UNIVERSITÉ DE BORDEAUX

Homological Algebra

Homework, due November 17th 2022

Execise 1. Let $\mathfrak{Rings}$ denote the category whose objects are unitary rings and where a morphism $f : A \to B$ is an homomorphism of unitary rings (so $f(1_A) = 1_B$).

1) Show that each surjective homomorphism of rings $f : A \to B$ is an epimorphism in this category.

2) Show that the natural inclusion $\mathbf{Z} \to \mathbf{Q}$ is an epimorphism in $\mathfrak{Rings}$.

Exercise 2. Let A be an integral domain (unitary commutative ring for which every non-zero element is cancellable under multiplication). Denote by $\mathcal{M}$ the category of A-modules. For any A-module M denote by $T(M)$ the torsion submodule of M, that is

$$T(M) = \{x \in M | ax = 0 \text{ for some nonzero } a \in A\}.$$

We say that M is torsion (respectively torsion free), if $T(M) = M$ (respectively $T(M) = 0$). Denote by $\mathcal{T}$ (resp. $\mathcal{TF}$) the full subcategory of $\mathcal{M}$ consisting of torsion (resp. torsion free) modules.

1) Show that $\mathcal{T}$ is an abelian category and $\mathcal{TF}$ is not necessarily abelian.

2) Denote by $I : \mathcal{T} \to \mathcal{M}$ the inclusion functor. Show that the assignement $M \to T(M)$ defines a functor $T : \mathcal{M} \to T$ which is right adjoint to I.

3) Denote by $J : \mathcal{TF} \to \mathcal{M}$ the inclusion functor. Show that the assignement $M \to M/T(M)$ defines a functor $F : \mathcal{M} \to \mathcal{TF}$ which is left adjoint to J.

Exercise 3. Let $C^\infty(0,1)$ denote the $\mathbf{R}$-vector space of infinitely differentiable functions on the open segment $]0,1[$. We denote by D the differentiation $D(f) = \frac{df(x)}{dx}$.

1) Consider the cochain complex

$$C \ : \ 0 \to C^\infty(0,1) \xrightarrow{D} C^\infty(0,1) \to 0,$$

where the two nonzero terms are placed in degrees 0 and 1. Compute the cohomology $H^i(C)$ of C.

2) We say that $f \in C^\infty(0,1)$ is compactly supported if its support $\mathrm{Supp}(f) = \{x \in]0,1[| f(x) \neq 0\}$ is contained in a compact subset of

$]0,1[$. Let $C_c^\infty(0,1) \subset C^\infty(0,1)$ denote the subspace of compactly supported functions of $C^\infty(0,1)$. Consider the complex

$$C_c \ : \ 0 \to C_c^\infty(0,1) \xrightarrow{D} C_c^\infty(0,1) \to 0,$$

where the two nonzero terms are placed in degrees 0 and 1. Show that $H^0(C_c) = 0$ and $H^1(C_c)$ is canonically isomorphic to $\mathbf{R}$.

Exercise 4. Let G be a finite cyclic group of order n. Fix a generator g of G. For any G-module A we define two morphisms $N : A \to A$ and $S : A \to A$ by

$$N(x) = x + gx + \cdots + g^{n-1}x, \qquad\qquad S(x) = gx - x.$$

1) Check that $N \circ S = S \circ N = 0$.

Therefore we can consider the complex

$$X \ : \qquad \cdots \xrightarrow{N} X^0 \xrightarrow{S} X^1 \xrightarrow{N} X^2 \xrightarrow{S} X^3 \xrightarrow{N} \cdots$$

where $X^i = A$ for all $i \in \mathbf{Z}$. If $H^0(X)$ and $H^1(X)$ are both finite, we set

$$h(A) = \frac{|H^0(X)|}{|H^1(X)|}$$

and say that $h(A)$ is well defined.

2) Show that $h(\mathbf{Z}) = n$ for the trivial G-module $\mathbf{Z}$.

3) Show that if A is finite, then $h(A) = 1$.

4) Let

$$0 \to A \to B \to C \to 0$$

be an exact sequence of G-modules. Show that, if two of the quotients $h(A)$, $h(B)$ and $h(C)$ are well defined, then the third is well defined and

$$h(B) = h(A)h(C).$$

(Hint: use an appropriate long exact cohomology sequence).

Exercise 5. Let $\mathcal{A}$ denote the category of left modules over a unitary ring A. Denote by $K(\mathcal{A})$ the category of chain complexes in $\mathcal{A}$. We know that $K(\mathcal{A})$ is an abelian category.

1) For any two complexes X and Y and $f, g \in \mathrm{Hom}_{K(\mathcal{A})}(X, Y)$ we write $f \simeq g$ if f and g are homotopic. Show in all detail that $\simeq$ is an equivalence relation on $\mathrm{Hom}_{K(\mathcal{A})}(X, Y)$ (see Proposition 2.2 of the lecture notes).

2) Show that there exists a category $H(\mathcal{A})$ whose objects are chain complexes and morphisms are given by

$$\mathrm{Hom}_{H(\mathcal{A})}(X,Y) = \mathrm{Hom}_{K(\mathcal{A})}(X,Y)/\simeq .$$

Assume that $\mathcal{A}$ is the category of abelian groups. Consider the following morphism of complexes $f : X \to Y$:

$$\begin{array}{lccccccccccc}
X: & \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mathbf{Z} & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & & & \\
Y: & \cdots & \longrightarrow & 0 & \longrightarrow & \mathbf{Z} & \xrightarrow{\mathrm{id}} & \mathbf{Z} & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}$$

3) Show that f is a monomorphism in $K(\mathcal{A})$ but $f = 0$ in $H(\mathcal{A})$.