$\mathrm{Hom}_{\mathcal{A}}(-,-) : \mathcal{A} \times \mathcal{A} \to \mathbf{Ab}, \quad (X, Y) \mapsto \mathrm{Hom}_{\mathcal{A}}(X, Y).$$

For any $X_1 \xrightarrow{f} X_2$, we have a natural map

$$\mathrm{Hom}_{\mathcal{A}}(X_2, Y) \xrightarrow{f^*} \mathrm{Hom}_{\mathcal{A}}(X_1, Y), \quad g \mapsto g \circ f.$$

Since $\mathcal{A}$ is abelian (and therefore additive), one has:

$$f^*(g_1 + g_2) = (g_1 + g_2) \circ f = g_1 \circ f + g_2 \circ f = f^*(g_1) + f^*(g_2).$$

Hence f^* is a morphism of groups. Similarly, for any $Y_1 \xrightarrow{f} Y_2$, the map

$$\mathrm{Hom}_{\mathcal{A}}(X, Y_1) \xrightarrow{f^*} \mathrm{Hom}_{\mathcal{A}}(X, Y_2), \quad g \mapsto f \circ g$$

is a morphism of abelian groups. From this observation it follows easily that $\mathrm{Hom}_{\mathcal{A}}(-,-)$ is an additive functor in two variables, which is contravariant with respect to the first argument and contravariant with respect to the second one. For each $A \in \mathbf{Obj}(\mathcal{A})$, we consider the functors of one variable $h_A : \mathcal{A} \to \mathbf{Ab}$ and $h^A : \mathcal{A} \to \mathbf{Ab}$ defined as follows:

$$h_A(X) := \mathrm{Hom}_{\mathcal{A}}(A, X), \qquad h^A(X) := \mathrm{Hom}_{\mathcal{A}}(X, A).$$

We remark that h_A is covariant and h^A is contravariant.

Theorem 6.5. *The functors h_A and h^A are left exact.*

Proof. Asuume that $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z$ is exact. We should check that the induced sequence

$$0 \to \mathrm{Hom}_{\mathcal{A}}(A, X) \xrightarrow{f^*} \mathrm{Hom}_{\mathcal{A}}(A, Y) \xrightarrow{g^*} \mathrm{Hom}_{\mathcal{A}}(A, Z)$$

is exact.

a) Injectivity of f^*. Assume that $\alpha_1, \alpha_2 \in \mathrm{Hom}_{\mathcal{A}}(A, X)$ are such that $f^*(\alpha_1) = f^*(\alpha_2)$. Then

$$f \circ \alpha_1 = f \circ \alpha_2.$$

Since f is monic, this implies that $\alpha_1 = \alpha_2$.

b) Since $g \circ f = 0$, we have $g^* \circ f^* = (g \circ f)^* = 0$. Hence $\mathrm{Im}(f^*) \subseteq \ker(g^*)$.

c) Assume that $\beta \in \ker(g^*)$. Then $\beta : A \to Y$ is such that $g \circ \beta = 0$. Since $(X \xrightarrow{f} Y)$ represents $\ker(g)$, from the universal property of the kernel it follows that there exists $\alpha : A \to X$ such that $\beta = f \circ \alpha$. The last formula can be written as

$$\beta = f^*(\alpha) \in \mathrm{Im} f^*).$$

Hence $\ker(g^*) \subseteq \mathrm{Im}(f^*)$. Together with b), this proves that $\ker(g^*) = \mathrm{Im}(f^*)$. To sum up, we have proved that h_A is left exact.

d) To prove the left exactness of h^A it is enough to remark that $h^A(X) = h_{A^\circ}(X^\circ)$. The left exactness of $h_{A^\circ} : \mathcal{A}^\circ \to \mathbf{Ab}$ is already proved. □

7. Yoneda lemma

In this section, we consider an arbitrary category $\mathcal{A}$. For any $A \in \mathbf{Obj}(\mathcal{A})$, we consider the covariant functor

$$\begin{aligned} h_A &: \mathcal{A} \to \mathbf{Sets}, \\ &X \mapsto \mathrm{Hom}_{\mathcal{A}}(A, X) \end{aligned}$$

and the contravariant functor

$$\begin{aligned} h^A &: \mathcal{A} \to \mathbf{Sets}, \\ &X \mapsto \mathrm{Hom}_{\mathcal{A}}(X, A). \end{aligned}$$

Let $\mathscr{F} : \mathcal{A} \to \mathbf{Sets}$ be a covariant functor. We denote by $\mathrm{Mor}(h_A, \mathscr{F})$ the natural transformations $\alpha : h_A \to \mathscr{F}$.

Lemma 7.1 (Yoneda lemma)**.** *There exists a natural one-to-one correspondence*

$$\mathrm{Mor}(h_A, \mathscr{F}) \simeq \mathscr{F}(A).$$

Proof. a) We construct a correspondence

$$\Phi : \mathrm{Mor}(h_A, \mathscr{F}) \to \mathscr{F}(A).$$

Let $\alpha \in \mathrm{Mor}(h_A, \mathscr{F})$. Then for each $X \in \mathbf{Obj}(\mathcal{A})$, we have a morphism $\alpha_X : h_A(X) \to \mathscr{F}(X)$. In particular, $\mathrm{id}_A \in h_A(A)$, and we set

$$\Phi(\alpha) := \alpha_A(\mathrm{id}_A) \in \mathscr{F}(A).$$

b) We construct a correspondence

$$\Psi : \mathscr{F}(A) \to \mathrm{Mor}(h_A, \mathscr{F}).$$

Let $a \in \mathscr{F}(A)$. Consider the composition

$$\alpha_X : h_A(X) = \mathrm{Mor}_{\mathcal{A}}(A, X) \xrightarrow{\mathscr{F}} \mathrm{Mor}_{\mathbf{Sets}}(\mathscr{F}(A), \mathscr{F}(X)) \xrightarrow{\mathrm{ev}_a} \mathscr{F}(X),$$

where the map ev_a is defined as $\mathrm{ev}_a(f) = f(a)$. The collection of maps $(\alpha_X)_{X \in \mathbf{Obj}(\mathcal{A})}$ defines a natural transformation $\alpha \in \mathrm{Mor}(h_A, \mathscr{F})$. Set $\Psi(a) = \alpha$.

c) It can be easily checked that Φ and Ψ are inverse to each other. Moreover, from the above constructions it follows that they are functorial with respect to the both arguments. Namely, if $f : A \to A'$ is a morphism and $\mathscr{F} \to \mathscr{F}'$ a natural transformation of functors, then the following diagrams commute:

$$\begin{array}{ccc} \mathrm{Mor}(h_A, \mathscr{F}) & \longrightarrow & \mathscr{F}(A) \\ \big\downarrow & & \big\downarrow \\ \mathrm{Mor}(h_{A'}, \mathscr{F}) & \longrightarrow & \mathscr{F}(A') \end{array} , \qquad \begin{array}{ccc} \mathrm{Mor}(h_A, \mathscr{F}) & \longrightarrow & \mathscr{F}(A) \\ \big\downarrow & & \big\downarrow \\ \mathrm{Mor}(h_A, \mathscr{F}') & \longrightarrow & \mathscr{F}'(A). \end{array}$$

The lemma is proved. □

We formulate the contravariant version of Yoneda lemma. Let $\mathcal{G} : \mathcal{A} \to \mathbf{Sets}$ be a contravariant functor. Let $\mathrm{Mor}(h^A, \mathcal{G})$ denote the natural transformations of contravariant functors $\alpha : h^A \to \mathcal{G}$.

Lemma 7.2 (Yoneda lemma)**.** *There exists a natural one-to-one correspondence*

$$\mathrm{Mor}(h^A, \mathcal{G}) \simeq \mathcal{G}(A).$$

Proof. The proof is analogous to the previous one and is omitted here. □

Corollary 7.3. *Let* $A, B \in \mathbf{Obj}(\mathcal{A})$. *Then*

$$\mathrm{Mor}(h_A, h_B) \simeq \mathrm{Mor}_{\mathcal{A}}(B, A),$$
$$\mathrm{Mor}(h^A, h^B) \simeq \mathrm{Mor}_{\mathcal{A}}(A, B).$$

Corollary 7.4. *Let* $\mathcal{F}unc(\mathcal{A}, \mathbf{Sets})$ *denote the category of covariant functors* $\mathcal{A} \to \mathbf{Sets}$. *The morphisms in this category are natural transformations of functors. Then the correspondence* $A \mapsto h_A$ *defines a contravariant functor:*

$$\mathcal{A} \to \mathcal{F}unc(\mathcal{A}, \mathbf{Sets}).$$

Corollary 7.3 shows that it is fully faithful.

8. Adjoint functors

8.1. Adjoint functors. In this section, we consider two categories $\mathcal{A}$ and $\mathcal{B}$ and a pair of functors:

$$\mathcal{A} \underset{\mathcal{F}}{\overset{\mathcal{G}}{\rightleftarrows}} \mathcal{B}$$

Definition. *We say that* $\mathcal{G}$ *is a right adjoint to* $\mathcal{F}$ *and* $\mathcal{F}$ *is a left adjoint to* $\mathcal{G}$ *if the functors*

$$\begin{array}{ccc} \mathcal{A} \times \mathcal{B} \to \mathbf{Sets}, & & \mathcal{A} \times \mathcal{B} \to \mathbf{Sets}, \\ (X, Y) \mapsto \mathrm{Mor}_{\mathcal{B}}(\mathcal{F}(X), Y) & \text{and} & (X, Y) \mapsto \mathrm{Mor}_{\mathcal{A}}(X, \mathcal{G}(Y)) \end{array}$$

are isomorphic.

This condition means that we have a system of bijections

$$\varphi : \mathrm{Mor}_{\mathcal{B}}(\mathcal{F}(X), Y) \simeq \mathrm{Mor}_{\mathcal{A}}(X, \mathcal{G}(Y)),$$

which are functorial in X and Y.

Example. Consider the functors

$$\mathbf{Sets} \underset{\mathcal{F}}{\overset{\mathcal{G}}{\rightleftarrows}} \mathbf{Groups}$$

defined as follows :

$$\mathcal{F}(X) := \text{free group generated by } X.$$
$$\mathcal{G}(Y) := Y \text{ viewed as a set (forgetful functor).}$$

It is easy to see that these functors are adjoint, namely

$$\mathrm{Hom}(\mathcal{F}(X), Y) \simeq \mathrm{Maps}(X, \mathcal{G}(Y)),$$

Properties 8.2. *Assume that $(\mathscr{F}, \mathscr{G})$ is a pair of adjoint functors. Then the following holds true:*

1) *$\mathscr{F}$ respects initial objects, zero objects, cokernels and coproducts. $\mathscr{G}$ respects final objects, zero objects, kernels and products.*
2) *If $\mathcal{A}$ and $\mathcal{B}$ are additive, then $\mathscr{F}$ and $\mathscr{G}$ are additive.*

Proof. a) We prove that $\mathscr{F}$ respects initial objects. Let X be an initial object in $\mathcal{A}$. For any $Y \in \mathbf{Obj}(\mathcal{B})$, we have a bijection

$$\mathrm{Mor}_{\mathcal{B}}(\mathscr{F}(X), Y) \simeq \mathrm{Mor}_{\mathcal{A}}(X, \mathscr{G}(Y)).$$

Since X is initial, there exactly one morphism $X \to \mathscr{G}(Y)$. Therefore there exists exactly one morphism $\mathscr{F}(X) \to Y$, and we proved that $\mathscr{F}(X)$ is initial.

b) We prove that $\mathscr{F}$ respect cokernels. Consider the diagram

$$X \xrightarrow{f} Y \xrightarrow{\beta} \mathrm{coker}(f).$$

It induces a diagram

$$\mathscr{F}(X) \xrightarrow{\mathscr{F}(f)} \mathscr{F}(Y) \xrightarrow{\mathscr{F}(\beta)} \mathscr{F}(\mathrm{coker}(f)).$$

We want to prove that $(\mathscr{F}(\mathrm{coker}(f)), \mathscr{F}(\beta))$ is a cokernel of $\mathscr{F}(f)$. Since $\mathscr{F}(\beta) \circ \mathscr{F}(f) = \mathscr{F}(\beta \circ f) = 0$, we only need to check the universal property. Consider the diagram

$$\begin{array}{ccccc} \mathscr{F}(X) & \xrightarrow{\mathscr{F}(f)} & \mathscr{F}(Y) & \xrightarrow{\mathscr{F}(\beta)} & \mathscr{F}(\mathrm{coker}(f)) \\ & & & \searrow^{\alpha} & \downarrow{\scriptstyle g} \\ & & & & Z \end{array}$$

Let $\alpha^* = \varphi(\alpha) : Y \to \mathscr{G}(Z)$. We have a diagram

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{\beta} & \mathrm{coker}(f) \\ & & & \searrow^{\alpha^*} & \downarrow{\scriptstyle g*} \\ & & & & \mathscr{G}(Z). \end{array}$$

By the universal property of cokernels, there exists a unique $g^* : \mathrm{coker}(f) \to \mathscr{G}(Z)$ such that $\alpha^* = g^* \circ \beta$. Let $g : \mathscr{F}(\mathrm{coker}(f)) \to Z$ be the unique morphism such that $\varphi(g) = g^*$. From the functoriality of morphisms φ it follows easily that $\alpha = g \circ \mathscr{F}(\beta)$. This shows that $\mathscr{F}(\mathrm{coker}(f))$ satisfies the universal property of cokernels.

c) The proof that $\mathscr{F}$ respects coproducts is analogous and is omitted here. Using dual categories $\mathcal{A}^\circ$ and $\mathcal{B}^\circ$, we see that $\mathscr{G}^\circ$ is the left adjoint of $\mathscr{F}^\circ$ and therefore respects initial objects, cokernels and coproducts. This implies that $\mathscr{F}$ respects final objects, kernels and products.

d) Assume that $\mathcal{A}$ and $\mathcal{B}$ are additive. The addition of morphisms in additive categories is defined using products, coproducts and the diagonal map. The additivity of $\mathscr{F}$ and $\mathscr{G}$ can be proved using properties 1). We omit the details here. □

Theorem 8.3. *Assume that $(\mathscr{F}, \mathscr{G})$ is a pair of adjoint functors between abelian categories $\mathcal{A}$ and $\mathcal{B}$. Then $\mathscr{F}$ is right exact and $\mathscr{G}$ is left exact.*

Proof. We prove that $\mathscr{G}$ is left exact. (The proof of the right exactness of $\mathscr{F}$ can be proved using duality). Consider an exact sequence

$$0 \to Y' \xrightarrow{f} Y \xrightarrow{g} Y''.$$

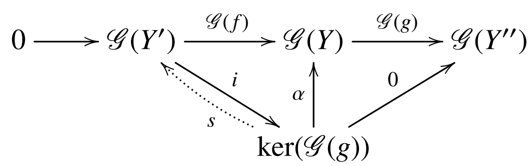
We will prove that the sequence

$$0 \to \mathscr{G}(Y') \xrightarrow{\mathscr{G}(f)} \mathscr{G}(Y) \xrightarrow{\mathscr{G}(g)} \mathscr{G}(Y'').$$

is exact. Since $\mathscr{G}$ respects kernels, the morphism $\mathscr{G}(f)$ is monic. Moreover from $g \circ f = 0$ it follows that $\mathscr{G}(g) \circ \mathscr{G}(f) = 0$. Therefore we have a unique (monic) map $i : \mathscr{G}(Y_1) \to \ker(\mathscr{G}(g))$ such that

$$\alpha \circ i = \mathscr{G}(f). \tag{1}$$

These data can be represented by the diagram



To prove that i is an isomorphism we will construct a section s of i. Consider the commutative diagram

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathrm{Hom}_{\mathscr{A}}(X, \mathscr{G}(Y')) & \xrightarrow{\mathscr{G}(f)^*} & \mathrm{Hom}_{\mathscr{A}}(X, \mathscr{G}(Y)) & \xrightarrow{\mathscr{G}(g)^*} & \mathrm{Hom}_{\mathscr{A}}(X, \mathscr{G}(Y'')) \\
& \downarrow \wr & & \downarrow \wr & & \downarrow \wr \\
0 \longrightarrow & \mathrm{Hom}_{\mathscr{B}}(\mathscr{F}(X), Y') & \longrightarrow & \mathrm{Hom}_{\mathscr{B}}(\mathscr{F}(X), Y) & \longrightarrow & \mathrm{Hom}_{\mathscr{B}}(\mathscr{F}(X), Y'')
\end{array}$$

The vertical morphisms are isomorphisms by the definition of adjoint functors. Moreover, the bottom row is exact by the left exactness of h_A with $A = \mathscr{F}(X)$. Therefore the upper row is exact. Take $X = \ker(\mathscr{G}(g))$. The map $\alpha : \ker(\mathscr{G}(g)) \to Y$ satisfies

$$\mathscr{G}(g)^*(\alpha) = \mathscr{G}(g) \circ \alpha = 0.$$

Then there exists $s : \ker(\mathscr{G}(g)) \to \mathscr{G}(Y')$ such that

$$\mathscr{G}(f) \circ s = \mathscr{G}(f)^*(s) = \alpha.$$

Together with (1) and the fact that α and $\mathscr{G}(f)$ are monic, it is easy to see that $i \circ s$ and $s \circ i$ are the identity morphisms. □

8.4. Tensor product. Let A be a ring (not necessarily commutative). For any right A-module M and left A-module N the tensor product $M \otimes_A N$ is the abelian group generated by the symbols $m \otimes n$ ($m \in M, n \in N$) with relations:

1) $(m_1 + m_2) \otimes n = m_1 \otimes n + m_2 \otimes n$;
2) $m \otimes (n_1 + n_2) = m \otimes n_1 + m \otimes n_2$;
3) $ma \otimes n = m \otimes an$, $a \in A$.

The assignment $(M,N) \mapsto M \otimes_A N$ define a functor

$$\otimes_A \;:\; (\mathbf{Mod} - A) \times (A - \mathbf{Mod}) \to \mathbf{Ab},$$

which is covariant in both arguments.

Definition. *Let A and B be two rings. Assume that N is an abelian group equipped with structures of a left A-module and a right B-module. We say that N is a (A,B)-bimodule if*

$$(an)b = a(nb), \qquad \forall a \in A,\ b \in B\,, n \in N.$$

Assume that N is an (A,B)-bimodule. Then for any right A-module M, the tensor product $M \otimes_A N$ has a natural structure of a right B-module:

$$(m \otimes n)b = m \otimes (nb).$$

Similarly, for any right B-module L, the group $\mathrm{Hom}_B(N,L)$ has a natural structure of a right A-module:

$$(fa)(x) = f(ax), \qquad f \in \mathrm{Hom}_B(N,L),\ x \in N,\ a \in A.$$

Proposition 8.5. *There exists a canonical and functorial isomorphism*

$$\varphi \;:\; \mathrm{Hom}_B(M \otimes_A N, L) \simeq \mathrm{Hom}_A(M, \mathrm{Hom}_B(N,L)).$$

Proof. Let $f \in \mathrm{Hom}_B(M \otimes_A N, L)$. We set $\varphi(f) := F \in \mathrm{Hom}_A(M, \mathrm{Hom}_B(N,L))$, where F is defined by the formula

$$(F(m)](n) = f(m \otimes n).$$

The same formula can be used to construct the converse map φ^{-1} setting $\varphi^{-1}(F) := f$. □

Fix a bimodule N and consider the functors

$$\begin{aligned} - \otimes_A N \;:&\quad \mathbf{Mod} - A \to \mathbf{Mod} - B \\ \mathrm{Hom}_B(N,-) \;:&\quad \mathbf{Mod} - B \to \mathbf{Mod} - A. \end{aligned}$$

Corollary 8.6. *For any bimodule N, the functor $- \otimes_A N$ is a left adjoint of* $\mathrm{Hom}_B(N,-)$ *(and therefore* $\mathrm{Hom}_B(N,-)$ *is a right adjoint of $- \otimes_A N$).*

Corollary 8.7. *The functor $- \otimes_A N$ is right exact.*

9. Some diagram lemmas

Let $\mathcal{A}$ be an abelian category.

Lemma 9.1 (five lemma)**.** *Assume that*

$$\begin{array}{ccccccccc} X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & X_3 & \xrightarrow{f_3} & X_4 & \xrightarrow{f_4} & X_5 \\ \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha_3} & & \downarrow{\scriptstyle \alpha_4} & & \downarrow{\scriptstyle \alpha_5} \\ Y_1 & \xrightarrow{g_1} & Y_2 & \xrightarrow{g_2} & Y_3 & \xrightarrow{g_3} & Y_4 & \xrightarrow{g_4} & Y_5 \end{array}$$

is a commutative diagram with exact rows. Then:

i) *If α_1 is epi and α_2 and α_4 are monic, then α_3 is monic.*

ii) *If* α_5 *is monic and* α_2 *and* α_4 *are epi, then* α_3 *is epi.*
iii) *If* $\alpha_1, \alpha_2, \alpha_4$ *and* α_5 *are isomorphisms, then* α_3 *is an isomorphism.*

Proof. We will prove this lemma for the category of modules.
i) (A diagram chase). Assume that $\alpha_3(x_3) = 0$. Then

$$\alpha_4(f_3(x_3)) = g_3(\alpha_3(x_3)) = 0.$$

Since α_4 is monic, $f_3(\alpha_3) = 0$. By the exactness of the upper row, there exists $x_2 \in X_2$ such that $f_2(x_2) = x_3$. We have

$$g_2(\alpha_2(x_2)) = \alpha_3(f_2(x_2)) = \alpha_3(x_3) = 0.$$

The exactness of the bottom row shows that there exists $y_1 \in Y_1$ such that $g_1(y_1) = \alpha_2(x_2)$. Since α_1 is epi, there exists $x_1 \in X_1$ such that $\alpha_1(x_1) = y_1$. Hence

$$\alpha_2(f_1(x_1)) = g_1(\alpha_1(y_1)) = \alpha_2(x_2).$$

Since α_2 is monic, this implies that $f_1(x_1) = x_2$. Therefore $x_3 = f_2(x_2) = f_2 \circ f_1(x_1) = 0$. To sum up, we proved that $\ker(\alpha_3) = 0$. Hence α_3 is monic.
ii) This statement can be deduced from i) using duality.
iii) It is clear that i) and ii) imply iii). □

Lemma 9.2 (snake lemma)**.** *Assume that we have a commutative diagram with exact rows*

$$\begin{array}{ccccccc} & & X_1 & \xrightarrow{f_1} & X & \xrightarrow{f} & X_2 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha_2} & & \\ 0 & \longrightarrow & Y_1 & \xrightarrow{g_1} & Y & \xrightarrow{g} & Y_2 & & \end{array}$$

Then there exists an exact sequence

$$\ker(\alpha_1) \xrightarrow{f_1} \ker(\alpha) \xrightarrow{f} \ker(\alpha_2) \xrightarrow{\delta} \operatorname{coker}(\alpha_1) \xrightarrow{f_1} \operatorname{coker}(\alpha) \xrightarrow{f} \operatorname{coker}(\alpha_2).$$

Proof. We prove this lemma for modules. It is not difficult to see that the morphisms f_1 and f induce morphisms $\ker(\alpha_1) \xrightarrow{f_1} \ker(\alpha)$, $\ker(\alpha) \xrightarrow{f} \ker(\alpha_2)$, $\operatorname{coker}(\alpha_1) \xrightarrow{f_1} \operatorname{coker}(\alpha)$, $\operatorname{coker}(\alpha) \xrightarrow{f} \operatorname{coker}(\alpha_2)$, which we denote by the same letters f_1 and f. A routine diagram chase shows that our exact sequence is exact at $\ker(f)$ and $\operatorname{coker}(\alpha)$.

We construct the map δ. Let $x_2 \in \ker(\alpha_2)$. Since f is epi, there exists $x \in X$ such that $f(x) = x_2$. We have

$$g(\alpha(x)) = \alpha_2(f(x)) = 0.$$

Frome the exactness of the bottm row it follows that there exists a unique $y_1 \in Y_1$ such that $g_1(y_1) = \alpha(x)$. Set

$$\delta(x_1) := \bar{y}_1 \in Y_1/\operatorname{Im}(\alpha_1) \simeq \operatorname{coker}(\alpha_1).$$

We omit the proof of the exactness at $\ker(\alpha_2)$ and $\operatorname{coker}(\alpha_1)$. □

We also have the following version of the snake lemma:

Lemma 9.3. *Assume that we have a commutative diagram with exact rows*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X_1 & \xrightarrow{f_1} & X & \xrightarrow{f} & X_2 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha_2} & & \\ 0 & \longrightarrow & Y_1 & \xrightarrow{g_1} & Y & \xrightarrow{g} & Y_2 & \longrightarrow & 0 \end{array}$$

Then the sequence

$$0 \to \ker(\alpha_1) \xrightarrow{f_1} \ker(\alpha) \xrightarrow{f} \ker(\alpha_2) \xrightarrow{\delta} \operatorname{coker}(\alpha_1) \xrightarrow{f_1} \operatorname{coker}(\alpha) \xrightarrow{f} \operatorname{coker}(\alpha_2) \to 0$$

is exact.

The following deep theorem can be used to reduce the proof of the previous lemmas for general abelian categories to the case of categories of modules:

Theorem 9.4 (Freyd-Mitchell embedding theorem)**.** *Let $\mathcal{A}$ be a small abelian category. Then there exists a ring A and a fully faithful exact functor*

$$\mathcal{A} \to A - \mathbf{Mod}.$$

CHAPTER 2

Complexes

1. Complexes

Let $\mathcal{A}$ be an abelian category.

Definition. *A chain complex $X_\bullet$ in $\mathcal{A}$ is a family $(X_n)_{n\in\mathbf{Z}}$ of objects $X_n \in$* **Obj**$(\mathcal{A})$ *together with morphisms $d_n : X_n \to X_{n-1}$ such that*

$$d_{n-1} \circ d_n = 0, \qquad \forall n \in \mathbf{Z}.$$

A chain complex can be represented by the diagram

$$\cdots \xrightarrow{d_{n+1}} X_n \xrightarrow{d_n} X_{n-1} \xrightarrow{d_{n-1}} X_{n-2} \xrightarrow{d_{n-2}} \cdots .$$

The morphisms d_n are called differentials. We will often write d instead d_n. To each complex we attach the following objects:

$Z_n := \ker(d_n)$ called n-cycles.
$B_n := \mathrm{Im}(d_{n+1})$ called n-boundaries.
$H_n(X) := Z_n/B_n$ called n-homology of X.

We define the category $\mathbf{K}(\mathcal{A})$ of complexes in $\mathcal{A}$. The objects of this category are complexes. A morphism of complexes $f : X_\bullet \to Y_\bullet$ is a family of morphisms $f_n : X_n \to Y_n$ such that the diagram

$$\begin{array}{ccccccccc}
\xrightarrow{d_{n+1}} & X_n & \xrightarrow{d_n} & X_{n-1} & \xrightarrow{d_{n-1}} & X_{n-2} & \xrightarrow{d_{n-2}} \\
 & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & & \downarrow{\scriptstyle f_{n-2}} & \\
\xrightarrow{d_{n+1}} & Y_n & \xrightarrow{d_n} & Y_{n-1} & \xrightarrow{d_{n-1}} & Y_{n-2} & \xrightarrow{d_{n-2}}
\end{array}$$

commutes. In other words

$$f_{n-1} \circ d_n = d_n \circ f_n, \qquad \forall n \in Z.$$

The proof of the following theorem is straightforward:

Theorem 1.1. $\mathbf{K}(\mathcal{A})$ *is an abelian category. In particular,*

i) $\ker(f) \simeq (\ker(f_n))_{n\in\mathbf{Z}}$;
ii) $\mathrm{coker}(f) \simeq (\mathrm{coker}(f_n))_{n\in\mathbf{Z}}$;
iii) *A short sequence of complexes*

$$0 \to X_\bullet \to Y_\bullet \to Z_\bullet \to 0$$

is exact if and only if the sequence

$$0 \to X_n \to Y_n \to Z_n \to 0$$

is exact for all $n \in \mathbf{Z}$.

We will also work with the dual notion of a cochain complex.

Definition. *A cochain complex* $X^\bullet$ *in* $\mathcal{A}$ *is a family* $(X^n)_{n\in\mathbf{Z}}$ *of objects* $X^n \in$ $\mathbf{Obj}(\mathcal{A})$ *together with morphisms* $d^n : X^n \to X^{n+1}$ *such that*

$$d^{n+1} \circ d_n = 0, \qquad \forall n \in \mathbf{Z}.$$

A cochain complex can be represented by the diagram

$$\cdots \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} X^{n+1} \xrightarrow{d^{n+1}} X_{n+2} \xrightarrow{d^{n+2}} \cdots .$$

To each cochain complex we attach:

$Z^n := \ker(d^n)$ called n-cocycles.
$B^n := \mathrm{Im}(d^{n-1})$ called n-coboundaries.
$H^n(X) := Z^n/B^n$ called n-cohomology of X.

Morphisms of cochain complexes are defined analogously to the case of chain complexes. We denote by $\mathbf{CK}(\mathcal{A})$ the the abelian category of cochain complexes in $\mathcal{A}$.

1.2. Let $f : X_\bullet \to Y_\bullet$ be a morphism of complexes. Considering the diagram

$$\begin{array}{ccccc} X_{n+1} & \xrightarrow{d_{n+1}} & X_n & \xrightarrow{d_n} & X_{n-1} \\ \downarrow{\scriptstyle f_{n+1}} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} \\ Y_{n+1} & \xrightarrow{d_{n+1}} & Y_n & \xrightarrow{d_n} & Y_{n-1} \end{array}$$

we see that the morphism f_n induces morphisms

$$Z_n(X_\bullet) \to Z_n(Y_\bullet);$$
$$B_n(X_\bullet) \to B_n(Y_\bullet);$$

Therefore in each degree n, we have a morphism

$$H_n(f) : H_n(X_\bullet) \to H_n(Y_\bullet).$$

It is easy to see that this defines covariant additive functors

$$H_n : \mathbf{K}(\mathcal{A}) \to \mathcal{A},$$
$$X_\bullet \to H_n(X_\bullet).$$

These functors are not exact, but they are related by the following property:

Theorem 1.3 (long exact sequence in homology)**.** *Let*

$$0 \to M_\bullet \xrightarrow{f} N_\bullet \xrightarrow{g} L_\bullet \to 0$$

be a short exact sequence in $\mathbf{K}(\mathcal{A})$*. Then there exists a long exact sequence of homology:*

$$\ldots \to H_{n+1}(L_\bullet) \xrightarrow{\delta} H_n(M_\bullet) \xrightarrow{H_n(f)} H_n(N_\bullet) \xrightarrow{H_n(g)} H_n(L_\bullet) \xrightarrow{\delta} H_{n-1}(M_\bullet)$$
$$\xrightarrow{H_{n-1}(f)} H_{n-1}(N_\bullet) \xrightarrow{H_{n-1}(g)} H_{n-1}(L_\bullet) \to \cdots$$

PROOF. We will apply repeatedly the snake lemma.

a) Consider the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M_n & \longrightarrow & N_n & \longrightarrow & L_n & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle d_n} & & \downarrow{\scriptstyle d_n} & & \downarrow{\scriptstyle d_n} & & \\
0 & \longrightarrow & M_{n-1} & \longrightarrow & N_{n-1} & \longrightarrow & L_{n-1} & \longrightarrow & 0.
\end{array}$$

The rows of this diagram are exact and the snake lemma implies that for all $n \in \mathbf{Z}$, the following sequences are exact:

$$(2)\qquad \begin{array}{l}
0 \to Z_n(M_\bullet) \to Z_n(N_\bullet) \to Z_n(L_\bullet), \\
M_{n-1}/B_{n-1}(M_\bullet) \to N_{n-1}/B_{n-1}(N_\bullet) \to L_{n-1}/B_{n-1}(L_\bullet) \to 0.
\end{array}$$

Consider the commutative diagram

$$\begin{array}{ccccccccc}
 & & M_{n+1}/B_{n+1}(M_\bullet) & \longrightarrow & N_{n+1}/B_{n+1}(N_\bullet) & \longrightarrow & L_{n+1}/B_{n+1}(L_\bullet) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle d^M} & & \downarrow{\scriptstyle d^N} & & \downarrow{\scriptstyle d^L} & & \\
0 & \longrightarrow & Z_n(M_\bullet) & \longrightarrow & Z_n(N_\bullet) & \longrightarrow & Z_n(L_\bullet). & &
\end{array}$$

From the exactness of sequences (2) it follows that the rows of this diagram are exact. Applying the snake lemma we obtain an exact sequence

$$\ker(d^M) \to \ker(d^N) \to \ker(d^L) \xrightarrow{\delta} \operatorname{coker}(d^M) \to \operatorname{coker}(d^N) \to \operatorname{coker}(d^L).$$

Il is easy to see that $\ker(d^M) \simeq Z_{n+1}(M_\bullet)/B_{n+1}(M_\bullet) =: H_{n+1}(M_\bullet)$ and $\operatorname{coker}(d^M) \simeq Z_n(M_\bullet)/B_n(M_\bullet) =: H_n(M_\bullet)$. Therefore this exact sequence reads:

$$H_{n+1}(M_\bullet) \xrightarrow{H_{n+1}(f)} H_{n+1}(N_\bullet) \xrightarrow{H_{n+1}(g)} H_{n+1}(L_\bullet) \xrightarrow{\delta} H_n(M_\bullet) \xrightarrow{H_n(f)} H_n(N_\bullet) \xrightarrow{H_n(g)} H_n(L_\bullet).$$

Gluing together these sequences for different n, we obtain the long exact sequence in homology. □

The following results shows the functoriality of the long exact sequence in homology.

Theorem 1.4. *Assume that we have a commutative diagram with exact rows:*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M_\bullet & \longrightarrow & N_\bullet & \longrightarrow & L_\bullet & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\
0 & \longrightarrow & M'_\bullet & \longrightarrow & N'_\bullet & \longrightarrow & L'_\bullet & \longrightarrow & 0.
\end{array}$$

Then the diagram

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & H_n(M_\bullet) & \longrightarrow & H_n(N_\bullet) & \longrightarrow & H_n(L_\bullet) & \xrightarrow{\delta} & H_{n-1}(M_\bullet) & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle H_n(\alpha)} & & \downarrow{\scriptstyle H_n(\beta)} & & \downarrow{\scriptstyle H_n(\gamma)} & & \downarrow{\scriptstyle H_{n-1}(\alpha)} & & \\
\cdots & \longrightarrow & H_n(M'_\bullet) & \longrightarrow & H_n(N'_\bullet) & \longrightarrow & H_n(L'_\bullet) & \xrightarrow{\delta} & H_{n-1}(N_\bullet) & \longrightarrow & \cdots .
\end{array}$$

is commutative.

Proof. The theorem can be proved by diagram chasing and we omit the details. □

We record the analogous results for cochain complexes.

Theorem 1.5 (long exact sequence in cohomology)**.** *Let*

$$0 \to M^\bullet \xrightarrow{f} N^\bullet \xrightarrow{g} L^\bullet \to 0$$

be a short exact sequence of cochain complexes. Then there exists a long exact sequence of cohomology:

$$\ldots \to H^{n-1}(L^\bullet) \xrightarrow{\delta} H^n(M^\bullet) \xrightarrow{H^n(f)} H^n(N^\bullet) \xrightarrow{H^n(g)} H^n(L^\bullet) \xrightarrow{\delta} H^{n+1}(M^\bullet) \xrightarrow{H^{n+1}(f)} H^{n+1}(N^\bullet) \xrightarrow{H^{n+1}(g)} H^{n+1}(L^\bullet) \to \cdots$$

Theorem 1.6. *Assume that we have a commutative diagram with exact rows:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M^\bullet & \longrightarrow & N^\bullet & \longrightarrow & L^\bullet & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & M_1^\bullet & \longrightarrow & N_1^\bullet & \longrightarrow & L_1^\bullet & \longrightarrow & 0. \end{array}$$

Then the diagram

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & H^n(M^\bullet) & \longrightarrow & H^n(N^\bullet) & \longrightarrow & H^n(L^\bullet) & \xrightarrow{\delta} & H^{n+1}(M^\bullet) & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle H^n(\alpha)} & & \downarrow{\scriptstyle H^n(\beta)} & & \downarrow{\scriptstyle H^n(\gamma)} & & \downarrow{\scriptstyle H^{n+1}(\alpha)} & & \\ \cdots & \longrightarrow & H^n(M_1^\bullet) & \longrightarrow & H^n(N_1^\bullet) & \longrightarrow & H^n(L_1^\bullet) & \xrightarrow{\delta} & H^{n+1}(N^\bullet) & \longrightarrow & \cdots . \end{array}$$

is commutative.

2. Homotopy

Definition. *i) Let $f, g : X_\bullet \to Y_\bullet$ be two morphisms of complexes. A chain homotopy from f to g is a collection of morphisms $s_n : X_n \to Y_{n+1}$ such that*

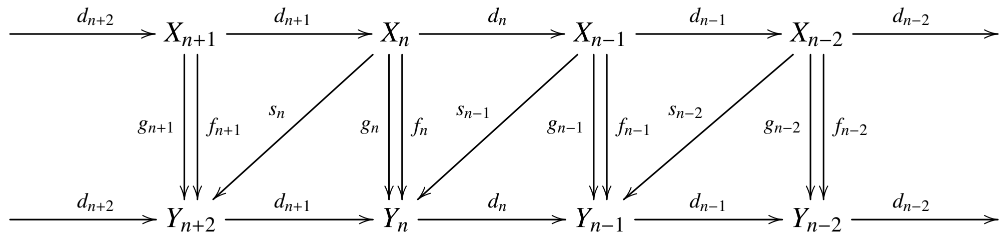
$$f_n - g_n = s_{n-1}d_n + d_{n+1}s_n, \qquad \forall n \in \mathbf{Z}.$$

We will write this property in the form $f - g = sd + ds$.

ii) We say that f and g are homotopic and write $f \simeq g$ if there exists a homotopy from f to g.

iii) A morphism $f : X \to Y$ is null homotopic if $f \simeq 0$. In this case there exists a homotopy s, called a contraction of f, such that $f = sd + ds$.

This data can be summarized by the following diagram:



Proposition 2.1. *If $f \simeq g$, then*

$$H_n(f) = H_n(g) \,:\, H_n(X_\bullet) \to H_n(Y_\bullet), \qquad \forall n \in \mathbf{Z}.$$

Proof. Let $x_n \in Z_n(X_\bullet)$. Then

$$f_n(x_n) - g_n(x_n) = d_{n+1}s_n(x_n) + s_{n-1}d_n(x_n) = d_{n+1}s_n(x_n) \in B_n(Y_\bullet).$$

Hence $\mathrm{cl}(f(x_n)) = \mathrm{cl}(g(x_n))$. □

Proposition 2.2. *The following properties hold true:*
i) $\simeq$ is an equivalence relation on the set $\mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(X_\bullet, Y_\bullet)$.
ii) Let $f_1, f_2 \,:\, X_\bullet \to Y_\bullet$ and $g \,:\, Y \to Z$. Assume that $f_1 \simeq f_2$ and $g_1 \simeq g_2$. Then $g_1 \circ f_1 \simeq g_2 \circ f_2$.

Proof. We leave the proof as an exercise. The proof of ii) can be divided into two parts:
a) Let $f_1, f_2 \,:\, X_\bullet \to Y_\bullet$ and $g \,:\, Y_\bullet \to Z_\bullet$. Assume that $f_1 \simeq f_2$. Then $g \circ f_1 \simeq g \circ f_2$.
b) Let $f \,:\, X_\bullet \to Y_\bullet$ and $g_1, g_2 \,:\, Y_\bullet \to Z_\bullet$. Assume that $g_1 \simeq g_2$. Then $g_1 \circ f \simeq g_2 \circ f$. □

Definition. *A morphism $f \,:\, X_\bullet \to Y_\bullet$ is a homotopy equivalence if there exists $g \,:\, Y_\bullet \to X_\bullet$ such that*

$$g \circ f \simeq \mathrm{id}_X, \qquad f \circ g \simeq \mathrm{id}_Y.$$

Proposition 2.3. *If $X_\bullet$ and $Y_\bullet$ are homotopically equivalent, then $H_n(X_\bullet) \simeq H_n(Y_\bullet)$ for all $n \in \mathbf{Z}$.*

Proof. From Proposition 2.1, it follows that $H_n(g) \circ H_n(f) = H_n(\mathrm{id}_X) = \mathrm{id}_{H_n(X)}$ and $H_n(f) \circ H_n(g) = H_n(\mathrm{id}_Y) = \mathrm{id}_{H_n(Y)}$. □

3. The mapping cone

In this section, we explain some important construction in the category of chain complexes. Let $f \,:\, X_\bullet \to Y_\bullet$ be a morphism of complexes. Set

$$c_n(f) := X_{n-1} \oplus Y_n, \qquad n \in \mathbf{Z}.$$

We define the morphisms

$$d_n \,:\, c_n(f) \to c_{n-1}(f),$$
$$d_n(x_{n-1}, y_n) = -(d_{n-1}(x_{n-1}), f(x_{n-1}) - d_n(y_n)).$$

It is easy to check that $c_\bullet(f) := (c_n(f), d_n)_{n\in\mathbf{Z}})$ is a cochain complex:

$$(3)\quad d\circ d(x,y) = -d(dx, f(x)-dy) = (d^2x, f\circ dx - d(f(x)-dy))$$
$$= (0, f\circ d(x) - d\circ f(x) + d^2y) = (0, f\circ d(x) - d\circ f(x)) = (0,0).$$

Definition. *The complex $c(f)$ is called the mapping cone of f.*

We will use the following notation: if $X_\bullet$ is a chain complex, we denote by $X[m]_\bullet$ and call the transition of $X_\bullet$ the complex defined as follows:

$$X[m]_n := X_{m+n}, \qquad d[m]_n = (-1)^m d_{m+n}.$$

It is clear that $H_n(X[m]_\bullet) = H_{m+n}(X_\bullet)$.

Proposition 3.1. *Let $f : X_\bullet \to Y_\bullet$ be a morphism of complexes.*
i) There is a short exact sequence

$$(4)\qquad 0 \to Y_\bullet \xrightarrow{\alpha} c_\bullet(f) \xrightarrow{\beta} X_\bullet[-1],$$

where $\alpha(y_n) = (0, y_n)$ and $\beta(x_{n-1}, y_n) = -x_{n-1}$.
ii) There exists a long exact sequence

$$\cdots \to H_n(Y_\bullet) \xrightarrow{H_n(\alpha)} H_n(c_\bullet(f)) \xrightarrow{H_n(\beta)} H_{n-1}(X_\bullet) \xrightarrow{H_{n-1}(f)} H_{n-1}(Y_\bullet) \to \cdots$$

Proof. i) The exactness of the exact sequence (4) is clear from definition.
ii) The long exact sequence in homology associated to the short exact sequence (4) reads:

$$\cdots \to H_n(Y_\bullet) \xrightarrow{H_n(\alpha)} H_n(c_\bullet(f)) \xrightarrow{H_n(\beta)} H_n(X_\bullet[-1]) \xrightarrow{\delta_n} H_{n-1}(Y_\bullet) \to \cdots$$

Note that $H_n(X_\bullet[-1]) = H_{n-1}(X_\bullet)$. Let $x_{n-1} \in Z_n(X_\bullet[-1]) = Z_{n-1}(X_\bullet)$. Take $z_n := (-x_{n-1}, 0) \in c_n(f)$. Then $\beta(z_n) = x_{n-1}$. We have $d_n(z_n) = (0, f(x_{n-1}))$ and therefore $\alpha(f(x_{n-1})) = d_n(z_n)$. By the definition of the connecting map δ, we obtain:

$$\delta_n(\mathrm{cl}(x_{n-1})) = \mathrm{cl}(f(x_{n-1})) = H_{n-1}(f)(\mathrm{cl}(x_{n-1})).$$

Hence $\delta_n = H_{n-1}(f)$ and the proposition is proved. □

Definition. *A morphism of complexes $f : X_\bullet \to Y_\bullet$ is a quasi-isomorphism if the induced morphisms $H_n(f) : H_n(X_\bullet) \to H_n(Y_\bullet)$ are isomorphisms for all n.*

In particular, a homotopy equivalence is a quasi-isomorphism by Proposition 2.3.

Corollary 3.2. *$f : X_\bullet \to Y_\bullet$ is a quasi-isomorphism if and only if $c_\bullet(f)$ is acyclic.*

4. Singular chain complexes

4.1. In this section, we discuss singular chain complexes and singular homology of topological spaces.

Definition.

1) *For each* $n \geqslant 0$, *the set*

$$\Delta^n = \{(t_0, t_1, \dots, t_n) \in \mathbf{R}^{n+1} \mid 0 \leqslant t_i \leqslant 1,\ \sum_{i=0}^{n} t_i = 1\}$$

is called the geometric n-simplex.

2) *The point*

$$e_k^{[n]} = (0, \dots, 0, 1, 0, \dots, 0), \qquad 0 \leqslant k \leqslant n$$

is called the k-vertice of Δ^n.

3) *The subset*

$$\{(t_0, t_1, \dots, t_n) \in \Delta^n \mid t_k = 0\}$$

is called the k-face of Δ^n.

For each integer $0 \leqslant k \leqslant n$, we have a map

$$\partial_n^k : \Delta^{n-1} \to \Delta^n,$$
$$(t_0, t_1, \dots, t_{n-1}) \mapsto (t_0, t_1, \dots, t_{k-1}, 0, t_k, \dots, t_{n-1}),$$

which identifies Δ^{n-1} with the k-face of Δ^n.

Let X be a topological space. For each $n \geqslant 0$ define:

$$C_n(X) := \text{free abelian group generated by all continuous } \varphi : \Delta^n \to X.$$

Set

$$d_n : C_n(X) \to C_{n-1}(X),$$
$$d_n(\varphi) := \sum_{k=0}^{n} (-1)^k \varphi \circ \partial_n^k.$$

Proposition 4.2. *One has* $d_{n-1} \circ d_n = 0$ *for all* $n \geqslant 1$.

Proof. Routine computation. □

Therefore we have the complex of abelian groups

$$\cdots \xrightarrow{d_{n+1}} C_n(X) \xrightarrow{d_n} C_{n-1}(X) \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} C_1(X) \xrightarrow{d_1} C_0(X) \xrightarrow{d_0} 0$$

called the singular chain complex of X.

Definition. *The n-th homology group of* $C_\bullet(X)$ *is called the n-th singular homology of X and is written* $H_n(X)$.

We summarize basic general properties of this construction.

1) (functoriality). For each n, the rule $X \to H_n(X)$ is a covariant functor

$$H_n : \mathbf{TSpaces} \to \mathbf{Ab}.$$

Namely for each continuous map $f : X \to Y$, we have natural maps

$$C_n(X) \to C_n(Y), \qquad \varphi \mapsto f \circ \varphi$$

which define a morphism of complexes $f_* : C_\bullet(X) \to C_\bullet(Y)$. Passing to homology, we obtain canonical morphisms of groups $H_n(f) : H_n(X) \to H_n(Y)$, which satisfy the required properties.

2) (dimension axiom or homology of a one-point set) One has:

$$H_n(\{\bullet\}) = \begin{cases} \mathbf{Z}, & \text{if } n = 0, \\ 0, & \text{if } n > 0. \end{cases}$$

3) (direct sum) If $X = X_1 \sqcup X_2$ is the disjoint union of X_1 and X_2, then

$$H_n(X) = H_n(X_1) \oplus H_n(X_2).$$

Recall that to continuous maps $f, g : X \to Y$ are homotopic if there exists a continuous map $F : X \times [0,1] \to Y$ such that $F(x,0) = f(x)$ and $F(x,1) = g(x)$. We write $f \simeq g$ if f and g are homotopic. A topological space X is contractible if the maps $\mathrm{id}_X : X \to X$ and $p : X \to \{x_0\} \subseteq X$ are homotopic for some (and therefore any) $x_0 \in X$.

4) (homotopy) If $f \simeq g : X \to Y$, then the maps $f_*, g_* : C_\bullet(X) \to C_\bullet(Y)$ are homotopic and therefore $H_n(f) = H_n(g) : H_n(X) \to H_n(Y)$ for all $n \geqslant 0$.

We formulate two corollaries of this property:

Corollary 4.3. *i) If X is contractible, then $H_0(X) = \mathbf{Z}$ and $H_n(X) = 0$ for $n \geqslant 1$.*

ii) One has $H_n(X \times [0,1]) = H_n(X)$ for all $n \geqslant 0$.

Proof. i) is clear.

ii) Consider the maps $i : X \to X \times [0,1]$, $i(x) = (x,0)$ and $f : X \times [0,1] \to X$, $f(x,t) = x$. Then $f \circ i = \mathrm{id}_X$ and therefore the composition

$$H_n(X) \xrightarrow{H_n(i)} H_n(X \times [0,1]) \xrightarrow{H_n(f)} H_n(X)$$

is the identity map. The composition

$$i \circ f : X \times [0,1] \to X \times [0,1],$$
$$i \circ f(x,t) = (x,0)$$

is homotopic to $\mathrm{id}_{X\times[0,1]}$. Namely the map

$$F : (X \times [0,1]) \times [0,1] \to X \times [0,1],$$
$$F((x,t),\xi) = (x, t\xi)$$

gives a homotopy $i \circ f \simeq \mathrm{id}_{X\times[0,1]}$. Therefore by the homotopy property the composition

$$H_n(X \times [0,1]) \xrightarrow{H_n(f)} H_n(X) \xrightarrow{H_n(i)} H_n(X \times [0,1])$$

is the identity map too. To sum up,

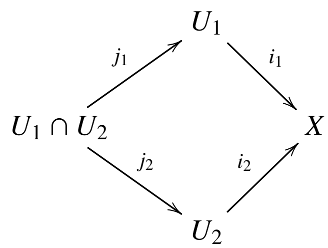
$$H_n(f) \circ H_n(i) = \mathrm{id}_X, \qquad H_n(i) \circ H_n(f) = \mathrm{id}_{X\times[0,1]},$$

and therefore $H_n(X \times [0,1]) = H_n(X)$. □

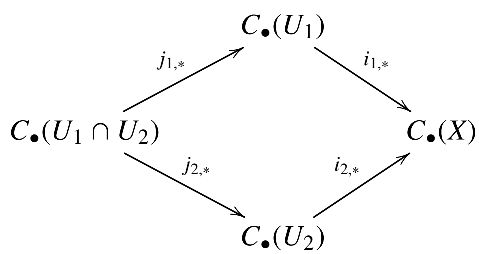
Assume now that $X = U_1 \cup U_2$. The following complexes can be naturally seen as subcomplexes of $C_\bullet(X)$:

$$C_\bullet(U_1 \cap U_2), \quad C_\bullet(U_1), \quad C_\bullet(U_2), \quad C_\bullet(U_1) + C_\bullet(U_2).$$

We have a diagram of inclusions



which induces a commutative diagram



The following sequence is exact:

$$0 \to C_\bullet(U_1 \cap U_2) \xrightarrow{\alpha} C_\bullet(U_1) \oplus C_\bullet(U_2) \xrightarrow{\beta} C_\bullet(U_1) + C_\bullet(U_2) \to 0,$$

where $\alpha(x) = (j_{1,*}(x), -j_{2,*}(x))$ and $\beta(x_1, x_2) = i_{1,*}(x_1) + i_{2,*}(x_2)$.

5) (Mayer–Vietoris exact sequence) Assume that $X \subseteq \mathring{U}_1 \cup \mathring{U}_2$. Then the inclusion $C_\bullet(U_1) + C_\bullet(U_2) \to C_\bullet(X)$ is a quasi-isomorphism, and we have a long exact sequence

$$\cdots \to H_{n+1}(X) \xrightarrow{\delta_{n+1}} H_n(U_1 \cap U_2) \xrightarrow{H_n(\alpha)} H_n(U_1) \oplus H_n(U_2) \xrightarrow{H_n(\beta)} H_n(X) \xrightarrow{\delta_n} \cdots$$
$$\cdots \to H_0(U_1 \cap U_2) \xrightarrow{H_0(\alpha)} H_0(U_1) \oplus H_0(U_2) \xrightarrow{H_0(\beta)} H_0(X) \to 0.$$

For the general theory, it is important to attach homology to each pair of topological spaces (A, X), where $A \subseteq X$. Set

$$C_\bullet(X, A) := C_\bullet(X)/C_\bullet(A)$$

and define the homology groups $H_n(X, A)$ as the homology of $C_\bullet(X, A)$. Then:

6) (exactness) We have a long exact sequence

$$\cdots \to H_{n+1}(X, A) \to H_n(A) \to H_n(X) \to H_n(X, A) \to \cdots$$
$$\cdots \to H_0(A) \to H_0(X) \to H_0(X, A) \to 0.$$

7) (excision) If (X, A) is a pair and $U \subset A$ is such that the closure of U is contained in $\mathring{A}$, then the inclusion map $(X \setminus U, A \setminus U) \to (X, A)$ induces isomorphisms

$$H_n(X \setminus U, A \setminus U) \simeq H_n(X, A).$$

We remark that property 6) follows directly frome definitions. Property 7) is more delicate.

Remark 4.4. *Properties 1-4), 6), 7) are known as Eilenberg–Steenrod axioms. It can be shown that they formally imply 5).*

4.5. In this section, we compute the homology of sphere and deduce from this computation a short proof of Brouwer fixed point theorem.

The d-dimensional sphere can be defined as the topological space

$$S_d := \{(x_0, x_1, \dots, x_d) \mid \sum_{i=0}^{d} x_i^2 = 1\}.$$

Note that S_d is the boundary of the $d+1$-dimensional disk

$$D_{d+1} := \{(x_0, x_1, \dots, x_d) \mid \sum_{i=0}^{d} x_i^2 \leqslant 1\}.$$

Theorem 4.6. *One has*

$$H_n(S_d) := \begin{cases} \mathbf{Z}, & \text{if } n = 0, d \\ 0, & \text{otherwise.} \end{cases}$$

Proof. We can write $S_d = U_1 \cup U_2$, where U_1 and U_2 are contractible and $U_1 \cap U_2$ is homeomorphic to $S_{d-1} \times [0, 1]$. Namely, we can take:

$$U_1 = \{(x_0, x_1, \dots, x_d) \in S_d \mid x_d > -\varepsilon\},$$
$$U_2 = \{(x_0, x_1, \dots, x_d) \in S_d \mid x_d < \varepsilon\}$$

for some small $\varepsilon > 0$. Hence

$$H_n(S_d) \simeq H_{n-1}(U_1 \cap U_2) \simeq H_{n-1}(S_{d-1}), \qquad n \geqslant 2. \tag{5}$$

For $n = 1$ we have an exact sequence

$$0 \to H_1(S_d) \to H_0(U_1 \cap U_2) \to H_0(U_1) \oplus H_0(U_2) \to H_0(S_d) \to 0,$$

which can be written as

$$0 \to H_1(S_d) \to H_0(S_{d-1}) \to \mathbf{Z} \oplus \mathbf{Z} \to \mathbf{Z} \to 0.$$

If $d > 1$, then $H^0(S_{d-1}) \simeq Z$ and this exact sequence shows that $H^1(S_d) = 0$. If $d = 1$, we have $H^0(S_0) = \mathbf{Z}^2$, and we obtain that $H^1(S_1) \simeq \mathbf{Z}$. The theorem can be eqsily deduce from this computation together with formula (5). □

As an application, we prove:

Theorem 4.7 (fixed point theorem)**.** *Each continuous map $\varphi : D_{d+1} \to D_{d+1}$ ($d \geqslant 0$) has a fixed point.*

Proof. We prove this theorem by contradiction. We will consider S_d as the boundary of D_{d+1}. Assume that $\varphi(x) \neq x$ for all $x \in D_{d+1}$. Consider the ray L with the initial point $\varphi(x)$ passing through x and consider the unique point $f(x) \in L \cap S_d$

such that $f(x) \neq \varphi(x)$. Then $x \mapsto f(x)$ defines a continuous map $f : D_{d+1} \to S_d$. We remark that f is a retraction of D_{d+1} on S_d, namely

$$f(x) = x, \qquad x \in S_d.$$

The composition $S_d \xrightarrow{i} D_{d+1} \xrightarrow{f} S_d$ is the identity map, and therefore the induced map on homology

$$H_d(S_d) \xrightarrow{H_d(i)} H_d(D_{d+1}) \xrightarrow{H_d(f)} H_d(S_d)$$

is also the identity morphism. If $d \geqslant 1$, then $H_d(D_{d+1}) = 0$ and we obtain that $H_d(S_d) = 0$, which contradicts Theorem 4.6. If $d = 0$, then $H_0(D_1) \simeq \mathbf{Z}$, $H_0(S_0) \simeq \mathbf{Z}^2$, and we obtain a contradiction again. □

5. Cohomology of groups

5.1. Let G be a group. We denote by $\mathbf{Z}[G]$ the group algebra of G over $\mathbf{Z}$. The elements of $\mathbf{Z}[G]$ are formal sums

$$\sum_{g \in G} a_g g, \qquad a_g \in \mathbf{Z}, \qquad \text{almost all } a_g \text{ are zero.}$$

The addition and multiplication are given by

$$\sum_{g\in G} a_g g + \sum_{g \in G} b_g g = \sum_{g\in G}(a_g + b_g) g,$$

$$\left(\sum_{s\in G} a_s s\right)\left(\sum_{t \in G} b_t t\right) = \sum_{g\in G} c_g g, \qquad \text{where } c_g := \sum_{st=g} a_s b_t.$$

Definition. *A (left) G-module is an abelian group M equipped with a left action $G \times M \to M$ of the group G satisfying the following properties:*

i) *$em = m$, for all $m \in M$.*

ii) *$(g_1g_2)m = g_1(g_2m)$ for all $g_1, g_2 \in G$ and $m \in M$.*

iii) *$g(m_1 + m_2) = gm_1 + gm_2$ for all $g \in G$ and $m_1, m_2 \in M$.*

If M is a G-module, it is equipped with a natural structure of a left $\mathbf{Z}[G]$-module given by

$$\left(\sum_{g\in G} a_g g\right) m = \sum_{g \in G} a_g (gm). \tag{6}$$

Conversely, each $\mathbf{Z}[G]$-module M can be considered as a G-module. Formula (6) shows that these structures are equivalent.

Definition. *We set*

$$M^G := \{m \in M \mid \forall g \in G, gm = m\}$$

and call it the invariant subgroup of M.

Let M be a left G-module. For each $n \geqslant 0$, set $G^n = \underbrace{G \times G \times \cdots \times G}_{n}$ and define:

$$C^n(G, M) = \{\text{maps } f : G^n \to M\}.$$

We remark that $C^n(G, M)$ have natural structure of abelian group. Set:

$$d^n : C^n(G, M) \to C^{n+1}(G, M),$$

$$\begin{aligned} d^n(f)(g_1, g_2, \ldots, g_{n+1}) = & g_1 f(g_2, \ldots, g_{n+1}) \\ & + \sum_{i=1}^{n} (-1)^i f(g_1, \ldots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \ldots, g_{n+1}) \\ & (-1)^{n+1} f(g_1, g_2, \ldots, g_n). \end{aligned}$$

Proposition 5.2. *For each $n \geqslant 0$, one has $d^{n+1} \circ d^n = 0$.*

Proof. The proof is omitted. □

From this proposition it follows that

$$0 \to C^0(G, M) \xrightarrow{d^0} C^1(G, M) \xrightarrow{d^1} C^2(G, M) \xrightarrow{d^2} \cdots$$

is a cochain complex.

Definition. *The n-th cohomology of the complex $C^\bullet(G, M)$ is called the n-th cohomology of G with coefficients in M and is written $H^n(G, M)$.*

We can write:

$$\begin{aligned} & Z^n(G, M) := \ker(d^n), \qquad B^n(G, M) := \operatorname{Im}(d^{n-1}), \\ & H^n(G, M) = Z^n(G, M)/B^n(G, M). \end{aligned}$$

Below, we summarize some properties of these groups.

1) $H^0(G, M) = M^G$.

Proof. One has $C^0(G, M) = M$. For each $m \in M$, the map $d^0(m) \in C^1(G, M)$ is given by

$$d^0(m)(g) = gm - m.$$

Therefore $\ker(d^0) = \{m \in M \mid \forall g \in G,\ gm - m = 0\} = M^G$. □

2) One has:

$$B^1(G, M) = \{f : G \to M \mid f(g) = gm - m \text{ for some } m \in M\},$$
$$Z^1(G, M) = \{f : G \to M \mid f(g_1 g_2) = g_1 f(g_2) + f(g_1)\}.$$

The elements of $Z^1(G, M)$ are called crossed homomorphisms.

Proof. The first formula follows from the computation of $d^0(m)$. The second formula follows directly from definitions. □

3) For any trivial G-module M one has:

$$H^1(G, M) = \mathrm{Hom}(G, M).$$

This follows from 2).

4) Let

$$0 \to M' \to M \to M'' \to 0$$

be a short exact sequence of G-modules. Then it induces a long exact sequence of cohomology:

$$0 \to H^0(G, M') \to H^0(G, M) \to H^0(G, M'') \xrightarrow{\delta} H^1(G, M') \to H^1(G, M) \to \cdots$$
$$\to H^{n-1}(G, M'') \xrightarrow{\delta} H^n(G, M') \to H^n(G, M) \to H^n(G, M'') \xrightarrow{\delta} \cdots$$

Proof. It is easy to see that the short exact sequence of modules induces a short exact sequence of complexes:

$$0 \to C^\bullet(G, M') \to C^\bullet(G, M) \to C^\bullet(G, M'') \to 0.$$

Now we can apply Theorem 1.5. □

5.3. In this section, we give some interpretation of the second cohomology group $H^2(G, -)$.

Definition. *Let G be a group and A be an abelian group. An extension of G by A is an exact sequence of groups*

$$0 \to A \xrightarrow{i} N \xrightarrow{\pi} G \to 1. \tag{7}$$

In other words, A ican be identified with a normal subgroup of N and $N/A \simeq G$.

Two extensions of G by A are equivalent if there exists an isomorphism φ such that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & N & \longrightarrow & G & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle\varphi} & & \| & & \\ 0 & \longrightarrow & A & \longrightarrow & N' & \longrightarrow & G & \longrightarrow & 0 \end{array}$$

commutes.

We will write the group law on A additively and the group law on G multiplicatively.

Definition. *Let $f : X \to Y$ be a surjective morphism in some category $\mathcal{A}$. A section of f is a morphism $s : Y \to X$ such that $f \circ s = \mathrm{id}_Y$.*

Each extension equips A with the structure of a left G-module defined as follows. Choose a set theoretic section $s : G \to N$ of π (i.e. s is a map of sets such that $\pi \circ s = \mathrm{id}_G$). The action of G on A will be defined by the formula:

$$ga := i^{-1}\left(s(g) \cdot i(a) \cdot s(g)^{-1}\right). \tag{8}$$

It is easy to see that the definition does not depend on the choice of s.

Definition. *The extension* (7) *is split if* π *has a section in the category of groups, i.e. if there exists a morphism of groups* $s : G \to N$ *such that* $\pi \circ s = \mathrm{id}_G$.

Proposition 5.4. *The following conditions are equivalent:*
i) *The extension* (7) *is split.*
ii) *The extension* (7) *is equivalent to the extension*

$$0 \to A \xrightarrow{\alpha} A \rtimes G \xrightarrow{\beta} G \to 1$$

with $\alpha(a) = (a, 0)$ *and* $\beta(a, g) = g$.

Proof. ii) $\Rightarrow$ i). Recall that

$$N = A \rtimes G = \{a, g) \mid a \in A, g \in G\},$$
$$(a_1, g_1)(a_2, g_2) = (a_1 + g_1 a_2, g_1 g_2).$$

It is clear that , the map $s : G \to A \rtimes G$, $s(g) = (0, g)$ is a morphism of groups such that $\pi \circ \beta = \mathrm{id}_G$.

i) $\Rightarrow$ ii). Conversely, assume that $s : G \to N$ is a morphism such that $\pi \circ s = \mathrm{id}_G$. Consider the maps

$$\varphi : N \to A \rtimes G,$$
$$\varphi(n) = (a, g), \qquad \text{where } g := \pi(n) \text{ and } a := i^{-1}(n \cdot (s \circ \pi)(n)^{-1}).$$

and

$$\psi : A \rtimes G \to N$$
$$\psi(a, g) = i(a)s(g).$$

Then is is easy to see that φ and ψ are morphisms of groups, which are inverse to each other. □

Let A be a left G-module. We want to classify the extensions of G by A (viewed as an abelian group) in which the induced G-module structure (8) coincides with the given G-module structure. Let $\mathrm{Ext}^1(G, A)$ denote the set of equivalence classes of such extensions.

For such extension (7), choose an arbitrary set-theoretic section $s : G \to N$ of π. For all $g_1, g_2 \in G$, $s(g_1)s(g_2)s(g_1g_2)^{-1} \in \ker(\pi)$ and we define $f(g_1, g_2) \in A$ setting

$$i \circ f(g_1, g_2) = s(g_1)s(g_2)s(g_1g_2)^{-1}.$$

Theorem 5.5. *The following statements hold true:*
i) *For each section s, one has* $f(g_1, g_2) \in Z^2(G, A)$. *The class* $\mathrm{cl}(f) \in H^2(G, A)$ *does not depend on the choice of s.*
ii) *Conversely, for any* $f(g_1, g_2) \in Z^2(G, A)$, *we denote by N the cartesian product* $A \times G$ *equipped with the composition law*

$$(a_1, g_1)(a_2, g_2) = (a_1 + g_1 a_2 + f(g_1, g_2), g_1 g_2), \qquad a_i \in A,\ g_i \in G.$$

Then N is a group and the morphisms $i : A \to N$, $i(a) = (a, 1)$ *and* $\pi : N \to G$, $\pi(a, g) = g$ *define and extension which depends only on the class of f in* $H^2(G, A)$.
iii) *The previous constructions define a bijection*

$$\mathrm{Ext}^1(G, A) \simeq H^2(G, A).$$

Proof. All statements of the theorem can be checked by straightforward computations. The details are omitted. □

5.6. In this subsection, we make a first step toward the study of Galois cohomology. Let K be a field and let L/K be a finite Galois extension. Then $\mathrm{Gal}(L/K)$ acts on the multiplicative group L^* of L.

Theorem 5.7 (Hilbert's theorem 90)**.** *One has*

$$H^1(\mathrm{Gal}(L/K), L^*) = \{1\}.$$

Proof. Set $G := \mathrm{Gal}(L/K)$. Let $f \in Z^1(G, L^*)$. Take $x \in L^*$ and consider the element

$$\theta(x) := \sum_{\tau\in G} f(\tau) \cdot \tau(x).$$

An easy computation using the property $f(g\tau) = gf(\tau) \cdot f(g)$ shows that

$$g(\theta(x)) = \sum_{\tau\in G} gf(\tau) \cdot g\tau(x) = \frac{1}{f(g)} \sum_{\tau\in G} f(g\tau) \cdot g\tau(x) = \frac{\theta(x)}{f(g)}.$$

Assume that $\theta(x) \neq 0$. Then setting $c := \theta(x)^{-1}$ we obtain that $f(g) = g(c)c^{-1}$ and therefore $f \in B^1(G, L^*)$. This proves the theorem.

It remains to show that $\theta(x) \neq 0$ for some $x \in L$. This can be easily proved by contradiction. Namely, let $\{x_1, \dots, x_n\}$ be a basis of L over K and let $G = \{\tau_1, \dots, \tau_n\}$. Assume that

$$\sum_{j=1}^{n} f(\tau_j) \cdot \tau_j(x_i) = 0, \qquad \forall i = 1, 2, \dots, n.$$

Then $(f(\tau_1), \dots, f(\tau_n))$ is a solution of the system of linear equations

$$\sum_{j=1}^{n} \tau_j(x_i) X_j = 0, \qquad \forall i = 1, 2, \dots, n.$$

Since the discriminant of a separable extension is nonzero, we have

$$\det\left(\tau_j(x_i)\right) \neq 0.$$

Therefore the above system has only the trivial solution and $f(\tau_i) = 0$ for all $1 \leqslant i \leqslant n$. This contradiction shows that $\theta(x) \neq 0$ for some $x \in L$. □

CHAPTER 3

Derived functors

1. Projective resolutions

Let $\mathcal{A}$ be an abelian category.

Definition. *An object $P \in \mathbf{Obj}(\mathcal{A})$ is projective if it satisfies the following property: given an epimorphism $g : Y \to Z$ and a morphism $\pi : P \to Z$, there exists a morphism $\pi' : P \to Y$ such that $g \circ \pi' = \pi$:*

$$\begin{array}{ccccc} & & P & & \\ & \swarrow^{\pi'} & \downarrow^{\pi} & & \\ Y & \xrightarrow{g} & Z & \longrightarrow & 0 \end{array}$$

Proposition 1.1. *The following assertions are equivalent:*

i) *P is projective.*
ii) *The functor $h_P := \mathrm{Hom}(P, -)$ is exact.*

Proof. i) $\Rightarrow$ ii).

Assume that P is projective. Let

$$0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$$

be a short exact sequence. By theorem 6.5, the functor $\mathrm{Hom}(P, -)$ is left exact. Hence

$$0 \to \mathrm{Hom}(P, X) \xrightarrow{f_*} \mathrm{Hom}(P, Y) \xrightarrow{g_*} \mathrm{Hom}(P, Z)$$

is exact. We only need to prove that g_* is surjective. For any $\pi \in \mathrm{Hom}(P, Z)$, there exists π' such that the diagram

$$\begin{array}{ccccc} & & P & & \\ & \swarrow^{\pi'} & \downarrow^{\pi} & & \\ Y & \xrightarrow{g} & Z & \longrightarrow & 0 \end{array}$$

commutes. Then $g_*(\pi') = \pi$, and the surjectivity of g_* is proved.

ii) $\Rightarrow$ i).

Assume that the functor h_P is exact. Consider the diagram

$$\begin{array}{ccccc} & & P & & \\ & & \downarrow^{\pi} & & \\ Y & \xrightarrow{g} & Z & \longrightarrow & 0 \end{array}$$

Set $X := \ker(g)$ and consider the exact sequence

$$0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0.$$

From the exactness of h_P, it follows that the map

$$\mathrm{Hom}(P, Y) \xrightarrow{g_*} \mathrm{Hom}(P, Z)$$

is surjective. Therefore there exists $\pi' : P \to Y$ such that $g \circ \pi' = g_*(\pi') = \pi$. This shows that P is projective. □

Definition. *An abelian category $\mathcal{A}$ has enough projectives if for any $X \in$ **Obj**$(\mathcal{A})$ there exists a projective $P \in$ **Obj**$(\mathcal{A})$ together with an epimorphism $\pi : P \to X$.*

Proposition 1.2. *Let A be a ring.*

i) *Each free A-module is projective. An A-module is projective if and only if it is a direct summand of a free module.*

ii) *The category $A -$ **Mod** has enough projectives.*

We will first prove an auxiliary lemma, which characterizes direct sums in terms of split exact sequences.

Definition. *A short exact sequence in an abelian category*

$$0 \longrightarrow X \xrightarrow{\alpha} Y \underset{s}{\overset{\beta}{\rightleftarrows}} Z \longrightarrow 0$$

splits if there exists a section s of the morphism β, i.e. a morphism $s : Z \to Y$ such that $\beta \circ s = \mathrm{id}_Z$.

Lemma 1.3. *The following conditions are equivalent:*

1) *A short exact sequence*

$$0 \longrightarrow X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \longrightarrow 0$$

splits.

2) *There exists an isomorphism $i : Y \simeq X \oplus Z$, such that the diagram*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X & \xrightarrow{\alpha} & Y & \xrightarrow{\beta} & Z & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle i} & & \| & & \\
0 & \longrightarrow & X & \xrightarrow{q_X} & X \oplus Z & \xrightarrow{p_Z} & Z & \longrightarrow & 0
\end{array}$$

commutes.

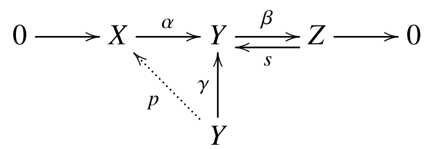
Proof of the lemma. 2) ⇒ 1). By the definition of the direct sum, there there exists $q_Z : Z \to X \oplus Z$ such that $p_Z \circ q_Z = \mathrm{id}_Z$. Then $s = i^{-1} \circ q_Z : Z \to Y$ is a section of β.

1) ⇒ 2). Assume that s is a section of β. We will show that Y equipped with the morphisms $q_X := \alpha : X \to Y$ and $q_Z := s : Z \to Y$ satisfies the universal property of a direct sum.

Consider the map

$$\gamma := \mathrm{id}_Y - s \circ \beta \;:\; Y \to Y.$$

This map sits in the diagram



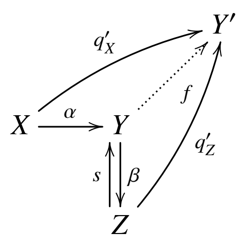
Then

$$\beta \circ \gamma = \beta \circ (\mathrm{id}_Y - s \circ \beta) = \beta - (\beta \circ s) \circ \beta = 0.$$

Since $(X, \alpha) = \ker(\beta)$, this implies that there exists a unique $p \;:\; Y \to X$ such that

$$\alpha \circ p = \gamma = \mathrm{id}_Y - s \circ \beta.$$

Assume that we have an object Y' with morphisms $q'_X \;:\; X \to Y'$ and $q'_Z \;:\; Z \to Y'$. Consider the diagram



Set $f = q'_X \circ p + q'_Z \circ \beta$. Then

(9) $\quad f \circ \alpha = (q'_X \circ p + q'_Z \circ \beta) \circ \alpha = q'_X \circ p \circ \alpha + q'_Z \circ \beta \circ \alpha = q'_X \circ p \circ \alpha = q'_X.$

Moreover,

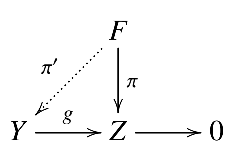
$$\alpha \circ (p \circ s) = (\alpha \circ p) \circ s = (id_Y - s \circ \beta) \circ s = s - s \circ (\beta \circ s) = 0.$$

Since α in monic, this implies that $p \circ s = 0$. Therefore

(10) $\quad f \circ s = (q'_X \circ p + q'_Z \circ \beta) \circ s = q'_X \circ p \circ s + q'_Z \circ \beta \circ s = q'_Z.$

Formulas (9) and (10) show that Y satisfies the universal property and therefore is a direct sum of X and Z. □

Proof of Proposition 1.2. a) Let F be a free module. Fix a base $\{e_i\}_{i \in I}$ of F. Then $F = \bigoplus_{i \in I} A e_i$. Consider the diagram



For each $i \in I$, set $z_i = \pi(e_i)$ and choose $y_i \in Y$ such that $g(y_i) = z_i$. Then the map $\pi' \;:\; F \to Y$ defined by

$$\pi'\left(\sum a_i e_i\right) = \sum a_i y_i$$

satisfies the property $g \circ \pi' = \pi$.

b) Let M be an A-module. By the universal property of free modules, there exists a free module F together with an epimorphism $\pi : F \to M$. Namely, choose a system $\{m_i\}_{i\in I}$ of generators of M and set $F = \bigoplus_{i\in I} Ae_i$. Then the map $\pi : F \to M$ defined by $\pi(\sum a_i e_i) = \sum a_i m_i$ is a well defined epimorphism. Since F is projective, part ii) is proved.

c) Assume that P is projective. Then there exists a free module F together with a surjection $\pi : F \to P$. Consider the diagram:

$$\begin{array}{ccccc} & & P & & \\ & \swarrow^{s} & \| & & \\ F & \xrightarrow{\pi} & P & \longrightarrow & 0 \end{array}$$

Since P is projective, there exists a morphism $s : P \to F$ such that $\pi \circ s = \mathrm{id}_P$. This implies that P is a direct summand of F.

d) Conversely, assume that P is a direct summand of a free module F. Then $F = P \oplus P'$ for some module P'. Assume that we have a diagram of the form

$$\begin{array}{ccccc} & & P & & \\ & & \downarrow{\scriptstyle \pi} & & \\ Y & \xrightarrow{g} & Z & \longrightarrow & 0 \end{array}$$

Consider the diagram

$$\begin{array}{ccccc} & & F = P \oplus P' & & \\ & \swarrow^{h} & \downarrow{\scriptstyle \pi_F} & & \\ Y & \xrightarrow{g} & Z & \longrightarrow & 0 \end{array}$$

where $\pi_F(x, x') = \pi(x)$ for any $(x, x') \in P \oplus P'$. Since F is projective, there exists $h : F \to Y$ such that $\pi_F = g \circ h$. Set $\pi' := h|_P$. Then it is easy to see that $\pi = g \circ \pi'$. □

Exercise 8. *Show that the category of torsion abelian groups has no projective nonzero objects.*

Definition. *i) A left resolution of $M \in$ **Obj**$(\mathcal{A})$ is a sequence*

$$P_\bullet : \qquad \ldots \to P_3 \to P_2 \to P_1 \to P_0$$

together with a morphism $\varepsilon : P_0 \to M$ such that the sequence

$$\ldots \to P_3 \to P_2 \to P_1 \to P_0 \xrightarrow{\varepsilon} M \to 0$$

is exact. The morphism ε is called an augmentation morphism.

ii) If all P_i are projective, $P_\bullet$ is called a projective resolution of M.

This data can be represented by the diagram

$$P_\bullet \xrightarrow{\varepsilon} M \to 0.$$

Proposition 1.4. *Assume that $\mathcal{A}$ has enough projectives. Then each $M \in$* **Obj**$(\mathcal{A})$ *has a projective resolution.*

Proof. Let $M \in \mathbf{Obj}(\mathcal{A})$. We construct a projective resolution of M inductively. Since $\mathcal{A}$ has enough projectives, there exists a projective object P_0 together with an epimorphism $\varepsilon : P_0 \to M$. Hence the sequence

$$P_0 \xrightarrow{\varepsilon} M \to 0$$

is exact. Now assume that we have an exact sequence

$$P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \ldots \to P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} M \to 0,$$

where all P_i are projective. Let $X := \ker(d_n)$. Then there exists an epimorphism $P_{n+1} \xrightarrow{\pi} X$, where P_{n+1} is projective. Consider the composition

$$d_{n+1} : P_{n+1} \xrightarrow{\pi} X \rightarrowtail P_n.$$

Then the upper row of the diagram

$$\begin{array}{ccccccccccccc} P_{n+1} & \xrightarrow{d_{n+1}} & P_n & \xrightarrow{d_n} & P_{n-1} & \xrightarrow{d_{n-1}} & \ldots & \xrightarrow{d_1} & P_0 & \xrightarrow{\varepsilon} & M & \longrightarrow & 0, \\ & {\scriptstyle\pi}\searrow & \uparrow & & & & & & & & & & \\ & & X & & & & & & & & & & \end{array}$$

is exact. □

Corollary 1.5. *In the category of left (respectively right) A-modules, each object has a free resolution.*

Proof. From the proof of Proposition 1.2, it follows that we can take P_i free in the above construction. □

Proposition 1.6. *Consider the diagram*

$$\begin{array}{ccccccccccc} \ldots & \longrightarrow & P_2 & \xrightarrow{d_2} & P_1 & \xrightarrow{d_1} & P_0 & \xrightarrow{\varepsilon} & M & \longrightarrow & 0 \\ & & \vdots{\scriptstyle f_2} & & \vdots{\scriptstyle f_1} & & \vdots{\scriptstyle f_0} & & \downarrow{\scriptstyle f} & & \\ \ldots & \longrightarrow & Q_2 & \xrightarrow{d_2} & Q_1 & \xrightarrow{d_1} & Q_0 & \xrightarrow{\varepsilon} & N & \longrightarrow & 0, \end{array}$$

where $P_\bullet$ is a projective resolution of M and $Q_\bullet$ is a (not necessarily projective) resolution of N. Then:

i) *There exists a morphism $f_\bullet = (f_n)_{n\geqslant 0} : P_\bullet \to Q_\bullet$ such that the resulting diagram commutes.*
ii) *The morphism $f_\bullet$ is unique up to a chain homotopy.*

Proof. i) We construct the morphisms $f_n : P_n \to Q_n$ inductively. Consider the diagram

$$\begin{array}{ccc} & & P_0 \\ & {\scriptstyle f_0}\swarrow & \downarrow{\scriptstyle f\circ\varepsilon} \\ Q_0 & \xrightarrow{\varepsilon} & N \end{array}$$

Since P_0 is projective, there exists $f_0 : P_0 \to Q_0$ which makes the diagram commute. This gives us a commutative diagram

$$\begin{array}{ccccc} P_0 & \xrightarrow{\varepsilon} & M & \longrightarrow & 0 \\ \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f} & & \\ Q_0 & \xrightarrow{\varepsilon} & N & \longrightarrow & 0. \end{array}$$

Assume that the morphisms $f_0, \ldots, f_n$ are constructed. We have a diagram

$$\begin{array}{ccccccc} \longrightarrow & P_{n+1} & \xrightarrow{d_{n+1}} & P_n & \xrightarrow{d_n} & P_{n-1} \\ & \downarrow{\scriptstyle f_{n+1}} & \searrow{\scriptstyle g} & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} \\ \longrightarrow & Q_{n+1} & \xrightarrow{d_{n+1}} & Q_n & \xrightarrow{d_n} & Q_{n-1} \end{array}$$

Here $g := f_n \circ d_{n+1}$. From the commutativity of the diagram it follows that $d_n \circ g = 0$, and therefore g factorizes through $\ker(d_n)$:

$$g : P_{n+1} \xrightarrow{\pi} \ker(d_n) \rightarrowtail Q_n.$$

Consider the diagram

$$\begin{array}{ccc} & & P_{n+1} \\ & \swarrow{\scriptstyle f_{n+1}} & \downarrow{\scriptstyle \pi} \\ Q_{n+1} & \twoheadrightarrow & \mathrm{Im}(d_{n+1}) \end{array}$$

Since P_{n+1} is projective, there exists $f_{n+1} : P_{n+1} \to Q_{n+1}$ which makes the diagram commute. This proves the existence of $f_\bullet$.

ii) Assume that we have another morphism $g : P_\bullet \to Q_\bullet$ such that the diagram

$$\begin{array}{ccccc} P_\bullet & \xrightarrow{\varepsilon} & M & \longrightarrow & 0 \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} & & \\ Q_\bullet & \xrightarrow{\varepsilon} & N & \longrightarrow & 0 \end{array}$$

commutes. We will construct a homotopy $f \simeq g$ inductively. Consider the diagram

$$\begin{array}{ccccccc} P_1 & \longrightarrow & P_0 & \xrightarrow{\varepsilon} & M & \longrightarrow & 0 \\ & \swarrow{\scriptstyle s_0} & {\scriptstyle f_0}\downarrow\downarrow{\scriptstyle g_0} & & \downarrow{\scriptstyle f} & & \\ Q_1 & \xrightarrow[d_1]{} & Q_0 & \xrightarrow{\varepsilon} & N & \longrightarrow & 0. \end{array}$$

Since $\varepsilon \circ f_0 = \varepsilon \circ g_0 = f \circ \varepsilon$, we have $\varepsilon \circ (f_0 - g_0) = 0$. Therefore we have a commutative diagram

$$\begin{array}{ccc} P_0 & \xrightarrow{f_0 - g_0} & \ker(\varepsilon) \\ \downarrow{\scriptstyle s_0} & & \downarrow{\scriptstyle \sim} \\ Q_1 & \overset{}{\underset{d_1}{\twoheadrightarrow}} & \mathrm{Im}(d_0), \end{array}$$

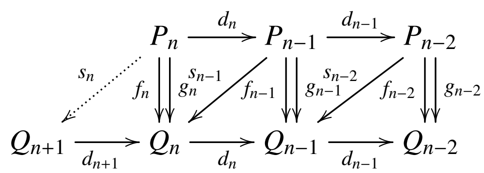
where the right vertical map is an isomorphism. Since P_0 is projective, there exists a map $s_0 : P_0 \to Q_1$ such that the diagram commutes, and we obtain that

$$f_0 - g_0 = d_0 \circ s_0.$$

Assume that we have morphisms $s_0 : P_0 \to Q_1$, $s_1 : P_1 \to Q_2$, …, $s_{n-1} : P_{n-1} \to Q_n$ such that

$$f_i - g_i = s_{i-1} \circ d_i + d_{i+1} \circ s_i, \qquad 0 \leqslant i \leqslant n - 1.$$

We summarize these data in the diagram



From the commutativity of this diagram, we have

$$d_n \circ g_n = g_{n-1} \circ d_n, \qquad d_n \circ f_n = f_{n-1} \circ d_n,$$

and therefore

$$d_n \circ (f_n - g_n) = (f_{n-1} - g_{n-1}) \circ d_n.$$

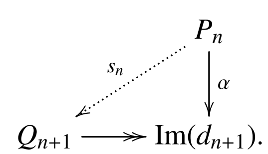
Using the identity $f_{n-1} - g_{n-1} = s_{n-2} \circ d_{n-1} + d_n \circ s_{n-1}$ we obtain that

$$d_n \circ (f_n - g_n) = s_{n-2} \circ d_{n-1} \circ d_n + +d_n \circ s_{n-1} \circ d_n = d_n \circ s_{n-1} \circ d_n.$$

Hence

$$d_n \circ (f_n - g_n - s_{n-1} \circ d_n) = 0.$$

Therefore the map $\alpha := f_n - g_n - s_{n-1} \circ d_n$ factorizes through $\ker(d_n) \simeq \mathrm{Im}(d_{n+1})$, and we have a diagram



Since P_n is projective, there exists $s_n : P_n \to Q_{n+1}$ such that

$$d_{n+1} \circ s_n = \alpha = f_n - g_n - s_{n-1} \circ d_n$$

Writing this formula in the form $f_n - g_n = d_{n+1} \circ s_n + s_{n-1} \circ d_n$ we see that s_n satisfies the required property. □

The following proposition can be proved by similar arguments, and we omit the proof:

Proposition 1.7. *Let*

$$0 \to M' \to M \to M'' \to 0$$

be a short exact sequence. Let $P'_\bullet$ and $P''_\bullet$ be projective resolutions of M' and M'' respectively. Then there exists a projective resolution $P_\bullet$ of M such that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P'_\bullet & \longrightarrow & P_\bullet & \longrightarrow & P''_\bullet & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \varepsilon} & & \downarrow{\scriptstyle \varepsilon} & & \downarrow{\scriptstyle \varepsilon} & & \\ 0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \end{array}$$

commutes.

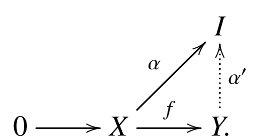
Exercise 9. *Let k be a field and $A := \mathrm{M}_n(k)$ the ring of $n \times n$ matrices with coefficients in k. Assume that $n \geqslant 2$. Give an example of an A-module that is projective but not free.*

Exercise 10. *Let $(P_i)_{i\in I}$ be a family of projective objects. Show that if the coproduct $\coprod_{i\in I} P_i$ exists, then it is projective.*

2. Injective resolutions

Let $\mathcal{A}$ be an abelian category.

Definition. *i) An object $I \in \mathbf{Obj}(\mathcal{A})$ is injective if it satisfies the following property: given an morphism $\alpha : X \to I$ and a monic morphism $f : X \to Y$, there exists a morphism $\alpha' : Y \to I$ such that $\alpha' \circ f = \alpha$:*



ii) The category $\mathcal{A}$ has enough injectives if for any $X \in \mathbf{Obj}(\mathcal{A})$ there exists an injective object I together with a monic morphism $X \to I$.

We remark that $I \in \mathbf{Obj}(\mathcal{A})$ is injective if and only if $I^\circ \in \mathbf{Obj}(\mathcal{A}^\circ)$ is projective.

Proposition 2.1. *The following assertions are equivalent:*

i) *$I \in \mathbf{Obj}(\mathcal{A})$ is injective;*

ii) *The contravariant functor $h^I := \mathrm{Hom}(-, I)$ is exact.*

Proof. $\mathrm{Hom}(-, I)$ is exact if and only if $\mathrm{Hom}^\circ(I^\circ, -)$ is exact. Using Proposition 1.1 and the above remark we obtain that $\mathrm{Hom}(-, I)$ is exact if and only if I is injective. □

Definition. *i) A right resolution of $M \in \mathbf{Obj}(\mathcal{A})$ is a sequence*

$$I^\bullet : \qquad I^0 \to I^1 \to I^2 \to \dots$$

together with a morphism $\varepsilon : M \to I^0$ such that the sequence

$$0 \to M \xrightarrow{\varepsilon} I^0 \to I^1 \to I^2 \to \dots$$

is exact.

ii) If all I^i are injective, $I^\bullet$ is called an injective resolution of M.

Proposition 2.2. *i) Assume that $\mathcal{A}$ has enough injectives. Then each $M \in$* **Obj**($\mathcal{A}$) *has an injective resolution.*

ii) Consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & J^0 & \longrightarrow & J^1 & \longrightarrow & \dots \\ & & \downarrow{\scriptstyle f} & & \vdots{\scriptstyle f^0} & & \vdots{\scriptstyle f^1} & & \\ 0 & \longrightarrow & N & \longrightarrow & I^0 & \longrightarrow & J^1 & \longrightarrow & \dots \end{array}$$

where $I^\bullet$ is an injective resolution of N and $J^\bullet$ is a right (not necessarily injective) resolution of M. Then there exists a morphism $f^\bullet : J^\bullet \to I^\bullet$ such that the resulting diagram commutes.

ii) The morphism $f^\bullet$ is unique up to a chain homotopy.

Proof. Use the duality argument. □

In the remainder of this section we prove that the category of left (resp. right) modules over a ring A has enough injectives. We will work with the category of left modules (the case of right modules is completely analogous and can be treated formally using duality).

Proposition 2.3 (Baer)**.** *Let I be an A-module. The following properties are equivalent:*

1) *I is injective.*

2) *For any ideal $\mathfrak{a} \subset A$ and any morphism $\beta : \mathfrak{a} \to I$ there exists a morphism $\beta' : A \to I$ such that $\beta'|_{\mathfrak{a}} = \beta$:*

$$\begin{array}{ccccc} & & & & I \\ & & & \nearrow^{\beta} & \vdots{\scriptstyle \beta'} \\ 0 & \longrightarrow & \mathfrak{a} & \longrightarrow & A. \end{array}$$

Proof. 1)⇒ 2) is clear.

2)⇒ 1). Consider the diagram

$$\begin{array}{ccccc} & & & & I \\ & & & \nearrow^{\alpha} & \vdots{\scriptstyle \alpha'} \\ 0 & \longrightarrow & M & \xrightarrow{f} & N, \end{array}$$

where $f : M \to N$ is an injective morphism of modules. We want to prove that α cn be extended to some morphism $\alpha' : N \to I$. Consider the set S of the pairs (X, h), where X is a submodule of N containing M and $h : X \to I$ is an extension of α to X. It is clear that the relation

$$(X, h) \leqslant (X', h') \iff X \subset X' \text{ and } h'\big|_X = h$$

is an order on S. It is also clear that each chain in S has an upper bound and by Kuratowski–Zorn lemma S has a maximal element (Z, h). We will show that $Z = N$ by contradiction. Assume that $Z \neq N$. Take $m \in N \setminus Z$ and set

$$Z' := Z + Am = (Z \oplus Am)/R,$$

where $R = \{(z, am) \mid z + am = 0\}$. Then

$$\mathfrak{a} := \{a \in \mathfrak{a} \mid am \in Z\}$$

is an ideal in A. Consider the map

$$\beta : \mathfrak{a} \to I, \qquad \beta(a) = h(am).$$

By our assumption, there exists $\beta' : A \to I$ such that $\beta'|_{\mathfrak{a}} = \beta$. Then the map

$$h' : Z' \to I,$$
$$h'(z + am) := h(z) + \beta'(a)$$

is well defined and extends h to Z'. This contradicts with the maximality of Z. □

Corollary 2.4. *Assume that A is a principal ideal domain (i.e. an integral domain in which every ideal is principal). For a A-module M, the following properties are equivalent*

1) *M is injective.*
2) *M is divisible, i.e. $aM = M$ for any $a \in A \setminus \{0\}$.*

Proof. 1)⇒ 2). Assume that M is injective. Let $a \in A \setminus \{0\}$ and let $(a) := Aa$ denote the principal ideal generated by a. For any $m \in M$ the assignment $\beta(ax) = xm$ is a well-defined morphism $\beta : (a) \to M$. Since M is injective, there exists an extension $\beta' : A \to M$ of β. Therefore $\beta'(1)$ satisfies the equation $a\beta'(1) = m$. This shows that $aM = M$.

2)⇒ 1). Assume that M is divisible. Let $\mathfrak{a}$ be an ideal of A and let $\beta : \mathfrak{a} \to M$ be a morphism of modules. Then $\mathfrak{a} = (a)$ for some a. We can assume that $a \neq 0$. Since M is divisible, there exists $m \in M$ such that $am = \beta(a)$. Set $\beta'(x) = xm$. Then $\beta' : A \to M$ is an extension of β to A and M is injective by Proposition 2.3. □

Corollary 2.5. *In the category of abelian groups, $\mathbf{Q}$, $\mathbf{Q}/\mathbf{Z}$ and $\mathbf{Q}_p/\mathbf{Z}_p$ (p is a prime number) are injective.*

Exercise 11. *Let $(I_j)_{j\in J}$ be a family of injective objects. Show that if the product $\prod_{j\in J} I_j$ exists, then it is injective.*

Proposition 2.6. *The category of abelian groups has enough injectives.*

Proof. Let M be an abelian group and let

$$I := \prod_{f\in \mathrm{Hom}(M,\mathbf{Q}/\mathbf{Z})} (\mathbf{Q}/\mathbf{Z})_f$$

be the direct product of copies of $\mathbf{Q}/\mathbf{Z}$ indexed by homomorphisms $f \in \mathrm{Hom}(M, \mathbf{Q}/\mathbf{Z})$. By Exercise 11, I is injective. We construct a morphism

$$\alpha : M \to I$$

setting:

$$\alpha(m) = (f(m))_{f\in \mathrm{Hom}(M,\mathbf{Q}/\mathbf{Z})} .$$

It is clear that α is a morphism of abelian groups. We only need to prove that α is monic i.e. that $\ker(\alpha) = \{0\}$. We will prove that $\alpha(m) \neq 0$ if $m \neq 0$.

Let $m \neq 0$. Set

$$\mathrm{Ann}(m) = \{w \in \mathbf{Z} \mid xm = 0\}.$$

Then $\mathrm{Ann}(m)$ is an ideal in $\mathbf{Z}$ and therefore $\mathrm{Ann}(m) = (a) := a\mathbf{Z}$ for some $a \in \mathbf{Z}$. We remark that $a \neq 1$ because $m \neq 0$. We consider the following two cases:

a) If $a = 0$, then $\mathbf{Z}m \simeq \mathbf{Z}$ and it is clear that there exists a nonzero morphism $h : \mathbf{Z}m \to \mathbf{Q}/\mathbf{Z}$. Since $\mathbf{Q}/\mathbf{Z}$ is injective, h extends to a morphism $f : M \to \mathbf{Q}/\mathbf{Z}$ and $f(m) = h(m) \neq 0$. Therefore $\alpha(m) \neq 0$.

b) If $a \neq 0$, we can assume that $a \geqslant 2$, and $\mathbf{Z}m \simeq \mathbf{Z}/a\mathbf{Z}$. Since $\mathbf{Z}/a\mathbf{Z} \simeq \frac{1}{a}\mathbf{Z}/Z \subset \mathbf{Q}/\mathbf{Z}$, it is clear that there exists a nonzero morphism $h : \mathbf{Z}m \to \mathbf{Q}/\mathbf{Z}$. Mimiking the arguments used above, we extend h to a morphism $f : M \to \mathbf{Q}/\mathbf{Z}$ and conclude that $f(m) \neq 0$. The proposition is proved. □

Now we can prove the main result of this section:

Theorem 2.7. *Let A be a ring.*
i) Let I be an abelian group. Consider

$$J := \mathrm{Hom}_{\mathbf{Z}}(A, I)$$

equipped with the following structure of a left A-module: if $a \in A$ and $f \in J$, then

$$(af)(x) := f(xa).$$

Then J is an injective A-module.
ii) The categories $A-$ **Mod** *and* **Mod** *$-A$ have enough injectives.*

Proof. i) We will use a particular case of the following version of Proposition 8.5. Let A and B be two rings and let N be a (B, A)-bimodule (i.e. a left B-module and a right A-module with the property $(bn)a = b(na)$). Then for any left A-module M and any left B-module L there exists a canonical isomorphism

$$\mathrm{Hom}_B(N \otimes_A M, L) \simeq \mathrm{Hom}_A(M, \mathrm{Hom}_B(N, L)).$$

Take $B :=_Z$, $N = A$ and $L := I$. Then

$$\mathrm{Hom}_{\mathbf{Z}}(M, I) \simeq \mathrm{Hom}_A(M, \mathrm{Hom}_{\mathbf{Z}}(A, I)) = \mathrm{Hom}_A(M, J).$$

We pass to the proof of the assertion i). Let

$$0 \to M' \to M \to M'' \to 0$$

be a short exact sequence of left A-modules. Then we have a commutative diagram, where the vertical maps are isomorphisms:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Hom}_A(M'', J) & \longrightarrow & \mathrm{Hom}_A(M, J) & \longrightarrow & \mathrm{Hom}_A(M', J) & \longrightarrow & 0 \\ & & \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq & & \\ 0 & \longrightarrow & \mathrm{Hom}_{\mathbf{Z}}(M'', I) & \longrightarrow & \mathrm{Hom}_{\mathbf{Z}}(M, I) & \longrightarrow & \mathrm{Hom}_{\mathbf{Z}}(M', I) & \longrightarrow & 0. \end{array}$$

Since I is injective in the category **Ab**, the bottow row is exact. Therefore the upper row is exact, and J is injective by Proposition 2.1.

ii) Let M be a left A-module. We consider M as an abelian group. By Proposition 2.6, there exists a monomorphism of abelian groups $i : M \to I$, where I is injective in **Ab**. Take $J := \mathrm{Hom}_{\mathbf{Z}}(A, I)$ and define

$$j : M \to J$$

by

$$j(m)(x) = i(xm), \quad x \in A, m \in M.$$

It is easy to check that j is a monomorphism of left A-modules. □

3. Derived functors

3.1. Let $\mathscr{F} : \mathcal{A} \to \mathcal{B}$ be a covariant additive functor between abelian categories. We will assume that $\mathscr{F}$ is right exact and that $\mathcal{A}$ has enough projectives. For any $X \in \mathbf{Obj}(\mathcal{A})$, choose a projective resolution $P_\bullet \to X$ of X and consider the sequence

$$\mathscr{F}(P_\bullet) : \qquad \ldots \to \mathscr{F}(P_2) \xrightarrow{d_2} \mathscr{F}(P_1) \xrightarrow{d_1} \mathscr{F}(P_0) \to 0.$$

Generally, the sequence $\mathscr{F}(P_\bullet)$ is far from being exact, but it is clearly a chain complex. Set

$$L_n\mathscr{F}(X) := H_n(\mathscr{F}(P_\bullet)).$$

Below we establish basic properties of this construction.

1) $L_0\mathscr{F}(X) \simeq \mathscr{F}(X)$.

Proof. Since $\mathscr{F}$ is right exact, the exact sequence

$$P_1 \to P_0 \to X \to 0$$

gives rise to an exact sequence

$$\mathscr{F}(P_1) \xrightarrow{d_1} \mathscr{F}(P_0) \to \mathscr{F}(X) \to 0.$$

Therefore $L_0\mathscr{F}(X) := \mathrm{coker}(d_1) \simeq \mathscr{F}(X)$. □

2) *$L_n\mathscr{F}(X)$ are well defined up to canonical isomorphisms.*

Proof. a) Assume that $Q_\bullet$ is another projective resolution of X. We have a diagram

$$\begin{array}{ccc} P_\bullet & \longrightarrow & X \\ {\scriptstyle f_\bullet}\downarrow\,\downarrow{\scriptstyle f'_\bullet} & & \downarrow{\scriptstyle \mathrm{id}} \\ Q_\bullet & \longrightarrow & X. \end{array}$$

By Proposition 1.6, there exists a morphism $f_\bullet$ which makes this diagram commute. It induces a morphism of complexes

$$\mathscr{F}(f_\bullet) : \mathscr{F}(P_\bullet) \to \mathscr{F}(Q_\bullet)$$

and therefore resulting morphisms on homology groups $H_n(\mathscr{F}(P_\bullet)) \to H_n(\mathscr{F}(Q_\bullet))$. We will show that these morphisms do not depend on the choice of $f_\bullet$. Let $f'_\bullet$:

$P_\bullet \to Q_\bullet$ be another morphism making the diagram above commute. By Proposition 1.6, $f'_\bullet$ and $f_\bullet$ are homotopic:

$$f'_\bullet \simeq f_\bullet.$$

Therefore the morphisms $\mathscr{F}(f_\bullet)$ and $\mathscr{F}(f'_\bullet)$ are homotopic. By Proposition 2.1, they induce the same morphisms $H_n(\mathscr{F}(P_\bullet)) \to H_n(\mathscr{F}(Q_\bullet))$. This shows that these morphisms do not depend on the choice of $f_\bullet$.

b) Now we prove that the above morphisms $H_n(\mathscr{F}(P_\bullet)) \to H_n(\mathscr{F}(Q_\bullet))$ are isomorphisms.

Consider the diagram

$$\begin{array}{ccc} P_\bullet & \longrightarrow & X \\ \Big\downarrow{\scriptstyle f_\bullet} & & \Big\downarrow{\scriptstyle \mathrm{id}} \\ Q_\bullet & \longrightarrow & X \\ \vdots{\scriptstyle g_\bullet} & & \Big\downarrow{\scriptstyle \mathrm{id}} \\ P_\bullet & \longrightarrow & X. \end{array} \qquad (\mathrm{id}: P_\bullet \to P_\bullet)$$

By Proposition 1.6, there exists a morphism $g_\bullet$ which make this diagram commute. Moreover $g_\bullet \circ f_\bullet$ and id are homotopic:

$$g_\bullet \circ f_\bullet \simeq \mathrm{id}.$$

Applying the functor $\mathscr{F}$, we obtain a diagram

$$\begin{array}{ccc} \mathscr{F}(P_\bullet) & \longrightarrow & \mathscr{F}(X) \\ \Big\downarrow{\scriptstyle \mathscr{F}(f_\bullet)} & & \Big\downarrow{\scriptstyle \mathrm{id}} \\ \mathscr{F}(Q_\bullet) & \longrightarrow & \mathscr{F}(X) \\ \Big\downarrow{\scriptstyle \mathscr{F}(g_\bullet)} & & \Big\downarrow{\scriptstyle \mathrm{id}} \\ \mathscr{F}(P_\bullet) & \longrightarrow & {}_cF(X), \end{array} \qquad (\mathrm{id}: \mathscr{F}(P_\bullet) \to \mathscr{F}(P_\bullet))$$

where $\mathscr{F}(g_\bullet) \circ \mathscr{F}(f_\bullet) \simeq \mathrm{id}$. Therefore, by Proposition 2.1, the composition map on homology

$$H_n(\mathscr{F}(P_\bullet)) \to H_n(\mathscr{F}(Q_\bullet)) \to H_n(\mathscr{F}(P_\bullet))$$

coincides with the identity map. Exchanging $P_\bullet$ and $Q_\bullet$ and mimiking the above arguments wh obtain that the composition map

$$H_n(\mathscr{F}(Q_\bullet)) \to H_n(\mathscr{F}(P_\bullet)) \to H_n(\mathscr{F}(Q_\bullet))$$

is the identity map. This shows that the morphisms $H_n(\mathscr{F}(P_\bullet)) \to H_n(\mathscr{F}(Q_\bullet))$ are canonical isomorphisms. □

3) *If X is projective, then $L_n\mathscr{F}(X) = 0$ for all $n \geqslant 1$.*

Proof. The complex

$$\ldots \to 0 \to 0 \to P_0$$

with $P_0 = X$ is a projective resolution of X. □

4) *Each morphism $f : X \to Y$ induces canonical morphisms*

$$L_n(f) : L_n\mathscr{F}(X) \to L_n\mathscr{F}(Y), \qquad n \geqslant 0.$$

Proof. We have a diagram

$$\begin{array}{ccc} P_\bullet & \longrightarrow & X \\ f_\bullet \Big\downarrow \Big\downarrow f'_\bullet & & \Big\downarrow f \\ Q_\bullet & \longrightarrow & Y, \end{array}$$

where $P_\bullet$ and $Q_\bullet$ are projective resolutions of X and Y respectively. By Proposition 1.6, there exists a morphism $f_\bullet$ which makes this diagram commute. Applying the functor $\mathscr{F}$ we obtain a diagram

$$\begin{array}{ccc} \mathscr{F}(P_\bullet) & \longrightarrow & \mathscr{F}(X) \\ \mathscr{F}(f_\bullet) \Big\downarrow \Big\downarrow \mathscr{F}(f'_\bullet) & & \Big\downarrow \mathscr{F}(f) \\ \mathscr{F}(Q_\bullet) & \longrightarrow & \mathscr{F}(Y). \end{array}$$

The morphism of complexes

$$\mathscr{F}(f_\bullet) : \mathscr{F}(P_\bullet) \to \mathscr{F}(Q_\bullet)$$

induces morphisms on homology groups

$$L_n\mathscr{F}(X) := H_n(\mathscr{F}(P_\bullet)) \to H_n(\mathscr{F}(Q_\bullet)) =: L_n\mathscr{F}(Y). \tag{11}$$

Assume that $f'_\bullet : P_\bullet \to Q_\bullet$ is another morphism between resolutions such that the above diagram commutes. Then $f'_\bullet \simeq f_\bullet$ and $\mathscr{F}(f_\bullet) \simeq \mathscr{F}(f'_\bullet)$. Therefore $\mathscr{F}(f'_\bullet)$ induces the same map on homology. This shows that the morphisms (11) do not depend on the choice of $f_\bullet$. □

5) *For each $n \leqslant 0$, the assignment*

$$\begin{aligned} L_n\mathscr{F} &: \mathcal{A} \to \mathcal{B}, \\ & X \to L_n\mathscr{F}(X) \end{aligned}$$

is an additive functor. Moreover $L_0\mathscr{F} \simeq \mathscr{F}$.

Proof. The proof is left as an exercise. □

Definition. *The functors $L_n\mathscr{F}$ are called the left derived functors of $\mathscr{F}$.*

6) *For each short exact sequence*

$$0 \to X' \xrightarrow{f} X \xrightarrow{g} X'' \to 0$$

there exists a long exact sequence

$$\ldots \to L_n\mathscr{F}(X') \to L_n\mathscr{F}(X) \to L_n\mathscr{F}(X'') \xrightarrow{\delta_n} L_{n-1}\mathscr{F}(X') \to L_{n-1}\mathscr{F}(X) \to \ldots$$

$$\ldots \to L_1\mathscr{F}(X) \to L_1\mathscr{F}(X'') \xrightarrow{\delta_1} L_0\mathscr{F}(X') \to L_0\mathscr{F}(X) \to L_0\mathscr{F}(X'') \to 0.$$

PROOF. By Proposition 1.7, we can choose projective resolutions $P'_\bullet$, $P_\bullet$ $P''_\bullet$ of X', X and X'' which sit in a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P'_\bullet & \longrightarrow & P_\bullet & \longrightarrow & P''_\bullet & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \varepsilon} & & \downarrow{\scriptstyle \varepsilon} & & \downarrow{\scriptstyle \varepsilon} & & \\ 0 & \longrightarrow & X' & \longrightarrow & X & \longrightarrow & X'' & \longrightarrow & 0 \end{array}$$

For each n, we have a commutative diagram

$$\begin{array}{ccccccccc} & & & & & & P''_n & & \\ & & & & & \swarrow & \| {\scriptstyle \mathrm{id}} & & \\ 0 & \longrightarrow & P'_n & \longrightarrow & P_n & \longrightarrow & P''_n & \longrightarrow & 0, \end{array}$$

which shows that this sequence splits i.e. $P_n \simeq P'_n \oplus P''_n$ (here we use the projectivity of P''_n). Since the functor $\mathscr{F}$ preserves direct sums, we obtain that $\mathscr{F}(P_n) \simeq \mathscr{F}(P'_n) \oplus \mathscr{F}(P''_n)$, i.e. that the sequence

$$0 \longrightarrow \mathscr{F}(P'_n) \longrightarrow \mathscr{F}(P_n) \longrightarrow \mathscr{F}(P''_n) \longrightarrow 0$$

is exact. Therefore we have an exact sequence of complexes

$$0 \longrightarrow \mathscr{F}(P'_\bullet) \xrightarrow{f_\bullet} \mathscr{F}(P_\bullet) \xrightarrow{g_\bullet} \mathscr{F}(P''_\bullet) \longrightarrow 0.$$

Applying to this exact sequence Theorem 1.3 (long exact homology sequence), we obtain our statement. □

7) *$\mathscr{F}$ is exact if and only if for all $X \in \mathbf{Obj}(\mathscr{A})$ and $n \geqslant 1$, one has $L_n\mathscr{F}(X) = 0$.*

PROOF. a) Assume that $\mathscr{F}$ is exact. For any object $X \in \mathbf{Obj}(\mathscr{A})$ we have an exact sequence $P_\bullet \to X \to 0$, where $P_\bullet$ is a projective resolution of X. Then the sequence

$$\mathscr{F}(P_\bullet) \to \mathscr{F}(X) \to 0$$

is exact, and from the definition of functors $L_n\mathscr{F}$ we obtain that $L_n\mathscr{F}(X) = 0$ for $n \geqslant 1$.

b) Assume that $L_n\mathscr{F} = 0$ for $n \geqslant 1$. For any short exact sequence

$$0 \to X' \xrightarrow{f} X \xrightarrow{g} X'' \to 0$$

the associated long exact sequence reads:

$$\cdots \to L_1\mathscr{F}(X'') \to \mathscr{F}(X') \xrightarrow{f} \mathscr{F}(X) \xrightarrow{g} \mathscr{F}(X'') \to 0.$$

Since $L_1\mathscr{F}(X'') = 0$, we obtain that the sequence

$$0 \to \mathscr{F}(X') \xrightarrow{f} \mathscr{F}(X) \xrightarrow{g} \mathscr{F}(X'') \to 0$$

is exact. □

8) (Dimension shifting) *Consider a short exact sequence of the form*

$$0 \to Y \xrightarrow{f} P \xrightarrow{g} X \to 0,$$

where P is projective. Then

$$L_n\mathscr{F}(X) \simeq L_{n-1}\mathscr{F}(Y), \qquad \forall n \geqslant 2$$

and

$$L_1\mathscr{F}(X) \simeq \ker(\mathscr{F}(Y) \to \mathscr{F}(P)).$$

Proof. We have an exact sequence

$$L_n\mathscr{F}(P) \to L_n\mathscr{F}(X) \to L_{n-1}\mathscr{F}(Y) \to L_{n-1}\mathscr{F}(P)$$

Since P is projective, $L_i(P) = 0$ if $i \geqslant 1$. This proves the assertion. □

Definition. *i) An object $Q \in \mathbf{Obj}(\mathcal{A})$ is acyclic (or $\mathscr{F}$-acyclic) if $L_n\mathscr{F}(Q) = 0$ for all $n \geqslant 1$.*
ii) A resolution $Q_\bullet$ of X is acyclic if all Q_i are acyclic.

From property 3) it follows that each projective object is acyclic.

9) *Let $Q_\bullet$ be an acyclic left resolution of X. Then*

$$L_n\mathscr{F}(X) \simeq H_n(\mathscr{F}(Q_\bullet)), \qquad n \geqslant 0.$$

Proof. We will prove this assertion by induction on n. For $n = 0$, we mimick the proof of property 1). Since ${}_cF$ is right exact, the exact sequence

$$Q_1 \to Q_0 \to X \to 0$$

induces an exact sequence

$$\mathscr{F}(Q_1) \to \mathscr{F}(Q_0) \to \mathscr{F}(X) \to 0.$$

This shows that $L_0\mathscr{F}(X) = \operatorname{coker}(\mathscr{F}(Q_1) \to \mathscr{F}(Q_0)) \simeq \mathscr{F}(X)$.

Assume that the statement holds for $n-1$ for all objects. Set $Y = \ker(Q_0 \to X)$. We have a short exact sequence

$$0 \to Y \to Q_0 \to X \to 0$$

and an exact sequence

$$\ldots \to Q_2 \to Q_1 \to Y \to 0.$$

The complex $\ldots \to Q_2 \to Q_1$ is an acyclic resolution of Y and by our induction assumption $L_{n-1}\mathscr{F}(Y) \simeq H_n(\mathscr{F}(Q_\bullet))$. From the short exact sequence and the acyclicity of Q_0 we obtain that $L_{n-1}\mathscr{F}(Y) \simeq L_n\mathscr{F}(X)$. Therefore $L_n\mathscr{F}(X) \simeq H_n(\mathscr{F}(Q_\bullet))$. □

3.2. Now we consider the case of a left exact functor. Assume that the category $\mathcal{A}$ has enough injectives. Let $\mathscr{F} : \mathcal{A} \to \mathcal{B}$ be covariant left exact functor. For any $X \in \mathbf{Obj}(\mathcal{A})$ take an injective resolution $X \to I^\bullet$ of X. The sequence

$$\mathscr{F}(I^\bullet) : \quad 0 \to \mathscr{F}(I^0) \to \mathscr{F}(I^1) \to \cdots$$

is a cochain complex, and we define:

$$R^i\mathscr{F}(X) := H^i(\mathscr{F}(I^\bullet)).$$

Let $\mathcal{A}^\circ$ and $\mathcal{B}^\circ$ denote the dual categories. Then

$$\mathscr{F}^{\circ\circ} : \mathcal{A}^\circ \to \mathcal{B}^\circ,$$
$$\mathscr{F}^{\circ\circ}(X^\circ) := \mathscr{F}(X)^\circ$$

is a covariant *right exact* functor. If $I^\bullet$ is an injective resolution of $M \in \mathbf{Obj}(\mathcal{A})$, then $I^{\bullet\,\circ}$ is a projective resolution of $M^\bullet$ in $\mathcal{A}^\bullet$. Since

$$\mathscr{F}(I^\bullet)^\circ \simeq \mathscr{F}^{\circ\circ}(I^{\bullet\,\circ}),$$

we obtain that

$$R^i\mathscr{F}(M)^\circ \simeq L_i\mathscr{F}^{\circ\circ}(M^\circ).$$

This allows to deduce general properties of $R^i\mathscr{F}(-)$ from general properties of Ł$_i\mathscr{F}^{\circ\circ}$. In particular, we see that $R^i\mathscr{F}(-)$ are well defined additive covariant functors.

Definition. *The functors $R^i\mathscr{F}(-)$ are called the right derived functors of the left exact functor $\mathscr{F}$.*

Below we summarize some basic properties of right derived functors.

Properties 3.3.

1) $R^0\mathscr{F}(M) = M$.
2) *If X is injective, then $R^i\mathscr{F}(I) = 0$ for all $i \geqslant 1$.*
3) *For each exact sequence*

$$0 \to X' \to X \to X'' \to 0$$

there exists a long exact sequence

$$0 \to R^0\mathscr{F}(X') \to R^0\mathscr{F}(X) \to R^0\mathscr{F}(X'') \xrightarrow{\delta^0} R^1\mathscr{F}(X')$$
$$\to R^1\mathscr{F}(X) \to R^1\mathscr{F}(X'') \to \cdots$$

4) (Dimension shifting) *Consider a short exact sequence of the form*

$$0 \to X \to I \to Y \to 0,$$

where I is injective. Then

$$R^i\mathscr{F}(X) \simeq R^{i-1}\mathscr{F}(Y), \qquad i \geqslant 2$$

and $R^1\mathscr{F}(X) \simeq \operatorname{coker}(\mathscr{F}(I) \to \mathscr{F}(Y))$.

Definition. *i) An object J is acyclic (or $\mathscr{F}$-acyclic) if $R^i\mathscr{F}(J) = 0$ for all $i \geqslant 1$.*
ii) A right resolution $J^\bullet$ is acyclic if all J^i are acyclic.

5) *Let $J^\bullet$ be an acyclic right resolution of X. Then*

$$R^i\mathscr{F}(X) \simeq H^i(\mathscr{F}(J^\bullet)).$$

3.4. We define derived functors of contravariant functors. If $\mathscr{F} : \mathcal{A} \to \mathcal{B}$ is a contravariant functor which is right (respectively left) exact, then

$$\mathscr{F}^\circ : \mathcal{A}^\circ \to \mathcal{B},$$
$$\mathscr{F}^\circ(X^\circ) := \mathscr{F}(X)$$

is a covariant functor which is right (respectively left) exact, and we define the left (respectively right) exact functors of $\mathscr{F}$ by $L_i\mathscr{F}(X) := L_i\mathscr{F}^\circ(X^\circ)$ (respectively $R^i\mathscr{F}(X) := R^i\mathscr{F}^\circ(X^\circ)$). Explicitely

$$L_i\mathscr{F}(X) := H_i(\mathscr{F}(I^\bullet)), \qquad \text{if } \mathscr{F} \text{ is contravariant right exact,}$$
$$R^i\mathscr{F}(X) := H^i(\mathscr{F}(P_\bullet)), \qquad \text{if } \mathscr{F} \text{ is contravariant left exact,}$$

where $I^\bullet$ (respectively $P_\bullet$) denotes the right injective (respectively left projective) resolution of X.

4. The functors Ext^i

4.1. Let $\mathcal{A}$ be an abelian category having enough injectives. To simplify notation, we will write $\mathrm{Hom}(-,-)$ instead $\mathrm{Hom}_{\mathcal{A}}(-,-)$. Fix an object $M \in \mathbf{Obj}(\mathcal{A})$ and consider the covariant left exact functor

$$h_M : \mathcal{A} \to \mathbf{Ab},$$
$$h_M(N) := \mathrm{Hom}(M, N).$$

Definition. *The right derived functors of h_M are called the* Ext*-groups and are denoted as*

$$\mathrm{Ext}^i(M, N) := R^ih_M(N).$$

Below we summarize some basic properties of these functors.

Properties 4.2.

i) $\mathrm{Ext}^0(M, N) = \mathrm{Hom}(M, N)$.

Proof. It's clear. □

ii) *The assignment*

$$\mathrm{Ext}^i(-,-) : \mathcal{A} \times \mathcal{A} \to \mathbf{Ab},$$
$$(M, N) \to \mathrm{Ext}^i(M, N)$$

is a functor which is contravariant in the first variable and covariant in the second variable.

Proof. a) From general properties of derived functors it follows that $\mathrm{Ext}^i(-,-)$ is covariant in the second variable.

b) Let $f : M' \to M$ be a morphism. Let $I^\bullet$ be an injective resolution of N. Then the morphism f induces a natural morphism of complexes

$$\begin{array}{ccc} \mathrm{Hom}(M, I^\bullet) & \longrightarrow & \mathrm{Hom}(M', I^\bullet) \\ \| & & \| \\ h_M(I^\bullet) & & h_{M'}(I^\bullet) \end{array}$$

This morphism of complexes induces morphisms between their groups of cohomology :

$$\mathrm{Ext}^i(M, N) \to \mathrm{Ext}^i(M', N).$$

Now it is easy to check that the assignment $\mathrm{Ext}^i(-, N)$ is a contravariant functor. □

iii) *A short exact sequence*

$$0 \to N' \to N \to N'' \to 0$$

induces a long exact sequence

$$(12) \quad 0 \to \mathrm{Hom}(M, N') \to \mathrm{Hom}(M, N) \to \mathrm{Hom}(M, N'') \to \mathrm{Ext}^1(M, N') \\ \to \mathrm{Ext}^1(M, N) \to \mathrm{Ext}^1(M, N'') \to \dots$$

Proof. This is the long exact sequence of derived functors $R^i h_M$. □

iv) *A short exact sequence*

$$0 \to M' \to M \to M'' \to 0$$

induces a long exact sequence

$$(13) \quad 0 \to \mathrm{Hom}(M'', N) \to \mathrm{Hom}(M, N) \to \mathrm{Hom}(M', N) \to \mathrm{Ext}^1(M'', N) \\ \to \mathrm{Ext}^1(M, N) \to \mathrm{Ext}^1(M', N) \to \dots$$

Proof. Let $I^\bullet$ be an injective resolution of N. Since the contravariant functor $h^J(-) = \mathrm{Hom}(-, J)$ is exact if J is injective, we have an exact sequence of complexes:

$$0 \to \mathrm{Hom}(M'', I^\bullet) \to \mathrm{Hom}(M, I^\bullet) \to \mathrm{Hom}(M', I^\bullet) \to 0.$$

Taking the long exact sequence of cohomology attached to this exact sequence, we obtain the sequence (13). □

Proposition 4.3. *The following properties are equivalent:*

1) *I is injective.*
2) *The functor $h^I(-) := \mathrm{Hom}(-, I)$ is exact.*
3) *For all $M \in \mathbf{Obj}(\mathcal{A})$,*

$$\mathrm{Ext}^i(M, I) = 0, \qquad \forall i \geqslant 1.$$

4) *For all $M \in \mathbf{Obj}(\mathcal{A})$,*

$$\mathrm{Ext}^1(M, I) = 0.$$

Proof. We already know that 1) $\Leftrightarrow$ 2) and 1) $\Rightarrow$ 3) (see Proposition 2.1 and Section). The implication 3) $\Rightarrow$ 4) is trivial. We only need to show that 4) $\Rightarrow$ 1). Let

$$0 \to M' \to M \to M'' \to 0$$

be a short exact sequence. The long exact sequence (13) for $N = I$ reads

$$0 \to \mathrm{Hom}(M'', I) \to \mathrm{Hom}(M, I) \to \mathrm{Hom}(M', I) \to \mathrm{Ext}^1(M'', I)$$

Since $\mathrm{Ext}^1(M'', I) = 0$, we obtain that the sequence

$$0 \to \mathrm{Hom}(M'', I) \to \mathrm{Hom}(M, I) \to \mathrm{Hom}(M', I) \to 0$$

is exact. Therefore the functor $h^I(-)$ is exact. □

Proposition 4.4. *The following properties are equivalent:*

1) *P is projective.*
2) *The functor $h_P(-) := \mathrm{Hom}(P, -)$ is exact.*
3) *For all $N \in \mathbf{Obj}(\mathcal{A})$,*

$$\mathrm{Ext}^i(P, N) = 0, \qquad \forall i \geqslant 1.$$

4) *For all $N \in \mathbf{Obj}(\mathcal{A})$,*

$$\mathrm{Ext}^1(P, N) = 0.$$

Proof. We already know that 1) $\Leftrightarrow$ 2) and 2) $\Rightarrow$ 3) (see Proposition 1.1 and Section). The implication 3) $\Rightarrow$ 4) is trivial. We only need to show that 4) $\Rightarrow$ 1). Let

$$0 \to N' \to N \to N'' \to 0$$

be a short exact sequence. The long exact sequence (12) for $M = P$ reads

$$0 \to \mathrm{Hom}(P, N') \to \mathrm{Hom}(P, N) \to \mathrm{Hom}(P, N'') \to \mathrm{Ext}^1(P, N').$$

Since $\mathrm{Ext}^1(P, N') = 0$, we obtain that the sequence

$$0 \to \mathrm{Hom}(P, N') \to \mathrm{Hom}(P, N) \to \mathrm{Hom}(P, N'') \to 0.$$

is exact. Hence the functor $h_P(-)$ is exact. □

4.5. Assume, in addition, that $\mathcal{A}$ has enough projectives. Fix $N \in \mathbf{Obj}(\mathcal{A})$ and consider the contravariant functor

$$h^N(-) : \mathcal{A} \to \mathbf{Ab},$$
$$h^N(M) := \mathrm{Hom}(M, N).$$

The functor $h^N(-)$ is left exact and we can consider its right derived functors:

$$R^i h^N(M) := H^i(\mathrm{Hom}(P_\bullet, N)), \quad \text{where } P_\bullet \text{ is a projective resolution of } M.$$

The following theorem will be proved in the next section:

Theorem 4.6. *There exist canonical and functorial isomorphisms*

$$R^i h^N(M) \simeq \mathrm{Ext}^i(M, N).$$

Example. If A is a ring, we write Ext^i_A for the Ext-functors in the category of left A-modules. Consider the category of $\mathbf{Z}$-modules (or, equivalently, the category of abelian groups). Choose a free $\mathbf{Z}$-module P_0 together with a surjective morphism $P_0 \xrightarrow{\varepsilon} M$. Since every non-null subgroup of a free $\mathbf{Z}$-module is free, $P_1 := \ker(\varepsilon)$ is free and

$$P_\bullet \,:\, 0 \to P_1 \to P_0 \to 0$$

is a projective resolution of M. Therefore

$$\mathrm{Ext}^i_{\mathbf{Z}}(M, N) = 0, \qquad i \geqslant 2.$$

If $M = \mathbf{Z}/m\mathbf{Z}$, we can take

$$P_\bullet \,:\, 0 \to \mathbf{Z} \xrightarrow{m} \mathbf{Z} \to 0.$$

The complex $\mathrm{Hom}_{\mathbf{Z}}(P_\bullet, N)$ is isomorphic to the complex

$$0 \to N \xrightarrow{m} N \to 0.$$

Therefore $\mathrm{Ext}^0_{\mathbf{Z}}(\mathbf{Z}/m\mathbf{Z}, N) \simeq {}_mN$, where ${}_mN = \{x \in N \mid mx = 0\}$ is the m-torsion submodule of N (we remark that ${}_mN = \mathrm{Hom}(\mathbf{Z}/m\mathbf{Z}, N)$), and

$$\mathrm{Ext}^1_{\mathbf{Z}}(\mathbf{Z}/m\mathbf{Z}, N) \simeq N/mN.$$

5. Double complexes

5.1. Double complexes. Let $\mathcal{A}$ be an abelian category.

Definition. *A double cochain complex concentrated in the first quadrant is a diagram in $\mathcal{A}$ of the form*

$$\begin{array}{ccccccccc}
 & & \uparrow{\scriptstyle d^v_{02}} & & \uparrow{\scriptstyle d^v_{12}} & & \uparrow{\scriptstyle d^v_{22}} & & \\
0 & \longrightarrow & X^{(0,2)} & \xrightarrow{d^h_{02}} & X^{(1,2)} & \xrightarrow{d^h_{12}} & X^{(2,2)} & \xrightarrow{d^h_{22}} & \\
 & & \uparrow{\scriptstyle d^v_{01}} & & \uparrow{\scriptstyle d^v_{11}} & & \uparrow{\scriptstyle d^v_{21}} & & \\
0 & \longrightarrow & X^{(0,1)} & \xrightarrow{d^h_{01}} & X^{(1,1)} & \xrightarrow{d^h_{11}} & X^{(2,1)} & \xrightarrow{d^h_{21}} & \\
 & & \uparrow{\scriptstyle d^v_{00}} & & \uparrow{\scriptstyle d^v_{10}} & & \uparrow{\scriptstyle d^v_{20}} & & \\
0 & \longrightarrow & X^{(0,0)} & \xrightarrow{d^h_{00}} & X^{(1,0)} & \xrightarrow{d^h_{10}} & X^{(2,0)} & \xrightarrow{d^h_{20}} & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where the morphisms satisfy the following properties:

$$d^h \circ d^h = d^v \circ d^v = 0, \qquad d^v \circ d^h + d^h \circ d^v = 0. \tag{14}$$

More explicitly, for all (i, j)

$$d^h_{i+1,j} \circ d^h_{ij} = 0, \qquad d^v_{i,j+1} \circ d^v_{ij} = 0, \qquad d^v_{i+1,j} \circ d^h_{ij} + d^h_{i,j+1} \circ d^v_{ij} = 0.$$

We remark, that in general, the above diagram is not commutative.

To each double cochain complex $X^{\bullet\bullet}$ we associate:

$$X^n := \bigoplus_{i+j=n} X^{(i,j)}, \qquad n \geqslant 0,$$
$$d = d^h + d^v \,:\, X^n \to X^{n+1}.$$

From the relations (14) we see that

$$d \circ d = (d^h + d^v) \circ (d^h + d^v) = d^h \circ d^h + d^v \circ d^h + d^h \circ d^v + d^v \circ d^v = 0,$$

and therefore $X^\bullet$ is a chain complex.

Definition. *The complex $X^\bullet$ is called the total complex attached to $X^{\bullet\bullet}$ and is denoted by* $\mathrm{Tot}(X^{\bullet\bullet})$.

The following proposition can be proved by a careful diagram chasing:

Proposition 5.2 (Acyclicity of the total complex)**.** *Assume that all rows (respectively all columns) of of $X^{\bullet\bullet}$ are exact. Then* $\mathrm{Tot}(X^{\bullet\bullet})$ *is acyclic.*

5.3. Proof of Theorem 4.6. In this subsection, we return to our study of the groups $\mathrm{Ext}^i(M, N)$. Assume that $\mathcal{A}$ has enough projectives and injectives. Choose a projective resolution $P_\bullet \xrightarrow{\varepsilon} M \to 0$ of M and an injective resolution $0 \to N \xrightarrow{\varepsilon} I^\bullet$ of N. Set

$$X^{(i,j)} = \mathrm{Hom}(P_i, I^j). \qquad i, j \geqslant 0.$$

The morphisms $P_{i+1} \to P_i$ and $I^j \to I^{j+1}$ induce morphisms

$$\begin{array}{ccc} X^{(i,j+1)} & & \\ \uparrow{\scriptstyle f^v_{ij}} & & \\ X^{(i,j)} & \xrightarrow{f^h_{ij}} & X^{(i+1,j)}, \end{array}$$

and we obtain a diagram

$$\begin{array}{ccccccccc}
 & & \uparrow{\scriptstyle f^v_{02}} & & \uparrow{\scriptstyle f^v_{12}} & & \uparrow{\scriptstyle f^v_{22}} & & \\
0 & \longrightarrow & X^{(0,2)} & \xrightarrow{f^h_{02}} & X^{(1,2)} & \xrightarrow{f^h_{12}} & X^{(2,2)} & \xrightarrow{f^h_{22}} & \\
 & & \uparrow{\scriptstyle f^v_{01}} & & \uparrow{\scriptstyle f^v_{11}} & & \uparrow{\scriptstyle d^v_{21}} & & \\
0 & \longrightarrow & X^{(0,1)} & \xrightarrow{f^h_{01}} & X^{(1,1)} & \xrightarrow{d^h_{11}} & X^{(2,1)} & \xrightarrow{d^h_{21}} & \\
 & & \uparrow{\scriptstyle f^v_{00}} & & \uparrow{\scriptstyle f^v_{10}} & & \uparrow{\scriptstyle f^v_{10}} & & \\
0 & \longrightarrow & X^{(0,0)} & \xrightarrow{f^h_{00}} & X^{(1,0)} & \xrightarrow{f^h_{10}} & X^{(2,0)} & \xrightarrow{f^h_{20}} & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

This diagram is not a double complex because the horizontal and the vertical morphisms commute, namely

$$f^v \circ f^h = f^h \circ f^v.$$

To remedy this problem, we define the differentials d^h and d^v as follows:

$$d^h_{ij} = f^h_{ij}, \qquad d^v_{ij} = (-1)^{i+j+1} f^v_{ij}.$$

It is easy to check that $(X^{\bullet\bullet}, d^h, d^v)$ is a double complex. We will prove the following strong version of Theorem 4.6 :

Theorem 5.4. *There exist natural an functorial quasi-isomorphisms :*

$$\mathrm{Hom}(P_\bullet, N) \xrightarrow{\sim} \mathrm{Tot}(X^{\bullet\bullet}) \xleftarrow{\sim} \mathrm{Hom}(M \otimes I^\bullet).$$

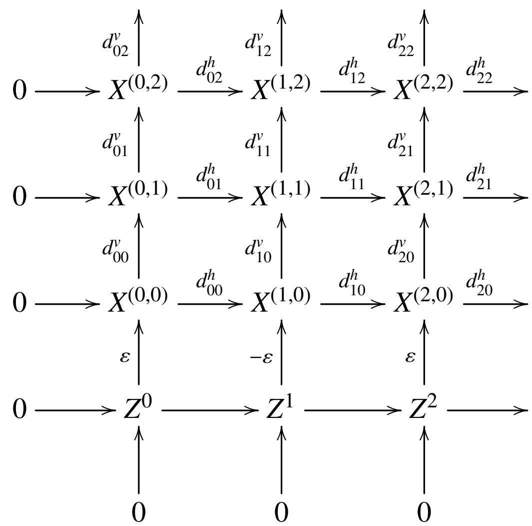
Proof. Since P_i are projective, the complexes

$$0 \to \mathrm{Hom}(P_i, N) \to \mathrm{Hom}(P_i, I^0) \to \mathrm{Hom}(P_i, I^1) \to \cdots$$

are exact. To simplify notation, set

$$Z^i := \mathrm{Hom}(P_i, N).$$

Consider the double complex



Let $Y^\bullet$ denote the total complex attached to this diagram. We make the following remarks :

a) The columns of this diagram are exact, and therefore $Y^\bullet$ is acyclic by Theorem 4.6.

b) The vertical maps $\pm\varepsilon$ induce a morphism of complexes

$$g \ : \ Z^\bullet \to \mathrm{Tot}(X^{\bullet\bullet}).$$

We use the version of Proposition 3.1 for cochains, which can be easily obtained by a renumbering of objects. There exists an exact sequence of complexes

$$0 \to Z^\bullet \to c(g) \to \mathrm{Tot}(X^{\bullet\bullet})[-1] \to 0,$$

where $c(g)$ is the cone of the morphism g. The complex $c(g)$ is defined explicitly as follows:

$$c(g)^n = Z^n \oplus X^{n-1}, \qquad X^n = \bigoplus_{i+j=n} X^{i,j},$$
$$d : c(g)^n \to c(g)^{n+1}, \qquad d(z_n, x_{n-1}) = (d(z_n), (-1)^n \varepsilon(z_n) + d(x_{n-1})).$$

and we can easily check that $c(g) = Y^\bullet$. By the remark a), this complex is acyclic, and from Corollary 3.2 it follows that the map

$$g : \mathrm{Hom}(P_\bullet, N) \to \mathrm{Tot}(X^{\bullet\bullet})$$

is a quasi-isomorphism.

Mimiking the previous arguments, we construct a quasi-isomorphism $\mathrm{Hom}(M, I^\bullet) \to \mathrm{Tot}(X^{\bullet\bullet})$. The theorem is proved. □

6. Extensions

Let $\mathcal{A}$ be an abelian category. Let M and N be two objects of $\mathcal{A}$.

Definition. *1) An extension of M by N in $\mathcal{A}$ is an exact sequence of the form*

$$E : \qquad 0 \to N \xrightarrow{\alpha} X \xrightarrow{\beta} M \to 0.$$

2) If E' : $\quad 0 \to N \xrightarrow{\alpha'} X' \xrightarrow{\beta'} M \to 0$ *is another extension of M by N, we say that they are equivalent if there exists a commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \xrightarrow{\alpha} & X & \xrightarrow{\beta} & M & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle g} & & \| & & \\ 0 & \longrightarrow & N & \xrightarrow{\alpha'} & X' & \xrightarrow{\beta'} & M & \longrightarrow & 0 \end{array}$$

By the five lemma, the morphism g is an isomorphism.

Recall that an exact sequence α splits if there exists a section $s : M \to X$ such that $g \circ s = \mathrm{id}_M$ or, equivalently, if α is equivalent to the extension

$$0 \longrightarrow N \xrightarrow{q_N} N \oplus M \xrightarrow{p_M} M \longrightarrow 0,$$

where q_M and p_M are the canonical morphisms (see Lemma 1.3).

Assume that $\mathcal{A}$ has enough injectives and therefore the functors Ext^i are defined.

Lemma 6.1. *Assume that* $\mathrm{Ext}^1(M, N) = 0$*. Then every extension of M by N splits.*

Proof. 1) ⇒ 2). Consider an exact sequence

$$0 \to N \to X \to M \to 0.$$

It induces a long exact sequence

$$0 \to \mathrm{Hom}(M, N) \to \mathrm{Hom}(M, X) \to \mathrm{Hom}(M, M) \xrightarrow{\delta} \mathrm{Ext}^1(M, N).$$

Assume that $\mathrm{Ext}^1(M,N)$. Then the sequence

$$0 \to \mathrm{Hom}(M,N) \to \mathrm{Hom}(M,X) \to \mathrm{Hom}(M,M) \to 0$$

is exact. Then the identity map $\mathrm{id} \in \mathrm{Hom}(M,M)$ lifts to a map $s \in \mathrm{Hom}(M,X)$ and we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & X & \longrightarrow & M & \longrightarrow & 0 \\ & & & & & {}^{s}\nwarrow & \Vert & & \\ & & & & & & M & & \end{array}$$

which shows that our sequence splits. □

More generally, to each extension

$$E: \qquad 0 \to N \to X \to M \to 0$$

we can associate the connection map

$$\mathrm{Hom}(M,M) \xrightarrow{\delta} \mathrm{Ext}^1(M,N)$$

and define $\theta(E) := \delta(\mathrm{id}_M)$.

Theorem 6.2. *The map θ establishes a one-to-one correspondence:*

$$\{\text{equivalence classes of extensions of } M \text{ by } N\} \xleftrightarrow{\theta} \mathrm{Ext}^1(M,N).$$

Sketch of the proof. We will only explain how to attach an extension to any element $x \in \mathrm{Ext}^1(M,N)$. Take an exact sequence of the form

$$0 \to L \xrightarrow{\lambda} P \xrightarrow{\pi} M \to 0,$$

where P is projective. This short exact sequence induces the exact sequence

$$\mathrm{Hom}(P,N) \to \mathrm{Hom}(L,N) \to \mathrm{Ext}^1(M,N) \to \mathrm{Ext}^1(P,N).$$

Since P is projective, the last term vanishes, and this sequence reads:

$$\mathrm{Hom}(P,N) \to \mathrm{Hom}(L,N) \to \mathrm{Ext}^1(M,N) \to 0.$$

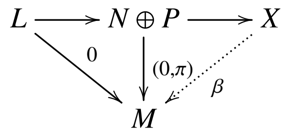
Take any lift $f \in \mathrm{Hom}(L,N)$ of $x \in \mathrm{Ext}^1(M,N)$. Using this morphism, we will construct a diagram of the form

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \xrightarrow{\lambda} & P & \xrightarrow{\pi} & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \Vert & & \\ 0 & \longrightarrow & N & \xrightarrow{\alpha} & X & \xrightarrow{\beta} & M & \longrightarrow & 0. \end{array}$$

Set

$$X := \mathrm{coker}(L \xrightarrow{(-f,\lambda)} N \oplus P).$$

Define the morphisms α and g as the composition of the morphisms $N \xrightarrow{q_N} N \oplus P$ and $P \xrightarrow{q_P} N \oplus P$ with the canonical map $N \oplus P \to X$. Finally the diagram



shows that there exists a unique morphism β making this diagram commute. It can be checked that the sequence

$$E : \quad 0 \to N \xrightarrow{\alpha} X \xrightarrow{\beta} M \to 0$$

is exact and its equivalence class does not depend on the choice of f. To sum up, this construction associates to each $x \in \operatorname{Ext}^1(M, N)$ a well defined equivalence class of extensions. Some additional work shows that this map is the inverse of θ. $\square$

Remark 6.3. 1) The bijection proved in Theorem 6.2 equips the set of equivalence classes of extensions with the structure of an abelian group. This structure can be defined directly in terms of extensions.

2) The theory sketched in this section can be extended to higher groups Ext^i $(i \geqslant 2)$.

7. The functors Tor_i

In this section, we fix a ring A and denote by $A - \mathbf{Mod}$ (respectively $\mathbf{Mod} - A$) the abelian category of left (respectively right) A-modules.

Fix a right module M and consider the covariant right exact functor

$$\mathscr{F}_M := M \otimes_A (-) : A - \mathbf{Mod} \to \mathbf{Ab},$$
$$\mathscr{F}_M(N) = M \otimes_A N.$$

Definition. *The left derived functors of $\mathscr{F}_M$ are called the* Tor*-groups and are denoted as*

$$\operatorname{Tor}_i^A(M, N) := L_i \mathscr{F}_M(N), \qquad i \geqslant 0.$$

We can also fix a left A-module N and consider the right exact functor

$$_N\mathscr{F} := (-) \otimes_A N : \mathbf{Mod} - A \to \mathbf{Ab},$$
$$_N\mathscr{F}(M) = M \otimes_A N.$$

Mimiking the agruments of Section 5 it is not difficult to prove that for all projective resolutions $P_\bullet$ and $Q_\bullet$ of M and N, there exist canonical and functorial quasi-isomorphisms

$$(P_\bullet \otimes_A N) \xrightarrow{\sim} \operatorname{Tot}(P_\bullet \otimes_A Q_\bullet) \xleftarrow{\sim} (M \otimes_A Q_\bullet).$$

In particular

$$L_i\, {}_N\mathscr{F}(M) \simeq L_i \mathscr{F}_M(N), \qquad \forall i \geqslant 0.$$

Proposition 7.1. *For any projective right A-module P the functor $\mathscr{F}_P := P \otimes_A (-)$ is exact. In particular,*

$$\operatorname{Tor}_i^A(P, N) = 0, \qquad \forall i \geqslant 1.$$

Proof. a) Since tensor product commutes with direct sums, the first assertion holds for free modules (which are direct sums of copies of A). Let P be an arbitrary projective module. By Proposition 1.2, there exists a free module F such that $F = P \oplus P'$ for some $P' \subset F$. Let

$$0 \to N' \to N \to N'' \to 0$$

be an exact sequence. Then we have an exact sequence

$$0 \to F \otimes_A N' \to F \otimes_A N \to F \otimes_A N'' \to 0$$

which can be also written in the form

$$0 \to (P \otimes_A N') \oplus (P' \otimes_A N') \to (P \otimes_A N) \oplus (P' \otimes_A N) \to (P \otimes_A N'') \oplus (P' \otimes_A N'') \to 0.$$

Therefore the sequence

$$0 \to P \otimes_A N' \to P \otimes_A N \to P \otimes_A N'' \to 0$$

is exact.

b) The vanishing of $\mathrm{Tor}_i^A(P, N)$ for $i \geqslant 1$ follows from a) and general properties of derived functors. □

Remark 7.2. *The same argument shows that for any projective right module Q the functor $(-) \otimes_A Q$ is exact and the derived functors $\mathrm{Tor}_i^A(-, Q)$ vanish for $i \geqslant 2$.*

Example. Take $A = \mathbf{Z}$. The complex

$$P_\bullet \ : \ 0 \to \mathbf{Z} \xrightarrow{m} \mathbf{Z} \to 0$$

is a projective resolution of $\mathbf{Z}/m\mathbf{Z}$. Therefore $P_\bullet \otimes_{\mathbf{Z}} N$ is isomorphic to the complex

$$0 \to N \xrightarrow{m} N \to 0.$$

We obtain that $\mathrm{Tor}_0^{\mathbf{Z}}(\mathbf{Z}/m\mathbf{Z}, N) = N/mN$ ($\simeq \mathbf{Z}/m\mathbf{Z} \otimes_{\mathbf{Z}} N$) and

$$\mathrm{Tor}_1^{\mathbf{Z}}(\mathbf{Z}/m\mathbf{Z}, N) = {}_mN,$$

where ${}_mN := \{x \in N \mid mx = 0\}$ is the m-torsion subgroup of N.

It can be proved that the functors Tor_i^A commute with direct limits, namely

$$\mathrm{Tor}_i^A(\varinjlim_j M_j, N) \simeq \varinjlim_j \mathrm{Tor}_i^A(M_j, N).$$

Using this property, we obtain the following:

Proposition 7.3. *For all $\mathbf{Z}$-modules M and N one has:*

i) $\mathrm{Tor}_1^{\mathbf{Z}}(M, N)$ *is a torsion group;*

ii)) $\mathrm{Tor}_i^{\mathbf{Z}}(M, N) = 0$ *for $i \geqslant 2$.*

Proof. Each module is a direct limit of the system of its finitely generated submodules. This reduces the proof to Proposition 7.1 and the above example. □

Definition. *A left (respectively right) A-module N is flat if the functor $(-) \otimes_A N$ (respectively $N \otimes_A (-)$) is exact.*

Proposition 7.4. *Let N be a left A-module. The following properties are equivalent:*

1) *N is flat;*
2) $\mathrm{Tor}^i_A(M, N) = 0$ *for all* M *and* $i \geqslant 1$;
3) $\mathrm{Tor}^1_A(M, N) = 0$ *for all* M.

Proof. The proof is straightforward and is left as an exercise. □

Exercise 12. Give an example of a non-projective flat module over **Z**.

CHAPTER 4

Cohomology of finite groups

1. Cohomology of groups: derived functors

1.1. Basic constructions. In this section, we redefine cohomology of groups using derived functiors and establish its basic properties.

Let G be a group. Each abelian group A can be considered as a trivial G-module:

$$g \cdot a = a, \qquad \forall g \in G, \quad a \in A.$$

This applies, in particular, to the group $\mathbf{Z}$. For any G-module M, we set

$$M^G := \{m \in M \mid gm = m, \quad \forall g \in G\}.$$

We will consider the assignment $M \to M^G$ as a functor

$$(-)^G \;:\; G - \mathbf{Mod} \to \mathbf{Ab}.$$

Lemma 1.2. *There exists an isomorphism of functors:*

$$(-)^G \simeq \mathrm{Hom}_{\mathbf{Z}}[G](\mathbf{Z}, -).$$

Proof. Let M be a G-module. It is easy to see that the map

$$\begin{aligned} &\mathrm{Hom}_{\mathbf{Z}[G]}(\mathbf{Z}, M) \to M^G, \\ &f \mapsto f(1) \end{aligned}$$

establishes an isomorphism

$$\mathrm{Hom}_{\mathbf{Z}[G]}(\mathbf{Z}, M) \simeq M^G.$$

Indeed, since $g(f(1)) = f(g(1)) = f(1)$, one has $f(1) \in M^G$, and the inverse map $M^G \to \mathrm{Hom}_{\mathbf{Z}[G]}(\mathbf{Z}, M)$ is given by

$$m \mapsto f \in \mathrm{Hom}_{\mathbf{Z}[G]}(\mathbf{Z}, M) \quad \text{such that } f(1) = m.$$

□

Corollary 1.3. *The functor $(-)^G$ is left exact.*

Of cause, this can be also easily checked directly.

Definition. *Let M be a G-module. The cohomology groups of G with coefficents in M are defined as*

$$H^i(G, M) := \mathrm{Ext}^i_{\mathbf{Z}[G]}(\mathbf{Z}, M).$$

Equivalently, $H^i(G, -)$ are the right derived functors of $(-)^G$.

For each short exact sequence of G-modules

$$0 \to M' \to M \to M'' \to 0$$

we have a long exact sequence of cohomology:

$$\begin{aligned} &0 \to H^0(G, M') \to H^0(G, M) \to H^0(G, M'') \xrightarrow{\delta} \\ &H^1(G, M') \to H^1(G, M) \to H^1(G, M'') \xrightarrow{\delta} H^2(G, M') \to \cdots \end{aligned} \tag{15}$$

1.4. The bar resolution. We now prove that our definition agrees with the definition given in Section 5 of Chapter 2. Set:

$$G^{i+1} = \underbrace{G \times G \times \cdots \times G}_{i+1}$$

and

$$P_i = \mathbf{Z}[G^{i+1}].$$

It is easy to see that P_i is a free $\mathbf{Z}[G]$-module generated by the elements of the form

$$(e, g_1, g_2, \ldots, g_i) \in G^{i+1} \tag{16}$$

(here e is the identity element of G). Define a map

$$\partial_i : P_i \to P_{i-1}$$

setting

$$\partial_i(g_0, g_1, \ldots, g_i) = \sum_{j=0}^{i} (-1)^j (g_0, g_1, \ldots, g_{j-1}, g_{j+1}, g_{j+2}, \ldots, g_i),$$

$$\text{if } (g_0, g_1, \ldots, g_i) \in G^{i+1}$$

and extending this formula to $\mathbb{Z}[G^{i+1}]$ by linearity. We also define the augmentation map:

$$\varepsilon : P_0 = \mathbf{Z}[G] \to \mathbf{Z},$$

$$\varepsilon\left(\sum_{g \in G} n_g g\right) = \sum_{g \in G} n_g.$$

Proposition 1.5. *The sequence*

$$\cdots \to P_{i+1} \xrightarrow{\partial_{i+1}} P_i \xrightarrow{\partial_i} \cdots \xrightarrow{\partial_1} P_0 \xrightarrow{\varepsilon} \mathbf{Z} \to 0$$

is a projective resolution of $\mathbf{Z}$.

Proof. The proof is routine and we omit it here. □

The complex $P_\bullet$ is called the bar resolution of $\mathbf{Z}$.
Using this resolution, we obtain the following

Theorem 1.6. *For any G-module M the complexes* $\mathrm{Hom}_{\mathbf{Z}[G]}(P_\bullet, M)$ *and* $C^\bullet(G, M)$ *are isomorphic.*

Proof. Recall that

$$C^i(G, M) = \{f : G^i \to M\}.$$

We see that the groups $\mathrm{Hom}_{\mathbf{Z}[G]}(P_i, M)$ and $C^i(G, M)$ are isomorphic: each morphism $\varphi \in \mathrm{Hom}_{\mathbf{Z}[G]}(P_i, M)$ is completely determined by its values

$$\varphi(e, g_1, g_2, \ldots, g_i).$$

Such a function is completely determined by its values on the elements of the form $(e, g_1, g_1g_2, \ldots, g_1g_2 \cdots g_i)$. Let $\varphi \mapsto f_\varphi$ be the map

$$\mathrm{Hom}_{\mathbf{Z}[G]}(P_i, M) \to C^i(G, M)$$

which to each φ associates the function

$$f_\varphi(g_1, g_2, \ldots, g_i) := \varphi(e, g_1, g_1g_2, \ldots, g_1g_2 \cdots g_i).$$

This gives isomorphisms

$$\mathrm{Hom}_{\mathbf{Z}[G]}(P_i, M) \simeq C^i(G, M), \qquad i \geqslant 0.$$

Writing the differentials explicitly, it is not difficult to check that $\mathrm{Hom}_{\mathbf{Z}[G]}(P_\bullet, M)$ is isomorphic to the complex

$$C^\bullet(G, M): \; C^0(G, M) \xrightarrow{d^0} C^1(G, M) \xrightarrow{d^1} C^2(G, M) \xrightarrow{d^2} \cdots$$

where

$$(d^i(f))(g_1, g_2, \ldots, g_{i+1}) = g_1(f(g_2, g_3, \ldots, g_{+1})+$$
$$+ \sum_{j=1}^{i} (-1)^j f(g_1, g_2, \ldots, g_jg_{j+1}, g_{j+2}, \ldots, g_{i+1}) + (-1)^{i+1} f(g_1, g_2, \ldots, g_i).$$

□

Recall that we set

$$Z^i(G, M) = \ker(d_i) \quad \text{(group of } i\text{-cocycles)},$$
$$B^i(G, M) = \mathrm{Im}(d_{i-1}) \quad \text{(group of } i\text{-coboundaries)}.$$

Then

$$H^i(G, M) \simeq Z^i(G, M)/B^i(G, M).$$

1.7. Coinduced modules. Let A be an abelian group. Set

$$A^* := \mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}[G], A).$$

Then A^* has a natural structure of a left G-module given by

$$\lambda(f)(\mu) := f(\mu\lambda), \qquad f \in A^*, \quad \lambda, \mu \in \mathbf{Z}[G].$$

For any $f \in A^*$, one has:

$$f\Big(\sum n_g g\Big) = \sum n_g f(g).$$

Therefore f is completely defined by the elements $f(g) \in A$, $g \in G$. Set

$$C^1(G, A) := \{f : G \to A\},$$

and equip $C^1(G,A)$ with a left action of G given by

$$(gf)(h) = f(hg), \qquad f \in C^1(G,A),\ g,h \in G.$$

Then $A^* \simeq C^1(G,A)$.

Definition. *The G-module A^* is called a coinduced module.*

Proposition 1.8. *One has:*

$$H^i(G,A^*) = \begin{cases} A, & \text{if } i = 0 \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Applying Proposition 8.5 to the rings $B = \mathbf{Z}$ and $A = \mathbf{Z}[G]$, we obtain an isomorphism

$$\mathrm{Hom}_{\mathbf{Z}[G]}(P, \mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}[G], A)) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{Z}}(P, A)$$

for any $\mathbf{Z}[G]$-module P. Using the definition of A^*, we can write it in the form

$$\mathrm{Hom}_{\mathbf{Z}[G]}(P, A^*) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{Z}}(P, A).$$

Let $P_\bullet$ be a free résolution of $\mathbf{Z}$. Then

$$H^n(G,A^*) = H^n(\mathrm{Hom}_{\mathbf{Z}[G]}(P_\bullet, A^*)) \simeq H^n(\mathrm{Hom}_{\mathbf{Z}}(P_\bullet, A)).$$

Since P_n are free $\mathbf{Z}$-modules, the sequence

$$0 \to \mathrm{Hom}_{\mathbf{Z}[G]}(\mathbb{Z}, A^*) \to \mathrm{Hom}_{\mathbf{Z}[G]}(P_\bullet, A^*)$$

is exact. Hence $H^i(G,A^*) = 0$ for $i \geqslant 1$, and for $i = 0$ one has:

$$H^0(G,A^*) = (A^*)^G = A$$

(see Exercise 14). □

Exercise 13. *Let G be an infinite cyclic group. Fix a generator g of G.*
1) Show that $\mathbf{Z}[G]$ is isomorphic to the ring $\mathbf{Z}[X,X^{-1}]$.
2) Show that the sequence

$$0 \to \mathbf{Z}[X,X^{-1}] \xrightarrow{\partial_0} \mathbf{Z}[X,X^{-1}] \xrightarrow{\varepsilon} \mathbf{Z} \to 0,$$

where $d_0(f(X)) = (X-1)f(X)$ and $\epsilon(f) = f(1)$, is a free resolution of $\mathbf{Z}$.
3) Let M be a G-module. Show that $H^0(G,M) = M^G$, $H^1(G,M) \simeq M/(g-1)M$ and $H^i(G,M) = 0$ for $i \geq 2$.

Exercise 14. *Let A be an abelian group and $f \in A^* = C^1(G,A)$. Show that if $gf = f$ for all $g \in G$, then f is a constant map, i.e. there exists $a \in A$ such that $f(g) = a$ for all $g \in G$. Therefore $(A^*)^G \simeq A$.*

2. Homology of groups

2.1. Let G be a group. We consider $\mathbf{Z}$ as a trivial right G-module.

Definition. *For any left G-module M, set*

$$H_i(G, M) := \mathrm{Tor}^i_{\mathbf{Z}[G]}(\mathbf{Z}, M).$$

The groups $H_i(G, M)$ are called the homology groups of G with coefficients in M.

Explicitly, let $P_\bullet$ be a projective resolution of $\mathbf{Z}$. Then $P_\bullet \otimes_{\mathbf{Z}[G]} M$ is a complex, and

$$H_i(G, M) = H_i(P_\bullet \otimes_{\mathbf{Z}[G]} M).$$

We compute $H_0(G, M)$. Let

$$\varepsilon : \mathbf{Z}[G] \to \mathbf{Z}$$

be the augmentation map

$$\varepsilon\left(\sum_{g\in G} n_g g\right) = \sum_{g\in G} n_g.$$

Set $I_G = \ker(\varepsilon)$. Explicitly,

$$I_G = \{\sum_{g\in G} n_g g \mid \sum_{g\in G} n_g = 0\}.$$

Lemma 2.2. *The ideal I_G is generated by the elements*

$$g - e, \qquad g \in G.$$

(here e is the identity element of G.)

Proof. It is clear that $g - e \in I_G$. Conversely, any $\sum_{g\in G} n_g g \in I_G$ can be written in the form:

$$\sum_{g\in G} n_g g = \sum_{g\in G} n_g(g - e) + \sum_{g\in G} n_g e = \sum_{g\in G} n_g(g - e).$$

Hence the elements $g - e$ generate I_G. □

We have a tautological exact sequence

$$0 \to I_G \to \mathbf{Z}[G] \xrightarrow{\varepsilon} \mathbf{Z} \to 0. \tag{17}$$

Let A be a ring. For any left A-module M and ideal $I \subset A$ we will write IM for the submodule of M generated by the elements am, $a \in I$, $m \in M$.

For a G-module M, we set

$$M_G := M/I_G M.$$

Proposition 2.3. *One has:*

$$H_0(G, M) \simeq M_G.$$

Proof. Since the tensor product is right exact, we have an exact sequence

$$I_G \otimes_{\mathbf{Z}[G]} M \to \mathbf{Z}[G] \otimes_{\mathbf{Z}[G]} M \to \mathbf{Z} \otimes_{\mathbf{Z}[G]} M \to 0.$$

Here $\mathbf{Z}[G] \otimes_{\mathbf{Z}[G]} M \simeq M$. On the other hand, the image of $I_G \otimes_{\mathbf{Z}[G]} M$ in M is $I_G M$. Therefore

$$H_0(G, M) = \mathbf{Z} \otimes_{\mathbf{Z}[G]} M \simeq M/I_G M.$$

□

For any short exact sequence of G-modules

$$0 \to M' \to M \to M'' \to 0$$

we have a long exact sequence of homology:

$$\cdots \to H_2(G, M') \to H_2(G, M) \to H_2(G, M'') \xrightarrow{\delta_2} H_1(G, M') \to$$

$$H_1(G, M) \to H_1(G, M'') \xrightarrow{\delta_1} H_0(G, M') \to H_0(G, M) \to H_0(G, M'') \to 0.$$

2.4. Induced modules. Let A be an abelian group. The tensor product $A_* = \mathbf{Z}[G] \otimes_{\mathbf{Z}} A$ is equipped with a natural structure of a left G-module:

$$g(h \otimes a) = (gh) \otimes a, \qquad g, h \in G, \quad a \in A.$$

Definition. *The G-module A_* is called an induced module.*

Proposition 2.5. *One has*

$$H_n(G, A_*) = \begin{cases} A, & \text{if } n = 0 \\ 0, & \textit{otherwise}. \end{cases}$$

Proof. Let $A_* = \mathbf{Z}[G] \otimes_{\mathbf{Z}} A$ and let $P_\bullet \to \mathbf{Z}$ be a free resolution of $\mathbf{Z}$. Then P_n are free abelian groups, and

$$H_n(G, A) = H_n(P_\bullet \otimes_{\mathbf{Z}[G]} A_*) = H_n(P_\bullet \otimes_{\mathbf{Z}[G]} (\mathbf{Z}[G] \otimes_{\mathbf{Z}} A)) = H_n(P_\bullet \otimes_{\mathbf{Z}} A).$$

Since P_n are free, the complex

$$P_\bullet \otimes_{\mathbf{Z}} A \to A \to 0$$

is exact. Hence $H_n(G, A_*) = 0$ for $n \geqslant 1$. On the other hand,

$$H_0(G, A_*) \simeq \mathbf{Z} \otimes_{\mathbf{Z}[G]} (\mathbf{Z}[G] \otimes_{\mathbf{Z}} A) \simeq A.$$

□

Proposition 2.6. *One has:*

$$H_{-1}(G, \mathbf{Z}) \simeq G/[G, G].$$

Proof. The short exact sequence (17) induces a long exact sequence of homology:

$$H_{-1}(G, \mathbf{Z}[G]) \to H_{-1}(G, \mathbf{Z}) \to H_0(G, I_G) \to H_0(G, \mathbf{Z}[G]).$$

Since $\mathbf{Z}[G]$ is free, $H_{-1}(G, \mathbf{Z}[G]) = 0$, and we have an isomorphism:

$$H_{-1}(G, \mathbf{Z}) \simeq I_G/I_G^2.$$

Now the proposition follows from Exercise 15 below. □

Exercise 15. *Show that the map*

$$G \to I_G/I_G^2, \qquad g \mapsto g - 1 \pmod{I_G^2}$$

induces an isomorphism $G/[G,G] \simeq I_G/I_G^2$.

Exercise 16. *Let G be an infinite cyclic group. Fix a generator g of G. Show that $H_0(G,M) = M/(g-1)M$, $H_1(G,M) = M^G$ and $H_i(G,M) = 0$ for $i \geqslant 2$.*

3. Tate (co)homology

3.1. Let

$$0 \to M' \to M \to M'' \to 0$$

be a short exact sequence of G-modules. We have long exact sequences:

$$\cdots \to H_1(G,M') \to H_1(G,M) \to H_1(G,M'') \xrightarrow{\delta_1} M'/I_G M' \to M/I_G M \to M''/I_G M'' \to 0$$

and

$$0 \to M'^G \to M^G \to M''^G \xrightarrow{\delta_0} H^1(G,M') \to H^1(G,M) \to H^1(G,M'') \to \cdots$$

If G is finite, we can glue these sequences.

Namely, assume that G is finite. For any G-module M, we define the norm map:

$$N : M \to M,$$
$$N(m) = \sum_{g \in G} gm.$$

Set ${}_N M := \ker(N)$ and $N(M) := \operatorname{Im}(N)$.

Lemma 3.2. *One has:*

$$N(M) \subset M^G,$$
$$I_G M \subset {}_N M.$$

Proof. a) Since

$$g(N(m)) = g \sum_{h \in G} hm = \sum_{h \in G} (gh)m = \sum_{h \in G} hm = N(m),$$

we have $N(M) \subset M^G$.

b) Let $x = (g-e)m$. Then

$$N(x) = \sum_{h \in G} h(g-e)x = \sum_{h \in G} (hg)x - \sum_{h \in G} h(x) = N(x) - N(x) = 0.$$

Hence $x \in {}_N M$. Since I_G is generated by the elements $g - e$, $g \in G$, the submodule $I_G M \subset M$ is generated by $x = (g-e)m$, $g \in G$, $m \in M$. Therefore $N(y) = 0$ for all $y \in I_G M$. □

Theorem 3.3. *Let G be a finite group. Then for any exact sequence of G-modules*

$$0 \to M' \to M \to M'' \to 0$$

there exists a long exact sequence:

$$\begin{aligned}&\cdots \to H_1(G, M') \to H_1(G, M) \to H_1(G, M'') \to {}_N M'/I_G M' \to \\ &\to {}_N M/I_G M \to {}_N M''/I_G M'' \xrightarrow{\delta_0} M'^G/N(M') \to M^G/N(M) \to \\ &\to M''^G/N(M'') \xrightarrow{\delta_0} H^1(G, M') \to H^1(G, M) \to H^1(G, M'') \to \cdots\end{aligned}$$

Comments on the proof. We only need to show that there exists an exact sequence

$$\begin{aligned}&H_1(G, M'') \to {}_N M'/I_G M' \to {}_N M/I_G M \to {}_N M''/I_G M'' \to M'^G/N(M') \\ &\to M^G/N(M) \to M''^G/N(M'') \to H^1(G, M').\end{aligned}$$

Applying the snake lemma to the diagram

$$\begin{array}{ccccccc} & & M'/I_G M' & \longrightarrow & M/I_G M & \longrightarrow & M''/I_G M'' & \longrightarrow & 0 \\ & & \downarrow N & & \downarrow N & & \downarrow & & \\ 0 & \longrightarrow & M'^G & \longrightarrow & M^G & \longrightarrow & M''^G & & \end{array}$$

we obtain an exact sequence:

$${}_N M'/I_G M' \to {}_N M/I_G M \to {}_N M''/I_G M'' \xrightarrow{\delta_0} M'^G/N(M') \to M^G/N(M) \to M''^G/N(M'')$$

So we only should prove that $\mathrm{Im}(\delta_1) \subset {}_N M'$ and $N(M'') \subset \ker(\delta_0)$. □

Definition. *The Tate cohomology of M is defined as*

$$\hat{H}^i(G, M) = \begin{cases} H^i(G, M), & \text{if } i \geq 1, \\ M^G/N(M), & \text{if } i = 0, \\ {}_N M/I_G M, & \text{if } i = -1, \\ H_{-i-1}(G, M), & \text{if } i \leq -2. \end{cases}$$

With this notation, the long exact sequence for Theorem 3.3 reads:

$$\begin{aligned}&\cdots \to \hat{H}^{-2}(G, M) \to \hat{H}^{-2}(G, M'') \to \hat{H}^{-1}(G, M') \to \hat{H}^{-1}(G, M) \to \\ &\to \hat{H}^{-1}(G, M'') \to \hat{H}^0(G, M') \to \hat{H}^0(G, M) \to \hat{H}^0(G, M'') \to \\ &\to \hat{H}^1(G, M') \to H^1(G, M) \to \hat{H}^1(G, M'') \to \cdots\end{aligned}$$

Example. We consider $\mathbf{Z}$ as a trivial G-module. Then:

$$\begin{aligned}&\hat{H}^0(G, \mathbf{Z}) = \mathbf{Z}/n\mathbf{Z}, \qquad \text{where } n = |G|; \\ &\hat{H}^1(G, \mathbf{Z}) = 0, \\ &\hat{H}^{-1}(G, \mathbf{Z}) = 0.\end{aligned}$$

PROOF. We have $N(x) = nx$ for all $x \in \mathbf{Z}$. Hence $N(\mathbf{Z}) = n\mathbf{Z}$ and ${}_N\mathbf{Z} = 0$. As a result, $\hat{H}^0(G, \mathbf{Z}) = \mathbf{Z}/n\mathbf{Z}$ and $\hat{H}^{-1}(G, \mathbf{Z}) = 0$. Moreover,

$$\hat{H}^1(G, \mathbf{Z}) = H^1(G, \mathbf{Z}) = \mathrm{Hom}(G, \mathbf{Z}),$$

where $\mathrm{Hom}(G, \mathbf{Z}) = 0$ for finite groups. □

3.4. Recall that for an abelian group A we set $A^* = \mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}[G], A)$ and $A_* = \mathbf{Z}[G] \otimes_{\mathbf{Z}} A$). If G if finite, the map

$$\begin{aligned} &\alpha : A^* \to A_*, \\ &\alpha(f) = \sum_{g \in G} g^{-1} \otimes f(g). \end{aligned} \tag{18}$$

is a well defined isomorphism (see Exercise 17 below).

Proposition 3.5. *One has:*

$$\hat{H}^i(G, A^*) = \hat{H}^i(G, A_*) = 0, \qquad \forall i \in \mathbf{Z}.$$

PROOF. In view of Propositions 1.8 and 2.5 and the isomorphism (18), we only need to show that

$$\hat{H}^0(G, A_*) = \hat{H}^{-1}(G, A_*) = 0.$$

a) Let $x \in \sum_{s \in G} s \otimes a_s \in A_*$. Then

$$g(x) = \sum_{s \in G} gs \otimes a_s = \sum_{s \in G} s \otimes a_{g^{-1}s}.$$

If $x \in (A_*)^G$, then $g(x) = x$ for all $g \in G$, and therefore $a_{g^{-1}s} = a_s$ for all $g \in G$. Hence all a_s are equal, and x is of the form

$$x = (\sum_{s \in G} s \otimes a), \qquad a \in A.$$

Then $x = N(e \otimes a)$, where $e \in G$ is the identity element, and we proved that $x \in N(A_*)$. Therefore $(A_*)^G \subset N(A_*)$, and $\hat{H}^0(G, A_*) = 0$.

b) Now assume that

$$N(x) = N\left(\sum_{s \in G} s \otimes a_s\right) = 0.$$

Writing the action of N explicitly, it is easy to see that

$$\sum_{s \in G} a_s = 0,$$

Therefore

$$x = \sum_{s \in G} s \otimes a_s - \sum_{s \in G} e \otimes a_s = \sum_{s \in G} (s - 1) \otimes a_s \in I_G A_*.$$

Therefore ${}_N A_* \subset I_G A_*$ and $\hat{H}_0(G, A_*) = 0$. The proposition is proved. □

Exercise 17. *Assume that G is finite. Show that for any abelian group A, the map* (18) *is an isomorphism of G-modules.*

4. Cyclic groups

4.1. Cohomology of cyclic groups. In this section, we compute Tate cohomology $\hat{H}^n(G, M)$ for a finite cyclic group G. Set $n := |G|$ and fix a generator g of G. Define:

$$s = \sum_{h \in G} h = \sum_{i=0}^{n-1} g^i \in \mathbf{Z}[G],$$
$$t = g - e \in \mathbf{Z}[G].$$

Let

$$N^* : \mathbf{Z}[G] \to \mathbf{Z}[G],$$
$$N^*(x) = sx, \qquad x \in \mathbf{Z}[G],$$

denote the multiplication by s map, and similarly

$$T^* : \mathbf{Z}[G] \to \mathbf{Z}[G],$$
$$T^*(x) = tx, \qquad x \in \mathbf{Z}[G],$$

denote the multiplication by t.

Let $\mathbf{Z}[X]$ denote the ring of polynomials over $\mathbf{Z}$. Since $g^n = e$, we have an isomorphism:

$$\mathbf{Z}[G] \xrightarrow{\sim} \mathbf{Z}[X]/(X^n - 1), \qquad g \leftrightarrow X \pmod{X^n - 1}.$$

Under this isomorphism, the maps N^* et T^* correspond to the multiplication by $1 + X + X^2 + \cdots X^{n-1}$ and $1 + X$ in $\mathbf{Z}[X]$ respectively. From the formula

$$(X - 1)(X^{n-1} + X^{n-2} + \cdots + X + 1) = X^n - 1$$

it follows that

$$\ker(T^*) = \operatorname{Im}(N^*),$$
$$\ker(N^*) = \operatorname{Im}(T^*).$$

Therefore the sequence

$$\cdots \to \mathbf{Z}[G] \xrightarrow{N^*} \mathbf{Z}[G] \xrightarrow{T^*} \mathbf{Z}[G] \xrightarrow{N^*} \mathbf{Z}[G] \xrightarrow{T^*} \mathbf{Z}[G] \xrightarrow{\varepsilon} \mathbf{Z} \to 0$$

is exact and gives us a projective resolution $P_\bullet$ of $\mathbf{Z}$.

Let M be a G-module. Then $\operatorname{Hom}_{\mathbf{Z}[G]}(\mathbf{Z}[G], M) \simeq M$, and the complex $\operatorname{Hom}_{\mathbf{Z}[G]}(P_\bullet, M)$ reads:

$$0 \to M \xrightarrow{T} M \xrightarrow{N} M \xrightarrow{T} M \xrightarrow{N} \cdots,$$

where $N(x) = \sum_{h \in G} hx$ and $T(x) = gx - x$. The groups $H^i(G, M)$ are isomorphic to the cohomology of this complex? Namely:

$$H^i(G, M) = \begin{cases} M^G, & \text{si } i = 0 \\ {}_N M/T(M), & \text{if } i \text{ is odd} \\ M^G/N(M), & \text{if } i \geq 2 \text{ is even.} \end{cases}$$

Now we compute the homology groups $H_i(G, M)$. Since $\mathbf{Z}[G] \otimes_{\mathbf{Z}[G]} M \xrightarrow{\sim} M$, the complex $P_\bullet \otimes_{\mathrm{Gal}} M$ reads:

$$\cdots \xrightarrow{N} M \xrightarrow{T} M \xrightarrow{N} M \xrightarrow{T} M \to 0.$$

Therefore:

$$H_i(G,M)=\begin{cases} M/T(M), & \text{if } i=0\\ M^G/N(M), & \text{if } n \text{ is odd}\\ {}_NM/T(M), & \text{if } i\geq 2 \text{ is even.}\end{cases}$$

For the Tate groups we obtain:

$$\hat{H}^i(G,M)=\begin{cases} M^G/N(M), & \text{if } i \text{ is even}\\ {}_NM/T(M), & \text{if } i \text{ is odd.}\end{cases}$$

4.2. Herbrand quotient. We continue to assume that G is a cyclic group of finite order n and M isa G-module. Assume that $\hat{H}^0(G,M)$ and $\hat{H}^1(G,M)$ are finite. We define the Herbrand quotient of M as

$$h(M)=\frac{|\hat{H}^0(G,M)|}{|\hat{H}^1(G,M)|}=\frac{(M^G:N(M))}{({}_NM:T(M))}.$$

Proposition 4.3. *i) Let*

$$0\to M'\to M\to M''\to 0$$

be an exact sequence. If two of three Herbrand quotients are defined, so is the third and

$$h(M)=h(M')\,h(M'').$$

ii) If M is finite, then $h(M)=1$.

iii) Assume that N is a submodule of M of finite index. Then

$$h(M)=h(N).$$

Proof. See the homework, exercise 4. □

5. Change of groups

5.1. Shapiro's lemma. In this section, G is a finite group and H is a subgroup of G. Let M be an H-module. Then $\mathbf{Z}[H]\subset\mathbf{Z}[G]$, and we can consider $\mathbf{Z}[G]$ as an $\mathbf{Z}[H]$-module. We apply a construction from Section 8.4, Chapter 1. Set

$$\mathrm{Ind}_H^G(M)=\mathrm{Hom}_{\mathbf{Z}[H]}(\mathbf{Z}[G],M).$$

The ring $\mathbf{Z}[G]$ has a natural structure of a *right* $\mathbf{Z}[G]$-module, and this allows to define a structure of a *left* $\mathbf{Z}[G]$-module on $\mathrm{Ind}_H^G(M)$:

$$(gf)(\sigma)=f(\sigma g),\qquad \forall f\in\mathrm{Hom}_{\mathbb{Z}[H]}(\mathbf{Z}[G],M),\quad \sigma,g\in G.$$

In particular, if $H=\{e\}$ then

$$\mathrm{Ind}_{\{e\}}^G(M)=M^*.$$

Mimiking the construction of the isomorphism (18), one can prove that

$$\mathrm{Ind}_H^G(M)\simeq\mathbf{Z}[G]\otimes_{\mathbb{Z}[H]}M.$$

Lemma 5.2 (Faddeev–Eckmann–Shapiro)**.** *There exist canonical and functorial isomorphisms:*

$$\hat{H}^i(G,\mathrm{Ind}_H^G(M))\xrightarrow{\sim}\hat{H}^i(H,M),\qquad i\in\mathbf{Z}.$$

Proof. Take a *free* resolution $P_\bullet$ of $\mathbf{Z}$ in the category of $\mathbf{Z}[G]$-modules. Then $P_\bullet$ is also a free resolution of $\mathbf{Z}$ in the category of $\mathbf{Z}[H]$-modules. Applying Exercise 18 below to $A = \mathbf{Z}[H]$ and $B = \mathbf{Z}[G]$ we have an isomorphism:

$$\mathrm{Hom}_{\mathbf{Z}[G]}(P_\bullet, \mathrm{Ind}_H^G(M)) \simeq \mathrm{Hom}_{\mathbf{Z}[G]}(P_\bullet, \mathrm{Hom}_{\mathbf{Z}[H]}(\mathbf{Z}[G], M)) \simeq \mathrm{Hom}_{\mathbf{Z}[H]}(P_\bullet, M).$$

Therefore

$$H^i(G, \mathrm{Ind}_H^G(M)) = H^i(\mathrm{Hom}_{\mathbf{Z}[G]}(P_\bullet, \mathrm{Ind}_H^G(M))) = H^i(\mathrm{Hom}_{\mathbf{Z}[H]}(P_\bullet, M)) = H^n(H, M)$$

for all $i \geq 0$.

On the other hand, since $\mathrm{Ind}_H^G(M)$ is isomorphic to $\mathbf{Z}[G] \otimes_{\mathbf{Z}[H]} M$,, we have:

$$P_\bullet \otimes_{\mathbf{Z}[G]} \mathrm{Ind}_H^G(M) \simeq P_\bullet \otimes_{\mathbf{Z}[G]} (\mathbf{Z}[G] \otimes_{\mathbf{Z}[H]} M) \simeq P_\bullet \otimes_{\mathbf{Z}[H]} M.$$

Hence

$$H_i(G, \mathrm{Ind}_H^G(M)) = H_i(P_\bullet \otimes_{\mathbf{Z}[G]} \mathrm{Ind}_H^G(M)) \simeq H_i(P_\bullet \otimes_{\mathbf{Z}[H]} M) = H_i(H, M)$$

for all $i \geq 0$. □

Exercise 18. Let $\alpha : A \to B$ be a morphism of rings. Each B-module M can be seen as an A-module: the action of A on M is given by $ax = \alpha(a)x$. In particular, B is an A-module. If N is an A-module,we equip $\mathrm{Hom}_A(B, N)$ with a B-module structure setting:

$$(bf)(x) = f(xb), \qquad f \in \mathrm{Hom}_A(B, N), \quad b, x \in B.$$

Show that for all B-module M there exists a natural isomorphism

$$\mathrm{Hom}_B(M, \mathrm{Hom}_A(B, N)) \simeq \mathrm{Hom}_A(M, N).$$

5.3. Restriction, corestriction, inflation.

5.3.1. We first consider the general case. Let $\varphi : H \to G$ be a morphism of (not necessarily finite) groups. Then each G-module M has a natural structure of an H-module, which is induced by φ. Let $P_\bullet^H$ and $P_\bullet^G$ denote the projective resolutions of $\mathbf{Z}$ in the categories of $\mathbf{Z}[H]$ and $\mathbf{Z}[G]$-modules respectively. Then $P_\bullet^G$ can be also seen as a (not necessarily projective) resolution of $\mathbf{Z}$ in the categoy of $\mathbf{Z}[H]$-modules. Therefore we have a map $\varphi^* : P_\bullet^H \to P_\bullet^G$, which makes the diagram

$$\begin{array}{ccc} P_\bullet^H & \xrightarrow{\varepsilon} & \mathbf{Z} \\ \downarrow{\scriptstyle \varphi^*} & & \| \\ P_\bullet^G & \xrightarrow{\varepsilon} & \mathbf{Z} \end{array}$$

commute and is unique up to a chain homotopy. If induces a morphism

$$\mathrm{Hom}_{\mathbf{Z}[G]}(P_\bullet^G, M) \to \mathrm{Hom}_{\mathbf{Z}[H]}(P_\bullet^H, M)$$

and functorial morphisms on cohomology:

$$\varphi^* : H^i(G, M) \to H^i(H, M), \qquad i \geqslant 0. \tag{19}$$

The maps φ^* can be naturally described in terms of complexes $C^\bullet(-,-)$. Namely, φ gives rise to the morphism

$$C^\bullet(G,M) \to G^\bullet(H,M),$$
$$f \mapsto f \circ \varphi,$$

which induces the maps (19) on cohomology.

Dually, we have a morphism

$$P_\bullet^H \otimes_{\mathbf{Z}[H]} M \to P_\bullet^G \otimes_{\mathbf{Z}[G]} M$$

which induces functorial morphisms on homology:

$$\varphi_* \ : \ H_i(H,M) \to H_i(G,M) \qquad i \geqslant 0.$$

We are mainly interested in the case where H is a subgroup of G.

Definition. *Let H be a subgroup of G and M a G-module. The canonical maps*

$$\text{res} \ : \ H^i(G,M) \to H^i(H,M), \tag{20}$$
$$\text{cor} \ : \ H_i(H,M) \to H_i(G,M) \tag{21}$$

are called the restriction and the corestriction maps respectively.

5.3.2. We continue to assume that H is a subgroup of G and M is a G-module. Then M^H has a natural structure of G/H-module. The projection morphism $\pi \ : \ G \to G/H$ induces a morphism $P_\bullet^G \to P_\bullet^{G/H}$ on projective resolutions. The inclusion $M^H \to M$ induces a morphism

$$\text{Hom}_{\mathbf{Z}[G/H]}(P_\bullet^{G/H}, M^H) \to \text{Hom}_{\mathbf{Z}[G]}(P_\bullet^G, M)$$

and the resulting morphisms on cohomology

$$\text{inf} \ : \ H^i(G/H, M^H) \to H^i(G,M) \tag{22}$$

called the inflation maps.

These maps can be naturally described in terms of complexes $C^\bullet(-,-)$. Let $f \in C^i(G/H, M^H)$. Composing f with ${}_pi \ : \ G \to G/H$ we obtain the map $f \circ \pi \in C^i(G,M)$. This defines a morphism of complexes

$$C^\bullet(G/H, M^H) \to C^\bullet(G,M),$$

which induces the morphisms (22) on cohomology.

Proposition 5.4. *i) There exists an exact sequence*

$$0 \to H^1(G/H, M^H) \xrightarrow{\text{inf}} H^1(G,M) \xrightarrow{\text{res}} H^1(H,M).$$

ii) Assume that $H^i(H,M) = 0$ for all $i = 1, 2, \dots, q-1$. Then there exists an exact sequence:

$$0 \to H^q(G/H, M^H) \xrightarrow{\text{inf}} H^q(G,M) \xrightarrow{\text{res}} H^q(H,M).$$

Proof. Part i) can be proved by a direct computation using the description of group cohomology in terms of cocycles and coboundaries. Part ii) can be deduced from part i) using the dimension shifting. □

5.5. The case of finite groups.

5.5.1. In this subsection, we assume that G is finite. Consider a free $\mathbf{Z}[G]$-resolution of $\mathbf{Z}$:

$$P_\bullet\ :\quad \cdots \to P_2 \to P_1 \to P_0 \xrightarrow{\varepsilon} \mathbf{Z} \to 0. \tag{23}$$

By Exercise 17, we have an isomorphism

$$\mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}[G], \mathbf{Z}) \simeq \mathbf{Z}[G],$$

which shows that $P^i_* = \mathrm{Hom}_{\mathbf{Z}}(P_i, \mathbf{Z})$ are free $\mathbf{Z}[G]$-modules. Therefore $P^\bullet_* = \mathrm{Hom}_{\mathbf{Z}}(P_\bullet, \mathbf{Z})$ is a right *projective* resolution of $\mathbf{Z}$:

$$0 \to \mathbf{Z} \xrightarrow{\varepsilon^*} P^*_0 \to P^*_1 \to \cdots\,. \tag{24}$$

We can glue the resolutions (23) and (24):

$$L_\bullet\ :\quad \cdots \to P_2 \to P_1 \to P_0 \xrightarrow{\varepsilon^*\circ\varepsilon} P^*_0 \to P^*_1 \to \cdots$$

Here we define:

$$L_i = \begin{cases} P_i & \text{if } i \geqslant 0, \\ P^*_{-i-1} & \text{if } i \leqslant -1. \end{cases}$$

The complex $L_\bullet$ is called a complete resolution of $\mathbf{Z}$.

Theorem 5.6. *There exist functorial isomorphisms*

$$\hat{H}^i(G, M) = H^i(\mathrm{Hom}_{\mathbf{Z}[G]}(L_\bullet, M)).$$

Proof. The proof is omitted. □

5.6.1. Let H be a subgroup of G. Each L_i is free over $\mathbf{Z}[H]$, and $L_\bullet$ is a full resolution of $\mathbf{Z}$ in the category of $\mathbf{Z}[H]$-modules. Therefore the natural map

$$\mathrm{Hom}_{\mathbf{Z}[G]}(L_\bullet, M) \to \mathrm{Hom}_{\mathbf{Z}[H]}(L_\bullet, M)$$

induces morphisms

$$\mathrm{res}\ :\ \hat{H}^i(G, M) \to \hat{H}^i(H, M)$$

for *all* $i \in \mathbf{Z}$. It is easy to see that this definition agrees with (20) for $i \geqslant 0$. Therefore our construction extends the definition of the restriction map to the case $i \leqslant -1$.

5.6.2. Write G as the union of left cosets of H: $G = \bigcup_{k=1}^{n} g_k H$. Consider the map

$$t\ :\ \mathrm{Hom}_{\mathbf{Z}[H]}(L_i, M) \to \mathrm{Hom}_{\mathbf{Z}[G]}(L_i, M)$$

defined as follows:

$$(t(f))\,(x) = \sum_{k=1}^{n} g_k f(g_k^{-1}x), \qquad f \in \mathrm{Hom}_{\mathbf{Z}[H]}(L_i, M),\ \ x \in L_i.$$

An easy computation shows that t is well defined and $t(f)$ does not depend on the choice of representatives g_i. Hence we have a morphism of complexes

$$t\ :\ \mathrm{Hom}_{\mathbf{Z}[H]}(L_\bullet, M) \to \mathrm{Hom}_{\mathbf{Z}[G]}(L_\bullet, M),$$

which induces morphisms on cohomology:

$$\mathrm{cor}\ :\ \hat{H}^i(H, M) \to \hat{H}^i(G, M), \qquad i \in \mathbf{Z}.$$

It can be checked that this definition agrees with (21) if $i \leqslant -1$. Therefore our construction extends the definition of the corestriction map to the case $i \geqslant 0$.

Proposition 5.7. *The map*

$$\mathrm{cor} : \hat{H}^0(H, M) \to \hat{H}^0(G, M)$$

is induced by the map

$$N_{G/H} : M^H \to M^G, \qquad x \mapsto \sum_{k=1}^{r} g_k(x).$$

Proof. We have $\hat{H}^0(H, M) = M^H/N_H(M)$ and $\hat{H}^0(G, M) = M^G/N_G(M)$. The formula follows directly from the definition of the map t. □

Proposition 5.8. *Let* $(G : H) = n$. *Then*

$$\mathrm{cor} \circ \mathrm{res} = n$$

i.e. for all $i \in \mathbf{Z}$ *the composition*

$$\hat{H}^i(G, M) \xrightarrow{\mathrm{res}} \hat{H}^i(H, M) \xrightarrow{\mathrm{cor}} \hat{H}^i(G, M)$$

coicides with the multiplication by n map.

Proof. a) We first prove this formula for $i = 0$. The map $\mathrm{res} : \hat{H}^0(G, M) \to \hat{H}^0(H, M)$ is induced by the natural inclusion $M^G \to M^H$. By Proposition 5.7, for all $x \in M^G$, we have:

$$\mathrm{cor} \circ \mathrm{res}(x) = \mathrm{cor}(x) = \sum_{i=1}^{n} g_i(x) = nx,$$

because $g_i(x) = x$ for all i.

b)In the general case, we prove the proposition by induction using the dimension shifting. Assume that the statement holds for som i. We have an exact sequence

$$0 \to M \xrightarrow{\alpha} M^* \to X \to 0,$$

where the map $\alpha : M \to \mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}[G], M)$ is

$$\alpha(m)(g) := gm.$$

This exact sequence provides us with a diagram:

$$\begin{array}{ccccc} H^i(G,X) & \xrightarrow{\mathrm{res}} & H^i(H,X) & \xrightarrow{\mathrm{cor}} & H^i(G,X) \\ \downarrow & & \downarrow & & \downarrow \\ H^{i+1}(G,M) & \xrightarrow{\mathrm{res}} & H^{i+1}(H,M) & \xrightarrow{\mathrm{cor}} & H^{i+1}(G,M). \end{array}$$

The vertical arrows of the diagram are isomorphisms because $H^q(G, M^*) = 0$ for all q. By the induction hypothesis, the composition $H^i(G, X) \to H^i(H, X) \to H^i(G, X)$ is the multiplication by n map. Therefore this is also true for the bottom row. This proves the proposition for $i \geqslant 0$. For $i < 0$, we consider the exact sequence

$$0 \to Y \to M_* \xrightarrow{\beta} M \to 0,$$

with $\beta(g \otimes x) = gx$ and use the dimension shifting argument. □

From this proposition, we deduce two important corollaries:

Corollary 5.9. *If $n = |G|$, the groups $\hat{H}^i(G, M)$ are annihilated by the multiplication by n.*

Proof. We apply Proposition 5.8 to $H = \{e\}$. Since $\hat{H}^i(\{e\}, M) = 0$, we have $n = \mathrm{cor} \circ \mathrm{res} = 0$ i.e. the multiplication by n annihilates $\hat{H}^i(G, M)$. □

Let p be a prime number and $G_p \subset G$ be a p-Sylow subgroup of G. For each abelian group A, we denote by $A(p)$ the p-primary component of A, namely:

$$A(p) = \{x \in A \mid \exists m \geqslant 1 \text{ such that } p^m x = 0\}.$$

Corollary 5.10. *The restriction map*

$$\mathrm{res} : \hat{H}^i(G, M) \to \hat{H}^i(G_p, M)$$

is injective on the p-primary component of $\hat{H}^i(G, M)$ i.e. the induced map

$$\hat{H}^i(G, M)\,(p) \to \hat{H}^i(G_p, M)$$

is injective.

Proof. The composition

$$\hat{H}^i(G, M)\,(p) \xrightarrow{\mathrm{res}} \hat{H}^i(G_p, M) \xrightarrow{\mathrm{cor}} \hat{H}^i(G, M)$$

concides with the multiplication by $m = (G : G_p)$. Since G_p is a p-Sylow subgroup, we have $(m, p) = 1$. Therefore the multiplication by m is injective on the p-primary component. □

Exercise 19. Let H be a subgroup of a finite index of a group G. Write $G = \bigcup_{k=1}^{n} Hg_k$.

a) Show that $\mathbf{Z}[G] = \oplus_{k=1}^{n} \mathbf{Z}[H]g_i$ and deduce that $\mathbf{Z}[G]$ is isomorphic to $\mathbf{Z}[H]^{(n)}$ as a $\mathbf{Z}[H]$-module.

b) Show that a free resolution of $\mathbf{Z}[G]$-modules is also a free resolution of $\mathbf{Z}[H]$-modules.

c) Let A be an abelian group. Show that the induced G-module $A_* = \mathbf{Z}[G] \otimes_{\mathbf{Z}} A$ is also an induced H-module.

Exercise 20. a) Let $f : M \to N$ be a morphism of G-modules. Show that it commutes with the restriction and the corestriction maps, i.e. that the diagrams

$$\begin{array}{ccc} \hat{H}^i(G, M) & \xrightarrow{f^*} & \hat{H}^i(G, N) \\ \downarrow{\scriptstyle \mathrm{res}} & & \downarrow{\scriptstyle \mathrm{res}} \\ \hat{H}^i(H, M) & \xrightarrow{f^*} & \hat{H}^i(H, N) \end{array}$$

and

$$\begin{array}{ccc} \hat{H}^i(H, M) & \xrightarrow{f^*} & \hat{H}^i(H, N) \\ \Big\downarrow \text{cor} & & \Big\downarrow \text{cor} \\ \hat{H}^i(G, M) & \xrightarrow{f^*} & \hat{H}^i(G, N) \end{array}$$

are commutative.

b) Let

$$0 \to M' \to M \to M'' \to 0$$

be a short exact sequence of G-modules and let H be a subgroup of G. Show that the restriction and the corestriction commute with the coboundary maps, i.e. that the diagrams

$$\begin{array}{ccc} \hat{H}^i(G, M') & \xrightarrow{\delta} & \hat{H}^{i+1}(G, M'') \\ \Big\downarrow \text{res} & & \Big\downarrow \text{res} \\ \hat{H}^i(H, M') & \xrightarrow{\delta} & \hat{H}^{i+1}(H, M'') \end{array}$$

$$\begin{array}{ccc} H^i(H, M') & \xrightarrow{\delta} & H^{i+1}(H, M'') \\ \Big\downarrow \text{cor} & & \Big\downarrow \text{cor} \\ H^i(G, M') & \xrightarrow{\delta} & H^{i+1}(G, M'') \end{array}$$

are commutative.

c) Assume that H is normal in G. Show that for any morphism of G-modules $f : M \to N$ the diagram

$$\begin{array}{ccc} H^n(G/H, M^H) & \xrightarrow{f^*} & H^n(G/H, N^H) \\ \Big\downarrow \text{inf} & & \Big\downarrow \text{inf} \\ H^n(G, M) & \xrightarrow{f^*} & H^n(G, N). \end{array}$$

commutes (functoriality of inflation).

6. Cohomological triviality

Definition. *Let G be a finite group. A G-module M is cohomologically trivial if for all subgroups $H \subset G$ one has:*

$$\hat{H}^i(H, M) = 0, \qquad \forall i \in \mathbf{Z}.$$

Examples. i) For any abelian group A, the les modules A_* et A^* are cohomologically trivial. In particular, $\mathbf{Z}[G]$ is cohomologically trivial.

ii) If two of the three modules in a short exact sequence

$$0 \to M' \to M \to M'' \to 0$$

are cohomologically trivial, so is the third. This follows from the long exact sequence for Tate groups.

iii) A G-module M is uniquely (or strongly) divisible if for all nonzero $n \in \mathbf{N}$ the multiplication by n is an isomorphism $n : M \simeq M$. Any uniquely divisible G-module is cohomologically trivial. In particular,

$$\hat{H}^i(G, \mathbf{Q}) = 0, \qquad \forall i \in \mathbf{Z},$$

Proof. Let H be a subgroup of G of order m. Then the isomorphism $n : M \simeq M$ induces the multiplication by m map on cohomology:

$$m : \hat{H}^i(H, M) \xrightarrow{\sim} \hat{H}^i(H, M).$$

On the other hand, $\hat{H}^i(H, M)$ are killed by the multiplication by m. Hence $\hat{H}^i(H, M) = 0$ for all i. □

Le but de ce paragraphe est d'établir un critère de trivialité cohomologique (voir les théorèmes 8.3 et 8.4). On commence par quelques resultats auxiliaires. Soit p un nombre premier. On dit que G est un p-groupe si G est d'ordre p^k. Nous allons utiliser la formule suivante qui est bien connue. Soit B un ensemble fini muni d'une action de G. Alors

$$|B| = \sum_i |Gx_i| = \sum_i (G : G_{x_i}),$$

où G_{x_i} désigne le stabilisateur de $x_i \in B$. Ici x_i parcourt un système de représentants des orbites.

Lemma 6.1. *Soit G un p-groupe et soit A un G-module vérifiant $pA = 0$. Alors les assertions suivantes sont équivalentes:*
i) $A = 0$;
ii) $H^0(G, A) = 0$;
iii) $H_0(G, A) = 0$.

Proof. (voir [**?**], lemme 9.1).
i) $\Rightarrow$ *ii), iii)* C'est clair.
ii) $\Rightarrow$ *i)* Preuve par l'absurde. Supposons que $A \neq 0$. Soit x un élément non-nul de A et soit $B = \mathbf{Z}[G]x$ le sous-module engendré par x. Comme G est fini et $pA = 0$, B est fini. Le groupe G opère sur B et on peut utiliser la formule (**). Posons $p^{\alpha_i} = (G : G_{x_i})$. Alors (**) s'écrit

$$p^k = \sum_i p^{\alpha_i}.$$

Si $x_i \notin B^G$, alors $G_{x_i} \neq G$ et $\alpha_i \geq 1$. Si $x \in B^G$, alors $\alpha_i = 0$ et $p^{\alpha_i} = 1$. Donc p divise $|B^G|$. Comme $0 \in B^G$ on en déduit que $B^G \neq 0$, d'où $H^0(G, A) = A^G \neq 0$. Contradiction.

iii) $\Rightarrow$ *i)* On a $H_0(G, A) = A/JA$, donc iii) signifie que $JA = A$. Posons $B = \mathrm{Hom}(A, \mathbf{Z}/p\mathbf{Z})$. On définit une action de G sur B en posant

$$(gf)(a) = f(g^{-1}a), \qquad f \in \mathrm{Hom}(A, \mathbf{Z}/p\mathbf{Z}),\ a \in A.$$

Soit $f \in B^G$. Alors $f(a) = f(ga)$, d'où $f((g-e)a) = 0$ pour tout $g \in G$. Comme les éléments $g - e$ engendrent J, on en déduit que $f(JA) = 0$. Alors l'hypothèse $JA = A$ implique $f = 0$ ce qui montre que $B^G = 0$.

En appliquant *ii)* $\Rightarrow$ *i)* à B on en déduit que $B = 0$. Donc $\mathrm{Hom}(A, \mathbf{Z}/p\mathbf{Z}) = 0$, d'où $A = 0$.

□

Le lemme suivant correspond au lemme 9.2 et au théorème 9.1 de [**?**], §9.

Lemma 6.2. *Soit G un p-groupe et soit M un G-module vérifiant $pM = 0$. Supposons qu'il existe $q \in \mathbf{Z}$ tel que $\hat{H}^q(G, M) = 0$. Alors M est un $\mathbf{Z}/p\mathbf{Z}[G]$-module libre.*

Proof. a) On étudie d'abord le cas $q = -2$. Comme $pM = 0$ on peut considérer M/JM comme un $\mathbf{Z}/p\mathbf{Z}$-espace vectoriel. Soit $\{a_i\}_{i\in I}$ une famille d'éléments de M telle que $\bar{a}_i = a_i + JM$ forment une base de M/JM sur $\mathbf{Z}/p\mathbf{Z}$. Soit N le $\mathbf{Z}[G]$-sous-module de M engendré par $\{a_i\}_{i\in I}$ et soit $A = M/N$. On a une suite exacte

$$0 \to N \to M \to A \to 0$$

qui induit une suite exacte

$$N/JN \to M/JM \to A/JA \to 0.$$

L'application $N/JN \to M/JM$ est surjective par construction, d'où $A/JA = 0$. En utilisant le lemme 8.1 on en déduit que $A = 0$. Donc M est engendré par $\{a_i\}_{i\in I}$. Nous allons montrer que $\{a_i\}_{i\in I}$ est une base de M. Soit

$$L = (\mathbf{Z}/p\mathbf{Z}[G])^{(I)} \simeq \bigoplus_{i\in I} \mathbf{Z}/p\mathbf{Z}[G]\, e_i$$

un $\mathbf{Z}/p\mathbf{Z}[G]$-module libre de rang I (I peut être infini). On note $\{e_i\}_{i\in I}$ une base de L. Alors l'application

$$f : L \to M,$$
$$f(\sum_i \alpha_i e_i) = \sum_i \alpha_i a_i, \qquad \alpha_i \in \mathbf{Z}/p\mathbf{Z}[G]$$

est un homomorphisme surjectif. On note X le noyau de f et on considère la suite exacte

$$0 \to X \to L \to M \to 0.$$

On va montrer que $X = 0$.

La suite exacte longue de homologie s'écrit (voir 4.5.2)

$$\cdots \to H_1(G, M) \to X/JX \to L/JL \to M/JM \to 0.$$

Par l'hypothèse, on a $H_1(G, M) = \hat{H}^{-2}(G, M) = 0$. D'autre part, comme $\mathbf{Z}[G]/J \xrightarrow{\sim} \mathbf{Z}$, on a

$$\mathbf{Z}/p\mathbf{Z}[G]/(J\mathbf{Z}/p\mathbf{Z}[G]) \xrightarrow{\sim} \mathbf{Z}/p\mathbf{Z},$$

d'où

$$L/JL \xrightarrow{\sim} \bigoplus_{i\in I} (\mathbf{Z}/p\mathbf{Z})\, e_i$$

Comme

$$M/JM = \bigoplus_{i\in I}(\mathbf{Z}/p\mathbf{Z})\bar{a}_i$$

l'application $L/JL \to M/JM$ est un isomorphisme et la suite exacte (*) s'écrit:

$$0 \to X/JX \to L/JL \xrightarrow{\sim} M/JM \to 0.$$

Alors $X/JX = 0$ d'où $X = 0$ (lemme 8.1).

Donc M et L sont isomorphes ce qui montre le lemme dans le cas $q = -2$.

b) Cas général. Supposons que $\hat{H}^q(G, M) = 0$. En utilisant le décalage (voir 7.1) on trouve un G-module N tel que

$$\hat{H}^i(G, N) \xrightarrow{\sim} \hat{H}^{i+q+2}(G, M), \qquad \forall i \in \mathbf{Z}.$$

En posant $i = -2$ on obtient

$$\hat{H}^{-2}(G, N) = \hat{H}^q(G, M) = 0.$$

Il résulte de la partie a) de la preuve que N est $\mathbf{Z}/p\mathbf{Z}[G]$-libre. Donc il est cohomologiquement trivial. La relation (**) implique maintenant la trivialité cohomologique de M. En particulier, $\hat{H}^{-2}(G, M) = 0$ et on applique encore a). Le lemme est démontré. □

Maintenant nous pouvons démontrer les résultats principaux de ce paragraphe.

Theorem 6.3 (critère de trivialité cohomologique pour les p-groupes)**.** *Soient G un p-groupe et M un G-module. S'il existe $q \in \mathbf{Z}$ tel que*

$$\hat{H}^q(G, M) = \hat{H}^{q+1}(G, M) = 0,$$

alors M est cohomologiquement trivial.

Proof. a) Supposons d'abord que M est sans p-torsion i.e. que $px = 0$ implique $x = 0$. Alors la multiplication par p est injective et on a une suite exacte

$$0 \to M \xrightarrow{p} M \to M/pM \to 0.$$

La suite exacte longue de cohomologie s'écrit:

$$\cdots \to \hat{H}^q(G, M) \to \hat{H}^q(G, M/pM) \to \hat{H}^{q+1}(G, M) \to \cdots$$

Comme $\hat{H}^q(G, M) = \hat{H}^{q+1}(G, M) = 0$, on en déduit que $\hat{H}^q(G, M/pM) = 0$. Comme $p(M/pM) = 0$, le lemme 8.2 montre que M/pM est $\mathbf{Z}/p\mathbf{Z}[G]$-libre. En particulier, pour tout sous groupe $H \subset G$ on a

$$\hat{H}^i(H, M/pM) = 0, \qquad \forall i \in \mathbf{Z}.$$

En revenant à la suite exacte longue de cohomologie

$$\to \underbrace{\hat{H}^{i-1}(H, M/pM)}_{=0} \to \hat{H}^i(H, M) \xrightarrow{p} \hat{H}^i(H, M) \to \hat{H}^i(H, M/pM) = 0$$

on obtient que la multiplication par p induit un isomorphisme

$$\hat{H}^i(H, M) \xrightarrow{p} \hat{H}^i(H, M).$$

Par récurrence, pour tout $k \geq 1$ la multiplication par p^k est un isomorphisme

$$\hat{H}^i(H, M) \xrightarrow{p^k} \hat{H}^i(H, M).$$

D'autre part, soit $|H| = p^s$. On sait que $\hat{H}^i(H, M)$ est annulé par multiplication par p^s. En posant $k = s$ on obtient que $\hat{H}^i(H, M) = 0$. Donc, M est cohomologiquement trivial.

b) Cas général. Tout G-module M est quotient d'un $\mathbf{Z}[G]$-module libre L. Donc, on a une suite exacte

$$0 \to X \to L \to M \to 0.$$

Comme L est libre, $X \subset L$ est sans p-torsion. D'autre part L est cohomologiquement trivial et la suite exacte longue de cohomologie

$$(25) \qquad \underset{=0}{\to} \hat{H}^q(G, M) \to \hat{H}^{q+1}(G, X) \to \underset{=0}{\to} \hat{H}^{q+1}(G, L) \to$$
$$\underset{=0}{\to} \hat{H}^{q+1}(G, M) \to \hat{H}^{q+2}(G, X) \to \underset{=0}{\to} \hat{H}^{q+2}(G, L)$$

donne $\hat{H}^{q+1}(G, X) = \hat{H}^{q+2}(G, X) = 0$. Alors, par a) X est cohomologiquement trivial, d'où la trivialité cohomologique de M. Le théorème est démontré. □

Nous considérons maintenant le cas d'un groupe fini quelconque.

Theorem 6.4 (criterion of cohomological triviality)**.** *Let Gbe a finite group and M a G-module. Assume that there exists $q \in \mathbf{Z}$ such that for all subgroup $H \subset G$,*

$$\hat{H}^q(H, M) = \hat{H}^{q+1}(H, M) = 0.$$

Then M is cohomologically trivial.

Proof. Soit H un sous-groupe de G. On va montrer que $\hat{H}^i(H, M) = 0$ pour tout i. Il suffit de montrer que pour nombre premier p la composante p-primaire $\hat{H}^i(H, M)\,(p)$ est $= 0$. Soit H_p un p-groupe de Sylow de H. Par hypothèse, on a

$$\hat{H}^q(H_p, M) = \hat{H}^{q+1}(H_p, M) = 0.$$

Le théorème 8.3 implique que M est cohomologiquement trivial en tant que H_p-module. Donc $\hat{H}^i(H_p, M) = 0$. Par le corollaire 7.6, $\hat{H}^i(H, M)\,(p)$ s'injecte dans $\hat{H}^i(H_p, M)$, d'où $\hat{H}^i(H, M)\,(p) = 0$. Le théorème est démontré.

□

Exercise 21. *Let L/K be a finite Galois extension of fields and let* $G = \mathrm{Gal}(L/K)$. *Show that the additive group of L is cohomologically trivial as G-module. Hint: use the normal basis theorem.*

Exercise 22. Let G be a finite group. Show that $H^1(G, \mathbf{Z}) = 0$ and $H^2(G, \mathbf{Z}) \simeq \mathrm{Hom}(G, \mathbf{Q}/\mathbf{Z})$.

Exercise 23. Let G be a finite group.

1) Using the exact sequence

$$0 \to I_G \to \mathbf{Z}[G] \xrightarrow{\varepsilon} \mathbf{Z} \to 0$$

show that for each subgroup $H \subset G$ there exists canonical and functorial isomorphisms

$$\hat{H}^i(H, \mathbf{Z}) \xrightarrow{\sim} \hat{H}^{i+1}(H, I_G).$$

2) Deduce that

$$\hat{H}^1(H, I_G) = \mathbf{Z}/m\mathbf{Z}, \qquad m = |H|,$$
$$\hat{H}^2(H, I_G) = 0.$$

7. Tate theorem

7.1. The fundamental class. We maintain previous notation and conventions. Let G be a finite group of order n and let $I_G := \ker(\mathbf{Z}[G] \xrightarrow{\varepsilon} \mathbf{Z})$.

Lemma 7.2. *i) I_G is the free abelian group of rank $n-1$ generated by the elements $y_\tau := \tau - e$, $\tau \in G \setminus \{e\}$.*

ii) The following formulas hold true:

$$\sigma y_\tau = y_{\sigma\tau} - y_\sigma, \qquad \text{if } \sigma \neq e, \tau^{-1}.$$
$$\tau^{-1} y_\tau = -y_{\tau^{-1}},$$
$$e y_\tau = y_\tau.$$

Proof. The proof is left as an exercise. □

Let M be a G-module and let $\alpha \in Z^2(G, M)$ be a 2-cocycle with values in M. Then:

$$g_1\alpha(g_2, g_3) - \alpha(g_1g_2, g_3) + \alpha(g_1, g_2g_3) - \alpha(g_1, g_2) = 0.$$

Let

$$I = \bigoplus_{\substack{\sigma \in G \\ \sigma \neq e}} \mathbf{Z}x_\sigma = \mathbf{Z}x_{\sigma_1} + \cdots + \mathbf{Z}x_{\sigma_{n-1}}$$

be a free abelian group of rank $n-1$ with basis $G \setminus \{e\}$. un groupe abélien libre de rang $n-1 = |G|-1$. Set

$$\overline{M}_\alpha = M \oplus I$$

and define an action of G on $\overline{M}_\alpha$ by the formulas:

$$\sigma(m \oplus x_\tau) = (\sigma m + \alpha(\sigma, \tau)) \oplus (x_{\sigma\tau} - x_\sigma), \text{ si } \sigma \neq \tau^{-1}, e,$$
$$\sigma(m \oplus x_\tau) = (\sigma m + \alpha(\sigma, \tau)) \oplus (-x_{\tau^{-1}}), \text{ if } \sigma = \tau^{-1},$$
$$e(m \oplus x_\tau) = m \oplus x_\tau.$$

Proposition 7.3. *i) The above formulas equip $\overline{M}_\alpha$ with the structure of a G-module.*

ii) The sequence

$$0 \to M \xrightarrow{f_1} \overline{M}_\alpha \xrightarrow{f_2} I_G \to 0, \tag{26}$$

where $f_1(m) = m \oplus 0$ and $f_2(m \oplus x_\tau) = y_\tau$, is exact.

Proof. The proof is left as an exercise. □

The exact sequence (26) induces a map

$$\delta_M \,:\, H^1(G, I_G) \to H^2(G, M).$$

Taking the composition of this map with the isomorphism $\delta_I \,:\, \mathbf{Z}/n\mathbf{Z} \simeq H^1(G, I_G)$ (see Exercise 23), we obtain a map

$$\varphi \,:\, \mathbf{Z}/n\mathbf{Z} \to H^2(G, M).$$

Claim: $\varphi(\bar{1}) = \mathrm{cl}(\alpha)$.

Proof of the claim. From the definition of the coboundary map it follows that the image of $\bar{1}$ under the map δ_I is the class of the cocycle

$$\beta \,:\, g \mapsto g - e \in I_G.$$

To compute the image of $\mathrm{cl}(\beta)$ under δ_M, take the lift $\widehat{\beta}$ of β in $\overline{M}_\alpha$ given by $\widehat{\beta}(g) = 0 \oplus \beta(g)$. Then $\delta_M(\mathrm{cl}(\beta))$ is represented by the cocycle

$$(g_1, g_2) \mapsto g_1\widehat{\beta}(g_2) - \widehat{\beta}(g_2) = g_1(0 \oplus y_{g_2}) - 0 \oplus y_{g_2} = \alpha(g_1, g_2).$$

This proves the claim. □

Definition. *Let M be a G-module such that*

$$H^1(H, M) = 0, \qquad \textit{for all subgroups } H \subset G.$$

We say that $x = cl(\alpha) \in H^2(G, M)$ is a fundamental class if for each $H \subset G$ the following conditions hold:

a) $H^2(H, M)$ is cyclic of order $|H|$;

b) $H^2(H, M)$ is generated by $\mathrm{res}(x)$.

Proposition 7.4. *Assume that $H^1(H, M) = 0$ for all $H \subset G$. Let* $\mathrm{cl}(\alpha) \in H^2(G, M)$. *The following properties are equivalent:*

a) $\mathrm{cl}(\alpha)$ *is a fundamental class;*

b) For each $H \subset G$, the short exact sequence

$$0 \to M \to \overline{M}_\alpha \to I_G \to 0$$

induces an isomorphism

$$H^1(H, I_G) \simeq H^2(H, M).$$

c) $\overline{M}_\alpha$ is cohomologically trivial.

Proof. The equivalence a) ⇔ b) follows directly from definitions.

b)⇒c). We have a long exact cohomology sequence

$$H^1(H, M) \to H^1(H, \overline{M}_\alpha) \to H^1(H, I_G) \xrightarrow{\sim} H^2(H, M) \to H^2(H, \overline{M}_\alpha) \to H^2(H, I_G).$$

Here $H^1(H, M) = 0$ by assumptions and $H^2(H, I_G) \simeq H^1(H, \mathbf{Z}) = 0$ by Exercise 23. Therefore $H^1(H, \overline{M}_\alpha) \simeq H^2(H, \overline{M}_\alpha) = 0$ and $\overline{M}_\alpha$ is cohomologically trivial.

c)⇒ b). If $\overline{M}_\alpha$ is cohomologically trivial, then in the exact sequence

$$H^1(H, \overline{M}_\alpha) \to H^1(H, I_G) \to H^2(H, M) \to H^2(H, \overline{M}_\alpha)$$

$H^1(H, \overline{M}_\alpha) = H^2(H, \overline{M}_\alpha) = 0$. Hence the map $H^1(H, I_G) \to H^2(H, M)$ is an isomorphism. □

7.5. Tate theorem.

Theorem 7.6 (Tate)**.** . *Soit G un groupe fini et soit M un G-module. Supposons que $cl(\alpha) \in H^2(G, M)$ est une classe fondamentale. Alors, pour tout $q \in \mathbb{Z}$ la classe $cl(\alpha)$ induit un isomorphisme canonique*

$$\hat{H}^i(G, M) \xrightarrow{\sim} \hat{H}^{i-2}(G, \mathbf{Z}).$$

Proof. La suite exacte courte

$$0 \to M \to \overline{M}_\alpha \to I_G \to 0$$

induit une suite exacte

$$\hat{H}^{q-1}(G, \overline{M}_\alpha) \to \hat{H}^{q-1}(G, I_G) \to \hat{H}^q(G, M) \to \hat{H}^q(G\overline{M}_\alpha).$$

Par l'exercice I.8.9, le module $\overline{M}_\alpha$ est cohomologiquement trivial. Donc on a

$$\hat{H}^{q-1}(G, \overline{M}_\alpha) = \hat{H}^q(G, \overline{M}_\alpha) = 0$$

est la suite (*) se réduit à un isomorphisme

$$\hat{H}^{q-1}(G, I_G) \xrightarrow{\sim} \hat{H}^q(G, M).$$

Comme

$$\hat{H}^{q-1}(G, I_G) \xrightarrow{\sim} \hat{H}^{q-2}(G, \mathbf{Z})$$

(voir exercice I.8.6) on en déduit le théorème. □

CHAPTER 5

Local class field theory

1. Local fields

1.1. Basic definitions. We recall basic facts about local fields. Let K be a field. A discrete valuation on K is a surjective map $v_K : K \to \mathbf{Z} \cup \{+\infty\}$ satisfying the following properties:

1) $v_K(xy) = v_K(x) + v_K(y), \qquad \forall x, y \in K^*$;
2) $v_K(x + y) \geqslant \min\{v_K(x), v_K(y)\}, \qquad \forall x, y \in K^*$;
3) $v_K(x) = +\infty \Leftrightarrow x = 0$.

In other words, v_K is a surjective morphism of groups $K^* \to \mathbf{Z}$ extended to K by the condition $v_K(0) = +\infty$. To each discrete valuation one can associate:

The ring of integers $O_K := \{x \in K \mid v_K(x) \geqslant 0\}$;
The maximal ideal $\mathfrak{m}_K := \{x \in K \mid v_K(x) > 0\}$;
The residue field $k_K := O_K/\mathfrak{m}_K$;
The group of units $U_K := \{x \in K \mid v_K(x) = 0\}$.

A element $\pi_K \in O_K$ is a uniformizer of K if $v_K(\pi_K) = 1$. Note that $\mathfrak{m}_K = (\pi_K)$ and

$$K^* = U_K \times \langle \pi_K \rangle .$$

The valuation v_K equips K with a topology characterized by the following property:

$$\lim_{n\to+\infty} x_n = x \quad \Leftrightarrow \quad v_K(x_n - x) \underset{n\to+\infty}{\to} +\infty.$$

Definition. *A discrete valuation field K is called a local field if the following conditions hold:*

1) *K is complete;*
2) *The residue field k_K is finite.*

Recall the classification of local fields:

Theorem 1.2. *i) Each local field of characteristic* 0 *is isomorphic to a finite extension of $\mathbf{Q}_p$ with $p = \mathrm{char}(k_K)$.*

ii) Each local field of characteristic p is isomorphic to $\mathbf{F}_q((t))$ where $\mathbf{F}_q$ is a finite field of q elements and the valuation on $\mathbf{F}_q((t))$ is given by

$$v_{\mathbf{F}_q((t))}\left(\sum_k a_k t^k\right) = \min\{k \mid a_k \neq 0\}.$$

Let L/K be a finite extension of a local field K. Then L is a local field. Let k_L denote the residue field of L and let π_L be a uniformizer of L. Set

$$f := [k_L : k_K] \qquad \text{the inertia degree of } L/K,$$
$$e := v_L(\pi_K) \qquad \text{the ramification index of } L/K.$$

We have the fundamental relation:

$$ef = [L : K].$$

An extension L/K is unramified (respectively totally ramified) if $e = 1$ (respectively $f = 1$). If L/K is a finite extension of local fields, then there exists a unique subextention $K \subset L_0 \subset L$ such that L_0/K is unramified and L/K is totally ramified.

Let L/K be a finite Galois extension. We have a natural surjective map

$$\mathrm{Gal}(L/K) \to \mathrm{Gal}(k_L/k_K)$$

which is an isomorphism if and only in L/K is unfamified. In that case $\mathrm{Gal}(L/K) \simeq \mathrm{Gal}(k_L/k_K)$ is cyclic of order $f = f(L/K)$ and we denote by $\mathrm{Fr}_{L/K} \in \mathrm{Gal}(L/K)$ the inverse image of the Frobenius automorphism

$$\mathrm{fr}_{L/K} \,:\, k_L \to k_L,$$
$$x \mapsto x^q, \qquad q = |k_K|.$$

1.3. The multiplicative group of a local field. Let K be a local field. Set

$$U_K^0 = U_K,$$
$$U_K^n = \{x \in U_K \mid v_K(x-1) \geq n\},$$

or, equivalently,

$$U_K^n = 1 + \pi_K^n O_K, \qquad n \geqslant 0.$$

Proposition 1.4. *For each $n \geqslant 1$, one has:*

$$(U_K : U_K^n) = q^{n-1}(q-1),$$

where $q = |k_K|$.

Proof. The proof is omitted. □

In the rest of this section, we assume that $\mathrm{char}(K) = 0$. Set $e_K = e(K/\mathbf{Q}_p)$. Define:

$$\exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!},$$
$$\log(1+x) = \sum_{k=0}^{\infty} (-1)^{k+1} \frac{x^k}{k}.$$

Proposition 1.5. *i) For each $x \in O_K$ such that*

$$v_K(x) > \frac{e_K}{p-1}$$

the series $\exp(x)$ *converges to an element of O_K and*

$$\exp(x+y) = \exp(x)\ \exp(y).$$

ii) For each $x \in \mathfrak{m}_K$, the series $\log(1+x)$ *converges to an element of K and*

$$\log(xy) = \log(x) + \log(y).$$

iii) For all $m \geq \left[\frac{2e_K}{p-1}\right] + 1$, the maps exp *and* log *define isomorphisms :*

$$\exp : \mathfrak{m}_K^m \xrightarrow{\sim} U_K^m,$$
$$\log : U_K^m \xrightarrow{\sim} \mathfrak{m}_K^m,$$

which are inverse to each other.

Proof. The proof is omitted.

□

2. Cohomology of the group of units

Let L/K be a finite Galois extension of local fields and let $G = \mathrm{Gal}(L/K)$. The main result of this section is the following (see [**?**], §1.4, Proposition 1.3):

Proposition 2.1. *The group of units U_L contains a cohomologically trivial G-submodule of finte index.*

Proof. We will prove this proposition for local fields of characteristic 0. This assumption allows to use the p-adic exponential. In the general case, the proof is slightly different (see [**?**], Proposition 1.3).

By the normal basis theorem, there exists $\alpha \in L$ such that each $x \in L$ can be written in a unique way in the form

$$x = \sum_{g \in G} a_g g(\alpha), \qquad a_g \in K.$$

In other words, L is the free $K[G]$-module generated by α. Multiplying, if necessary, α by some $c \in O_K$, one can assume that $\alpha \in O_L$. Then

$$M := \sum_{g \in G} O_K g(\alpha)$$

is a G-submodule of O_L, which is isomorphic to $O_K[G]$. Therefore M is induced, hence cohomologically trivial. This implies that for all $m \geqslant 0$, the module $\pi_K^m M$ is cohomologically trivial. Set $e = e(L/K)$ and fix m such that $v_L(\pi_K^m) = me \geqslant \left[\frac{2e_L}{p-1}\right] + 1$. Set:

$$N = \exp(\pi_K^m M).$$

By Proposition 1.5, N is a submodule of $U_L^{(me)}$. Since exp defines an isomorphism between $\mathfrak{m}_L^{me}$ and $U_K^{(me)}$, N and M are isomorphic. In particular, N is cohomologically trivial. It remains to prove that $(U_L : N) < +\infty$. Note that O_L is a free O_K-module of rank $n = [L : K]$. Fix a base $x_1, \ldots, x_n$ of O_L over O_K. Then

$$x_i = b_{i1} g_1(\alpha) + b_{i2} g_2(\alpha) + \cdots + b_{in} g_n(\alpha), \qquad b_{ij} \in K.$$

Fix k such that $\pi_K^k b_{ij} \in O_K$ for all $1 \leqslant i, j \leqslant n$. Then $\pi_K^k x_i \in M$, and therefore $\pi_K^k O_L \subset M$. Hence

$$\mathfrak{m}_L^{e(k+m)} = \pi_K^{k+m} O_L \subset \pi_K^m M,$$

and

$$U_L^{(e(k+m))} = \exp(\mathfrak{m}_L^{e(k+m)}) \subset \exp(\pi_K^m M) = N.$$

Since $U_L^{(e(k+m))}$ has finite index in U_L, we obtain that $(U_L : N) < +\infty$. □

We will deduce from this proposition several important corollaries.

Definition. *Let G be a cyclic group and M a G-module. If the groups $\hat{H}^0(G, M)$ and $\hat{H}^1(G, M)$ are finite, we set*

$$h(M) := \frac{\left|\hat{H}^0(G, M)\right|}{\left|\hat{H}^1(G, M)\right|}$$

and call it the Herbrand index of M.

We recall the main properties of Herbrand index:

Proposition 2.2. *i) If M is finite, then $h(M) = 1$.*
ii) Let

$$0 \to M' \to M \to M'' \to 0$$

be a short exact sequence of G-modules. If two of the three terms of this sequence have finite Herbrand indexes, so the third and

$$h(M) = h(M')\, h(M'').$$

Recall that an extension L/K is called cyclic if G is a cyclic group.

Corollary 2.3. *Assume that L/K is a cyclic extension of local fields of degree n. Then:*
i) $h(U_L) = 1$.
ii) $h(L^*) = n$;
iii) $\left|\hat{H}^0(G, L^*)\right| = \left|H^2(G, L^*)\right| = n$.

Proof. i) Let $N \subset U_L$ be a cohomologically trivial G-submodule of finite index. Then I.6.3, on a

$$h(N) = 1,$$
$$h(U_L/N) = 1,$$
$$h(U_L) = h(U_L/N)\, h(N) = 1.$$

ii) The valuation map $v_L : L^* \to \mathbf{Z}$ gives rise to an exact sequence

$$0 \to U_L \to L^* \xrightarrow{v_L} \mathbf{Z} \to 0.$$

Since $h(\mathbf{Z}) = n$, we get

$$h(L^*) = h(U_L) h(\mathbf{Z}) = n.$$

iii) Recall that $\hat{H}^0(G, L^*) \simeq H^2(G, L^*)$. By definition,

$$h(L^*) = \frac{|\hat{H}^0(G, L^*)|}{|H^1(G, L^*)|}.$$

On the other hand, by Hilbert theorem 90

$$H^1(G, L^*) = 0.$$

hence

$$|\hat{H}^0(G, L^*)| = h(L^*) = n.$$

□

Corollary 2.4. *Let L/K be an unramified extension of local fields. Then U_L is cohomologically trivial.*

Proof. We use our criterion of cohomological triviality. Recall that
a) Each unramified extension is unramified;
b) If $H \subset G$ is a subgroup, then $H = \mathrm{Gal}(L/F)$ with $F = L^H$ (Galois theory).
It is sufficient to show that

$$H^1(G, U_L) = \hat{H}^0(G, U_L) = 0.$$

The short exact sequence

$$0 \to U_L \to L^* \to \mathbf{Z} \to 0$$

induces a long exact cohomology sequence

$$0 \to H^0(G, U_L) \to H^0(G, L^*) \to H^0(G, \mathbf{Z}) \to H^1(G, U_L) \to H^1(G, L^*) \to \dots$$

One has $H^0(G, L^*) = K^*$, $H^0(G, \mathbf{Z}) = \mathbf{Z}$ and $H^1(G, L^*) = 0$. Hence our sequence reads:

$$0 \to U_K \to K^* \xrightarrow{v_L} \mathbf{Z} \to H^1(G, U_L) \to 0.$$

Since L/K is unramified, $v_L(K^*) = v_L(L^*) = \mathbf{Z}$. Hence $H^1(G, U_L) = 0$. By Corollary 2.3, we have $h(U_L) = 1$, and therefore $\hat{H}^0(G, U_L) = 0$. □

Corollary 2.5. *Let L/K be a finite unramified extension. Then:*

$$\hat{H}^i(G, L^*) \xrightarrow{\sim} \hat{H}^i(G, \mathbf{Z}), \qquad \forall i \in \mathbf{Z}.$$

Proof. The short exact sequence

$$0 \to U_L \to L^* \to \mathbf{Z} \to 0$$

induces a long exact sequence

$$\hat{H}^i(G, U_L) \to \hat{H}^i(G, L^*) \to \hat{H}^i(G, \mathbf{Z}) \to \hat{H}^{i+1}(G, U_L)$$

Since U_L is cohomologically trivial, the middle map is an isomorphism. □

Exercise 24. *Assume that L/K is unramified. Show that que*

$$N_{L/K}(U_L) = U_K.$$

3. The Brauer group of a local field

3.1. The Brauer group. Let K be a field and let $K \subset L \subset E$ be Galois extensions of K. Since $H^1(\mathrm{Gal}(E/L), E^*) =, 0$ the inflation-restriction exact sequence reads:

$$0 \to H^2(\mathrm{Gal}(L/K), L^*) \to H^2(\mathrm{Gal}(E/K), E^*) \xrightarrow{\text{res}} H^2(\mathrm{Gal}(E/L), E^*).$$

Therefore the map inf is an injection.

Definition. *The direct limit*

$$\mathrm{Br}(K) = \varinjlim_{L/K} H^2(\mathrm{Gal}(L/K), E^*) = \bigcup_{L/K} H^2(\mathrm{Gal}(L/K), L^*).$$

where E/K runs all Galois extensions of K, is called the Brauer group of K.

Example. *let $K = \mathbf{R}$. Then*

$$\mathrm{Br}(\mathbf{R}) = H^2(\mathrm{Gal}(\mathbf{C}/\mathbf{R}), \mathbf{C}^*).$$

The group $\mathrm{Gal}(\mathbf{C}/\mathbf{R})$ *is cyclic of order* 2 :

$$\mathrm{Gal}(\mathbf{C}/\mathbf{R}) = \{\mathrm{id}, \sigma\}.$$

Hence

$$\mathrm{Br}(\mathbf{R}) = (\mathbf{C}^*)^{\sigma=1}/N(\mathbf{C}^*) = \mathbf{R}^*/(\mathbf{R}^*)^+ \simeq \{1, -1\}.$$

3.2. The Brauer group of a local field. In this section, K is a local field. We define the unramified part of the Brauer group of K as

$$\mathrm{Br}(K)_{\mathrm{ur}} = \bigcup_{L/K \text{ unram.}} H^2(\mathrm{Gal}(L/K), L^*).$$

Let L/K be a finite unramified extension. Then

$$H^2(\mathrm{Gal}(L/K), L^*) \simeq H^2(\mathrm{Gal}(L/K), \mathbf{Z}) \simeq \mathrm{Hom}(\mathrm{Gal}(L/K), \mathbf{Q}/\mathbf{Z}).$$

The group $\mathrm{Gal}(L/K)$ is generated by the Frobenius automorphism $\mathrm{Fr}_{L/K}$. The map

$$\begin{aligned} \mathrm{Hom}(\mathrm{Gal}(L/K), \mathbf{Q}/\mathbf{Z}) &\to \frac{1}{n}\mathbf{Z}/\mathbf{Z}, \\ \chi &\mapsto \chi(\mathrm{Fr}_{L/K}). \end{aligned}$$

is an isomorphism.

The following lemma shows that these isomorphisms are compatible:

Lemma 3.3. *Let $K \subset L \subset E$ be a tower of Galois extensions. The following diagram is commutative*

$$\begin{array}{ccccccc}
H^2(\mathrm{Gal}(L/K), L^*) & = & H^2(\mathrm{Gal}(L/K), \mathbf{Z}) & = & \mathrm{Hom}(\mathrm{Gal}(L/K), \mathbf{Q}/\mathbf{Z}) & \longrightarrow & \mathbf{Q}/\mathbf{Z} \\
\downarrow{\scriptstyle \mathrm{inf}} & & \downarrow{\scriptstyle \mathrm{inf}} & & \downarrow{\scriptstyle \mathrm{inf}} & & \Vert \\
H^2(\mathrm{Gal}(E/K), E^*) & = & H^2(\mathrm{Gal}(E/K), \mathbf{Z}) & = & \mathrm{Hom}(\mathrm{Gal}(E/K), \mathbf{Q}/\mathbf{Z}) & \longrightarrow & \mathbf{Q}/\mathbf{Z}
\end{array}$$

Proof. The commutativity of the left squares follows from the functoriality of inflation. The commutativity of the third square follows from the fact that the projection $\mathrm{Gal}(E/K) \to \mathrm{Gal}(L/K)$ sends $\mathrm{Fr}_{E/K}$ to $\mathrm{Fr}_{L/K}$. □

Passing to the direct limit, we obtain an isomorphism

$$\mathrm{inv}_K : \mathrm{Br}(K)_{\mathrm{ur}} = \bigcup_{L/K\ \mathrm{unram.}} H^2(\mathrm{Gal}(L/K), L^*) \xrightarrow{\sim} \bigcup_n \frac{1}{n}\mathbf{Z}/\mathbf{Z} = \mathbf{Q}/\mathbf{Z}.$$

Let F/K be a finite extension of degree n. For any unramified extension L/K set $L' = LF$. Then L'/F is unramified. Set $G = \mathrm{Gal}(L/K)$ and $H = \mathrm{Gal}(L'/F)$. By Galois theory, $H \subset G$, and we have a map

$$H^2(G, L^*) \xrightarrow{\mathrm{res}} H^2(H, L^*) \to H^2(H, L'^*).$$

To simplify the notation, we denote it by res. Passing to the direct limit, we obtain a map

$$\mathrm{res} : \mathrm{Br}(K)_{\mathrm{ur}} \to \mathrm{Br}(F)_{\mathrm{ur}}.$$

Proposition 3.4. *Let F/K be a finite extension of degree n. Then the diagram*

$$\begin{array}{ccc} \mathrm{Br}(K)_{\mathrm{ur}} & \xrightarrow{\mathrm{inv}_K} & \mathbf{Q}/\mathbf{Z} \\ \downarrow{\scriptstyle \mathrm{res}} & & \downarrow{\scriptstyle n} \\ \mathrm{Br}(F)_{\mathrm{ur}} & \xrightarrow{\mathrm{inv}_F} & \mathbf{Q}/\mathbf{Z}. \end{array}$$

commutes.

Proof. On pose $e = e(F/K)$ et $f = f(F/K)$. Soit L/K une extension non-ramifiée. Alors $f = f(L'/F)$. Nous allons démontrer que le diagramme suivant est commutatif:

$$\begin{array}{ccccccc} H^2(G, L^*) & = & H^2(G, \mathbb{Z}) & = & \mathrm{Hom}(G, \mathbb{Q}/\mathbb{Z}) & \longrightarrow & \mathbb{Q}/\mathbb{Z} \\ \downarrow{\scriptstyle \mathrm{res}} & & \downarrow{\scriptstyle e\,\mathrm{res}} & & \downarrow{\scriptstyle e\,\mathrm{res}} & & \downarrow{\scriptstyle n=ef} \\ H^2(H, L'^*) & = & H^2(H, \mathbb{Z}) & = & \mathrm{Hom}(H, \mathbb{Q}/\mathbb{Z}) & \longrightarrow & \mathbb{Q}/\mathbb{Z} \end{array}$$

En effet, comme L/K est non-ramifiée, on a $e(L'/L) = e(F/K) = e$. Donc, on a un diagramme commutatif

$$\begin{array}{ccc} L & \xrightarrow{v_L} & \mathbb{Z} \\ \downarrow & & \downarrow{\scriptstyle e} \\ L' & \xrightarrow{v_{L'}} & \mathbb{Z} \end{array}$$

qui donne la commutativité du premier carré de (*):

$$\begin{array}{ccc} H^2(G, L^*) & = & H^2(G, \mathbb{Z}) \\ \downarrow & & \downarrow \\ H^2(H, L'^*) & \longrightarrow & H^2(H, \mathbb{Z}) \end{array}$$

Comme l'application "restriction" commute avec l'application "cobord" le deuxième carré est commutatif. Pour démontrer la commutativité du dernier carré, posons $q_K = |k_K|$, $q_F = |k_F|$, $q_L = |k_L|$ et $q_{L'} = |k_{L'}|$. Alors

$$fr_{k_{L'}/k_F}(x) = x^{q_F},$$
$$fr_{k_L/k_K}(x) = x^{q_K}.$$

Comme $q_F = q_K^f$, on voit que la restriction de $fr_{k_{L'}/k_F}$ sur k_L coïncide avec $fr^f_{k_L/k_K}$. Donc la restriction de $Fr_{L'/F}$ à L coïncide avec $Fr^f_{L/K}$. Ceci donne la commutativité du diagramme:

$$\begin{array}{ccc} H^1(G,\mathbf{Q}/\mathbf{Z}) & \longrightarrow & \mathbf{Q}/\mathbf{Z} \\ \downarrow{\scriptstyle \mathrm{res}} & & \downarrow{\scriptstyle f} \\ H^1(H,\mathbf{Q}/\mathbf{Z}) & \longrightarrow & \mathbf{Q}/\mathbf{Z}. \end{array}$$

Comme $n = ef$, on obtient la commutativité du troisième carré de (*). La proposition s'en déduit. □

Corollary 3.5. *Let F/K be a Galois extension of degree n and $G = \mathrm{Gal}(F/K)$. Set*

$$H^2(G, F^*)_{\mathrm{ur}} = H^2(G, F^*) \cap \mathrm{Br}(K)_{\mathrm{ur}}.$$

Then $H^2(G, F^*)_{\mathrm{ur}}$ is cyclic of order n. In particular,

$$\left|H^2(G, F^*)\right| \geqslant n.$$

Proof. Consider the exact sequence

$$0 \to H^2(G, F^*) \xrightarrow{\mathrm{inf}} \mathrm{Br}(K) \xrightarrow{\mathrm{res}} \mathrm{Br}(F).$$

Therefore, we have an exact sequence

$$0 \to H^2(G, F^*)_{\mathrm{ur}} \to \mathrm{Br}(K)_{\mathrm{ur}} \xrightarrow{\mathrm{res}} \mathrm{Br}(F)_{\mathrm{ur}}.$$

and a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^2(G,F^*)_{\mathrm{ur}} & \longrightarrow & \mathrm{Br}(K)_{\mathrm{ur}} & \xrightarrow{\mathrm{res}} & \mathrm{Br}(F)_{\mathrm{ur}} \\ & & \vdots & & \downarrow{\scriptstyle \mathrm{inv}_K} & & \downarrow{\scriptstyle \mathrm{inv}_F} \\ 0 & \longrightarrow & \frac{1}{n}\mathbf{Z}/\mathbf{Z} & \longrightarrow & \mathbf{Q}/\mathbf{Z} & \xrightarrow{n} & \mathbf{Q}/\mathbf{Z}. \end{array}$$

Hence, $H^2(G, F^*)_{\mathrm{ur}} \simeq \frac{1}{n}\mathbf{Z}/\mathbf{Z}$. □

We need the following technical result:

Lemma 3.6 (the "ugly" lemma)**.** *Let G be a finite group and M a G-module. Let q and ρ be two integers ≥ 0. Assume that the following conditions hold:*

a) $H^i(H, M) = 0$ for each subgroup $H \subset G$ and all $i = 1, 2, \dots, q-1$.

b) For each chain of subgroups $H \subset K \subset G$ such that H is normal in K and $(K : H)$ is a prime number, one has:

$$|H^q(K/H, M^H)| \text{ divise } (K : H)^\rho.$$

Then $|H^q(G, M)|$ *divides* $|G|^\rho$.

Proof. a) First assume that $|G| = p^k$, where p is a prime number. We show the lemma by induction on k. For $k = 1$, the statement is clearly true. Now assume that $k \geq 2$. Since the center $Z(G)$ of G is no trivial, G has a normal subgroup N such that $1 < |N| < |G|$. We have the inflation-restriction exact sequence

$$0 \to H^q(G/N, M^N) \xrightarrow{\text{inf}} H^q(G, M) \xrightarrow{\text{res}} H^q(N, M).$$

By the induction hypothesis, $|H^q(G/N, M^N)|$ divides $(G : N)^\rho = |G/N|^\rho$ and $|H^q(N, M)|$ divides $|N|^\rho$. Hence $|H^q(G, M)|$ divides

$$(G : N)^\rho |N|^\rho = |G|^\rho.$$

b) We prove the lemma for an arbitrary group G. For each p, let G_p denote the Sylow p-subgroup of G. Then

$$|G| = \prod_p |G_p|.$$

Applying part a) to the groups G_p, we obtain that $|H^q(G_p, M)|$ divides $|G_p|^\rho$ for each p. On the other hand, the maps

$$\text{res} : H^q(G, M)(p) \to H^q(G_p, M)$$

are injective. Hence $|H^q(G, M)(p)|$ divides $|G_p|^\rho$. Since

$$H^q(G, M) = \bigoplus_p H^q(G, M)(p),$$

we obtain that $|H^q(G, M)|$ divides $|G|^\rho$. □

We prove the main result of this section:

Theorem 3.7. *i) Let L/K be a finite Galois extension with the Galois group* $G = \text{Gal}(L/K)$. *Then*

a) $H^2(G, L^*)_{nr} = H^2(G, L^*)$ *i.e.* $H^2(G, L^*) \subset \text{Br}(K)_{nr}$.

b) $H^2(G, L^*)$ *is a cyclic group of order* $n = [L : K]$.

ii) $\text{Br}(K) = \text{Br}(K)_{\text{ur}}$ *and the map* inv_K *induces an isomorphism*

$$\text{inv}_K : \text{Br}(K) \simeq \mathbf{Q}/\mathbf{Z}.$$

Proof. i)We apply Lemma 3.6 to the group. $H^2(G, L^*)$. For all $H \subset G$, we have $H^1(H, L^*) = 0$ by Hilbert theorem 90. Let $K \subset G$ tbe a subgroup such that H is normal in K. Set $M = L^K$. Then $\text{Gal}(F/M) = K/H$. If $(K : H)$ is a prime number, then, by Corollary 2.3, the group $H^2(K/H, (L^*)^H) = H^2(\text{Gal}(F/M), F^*)$ is of order $(K : H)$. By Lemma 3.6, this implies that

$$|H^2(G, L^*)| \textit{ divides } |G| = [L : K].$$

On the other hand, by Corollary 3.5, we have:

$$|H^2(G, L^*)| \geq |H^2(G, L^*)_{\text{ur}}| = [L : K].$$

Hence $|H^2(G, L^*)| = [L : K]$ and $H^2(G, L^*)_{\text{ur}} = H^2(G, L^*)$. This proves a) and b).

ii) One has

$$\mathrm{Br}(K) = \cup_{L/K} H^2(\mathrm{Gal}(L/K), L^*) = \cup_{L/K} H^2(\mathrm{Gal}(L/K), L^*)_{\mathrm{ur}} = \mathrm{Br}(K)_{\mathrm{ur}},$$

This proves ii). □

Corollary 3.8. *Let L/K be a finite extension of degree n. Then the diagram*

$$\begin{array}{ccc} \mathrm{Br}(K) & \xrightarrow{\mathrm{inv}_K} & \mathbf{Q}/\mathbf{Z} \\ \downarrow{\scriptstyle \mathrm{res}} & & \downarrow{\scriptstyle n} \\ \mathrm{Br}(L) & \xrightarrow{\mathrm{inv}_L} & \mathbf{Q}/\mathbf{Z} \end{array}$$

commutes.

Proof. This follows from Proposition 3.4. □

4. The reciprocity map

4.1. The fundamental class. We apply Tate's theorem. Let L/K be a finite extension of degree n and let $G = \mathrm{Gal}(L/K)$. The restriction of inv_K on $H^2(G, L^*)$ gives an isomorphism:

$$\mathrm{inv}_{L/K},: \ H^2(G, L^*) \simeq \frac{1}{n}\mathbf{Z}/\mathbf{Z}.$$

Let $u_{L/K} \in H^2(G, L^*)$ be such that

$$\mathrm{inv}_{L/K}(u_{L/K}) = \frac{1}{n}.$$

Then $u_{L/K}$ is a canonical generator of $H^2(G, L^*)$. For any subextension $K \subset F \subset L$, we have:

$$\mathrm{inv}_{L/F}(\mathrm{res}(u_{L/K})) = [F : K]\,\mathrm{inv}_{L/K}(u_{L/K}) = \frac{[F : K]}{[L : K]} = \frac{1}{[L : F]}.$$

Hence $u_{F/K} = \mathrm{res}(u_{L/K})$. Since $H^1(\mathrm{Gal}(L/F), L^*) = 0$ for each intermediate extension, $u_{L/K}$ is a fundamental class. By Tate's theorem, we have canonical isomorphisms

$$\hat{H}^i(G, L^*) \simeq \hat{H}^{i-2}(G, \mathbf{Z}), \qquad i \in \mathbf{Z}.$$

4.2. The reciprocity map. Take $i = 0$ in the above isomorphism:

$$\hat{H}^0(G, L^*) \simeq \hat{H}^{-2}(G, \mathbf{Z}), \qquad i \in \mathbf{Z}.$$

Since $\hat{H}^0(G, L^*) \simeq K^*/N_{L/K}(L^*)$ and $\hat{H}^{-2}(G, \mathbb{Z}) \simeq G/[G, G]$, we obtain an isomorphism:

$$\theta_{L/K} \ : \ K^*/N_{L/K}(L^*) \xrightarrow{\sim} G/[G, G].$$

If L/K is abelian, $[G, G] = 1$ and we obtain an isomorphism

$$\theta_{L/K} \ : \ K^*/N_{L/K}(L^*) \simeq \mathrm{Gal}(L/K).$$

called the reciprocity map.