UNIVERSITÉ DE BORDEAUX

Homological Algebra

Exam, December 13th 2022. Duration 3h

Hard copies of the lecture notes are allowed.

You can use all results explicitly stated in the chapters 1-3 of the course.

Exercise 1. Let $\mathscr{F} : \mathcal{A} \to \mathcal{B}$ and $\mathscr{G} : \mathcal{B} \to \mathcal{A}$ be two adjoint functors between categories of modules (with $\mathscr{F}$ left adjoint and $\mathscr{G}$ right adjoint). Show that the following properties are equivalent:

1) $\mathscr{G}$ is exact.

2) For each projective $P \in \mathrm{Obj}(\mathcal{A})$, the object $\mathscr{F}(P)$ is projective in $\mathcal{B}$.

Solution. 1) $\implies$ 2). Consider the diagram

$$\begin{array}{ccccc} & & \mathscr{F}(P) & & \\ & & \downarrow{\scriptstyle \pi} & & \\ Y & \xrightarrow{g} & Z & \longrightarrow & 0 \end{array}$$

Let $X = \ker(g)$. By the definition of adjoint functors, we have a commutative diagram

$$\begin{array}{ccccccccc} & & \mathrm{Hom}_{\mathcal{B}}(\mathscr{F}(P), X) & \longrightarrow & \mathrm{Hom}_{\mathcal{B}}(\mathscr{F}(P), Y) & \longrightarrow & \mathrm{Hom}_{\mathcal{B}}(\mathscr{F}(P), Z) & \dashrightarrow & 0 \\ & & \downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \simeq} & & \\ 0 & \longrightarrow & \mathrm{Hom}_{\mathcal{A}}(P, \mathscr{G}(X)) & \longrightarrow & \mathrm{Hom}_{\mathcal{A}}(P, \mathscr{G}(Y)) & \longrightarrow & \mathrm{Hom}_{\mathcal{A}}(P, \mathscr{G}(Z)) & \longrightarrow & 0 \end{array}$$

The vertical arrows are isomorphisms. Since P is projective, the functor $\mathrm{Hom}_{\mathcal{A}}(P, -)$ is exact. Since the functor $\mathscr{G}$ is exact, this implies that the bottom row is exact. Therefore the map $\mathrm{Hom}_{\mathcal{B}}(\mathscr{F}(P), Y) \to \mathrm{Hom}_{\mathcal{B}}(\mathscr{F}(P), Z)$ is surjective. This shows that $\mathscr{F}(P)$ is projective.

2) $\implies$ 1). Let

$$0 \to X \to Y \to Z \to 0$$

be an exact sequence in $\mathscr{B}$. By a general property of adjoint functors, $\mathscr{G}$ is left exact. For any projective P, consider the diagram

$$\begin{array}{ccccccccc} & & \mathrm{Hom}_{\mathcal{B}}(\mathscr{F}(P), X) & \longrightarrow & \mathrm{Hom}_{\mathcal{B}}(\mathscr{F}(P), Y) & \longrightarrow & \mathrm{Hom}_{\mathcal{B}}(\mathscr{F}(P), Z) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \simeq} & & \\ 0 & \longrightarrow & \mathrm{Hom}_{\mathcal{A}}(P, \mathscr{G}(X)) & \longrightarrow & \mathrm{Hom}_{\mathcal{A}}(P, \mathscr{G}(Y)) & \longrightarrow & \mathrm{Hom}_{\mathcal{A}}(P, \mathscr{G}(Z)) & \dashrightarrow & 0 \end{array}$$

Since $\mathscr{F}(P)$ is projective, the upper row is exact. Therefore the bottom row is also exact, i.e. for any projective P, the sequence

$$0 \to \mathrm{Hom}_{\mathcal{A}}(P, \mathscr{G}(X)) \to \mathrm{Hom}_{\mathcal{A}}(P, \mathscr{G}(Y)) \to \mathrm{Hom}_{\mathcal{A}}(P, \mathscr{G}(Z)) \to 0$$

is exact. If $\mathcal{A}$ is the category of modules over a ring A, take $P = A$. Then P is free, hence projective. For any A-modume M, one has $\mathrm{Hom}_{A-\mathbf{Mod}}(A, M) = M$. Therefore the above exact sequence reads

$$0 \to \mathscr{G}(X) \to \mathscr{G}(Y) \to \mathscr{G}(Z) \to 0,$$

and we have proved the exacteness of $\mathscr{G}$.

Exercise 2. Let $\mathscr{F} : \mathcal{A} \to \mathcal{B}$ be a covariant functor. Assume that $\mathscr{F}$ is right exact and $\mathcal{A}$ has enough projectives, and denote by $L_n\mathscr{F}$ the derived functors. Show that if for some $n \geqslant 1$ the functor $L_n\mathscr{F}$ is right exact, then $L_m\mathscr{F} = 0$ for all $m \geqslant n$.

Solution. Any object Z of $\mathcal{A}$ can be inserted in an exact sequence

$$0 \to X \to P \to Z \to 0$$

where P is projective. Assume that $L_n\mathscr{F}$ is right exact for some $n \geqslant 1$. Then

$$L_n\mathscr{F}(X) \to L_n\mathscr{F}(P) \to L_n\mathscr{F}(Z) \to 0.$$

Since P is projective, $L_n\mathscr{F}(P) = 0$, and we obtain that $L_n\mathscr{F}(Z) = 0$. Therefore $L_n\mathscr{F} = 0$. Assume now that $L_m\mathscr{F} = 0$ for some $m \geqslant n$. We have a long exact sequence

$$L_{m+1}\mathscr{F}(X) \to L_{m+1}\mathscr{F}(P) \to L_{m+1}\mathscr{F}(Z) \to L_m\mathscr{F}(X).$$

Here $L_m\mathscr{F}(X)$ by assumption, and mimiking the previous argument, we conclude that $L_{m+1}\mathscr{F}(Z) = 0$.

By induction, $L_m\mathscr{F} = 0$ for all $m \geqslant n$.

Exercise 3. 1) Let $A = \mathbf{Z}[X]/(X^2)$

1a) Give a simple projective resolution of $\mathbf{Z}$ in the category of A-modules.

Solution. We have an exact sequence

$$\cdots \xrightarrow{X} \mathbf{Z}[X]/(X^2) \xrightarrow{X} \mathbf{Z}[X]/(X^2) \xrightarrow{X} \mathbf{Z}[X]/(X^2) \xrightarrow{\varepsilon} \mathbf{Z} \to 0,$$

where ε is the projection of $\mathbf{Z}[X]/(X^2)$ onto $\mathbf{Z}$ and X denotes the multiplication by X map. Therefore

$$P_\bullet : \quad \cdots \xrightarrow{X} \mathbf{Z}[X]/(X^2) \xrightarrow{X} \mathbf{Z}[X]/(X^2) \xrightarrow{X} \mathbf{Z}[X]/(X^2) \to 0$$

is a projective resolution of $\mathbf{Z}$.

1b) For any A-module M, compute $\operatorname{Tor}_i^A(\mathbf{Z}, M)$ and $\operatorname{Ext}_A^i(\mathbf{Z}, M)$ in terms of M.

Solution. We have $A \otimes_A M \simeq M$, and

$$P_\bullet \otimes_A M : \qquad \cdots \xrightarrow{X} M \xrightarrow{X} M \xrightarrow{X} M \xrightarrow{X} M \to 0.$$

Therefore $\operatorname{Tor}_0^A(\mathbf{Z}, M) = M/XM$ and

$$\operatorname{Tor}_i^A(\mathbf{Z}, M) = M_X/XM, \qquad i \geqslant 1,$$

where we write M_X for the kernel of the multiplication by X map $X : M \to M$. Note that $M_X/XM = \mathbf{Z} \otimes_A M$.

Also, $\operatorname{Hom}_A(A, M) \simeq M$, and

$$\operatorname{Hom}_A(P_\bullet, M) : \qquad 0 \to M \xrightarrow{X} M \xrightarrow{X} M \xrightarrow{X} \cdots .$$

Therefore $\operatorname{Ext}_A^0(\mathbf{Z}, M) = M_X$ and

$$\operatorname{Ext}_A^i(\mathbf{Z}, M) = M_X/XM, \qquad i \geqslant 1.$$

2) Can you generalize this computation to the case of the ring $A = \mathbf{Z}[X]/(X^n)$ where $n \geqslant 2$?

Solution. We have an exact sequence

$$\cdots \xrightarrow{X} \mathbf{Z}[X]/(X^n) \xrightarrow{X^{n-1}} \mathbf{Z}[X]/(X^n) \xrightarrow{X} \mathbf{Z}[X]/(X^n) \xrightarrow{X^{n-1}} \mathbf{Z}[X]/(X^n) \xrightarrow{X} \mathbf{Z}[X]/(X^n) \xrightarrow{\varepsilon} \mathbf{Z} \to 0,$$

Therefore

$$P_\bullet : \qquad \cdots \xrightarrow{X} \mathbf{Z}[X]/(X^n) \xrightarrow{X^{n-1}} \mathbf{Z}[X]/(X^n) \xrightarrow{X} \mathbf{Z}[X]/(X^n) \xrightarrow{X^{n-1}} \mathbf{Z}[X]/(X^n) \xrightarrow{X} \mathbf{Z}[X]/(X^n) \to 0$$

is a projective resolution of $\mathbf{Z}$ as A-module. We easily check that again $\operatorname{Tor}_0^A(\mathbf{Z}, M) = M/XM$ and

$$\operatorname{Tor}_i^A(\mathbf{Z}, M) = \begin{cases} M/XM, & \text{if } i = 0, \\ M_{X^{n-1}}/XM, & \text{if } i > 0 \text{ is even}, \\ M_X/X^{n-1}M, & \text{if } i \text{ is odd} \end{cases}$$

and

$$\operatorname{Ext}_A^i(\mathbf{Z}, M) = \begin{cases} M_X, & \text{if } i = 0, \\ M_X/X^{n-1}M, & \text{if } i > 0 \text{ is even}, \\ M_{X^{n-1}}/XM, & \text{if } i \geqslant 1 \text{ is odd}. \end{cases}$$

Here $M_{X^{n-1}}$ denotes the kernel of the multiplication by X^{n-1} map $X^{n-1} : M \to M$.

Exercise 4. Let G be a finite group of order n and M a G-module. We denote by $C^\bullet(G, M)$ the standard complex computing the cohomology of G.

1) Let $f : G \to M$ be a 1-cocycle. Set $m = \sum_{h\in G} f(h) \in M$. Show that $d_0(m) = -nf$ and deduce that $H^1(G, M)$ is killed by the multiplication by n.

Solution. Let $F = d_0(m) \in C^1(G, M)$. By definition of the map $d_0 : C^0(G, M) \to C^1(G, M)$, we have

$$F(g) = g(m) - m = \sum_{h\in G} (gf(h) - f(h)).$$

Since $f(gh) = gf(h) + f(g)$, this gives

$$F(g) = \sum_{h\in G} f(gh) - \sum_{h\in G} f(h) - nf(g) = -nf(g).$$

2) Can you generalize this argument and prove that $H^i(G, M)$ is killed by the multiplication by n for all $i \geqslant 1$?

Solution. Let $f \in Z^i(G, M)$. Set

$$F(g_1, g_2, \dots, g_{i-1}) = \sum_{h\in G} f(g_1, \dots, g_{i-1}, h).$$

The map f satisfies the cocycle condition $d^i(f) = 0$, where

$$\begin{aligned}(d^i f)(g_0, g_1, \dots, g_i) &= g_0 f(g_1, \dots, g_i)\\ &+ \sum_{k=0}^{i-1} (-1)^{k+1} f(g_0, \dots, g_k g_{k+1}, \dots, g_i) + (-1)^{i+1} f(g_0, \dots, g_{i-1}).\end{aligned}$$

We compute $d^{i-1}F$:

$$\begin{aligned}&(d^{i-1}F)(g_0, \dots, g_{i-1}) =\\ &= g_0 F(g_1, \dots, g_{i-1}) + \sum_{k=0}^{i-2} (-1)^{k+1} F(g_0, \dots, g_k g_{k+1}, \dots, g_{i-1}) + (-1)^i F(g_0, \dots, g_{i-2}) =\\ &= \sum_{h\in G} g_0 f(g_1, \dots, g_{i-1}, h) + \sum_{k=0}^{i-2} (-1)^{k+1} \left(\sum_{h\in G} f(g_0, \dots, g_k g_{k+1}, \dots, h) \right)\\ &\qquad + (-1)^i \sum_{h\in G} f(g_0, \dots, g_{i-2}, h).\end{aligned}$$

From the cocycle condition, we have:

$$g_0 f(g_1,\dots,g_{i-1},h)+\sum_{k=0}^{i-2}(-1)^{k+1}f(g_0,\dots,g_k g_{k+1},\dots,h)=$$
$$=(-1)^{i-1}f(g_0,g_1,\dots,g_{i-1}h)+(-1)^i f(g_0,g_1,\dots,g_{i-1}).$$

Hence

$$(d^{i-1}F)(g_0,\dots,g_{i-1})=(-1)^{i-1}\sum_{h\in G}f(g_0,g_1,\dots,g_{i-1}h)+$$
$$(-1)^i\sum_{h\in G}f(g_0,g_1,\dots,g_{i-1})+(-1)^i\sum_{h\in G}f(g_0,\dots,g_{i-2},h).$$

Since

$$\sum_{h\in G}f(g_0,g_1,\dots,g_{i-1}h)=\sum_{h\in G}f(g_0,\dots,g_{i-2},h),$$

we obtain that

$$(d^{i-1}F)(g_0,\dots,g_{i-1})=n\cdot(-1)^i f(g_0,g_1,\dots,g_{i-1}).$$

3) Let

$$0\to A\to N\to G\to 0$$

be an extension of G by a finite abelian group A of order m. Show that if $\gcd(m,n)=1$, then N is a semidirect product of G and A.

Solution. The above extension equips A with a G-module structure, and we can consider $H^2(G,A)$. By 1), $H^2(G,A)$ is an abelian group killed by the multiplication by n. On the other hand, it is killed by the multiplication by $m=|A|$. Since $\gcd(m,n)=1$, this implies that $H^2(G,A)=0$. Therefore each extension of G by A is a semidirect product (see Theorem 5.5).

Exercise 5. Let A be a ring and let I (respectively J) be a right (respectively left) ideal of A. We denote by IJ the abelian group generated by the products xy, $x\in I$, $y\in J$.

1) Show that the sequence

$$0\to IJ\xrightarrow{\alpha} I\xrightarrow{\beta} I\otimes_A(A/J)\to 0,$$

where α is the inclusion and $\beta(x):=x\otimes 1$, is exact.

Solution. Consider the exact sequence

$$0\to J\to A\to A/J\to 0.$$

Since the tensor product is right exact, the sequence

$$I \otimes_A J \to I \otimes_A A \to I \otimes_A (A/J) \to 0$$

is exact. It remains to remark that $I \otimes_A A = I$ and the image of $I \otimes_A J$ in I is IJ.

2) Let $\gamma : I \otimes_A (A/J) \to A/J$ be the map defined by $\gamma(x \otimes \overline{a}) = \overline{xa}$. Show that

$$\ker(\gamma) \simeq (I \cap J)/(IJ).$$

Hint: consider the diagram

$$\begin{CD} 0 @>>> IJ @>>> I @>>> I \otimes_A (A/J) @>>> 0 \\ @. @VVV @VVV @VV{\gamma}V \\ 0 @>>> J @>>> A @>>> A/J @>>> 0. \end{CD}$$

Solution. Apply the shake lemma to the diagram

$$\begin{CD} 0 @>>> IJ @>>> I @>>> I \otimes_A (A/J) @>>> 0 \\ @. @VV{f}V @VV{g}V @VV{\gamma}V \\ 0 @>>> J @>>> A @>>> A/J @>>> 0. \end{CD}$$

We obtain an exact sequence

$$\ker(g) \to \ker(\gamma) \to \operatorname{coker}(f) \to \operatorname{coker}(g).$$

Here $\ker(g) = 0$, $\operatorname{coker}(f) = J/IJ$ and $\operatorname{coker}(g) = A/I$. Therefore

$$\ker(\gamma) = \ker\Big(J/IJ \to A/I\Big) = (J \cap I)/(IJ).$$

3) Using question 2), show that $\operatorname{Tor}_1^A(R/I, R/J) = (I \cap J)/(IJ)$.

Solution. The exact sequence

$$0 \to I \to A \to A/I \to 0$$

induces a long exact sequence

$$\to \operatorname{Tor}_A^1(I, A/J) \to \operatorname{Tor}_A^1(A, A/J) \to \operatorname{Tor}_A^1(A/I, A/J) \to I \otimes_A A/J \to A/J.$$

Since A is free, $\operatorname{Tor}_A^1(I, A/J)$, and

$$\operatorname{Tor}_A^1(A/I, A/J) = \ker\Big(I \otimes_A A/J \to A/J\Big) = (I \cap J)/(IJ)$$

by question 2).

Exercise 6. Let A be a ring.

1) Show that a left A-module I is injective if and only if $\mathrm{Ext}^1(A/\mathfrak{a}, I) = 0$ for any left ideal $\mathfrak{a}$ of A.

Solution. If I is injective, $\mathrm{Ext}^i_A(-, I)$ are zero for $i \geqslant 1$ (see Proposition 4.3). Conversely, each short exact sequence of the form

$$0 \to \mathfrak{a} \to A \to A/\mathfrak{a} \to 0$$

induces a long exact sequence

$$0 \to \mathrm{Hom}_A(A/\mathfrak{a}, I) \to \mathrm{Hom}_A(A, I) \to \mathrm{Hom}_A(\mathfrak{a}, I) \to \mathrm{Ext}^1_A(A/\mathfrak{a}, I)$$

If $\mathrm{Ext}^1(A/\mathfrak{a}, I) = 0$, then the map $\mathrm{Hom}_A(A, I) \to \mathrm{Hom}_A(\mathfrak{a}, I)$ is surjective. Applying Baer criterion (Proposition 2.3), we obtain that I is injective.

2) Let M be a left A-module. Show that the following conditions are equivalent:

a) M has an injective resolution $I^\bullet$ of length 2:

$$0 \to M \to I_0 \to I_1 \to 0.$$

b) For any left A-module N,

$$\mathrm{Ext}^i_A(N, M) = 0, \qquad \forall i \geqslant 2.$$

Solution. *a*) $\implies$ *b*). This follows directly from the definition of the derived functor:

$$\mathrm{Ext}^i_A(N, M) = H^i(\mathrm{Hom}_A(N, I_\bullet)) = 0, \qquad i \geqslant 2.$$

b) $\implies$ *a*). There exists a monomorphism $M \to I_0$, where I_0 is injective. Set $I_1 = I_0/M$. We have an exact sequence

$$0 \to M \to I_0 \to I_1 \to 0.$$

We only need to prove that I_1 is injective. For any ideal $\mathfrak{a}$ of A, we have a long exact sequence

$$\cdots \to \mathrm{Ext}^1(A/\mathfrak{a}, I_0) \to \mathrm{Ext}^1(A/\mathfrak{a}, I_1) \to \mathrm{Ext}^2(A/\mathfrak{a}, M) \to \cdots$$

Here $\mathrm{Ext}^1(A/\mathfrak{a}, I_0) = 0$ by 1), and $\mathrm{Ext}^2(A/\mathfrak{a}, M) = 0$ by assumption b). Hence $\mathrm{Ext}^1(A/\mathfrak{a}, I_1) = 0$ and applying 1), we obtain the injectivity of I_1.