Homological Algebra

Solutions and hints for some exercises

Exercise 3. X is a direct product of $(X_i)_{i\in I}$ in $\mathcal{A}$ if and only if X° is a direct sum of $(X_i^\circ)_{i\in I}$ in $\mathcal{A}^\circ$.

Solution. Dualize diagrams in the definition of the direct product.

Exercise 4. Let K be a field. A filtered finite-dimensional vector space $X = (V, (V_i)_{i\in\mathbf{Z}})$ over K is a finite dimensional K-vector space K equipped with an increasing filtration by K-subspaces:

$$\ldots \subseteq V_{i-1} \subseteq V_i \subseteq V_{i+1} \subseteq \ldots$$

Let $Y = (W, (W_i)_{i\in\mathbf{Z}})$. A morphism $f : X \to Y$ is a linear map $f : V \to W$ such that $f(V_i) \subseteq W_i$ for all $i \in \mathbf{Z}$. Let $\mathbf{FVect}_K$ denote the category of filtered finite-dimensional vector spaces over K.

1) Show that $\mathbf{FVect}_K$ is additive.
   Solution. Straightforward.

2) Show that each morphism in $\mathbf{FVect}_K$ has a kernel and a cokernel.
   Solution. Let $f : X \to Y$ be a morphism. Write $\varphi : V \to W$ for the associated linear map (such that $\varphi(V_i) \subseteq W_i$ for all $i \in \mathbf{Z}$.) Set $\ker(\varphi)_i := \ker(\varphi) \cap V_i$ and $\operatorname{coker}(\varphi_i) := Y_i/(Y_i \cap \varphi(X))$. It is not difficult to check that

$$\ker(f) = (\ker(\varphi), (\ker(\varphi)_i)_{i\in\mathbf{Z}}).$$

   and

$$\operatorname{coker}(f) = (\operatorname{coker}(\varphi), (\operatorname{coker}(\varphi_i))_{i\in\mathbf{Z}})$$

   are kernel and cokernel of f.

3) Let V be a nonzero vector space and let $X = (V, (V_i)_{i\in\mathbf{Z}})$ and $Y = (V, (V_i')_{i\in\mathbf{Z}})$ be the objects defined as:

$$V_i = \begin{cases} 0, & \text{if } i \leqslant 0, \\ V, & \text{if } i \geqslant 1, \end{cases} \qquad V_i' = \begin{cases} 0, & \text{if } i \leqslant -1, \\ V, & \text{if } i \geqslant 0, \end{cases}$$

Show that the identity map on V induces a morphism $f : X \to Y$ which is monic and epi, but is not an isomorphism. Deduce that $\mathbf{FVect}_K$ is not abelian.

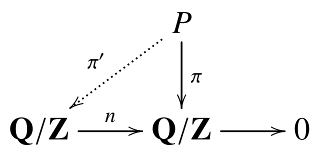
Solution. From part 2), it follows that $\ker(f) = \operatorname{coker}(f) = 0$. Therefore f is monic and epi. However f is not an isomorphism (compare the filtrations of X and Y).

Exercise 7. In a an additive category, the zero map $X \xrightarrow{0} Y$ is monic (resp. epi) if and only if $X = 0$ (resp. $Y = 0$).

Solution. Assume that $f_1, f_2 : Z \to X$ be two morphisms. Then $0 \circ f_1 = 0 \circ f_2$. If the zero map if monic, then $f_1 = f_2$. Therefore X is initial.

Exercise 8. Show that the category of torsion abelian groups has no projective nonzero objects.

Solution. Assume that P is a projective object in this category. Let $x \in P$ be an element of order $n \geqslant 2$. Let $f : \langle x \rangle \to \mathbf{Q}/\mathbf{Z}$ denote the map defined by $f(x) = 1/n \pmod{\mathbf{Z}}$. The abelian group $\mathbf{Q}/\mathbf{Z}$ is an injective object in the category of abelian groups. Therefore there exists a map $\pi : P \to \mathbf{Q}/\mathbf{Z}$ such that $g(x) = 1/n \pmod{\mathbf{Z}}$. Consider the diagram



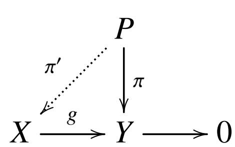
Since P is projective, there exists π' such that $n\pi'(x) = \pi(x) = 1/n \pmod{\mathbf{Z}}$. On the other hand, $n\pi'(x) = \pi'(nx) = 0$. This gives a contradiction.

Exercise 9. Let k be a field and $A := \mathrm{M}_n(k)$ the ring of $n \times n$ matrices with coefficients in k. Assume that $n \geqslant 2$. Give an example of an A-module which is projective but not free.

Solution. Take $P = k^n$ (the module of columns equipped with the natural left action of A) and remark that A is a direct sum of n copies of P.

Exercise 10. Let $(P_i)_{i \in I}$ be a family of projective objects. Show that if the coproduct $\coprod_{i \in I} P_i$ exists, then it is projective.

Solution. Set $P := \coprod_{i \in I} P_i$. We have canonical morphisms $q_i : P_i \to P$. Assume that we have a diagram



Set $\pi_i = \pi \circ q_i : P_i \to Y$. By the projectivity of P_i, there exist $\pi'_i : P_i \to X$ such that $g \circ \pi'_i = \pi_i$. By the definition of coproducts, there exists a unique $\pi' : P \to X$ such that $\pi' \circ q_i = \pi'_i$ for all i. Then the composition $(g \circ \pi')$ satisfies

$$(g \circ \pi') \circ q_i = g \circ \pi'_i = \pi_i = \pi \circ q_i, \qquad \forall i \in I.$$

Hence $g \circ \pi' = \pi$.

Exercise 11. Let $(I_j)_{j\in J}$ be a family of injective objects. Show that if the product $\prod_{j\in J} I_j$ exists, then it is injective.

Solution. Use Exercise 10 and duality.

Exercise 12. Give an example of a non-projective flat module over $\mathbf{Z}$.

Solution. The $\mathbf{Z}$-module $\mathbf{Q}$ is divisible and therefore not projective. Let $f : A \to B$ be an injective morphism of abelian groups. Let A_{tor} and B_{tor} denote the torsion subgroups of A and B. Then A/A_{tor} and B/B_{tor} are torsion free and the map $A/A_{\mathrm{tor}} \to B/B_{\mathrm{tor}}$ is injective.

$$\mathbf{Q}\otimes A \simeq \mathbf{Q}\otimes (A/A_{\mathrm{tor}}), \qquad \mathbf{Q}\otimes B \simeq \mathbf{Q}\otimes (B/B_{\mathrm{tor}}).$$

Therefore the induced map $f_{\mathbf{Q}} : \mathbf{Q}\otimes A \to \mathbf{Q}\otimes B$ is injective. This implies that $\mathbf{Q}$ is flat.

Exercise 13. Let G be an infinite cyclic group. Fix a generator g of G.

1) Show that $\mathbf{Z}[G]$ is isomorphic to the ring $\mathbf{Z}[X, X^{-1}]$.

Solution. Check that the map

$$\mathbf{Z}[X, X^{-1}] \to \mathbf{Z}[G],$$
$$\sum_k a_k X^k \mapsto \sum_k a_k g^k$$

is an isomorphism.

2) Show that the sequence

$$0 \to \mathbf{Z}[X, X^{-1}] \xrightarrow{\partial_0} \mathbf{Z}[X, X^{-1}] \xrightarrow{\varepsilon} \mathbf{Z} \to 0,$$

where $\partial_0(f(X)) = (X-1)f(X)$ and $\epsilon(f) = f(1)$, is a free resolution of $\mathbf{Z}$.

Solution. Compute $\ker(\epsilon)$. If

$$f = \sum_k a_k X^k \in \ker(\epsilon),$$

then $\sum_k a_k = 0$ and one can write

$$f = \sum_k a_k(X^k - 1) = (X-1)g$$

for some $g \in \mathbf{Z}$. This shows that $\ker(\epsilon)$ is the principal ideal generated by $(X-1)$. Therefore the sequence

$$0 \to \mathbf{Z}[X, X^{-1}] \xrightarrow{\partial_0} \mathbf{Z}[X, X^{-1}] \xrightarrow{\varepsilon} \mathbf{Z} \to 0,$$

is exact. The module $\mathbf{Z}[X, X^{-1}]$ is clearly free over $\mathbf{Z}[X, X^{-1}]$.

3) Let M be a G-module. Show that $H^0(G,M) = M^G$, $H^1(G,M) \simeq M/(g-1)M$ and $H^i(G,M) = 0$ for $i \geq 2$.

Solution. Note that $\mathrm{Hom}_{\mathbf{Z}[G]}(\mathbf{Z}[G], M) \simeq M$. Taking $\mathrm{Hom}_{\mathbf{Z}[G]}(P_\bullet, M)$ where

$$P_\bullet \ : \ 0 \to \mathbf{Z}[X, X^{-1}] \xrightarrow{\partial_0} \mathbf{Z}[X, X^{-1}] \to 0,$$

we obtain the complex

$$0 \to M \xrightarrow{d^0} M \to 0,$$

with $d^0(m) = (X-1)m$. Hence

$$H^0(G,M) = \ker(g-1 \ : \ M \to M) = M^{g=1} = M^G,$$

and

$$H^1(G,M) = M/(g-1)M.$$

Also $H^i(G,M) = 0$ for $i \geqslant 2$.

Exercise 14. Let A be an abelian group and $f \in A^* = C^1(G,A)$. Show that if $gf = f$ for all $g \in G$, then f is a constant map, i.e. there exists $a \in A$ such that $f(g) = a$ for all $g \in G$. Therefore $(A^*)^G \simeq A$.

Solution. Set $a := f(e)$. Since $gf = f$, we have

$$f(g) = gf(e) = f(e), \qquad \forall g \in G.$$

The map $f \mapsto f(e)$ gives an isomorphism $(A^*)^G \simeq A$.

Exercise 16. Let G be an infinite cyclic group. Fix a generator g of G. Show that $H_0(G,M) = M/(g-1)M$, $H_1(G,M) = M^G$ and $H_i(G,M) = 0$ for $i \geqslant 2$.

Solution. Take the projective resolution $P_\bullet$ from Exercise 13. Since $\mathbf{Z}[G] \otimes_{\mathbf{Z}[G]} M \simeq M$, the complex $P_\bullet \otimes_{\mathbf{Z}[G]} M$ reads:

$$0 \to M \xrightarrow{d_0} M \to 0,$$

where $d_0(m) = (X-1)m$ and the nonzero terms of the complex are concentrated in degrees 1 and 0.