UNIVERSITÉ DE BORDEAUX

# Homological Algebra

## Homework solution

**Execise 1.** Let $\mathfrak{Rings}$ denote the category whose objects are unitary rings and where a morphism $f : A \to B$ is an homomorphism of unitary rings (so $f(1_A) = 1_B$).

1) Show that each surjective homomorphism of rings $f : A \to B$ is an epimorphism in this category.

**Solution.** Let $g_1, g_2 : B \to C$ be two morphisms of rings. Assume that $g_1 \circ f = g_2 \circ f$. Since $f$ is surjective, this implies that $g_1 = g_2$. Therefore $f$ is an epimorphism.

2) Show that the natural inclusion $\mathbf{Z} \to \mathbf{Q}$ is an epimorphism in $\mathfrak{Rings}$.

**Solution.** a) Let $g : \mathbf{Q} \to A$ be a morphism of rings. Then $\ker(g)$ is an ideal in $\mathbf{Q}$. Since $g(1) = 1_A$, we have $\ker(g) \neq \mathbf{Q}$, and therefore $\ker(g) = \{0\}$. Hence the map $g$ is injective.

b) Let $n \geqslant 1$. Assume that $a \in A$ be an $n$-torsion element of $A$ i.e. $na = 0$. The the equality

$$a = 1_A \cdot a = g\left(\frac{1}{n} \cdot n\right) = g\left(\frac{1}{n}\right) \cdot na = 0$$

shows that $a = 0$. Hence for all $n$, the ring $A$ has no $n$-torsion.

c) Let $g_1, g_2 : \mathbf{Q} \to A$ be two morphisms such that $g_1 \circ f = g_2 \circ f$. Then $g_1(n) = g_2(n)$ for all $n \in \mathbf{Z}$. Then for any $x = m/n \in \mathbf{Q}$ we have

$$ng_1(x) = g_1(nx) = g_1(m) = g_2(m) = ng_2(x).$$

Hence $n(g_1(x) - g_2(x)) = 0$, and we conclude that $g_1(x) = g_2(x)$.

*Remark.* For any morphism $g : \mathbf{Q} \to A$, one has $g(m) = m \cdot 1_A$ for all $m \in \mathbf{Z}$. Therefore our proof shows that for each $A$, there exists at most one morphism $\mathbf{Q} \to A$.

**Exercise 2.** Let $A$ be an integral domain (unitary commutative ring for which every non-zero element is cancellable under multiplication). Denote by $\mathcal{M}$ the category of $A$-modules. For any $A$-module $M$ denote by $T(M)$ the torsion submodule of $M$, that is

$$T(M) = \{x \in M | ax = 0 \text{ for some nonzero } a \in A\}.$$

We say that $M$ is torsion (respectively torsion free), if $T(M) = M$ (respectively $T(M) = 0$). Denote by $\mathcal{T}$ (resp. $\mathcal{TF}$) the full subcategory of $\mathcal{M}$ consisting of torsion (resp. torsion free) modules.

1) Show that $\mathcal{T}$ is an abelian category and $\mathcal{TF}$ is not necessarily abelian.

**Solution.** It is easy to see that if a full category $\mathcal{B}$ of an abelian category $\mathcal{A}$ satisfies the following properties, it is also abelian : a) It contains zero objects of $\mathcal{A}$; b) For any $X, Y \in \mathcal{B}$, the coproduct (in $\mathcal{A}$) of $X$ and $Y$ lies in $\mathcal{B}$; c) For any $X, Y \in \mathcal{B}$ and any $f : X \to Y$ the kernel $\ker(f)$ (in $\mathcal{A}$) and $\mathrm{coker}(f)$ (in $\mathcal{A}$) are objects of $\mathcal{B}$. Indeed, directly from definitions it follows that in this case for any morphism $f : X \to Y$, $\ker(f)$ and $\mathrm{coker}(f)$ are the kernel and the cokernel of $f$ in $\mathcal{B}$. The objects $\mathrm{coim}(f)$ and $\mathrm{Im}(f)$ are in $\mathcal{B}$ and the canonical map $\bar{f} : \mathrm{coim}(f) \to \mathrm{Im}(f)$ is an isomorphism both in $\mathcal{A}$ and $\mathcal{B}$ because $\mathcal{B}$ is a full subcategory of $\mathcal{A}$. Applying this observation to the categories $\mathcal{M}$ and $\mathcal{T}$, we get the first statement.

In $\mathcal{TF}$, we can consider the morphism $f : \mathbf{Z} \to \mathbf{Z}$ given by $f(x) = 2x$. It is clear that $\ker(f) = 0$ and therefore $\mathrm{coim}(f) = \mathbf{Z}$. If $g : \mathbf{Z} \to X$ is a morphism in $\mathcal{TF}$ satisfying $g \circ f = 0$, then $g(2\mathbf{Z}) = 0$ and therefore $g(\mathbf{Z}) = 0$ because $X$ is torsion free. This implies that 0 is the cokernel of $f$ in $\mathcal{TF}$. Then $\mathrm{Im}(f) = \mathbf{Z}$. Therefore the canonical map $\bar{f} : \mathrm{coim}(f) \to \mathrm{Im}(f)$ coincides with $f$, but it is not an isomorphism (it has no inverse).

2) Denote by $I : \mathcal{T} \to \mathcal{M}$ the inclusion functor. Show that the assignement $M \to T(M)$ defines a functor $T : \mathcal{M} \to T$ which is right adjoint to $I$.

**Solution.** The property we need to prove reads

$$\mathrm{Hom}_A(N, T(M)) \simeq \mathrm{Hom}_A(N, M)$$

for any torsion module $N$ and any $A$-module $M$. This follows from the observation that for any morphism $f : X \to Y$ of $A$-modules $f(T(X)) \subset T(Y)$.

3) Denote by $J : \mathcal{TF} \to \mathcal{M}$ the inclusion functor. Show that the assignement $M \to M/T(M)$ defines a functor $F : \mathcal{M} \to \mathcal{TF}$ which is left adjoint to $J$.

**Solution.** The property we need to prove reads

$$\mathrm{Hom}_A(M/T(M), N) \simeq \mathrm{Hom}_A(M, N)$$

for any $A$-module $M$ and any torsion free $A$-module $N$. This follows from the observation that for any morphism $f : M \to N$ we have $f(T(M)) \subset T(N) = 0$ and therefore $f$ factorizes through $M/T(M)$.

**Exercise 3.** Let $C^\infty(0,1)$ denote the $\mathbf{R}$-vector space of infinitely differentiable functions on the open segment $]0,1[$. We denote by $D$ the differentiation $D(f) = \frac{df(x)}{dx}$.

1) Consider the cochain complex

$$C \ : \ 0 \to C^\infty(0,1) \xrightarrow{D} C^\infty(0,1) \to 0,$$

where the two nonzero terms are placed in degrees 0 and 1. Compute the cohomology $H^i(C)$ of $C$.

**Solution.** It is clear that $\ker(D) = \mathbf{R}$ (constant functions) and that the map $D$ is surjective (for each $f \in C^\infty(0,1)$, take a primitive $F$ of $f$; then $D(F) = f$). Hence $H^0(C) \simeq \mathbf{R}$ and $H^1(C) = 0$.

2) We say that $f \in C^\infty(0,1)$ is compactly supported if its support $\mathrm{Supp}(f) = \{x \in ]0,1[ | f(x) \neq 0\}$ is contained in a compact subset of $]0,1[$. Let $C_c^\infty(0,1) \subset C^\infty(0,1)$ denote the subspace of compactly supported functions of $C^\infty(0,1)$. Consider the complex

$$C_c \ : \ 0 \to C_c^\infty(0,1) \xrightarrow{D} C_c^\infty(0,1) \to 0,$$

where the two nonzero terms are placed in degrees 0 and 1. Show that $H^0(C_c) = 0$ and $H^1(C_c)$ is canonically isomorphic to $\mathbf{R}$.

**Solution.** Since the only constant function in $C_c$ is the zero function, we have $H^0(C_c) = 0$. Consider the map

$$\alpha \ : \ C_c^\infty(0,1) \to \mathbf{R},$$
$$\alpha(f) = \int_0^1 f(t)dt.$$

It is clear that $\alpha \circ D = 0$. Conversely, if $f_in \ker(\alpha)$, then the primitive $F(x) := \int_0^x f(t)dt$ satisfies $F(0) = F(1) = 0$. This implies that

$$H^1(C_c) \simeq \mathrm{Im}(\alpha) = \mathbf{R}.$$

**Exercise 4.** Let $G$ be a finite cyclic group of order $n$. Fix a generator $g$ of $G$. For any $G$-module $A$ we define two morphisms $N \ : \ A \to A$ and $S \ : \ A \to A$ by

$$N(x) = x + gx + \cdots + g^{n-1}x, \qquad S(x) = gx - x.$$

1) Check that $N \circ S = S \circ N = 0$.

**Solution.** In the commutative ring $\mathbf{Z}[G]$ we have

$$SN = NS = (1 + g + \cdots + g^{n-1})(g-1) = g^n - 1 = 0$$

Therefore we can consider the complex

$$X: \qquad \cdots \xrightarrow{N} X^0 \xrightarrow{S} X^1 \xrightarrow{N} X^2 \xrightarrow{S} X^3 \xrightarrow{N} \cdots$$

where $X^i = A$ for all $i \in \mathbf{Z}$. If $H^0(X)$ and $H^1(X)$ are both finite, we set

$$h(A) = \frac{|H^0(X)|}{|H^1(X)|}$$

and say that $h(A)$ is well defined.

2) Show that $h(\mathbf{Z}) = n$ for the trivial $G$-module $\mathbf{Z}$.

**Solution.** If $G$ acts trivially on $\mathbf{Z}$ we have $gx = x$ for all $x \in \mathbf{Z}$. The $Sx = 0$ and $Nx = nx$ where $n$ is the order of $G$. Thus $\ker(S) = \mathbf{Z}$, $\mathrm{Im}(S) = 0$, $\ker(N) = 0$ and $\mathrm{Im}(N) = n\mathbf{Z}$. We get $H^0(\mathbf{Z}) = \mathbf{Z}/n\mathbf{Z}$ and $H^1(\mathbf{Z}) = 0$. Therefore $h(\mathbf{Z}) = n$.

3) Show that if $A$ is finite, then $h(A) = 1$.

**Solution.** Since $A$ is finite,

From the tautological exact sequences

$$0 \to \ker(S) \to A \xrightarrow{S} \mathrm{Im}(S) \to 0$$

$$0 \to \ker(N) \to A \xrightarrow{N} \mathrm{Im}(N) \to 0$$

(or simply from the first isomorphism theorem) it follows that

$$|\ker(S)| \cdot |\mathrm{Im}(S)| = |A|, \qquad |\ker(N)| \cdot |\mathrm{Im}(N)| = |A|.$$

This implies that $h(A) = 1$.

4) Let

$$0 \to A \to B \to C \to 0$$

be an exact sequence of $G$-modules. Show that, if two of the quotients $h(A)$, $h(B)$ and $h(C)$ are well defined, then the third is well defined and

$$h(B) = h(A)h(C).$$

(Hint: use an appropriate long exact cohomology sequence).

**Solution.** Let $Y$ (resp. $Z$) denote the complex $X$ associated to the module $B$ (resp. $C$). It is easy to see that the complexes $X$, $Y$ and $Z$ form an exact sequence of complexes

$$0 \to X \to Y \to Z \to 0.$$

Therefore we have an exact cohomology sequence

$$\cdots \to H^{-1}(Z) \xrightarrow{\delta^{-1}} H^0(X) \to H^0 \to H^0(Z) \to H^1(X)$$
$$\to H^1(Y) \to H^1(Z) \xrightarrow{\delta^1} H^2(X) \to \cdots$$

Note that from the definition of the complex $X$ it follows that $H^{i+2}(X) = H^i(X)$ for all $i \in \mathbf{Z}$ (and the same holds for $Y$ and $Z$) and that $\delta^{i+2} = \delta^i$ for all $i \in \mathbf{Z}$. Tautologically, we have a finite exact sequence

$$(1) \quad 0 \to \mathrm{Im}(\delta^{-1}) \to H^0(X) \to H^0(Y) \to H^0(Z)$$
$$\to H^1(X) \to H^1(Y) \to H^1(Z) \xrightarrow{\delta^1} \mathrm{Im}(\delta^1) \to 0.$$

Note that for any exact sequence of *finite* modules $K^i$ of the form

$$0 \xrightarrow{f^{-1}=0} K^0 \xrightarrow{f^0} K^1 \xrightarrow{f^1} K^2 \xrightarrow{f^2} \cdots \xrightarrow{f^{n-1}} K^n \xrightarrow{f^n=0} 0$$

one has

$$\prod_{i=0}^{n} |K^i|^{(-1)^i} = 1.$$

(This follows from the formula $|K^i| = |\ker(f^i)| \cdot |\mathrm{Im}(f^i)|$ already used in question 3) and the exactness of the sequence which implies that $|\ker(f^i)| = |\mathrm{Im}(f^{i-1})|$ for all $i$. ) Applying this formula to the exact sequence (1) and using the fact that $\mathrm{Im}(\delta^{-1}) = \mathrm{Im}(\delta^1)$, we get that $h(B) = h(A)h(C)$.

**Exercise 5.** Let $\mathcal{A}$ denote the category of left modules over a unitary ring $A$. Denote by $K(\mathcal{A})$ the category of chain complexes in $\mathcal{A}$. We know that $K(\mathcal{A})$ is an abelian category.

1) For any two complexes $X$ and $Y$ and $f, g \in \mathrm{Hom}_{K(\mathcal{A})}(X, Y)$ we write $f \simeq g$ if $f$ and $g$ are homotopic. Show in all detail that $\simeq$ is an equivalence relation on $\mathrm{Hom}_{K(\mathcal{A})}(X, Y)$ (see Proposition 2.2 of the lecture notes).

**Solution.** For each $f$ we have $f \simeq f$ with the zero homotopy map. If $f-g = d\circ s+s\circ d$, with a homotopy $s$ then $g-f = d\circ(-s)+(-s)\circ d$. This shows that $\simeq$ is a symmetric relation. Finally, if $f - g = d \circ s_1 + s_1 \circ d$ and $g - h = d \circ s_2 - 1 + s_2 \circ d$ with homopopies $s_1$ and $s_2$, then $f - h = d \circ s_1 + s \circ d$ with $s = s_1 + s_2$.

2) Show that there exists a category $H(\mathcal{A})$ whose objects are chain complexes and morphisms are given by

$$\mathrm{Hom}_{H(\mathcal{A})}(X, Y) = \mathrm{Hom}_{K(\mathcal{A})}(X, Y)/\simeq .$$

**Solution.** Let $\alpha \in \mathrm{Hom}_{H(\mathcal{A})}(X,Y)$ and $\beta \in \mathrm{Hom}_{H(\mathcal{A})}(Y,Z)$. Take any representatives $f$ and $g$ of $\alpha$ and $\beta$ and define $\beta \circ \alpha$ as the class of $g \circ f$. We should check that this class does not depend on the choice of representatives. Let $f' \simeq f$ and $g' \simeq g$. We show that

$$g \circ f \simeq g' \circ f \simeq g' \circ f'.$$

Let $g' - g = d \circ s + s \circ d$ with a homotopy $s = (s_n : Y_n \to Z_{n+1})$. Then

$$g' \circ f - g \circ f = (g' - g) \circ f = (d \circ s + s \circ d) \circ f =$$
$$d \circ s \circ f + s \circ d \circ f = d(\circ s \circ f) + (s \circ f) \circ d$$

because $d \circ f = f \circ d$. Thus $s \circ f$ is a homotopy and $g' \circ f \simeq g \circ f$. The same argument shows that $g' \circ f \simeq g' \circ f'$.

All other properties follow directly from the definitions.

Assume that $\mathcal{A}$ is the category of abelian groups. Consider the following morphism of complexes $f : X \to Y$:

$$\begin{array}{lccccccccccc}
X: & \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mathbf{Z} & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & & & \\
Y: & \cdots & \longrightarrow & 0 & \longrightarrow & \mathbf{Z} & \xrightarrow{\mathrm{id}} & \mathbf{Z} & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}$$

3) Show that $f$ is a monomorphism in $K(\mathcal{A})$ but $f = 0$ in $H(\mathcal{A})$.

**Solution.** Let $X = (X_n)$ and $Y = (Y_n)$ be two complexes of abelian groups. In the category $K(\mathcal{A})$ the kernel of a morphism $f : X \to Y$ is $\ker(f) = (\ker(f_n))$, where $f_n : X_n \to Y_n$. In our case $f_0 = \mathrm{id}$ and in all other degrees the source object is 0. Thus it is clear that $f$ is a monomorphism.

Define $s_0 : X_0 = \mathbf{Z} \to Y_1 = \mathbf{Z}$ by $s_0 = \mathrm{id}$ and set $s_n = 0$ for all $n \neq 0$. It is easy to check that $f = d \circ s + s \circ d$. Thus $f \simeq 0$.