UNIVERSITÉ DE BORDEAUX

Homological Algebra

Exam, December 13th 2022. Duration 3h

Hard copies of the lecture notes are allowed.

You can use all results explicitly stated in the chapters 1-3 of the course.

Exercise 1. Let $\mathscr{F} : \mathcal{A} \to \mathcal{B}$ and $\mathscr{G} : \mathcal{B} \to \mathcal{A}$ be two adjoint functors between abelian categories (with $\mathscr{F}$ left adjoint and $\mathscr{G}$ right adjoint). Show that the following properties are equivalent:

1) $\mathscr{G}$ is exact.

2) For each projective $P \in \mathrm{Obj}(\mathcal{A})$, the object $\mathscr{F}(P)$ is projective in $\mathcal{B}$.

Exercise 2. Let $\mathscr{F} : \mathcal{A} \to \mathcal{B}$ be a covariant functor. Assume that $\mathscr{F}$ is right exact and $\mathcal{A}$ has enough projectives, and denote by $L_n\mathscr{F}$ the derived functors. Show that if for some $n \geqslant 1$ the functor $L_n\mathscr{F}$ is right exact, then $L_m\mathscr{F} = 0$ for all $m \geqslant n$.

Exercise 3. 1) Let $A = \mathbf{Z}[X]/(X^2)$

1a) Give a simple projective resolution of $\mathbf{Z}$ in the category of A-modules.

1b) For any A-module M, compute $\mathrm{Tor}_i^A(\mathbf{Z}, M)$ and $\mathrm{Ext}_A^i(\mathbf{Z}, M)$ in terms of M.

2) Can you generalize this computation to the case of the ring $A = \mathbf{Z}[X]/(X^n)$ where $n \geqslant 2$?

Exercise 4. Let G be a finite group of order n and M a G-module. We denote by $C^\bullet(G, M)$ the standard complex computing the cohomology of G.

1) Let $f : G \to M$ be a 1-cocycle. Set $m = \sum\limits_{h\in G} f(h) \in M$. Show that $d_0(m) = -nf$ and deduce that $H^1(G, M)$ is killed by the multiplication by n.

2) Can you generalize this argument and prove that $H^i(G, M)$ is killed by the multiplication by n for all $i \geqslant 1$?

3) Let

$$0 \to A \to N \to G \to 0$$

be an extension of G by a finite abelian group A of order m. Show that if $\gcd(m, n) = 1$, then N is a semidirect product of G and A.

Exercise 5. Let A be a ring and let I (respectively J) be a right (respectively left) ideal of A. We denote by IJ the abelian group generated by the products xy, $x \in I$, $y \in J$.

1) Show that the sequence

$$0 \to IJ \xrightarrow{\alpha} I \xrightarrow{\beta} I \otimes_A (A/J) \to 0,$$

where α is the inclusion and $\beta(x) := x \otimes 1$, is exact.

2) Let $\gamma : I \otimes_A (A/J) \to A/J$ be the map defined by $\gamma(x \otimes \overline{a}) = \overline{xa}$. Show that

$$\ker(\gamma) \simeq (I \cap J)/(IJ).$$

Hint: consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & IJ & \longrightarrow & I & \longrightarrow & I \otimes_A (A/J) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \gamma} & & \\ 0 & \longrightarrow & J & \longrightarrow & A & \longrightarrow & A/J & \longrightarrow & 0. \end{array}$$

3) Using question 2), show that $\mathrm{Tor}_1^A(R/I, R/J) = (I \cap J)/(IJ)$.

Exercise 6. Let A be a ring.

1) Show that a left A-module I is injective if and only if $\mathrm{Ext}^1(A/\mathfrak{a}, I) = 0$ for any left ideal $\mathfrak{a}$ of A.

2) Let M be a left A-module. Show that the following conditions are equivalent:

a) M has an injective resolution $I^\bullet$ of length 2:

$$0 \to M \to I_0 \to I_1 \to 0.$$

b) For any left A-module N,

$$\mathrm{Ext}^i_A(N, M) = 0, \qquad \forall i \geqslant 2.$$