UNIVERSITÉ DE BORDEAUX
M2, p-adic Hodge Theory
2020-2021

Solutions to the midterm homework.

Exercice 1. Let $K = \mathbf{F}_p((t))$, thus K is a local field of characteristic p. Set $f(X) = X^p - X - \dfrac{1}{t} \in K[X]$.

1) Show that $f(X)$ has no roots in K.

Solution. Let K^{alg} denote an algebraic closure of K. We consider the discrete valuation v_K on K and denote again by v_K its continuation to K^{alg}. Let $\alpha \in K^{\mathrm{alg}}$ be a root of $f(X)$. Since $v_K(t^{-1}) = 1$, and $\alpha^p - \alpha = 1/t$, one has $v_K(\alpha) < 0$. Moreover $v_K(\alpha^p) = pv_K(\alpha) < v_K(\alpha)$. This implies that $v_K(\alpha^p - \alpha) = -pv_K(\alpha)$. On the other hand $v_K(\alpha^p - \alpha) = -v_K(t) = -1$ (t is a uniformizer of K). Therefore $v_K(\alpha) = -1/p$. This implies that $\alpha \notin K$.

2) Let $L = K(\alpha)$, where α is a root of $f(X)$. Express the roots of $f(X)$ in terms of α. Show that L is a splitting field of $f(X)$ *i.e.* that $f(X)$ decomposes over L into linear factors.

Solution. Let β be another root of $f(X)$. Set $a = \beta - \alpha$. Then $a^p = \beta^p - \alpha^p$ (note that K is of characteristic p). Therefore a is a root of the polynomial $X^p - X$. Since the roots of $X^p - X$ are the elements of $\mathbf{F}_p$, we obtain that the roots of $f(X)$ are

$$\alpha + a, \qquad a \in \mathbf{F}_p.$$

Therefore

$$f(X) = \prod_{a \in \mathbf{F}_p} (X - (\alpha + a)).$$

This implies that $f(X)$ decomposes over $L = K(\alpha)$ into the product of linear factors.

3) Show that L/K is a Galois extension and that the map

$$\begin{cases} \varphi \,:\, \mathrm{Gal}(L/K) \to \mathbf{F}_p, \\ \varphi(g) = g(\alpha) - \alpha \end{cases}$$

is an injective homomorphism. Deduce that $[L : K] = p$.

Solution. For each $g \in \mathrm{Gal}(L/K)$, $g(\alpha) = \alpha + \varphi(g)$, where $\varphi(g) \in \mathbf{F}_p$. Therefore $\forall g_1, g_2 \in \mathrm{Gal}(L/K)$, one has:

$$g_1g_2(\alpha) = g_1(\alpha + \varphi(g_2)) = g_1(\alpha) + \varphi(g_2) = \alpha + \varphi(g_1) + \varphi(g_2).$$

This shows that $\varphi(g_1g_2) = \varphi(g_1) + \varphi(g_2)$. Moreover, g is completely defined by $g(\alpha)$ and therefore by $\varphi(g)$. Hence φ is an injective homomorphism, and $\mathrm{Im}(\varphi)$ is a nontrivial subgroup of $\mathbf{F}_p$. Since $\mathbf{F}_p$ is of prime order p, this implies that φ is an isomorphism. In particular, $[L : K] = p$.

4) Show that L/K is totally ramified and give an uniformizer of L.

Solution. Set $\pi_L = 1/\alpha$. Then $v_K(\pi_L) = -v_K(\alpha) = 1/p = 1/[L : K]$. This implies that $[L : K]$ is totally ramified and π_L is a uniformizer of L.

5) Describe the ramification subgroups of $G = \mathrm{Gal}(L/K)$ in low enumeration.

Solution. For any $g \in \mathrm{Gal}(L/K)$, one has:

$$g(\pi_L) - \pi_L = \frac{1}{\alpha + a(g)} - \frac{1}{\alpha} = -\frac{a(g)}{\alpha(\alpha + a(g))}.$$

If $a(g) \neq 0$, then

$$v_L(g(\pi_L) - \pi_L) = -v_L(\alpha(\alpha + a(g))) = -2v_L(\alpha) = 2.$$

Therefore

$$G = G_0 = G_1, \qquad G_2 = \{e\}.$$

Exercise 2. Let π_1 be a root of the polynomial $X^p - p$. For each $n \geqslant 1$, let $\pi_{n+1} = \sqrt[p]{\pi_n}$. Let $F_n = \mathbf{Q}_p(\pi_n)$ and $F_\infty = \bigcup_{n=0}^{\infty} F_n$. Show that F_∞/F is deeply ramified.

Solution. The minimal polynomial of π_n over $\mathbf{Q}_p$ is $f_n(X) = X^{p^n} - p$. Since f_n is Eisenstein, $O_{F_n} = \mathbf{Z}_p[\pi_n]$ and we can compute its different:

$$\mathfrak{D}_{F_n/\mathbf{Q}_p} = (f_n'(\pi_n)) = (p^n \pi_n^{p^n-1}).$$

Therefore $v_{\mathbf{Q}_p}(\mathfrak{D}_{F_n/\mathbf{Q}_p}) = n + 1 - v_{\mathbf{Q}_p}(\pi_n) = n + 1 - \frac{1}{p^n}$. This implies that

$$v_{\mathbf{Q}_p}(\mathfrak{D}_{F_n/\mathbf{Q}_p}) \to +\infty \qquad \text{when } n \to +\infty.$$

Hence $F_\infty/\mathbf{Q}_p$ is deeply ramified.

Exercise 3. 1) Let ζ_{p^n} be a p^nth primitive root of unity. Set $K = \mathbf{Q}_p(\zeta_{p^n})$ and $G = \mathrm{Gal}(K/\mathbf{Q}_p)$. We have an isomorphism

$$\chi_n : G \simeq (\mathbf{Z}/p^n\mathbf{Z})^*, \qquad g(\zeta_{p^n}) = \zeta_{p^n}^{\chi_n(g)}.$$

Set $\Gamma = (\mathbf{Z}/p^n\mathbf{Z})^*$. Let $\Gamma^{(m)} = \{\bar{a} \in (\mathbf{Z}/p^n\mathbf{Z})^* \mid a \equiv 1 \pmod{p^m}\}$ (in particular $\Gamma^{(0)} = (\mathbf{Z}/p^n\mathbf{Z})^*$ and $\Gamma^{(n)} = \{1\}$).

1) Show that

$$\chi_n(G_i) = \Gamma^{(m)}, \quad \text{where } m \text{ is the unique integer such that } p^{m-1} \leqslant i < p^m.$$

Solution. Recall that for any n, $\mathbf{Q}_p(\zeta_{p^n})/\mathbf{Q}_p$ is totally ramified of degree $p^{n-1}(p-1)$, and $\zeta_{p^n} - 1$ is a uniformizer of $\mathbf{Q}_p(\zeta_{p^n})$. Set $\pi = \zeta_{p^n} - 1$. For any $g \in G$, one has:

$$v_K(g(\pi) - \pi) = v_K(\zeta_{p^n}^{\chi_n(g)} - \zeta_{p^n}) = v_K(\zeta_{p^n}^{\chi_n(g)-1} - 1).$$

If $g \neq e$, one can write:

$$\chi_n(g) - 1 = p^k \bar{c}, \qquad p \not| c, \quad 0 \leqslant k \leqslant n-1.$$

Then $\zeta_{p^n}^{\chi_n(g)-1} = \zeta_{p^{n-k}}^c - 1$ is a uniformizer of $\mathbf{Q}_p(\zeta_{p^{n-k}})$, and

$$v_K(g(\pi) - \pi) = v_K(\zeta_{p^{n-k}}^c - 1) = p^k$$

(note that $[K : \mathbf{Q}_p(\zeta_{p^{n-k}})] = p^k$). By the definition of ramification subgroups, $g \in G_i$ if and only if

$$v_K(g(\pi) - \pi) \geqslant i + 1.$$

Therefore $g \in G_i$ if and only if $p^k \geqslant i + 1$ if and only if $i < p^k$. Hence $\chi_n(G_i) = \Gamma^{(m)}$, where m is the smallest integer such that $i < p^m$. This proves the statement.

We remark that we proved that the ramification jumps of $K/\mathbf{Q}_p$ (in low enumeration) are $0, p-1, p^2-1, \ldots, p^{n-1}-1$.

2) Give Hasse–Herbrand's functions $\phi_{K/\mathbf{Q}_p}$ and $\psi_{K/\mathbf{Q}_p}$.

Solution. By part 1), for any integer $i \geqslant 0$,

$$(G : G_i) = (p-1)p^{m-1}, \qquad p^{m-1} \leqslant i \leqslant p^m - 1.$$

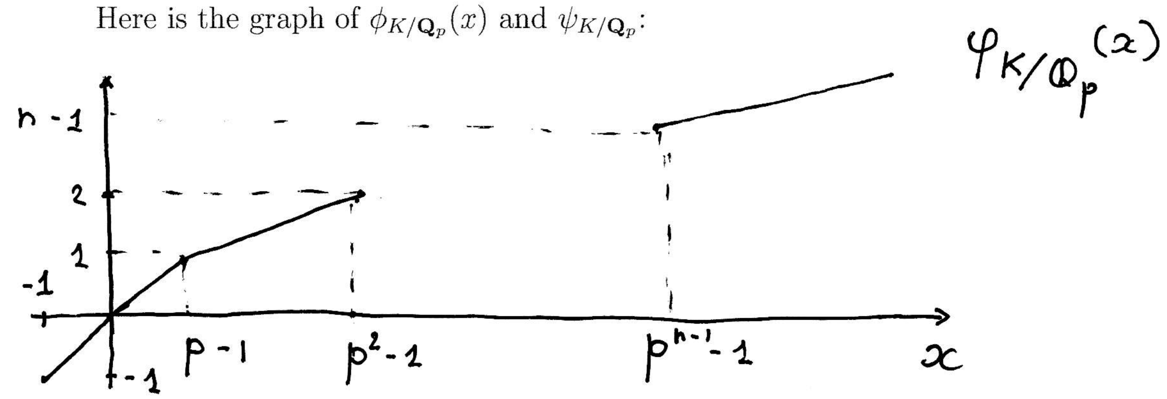
By definition and by part 1) of the exercise, for $0 \leqslant x \leqslant p^{n-1} - 1$, one has:

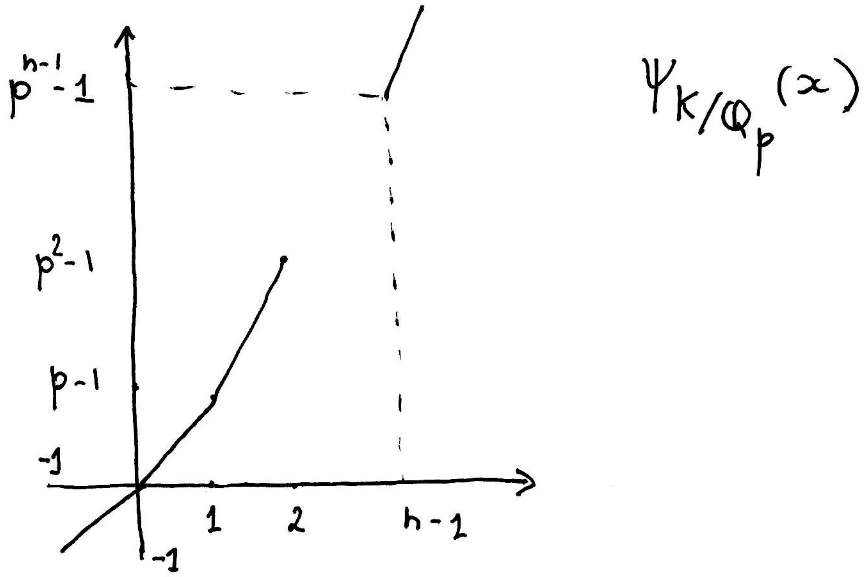
$$\phi'_{K/\mathbf{Q}_p}(x) = \frac{1}{(G : G_x)} = \frac{1}{p^{m-1}(p-1)}, \qquad p^{m-1} - 1 < x \leqslant p^m - 1.$$

For $x > p^{n-1} - 1$, one has:

$$\phi'_{K/\mathbf{Q}_p}(x) = \frac{1}{(G : G_x)} = \frac{1}{p^{n-1}(p-1)}.$$

Here is the graph of $\phi_{K/\mathbf{Q}_p}(x)$ and $\psi_{K/\mathbf{Q}_p}$:





3) Set

$$\Gamma^{(v)} = \Gamma^{(m)} \qquad \text{where } m \text{ is the smallest integer} \geqslant v.$$

Show that the upper ramification filtration on G is given by

$$\chi_n(G^{(v)}) = \Gamma^{(v)}.$$

Solution. Recall that $G^{(\phi_{K/\mathbf{Q}_p}(x))} = G_x$. Assume that $p^{m-1} - 1 < x \leqslant p^m - 1$, where $m \leqslant n - 1$. Then

$$m - 1 < \phi_{K/\mathbf{Q}_p}(x) \leqslant m.$$

Therefore for $m - 1 < v \leqslant m$, and by part 1) one has:

$$\chi_n(G^{(v)}) = \Gamma^{(m)} = \Gamma^{(v)}.$$

4) Let $(\zeta_{p^n})_{n\geqslant 1}$ denote a system of p^nth primitive roots of unity such that $\zeta_{p^n}^p=\zeta_{p^{n-1}}$. Set $K_n=\mathbf{Q}_p(\zeta_{p^n})$, $K_\infty=\underset{n\geqslant 1}{\cup}K_n$ and $G_\infty=\mathrm{Gal}(K_\infty/\mathbf{Q}_p)$. Let $U_{\mathbf{Q}_p}=\mathbf{Z}_p^*$ be the group of units of $\mathbf{Q}_p$. We have the isomorphism:

$$\chi\,:\,G_\infty\simeq U_{\mathbf{Q}_p},\qquad g(\zeta_{p^n})=\zeta_{p^n}^{\chi(g)},\quad \forall n\geqslant 1.$$

For any $v\geqslant 0$ set

$$U_{\mathbf{Q}_p}^{(v)}=U_{\mathbf{Q}_p}^{(m)},\qquad \text{where } m \text{ is the smallest integer} \geqslant v.$$

Show that

$$\chi(G_\infty^{(v)})=U_{\mathbf{Q}_p}^{(v)},\qquad \forall v\geqslant 0.$$

Solution. One has:

$$G_\infty^{(v)}=\varprojlim_n \mathrm{Gal}(K_n/\mathbf{Q}_p)^{(v)}.$$

We have a commutative diagram, where by part 3), w the horizonal maps are isomorphisms:

$$\begin{array}{ccc}
\mathrm{Gal}(K_n/\mathbf{Q}_p)^{(v)} & \xrightarrow{\chi_n} & (\mathbf{Z}/p^n\mathbf{Z})^{(v)}\\
\downarrow & & \downarrow\\
\mathrm{Gal}(K_{n-1}/\mathbf{Q}_p)^{(v)} & \xrightarrow{\chi_{n-1}} & (\mathbf{Z}/p^{n-1}\mathbf{Z})^{(v)}.
\end{array}$$

Hence

$$G_\infty^{(v)}\simeq \varprojlim_n(\mathbf{Z}/p^n\mathbf{Z})^{(v)}\simeq U_{\mathbf{Q}_p}^{(v)}.$$

Exercise 4. Let K_∞/K be a totally ramified $\mathbf{Z}_p$-extension of local fields of characteristic 0. Recall some standard notation. For each n, we denote by $K_n\subset K_\infty$ the unique subextension of degree p^n over K. Let $\Gamma=\mathrm{Gal}(K_\infty/K)$ and $\gamma\in\Gamma$ a fixed topological generator of Γ. Let $\widehat{K}_\infty$ denote the completion of K_∞ and $|\ |_K$ the extension of the absolute value on K to $\widehat{K}_\infty$.

We consider the normalized traces $\mathrm{T}_{K_\infty/K_n}\,:\,\widehat{K}_\infty\to K_n$ for the ground fields K_n $(n\geqslant 0)$. Recall that from Proposition 4.2 it follows that there exists a constant $c>0$ *which does not depend on* n and such that

$$|\mathrm{T}_{K_\infty/K_n}(x)-x|_K\leqslant c|\gamma^{p^n}(x)-x|_K,\qquad \forall x\in\widehat{K}_\infty.$$

1) Show that $|\mathrm{T}_{K_\infty/K_n}(x)|_K\leqslant \max\{1,c\}\cdot|x|_K$ and deduce that the map $\mathrm{T}_{K_\infty/K_n}$ is continuous.

Solution. Since

$$\mathrm{T}_{K_\infty/K_n}(x)=(\mathrm{T}_{K_\infty/K_n}(x)-x)+x,$$

one has:

$$|\mathrm{T}_{K_\infty/K_n}(x)|_K\leqslant \max\{|\mathrm{T}_{K_\infty/K_n}(x)-x|_K,|x|_K\}.$$

Moreover

$$|\gamma^{p^n}(x)-x|_K\leqslant |x|_K.$$

This implies 1).

2) Let $x\in\widehat{K}_\infty$. Show that the sequence $\left(T_{K_\infty/K_n}(x)\right)_{n\geqslant 1}$ converges to x.

Solution. There exists a sequence $(x_n)_{n\geqslant 0}$ such that $x_n\in K_n$ and $\lim_{n\to+\infty}x_n=x$. Write

$$|T_{K_\infty/K_n}(x)-x|_K=|(T_{K_\infty/K_n}(x)-x_n)+(x_n-x)|_K\leqslant\max\{|T_{K_\infty/K_n}(x)-x_n|_K,|x-x_n|_K\}.$$

Since $T_{K_\infty/K_n}(x_n)=x_n$, one has:

$$|T_{K_\infty/K_n}(x)-x_n|_K=|T_{K_\infty/K_n}(x-x_n)|_K\leqslant \max\{1,c\}\cdot|x-x_n|_K$$

(here we use part 1)). Hence

$$|T_{K_\infty/K_n}(x)-x|_K\leqslant \max\{1,c\}\cdot|x-x_n|_K.$$

This implies 2).

3) This question can be treated independently. Assume that $W\subset \widehat{K}_\infty$ is a finite dimensional K-vector space which is stable under the action of Γ (i.e. $\gamma(x)\in W$ for all $x\in W$). Show that $W\subset K_\infty$. Hint: consider γ as a linear operator on W. First consider the case where the eigenvalues of γ are in K. Next reduce the general case to this particular case.

Solution. a) Assume that the eigenvalues of γ belong to K. We can decompose W into the direct sum of generalized eigenspaces $W=W_1\oplus\cdots\oplus W_m$. Therefore without loss of generality one can assume that $\lambda\neq 0$ is the unique eigenvalue of γ on W. Then there exists $x\in W\setminus\{0\}$ such that $\gamma(x)=\lambda x$. Recall that $\gamma-\lambda$ is invertible on $\widehat{K}_\infty$ if λ is not a p^nth root of unity. Hence $\lambda^{p^n}=1$ for some n. Recall the decomposition

$$\widehat{K}_\infty=K_n\oplus(\widehat{K}_n)^\circ_\infty.$$

Let $w\in W$. Then $(\gamma-\lambda)^m(w)=0$ for some $m\geqslant 1$. Since

$$\gamma^{p^n}-1=(\gamma-\lambda)(\gamma^{p^n-1}+\cdots+\lambda^{p^n-1}),$$

one has $(\gamma^{p^n}-1)^m(w)=0$. Write $w=\alpha+\beta$, where $\alpha\in K_n$ and $\beta\in(\widehat{K}_n)^\circ_\infty$. Since $\gamma^{p^n}-1$ is invertible on $(\widehat{K}_n)^\circ_\infty$, this implies that $\beta=0$, and $w=\alpha\in K_n$.

b) In the general case, assume that λ is an eigenvalue of γ. Then $\gamma(x)=\lambda x$. Therefore $\lambda\in\widehat{K}_\infty$. Moreover λ is algebraic over K. Hence

$$\lambda\in\overline{K}^{G_{K_\infty}}=K_\infty.$$

Therefore there exists n such that all eigenvalues of γ belong to K_n. Replacing W by $W\cdot K_n$, we reduce the general case to the case a).