ON EXTRA ZEROS OF p-ADIC RANKIN–SELBERG L-FUNCTIONS

by

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0. Introduction

0.1. The extra-zero conjecture. — Let $M/\mathbf{Q}$ be a pure motive of weight $\mathrm{wt}(M) \leqslant -2$. Fix an odd prime number p and denote by V the p-adic realization of M. So V is a finite dimensional vector space over a finite extension E of $\mathbf{Q}_p$, equipped with a continuous Galois action. We will always assume that M has a good reduction at p. Let $\mathbf{D}_{\mathrm{cris}}(V)$ denote the Dieudonné module associated to the restriction of V on the decomposition group at p and $t_V(\mathbf{Q}_p) = \mathbf{D}_{\mathrm{cris}}(V)/\mathrm{Fil}^0\mathbf{D}_{\mathrm{cris}}(V)$ be the corresponding tangent space. The Bloch–Kato logarithm is an isomorphism [(1)]

$$\log_V \,:\, H^1_f(\mathbf{Q}_p, V) \to t_V(\mathbf{Q}_p).$$

Let $H^1_f(\mathbf{Q}, V)$ denote the Bloch–Kato Selmer group of V. We have a commutative diagram

$$\begin{array}{ccc} H^1_f(\mathbf{Q},V) & \xrightarrow{\mathrm{res}_p} & H^1_f(\mathbf{Q}_p,V) \\ & \searrow^{r_V} & \downarrow{\scriptstyle \log_V} \\ & & t_V(\mathbf{Q}_p), \end{array}$$

where res_p is the restriction map, and r_V denotes the resulting map. Note that r_V is closely related to the syntomic regulator. Assume that res_p is injective. One expects that this always holds under our assumptions [**39**]. A φ-submodule $D \subset \mathbf{D}_{\mathrm{cris}}(V)$ is called *regular*, if $D \cap \mathrm{Fil}^0\mathbf{D}_{\mathrm{cris}}(V) = \{0\}$ and

$$t_V(\mathbf{Q}_p) = D \oplus r_V\left(H^1_f(\mathbf{Q}, V)\right),$$

where we identify D with its image in $t_V(\mathbf{Q}_p)$. If D is regular, the composition of r_V with the projection $t_V(\mathbf{Q}_p) \to \mathbf{D}_{\mathrm{cris}}(V)/\left(\mathrm{Fil}^0\mathbf{D}_{\mathrm{cris}}(V) + D\right)$ is an isomorphism

$$r_{V,D} \,:\, H^1_f(\mathbf{Q}, V) \to \mathbf{D}_{\mathrm{cris}}(V)/\left(\mathrm{Fil}^0\mathbf{D}_{\mathrm{cris}}(V) + D\right). \tag{1}$$

We call p-adic regulator and denote by $R_p(V,D)$ the determinant of this map. Of course, it depends on the choice of bases, but we omit them from notation in this general discussion.

In [**52**], Perrin-Riou conjectured that to each regular D one can associate a p-adic L-function $L_p(M,D,s)$ satisfying some precise interpolation property. At $s=0$, the conjectural interpolation formula reads

$$L_p(M,D,0) = \mathscr{E}(V,D)R_p(V,D)\frac{L(M,0)}{R_\infty(M)}, \tag{2}$$

where $L(M,s)$ is the complex L-function associated to M, $R_\infty(M)$ and $R_p(V,D)$ are the archimedean and p-adic regulators respectively, computed in the compatible bases[(2)] and $\mathscr{E}(V,D)$ is the Euler-like factor given by

$$\mathscr{E}(V,D) = \det(1 - p^{-1}\varphi^{-1}|D)\,\det(1 - \varphi|\mathbf{D}_{\mathrm{cris}}(V)/D). \tag{3}$$

(1) Since $\mathrm{wt}(M) \leqslant -2$, one has $\mathbf{D}_{\mathrm{cris}}(V)^{\varphi=1} = 0$.

(2) See, for example [**8**, Section 4.2.1].

We say that $L_p(M,D,s)$ has an extra-zero at $s=0$ if $\mathscr{E}(V,D)=0$. By the weight argument, this can occur only if $\mathrm{wt}(M)=-2$, and in this case we define

$$\mathscr{E}^+(V,D)=\det\left(1-p^{-1}\varphi^{-1}|D/D^{\varphi=p^{-1}}\right)\cdot\det\bigl(1-\varphi|\mathbf{D}_{\mathrm{cris}}(V)/D\bigr).$$

If, in addition, we assume that the action of φ on $\mathbf{D}_{\mathrm{cris}}(V)$ is semisimple at p^{-1}, then $\mathscr{E}^+(V,D)\neq 0$. In [**8**], the first named author proposed the following conjecture.

Extra-zero conjecture. — *The p-adic L-function $L_p(M,D,s)$ has a zero of order $e=\dim_E D^{\varphi=p^{-1}}$ at $s=0$ and*

$$L_p(M,D,0)=\mathscr{L}(V,D)\,\mathscr{E}^+(V,D)\,R_p(V,D)\frac{L(M,0)}{R_\infty(M)},$$

where $\mathscr{L}(V,D)$ is the $\mathscr{L}$-invariant constructed in [**8**].

0.2. Examples. — 1) Let $\mathbf{Q}(\eta)$ be the motive associated to an odd Dirichlet character $\eta:(\mathbf{Z}/N\mathbf{Z})^*\to\overline{\mathbf{Q}}^*$ such that $(p,N)=1$ and let $\mathbf{Q}(\eta\chi)$ be its twist by the cyclotomic character χ. In this case, the extra-zero conjecture follows from the explicit formula for the derivative of Kubota–Leopoldt p-adic L-functions proved by Ferrero and Greenberg [**30**] and Gross–Koblitz [**35**] (see also [**7**]).

2) More generally, assume that F is either a totally real or a CM-field and $\mathbf{Q}(\rho)$ is the Artin motive over F associated to an Artin representation ρ of $G_F=\mathrm{Gal}(\overline{F}/F)$. In this case, the extra-zero conjecture for $M=\mathbf{Q}(\rho\chi)$ generalizes the Gross–Stark conjecture for abelian characters of totally real fields proved by Dasgupta, Kakde and Ventullo [**27**]. However, our methods are also applicable beyond the totally real and CM cases, and should provide some insight on a computation of Betina and Dimitrov [**18**]. On the other hand, it seems interesting to compare the formalism of [**8**] with the approach of Büyükboduk and Sakamoto [**21**].

3) Let M_f be the motive associated to a modular form f of odd weight $k\geqslant 3$ and level N_f. We assume that $(p,N_f)=1$. Then the Tate twist $M_f\left(\frac{k+1}{2}\right)$ of M_f is a motive of weight -2 which has a good reduction at p. In this case, the extra-zero conjecture was proved in [**7**].

4) The $\mathscr{L}$-invariant of the adjoint weight one modular form introduced and studied in [**55**] is covered by the general formalism of [**8**].

We remark that in the cases 1) and 3) the motive M is *critical* and $H^1_f(\mathbf{Q},V)=0$.

0.3. Rankin–Selberg L-functions. — In this paper, we prove some results toward the extra-zero conjecture for Rankin–Selberg convolutions of modular forms. This provides some evidences for the Extra-zero conjecture in a *non-critical* setting.

Let $f=\sum\limits_{n=1}^{\infty}a_nq^n$ and $g=\sum\limits_{n=1}^{\infty}b_nq^n$ be two newforms of weights k_0 and l_0, levels N_f and N_g and nebentypus ε_f and ε_g respectively. Let S denote the set of primes dividing N_fN_g. The Rankin–Selberg L-function $L(f,g,s)$ is defined by

$$L(f,g,s)=L_{(N_fN_g)}(\varepsilon_f\varepsilon_g,2s-k_0-l_0+2)\sum_{n=1}^{\infty}\frac{a_nb_n}{n^s},$$

where $L_{(N_fN_g)}(\varepsilon_f\varepsilon_g,2s-k_0-l_0+2)$ is the Dirichlet L-function with removed Euler factors at the primes $q\in S$. Note that, up to Euler factors at the bad primes, $L(f,g,s)$ coincides with the L-function of the motive $M_{f,g}=M_f\otimes M_g$.

Fix an odd prime number p such that $(p,N_fN_g)=1$ and denote by $\alpha_p(f)$ and $\beta_p(f)$ (respectively by $\alpha_p(g)$ and $\beta_p(g)$) the roots of the Hecke polynomial of f (respectively g) at p. We will always assume that

M1) $\alpha(f)\neq\beta(f)$ and $\alpha(g)\neq\beta(g)$.

M2) $v_p(\alpha(f))<k_0-1$ and $v_p(\alpha(g))<l_0-1$, where v_p denotes the p-adic valuation normalized by $v_p(p)=1$.

Denote by f_α and g_α the p-stabilizations of the forms f and g with respect to $\alpha(f)$ and $\alpha(g)$. Let $\mathbf{f}$ and $\mathbf{g}$ be Coleman families passing through f_α and g_α respectively. We denote by $\mathbf{f}_x$ and $\mathbf{g}_y$ the specializations of $\mathbf{f}$ and $\mathbf{g}$ at x and y respectively and by $\mathbf{f}_x^0$ (respectively $\mathbf{g}_y^0$) the primitive modular form of weight x (respectively y) whose p-stabilization is $\mathbf{f}_x$ (respectively $\mathbf{g}_y$).

Theorem 0.3.1. — *For each $0 \leqslant a \leqslant p-2$, there exists a three variable p-adic analytic function $L_p(\mathbf{f},\mathbf{g},\omega^a)(x,y,s)$ defined on $U_{f,g}\times \mathbf{Z}_p$, where $U_{f,g}$ is a sufficiently small neighborhood of (k_0,l_0) in the weight space, such that for each triple of integers $(x,y,j)\in U_{f,g}\times \mathbf{Z}_p$ satisfying*

$$\begin{aligned} &x \equiv k_0 \mod (p-1), \quad y\equiv l_0 \mod (p-1),\\ &j\equiv a \mod (p-1), \quad 2\leqslant y\leqslant j<x, \end{aligned}$$

one has

$$L_p(\mathbf{f},\mathbf{g},\omega^a)(x,y,j)=\frac{\mathscr{E}(\mathbf{f}_x^0,\mathbf{g}_y^0,j)}{C(\mathbf{f}_x^0)}\cdot \frac{\Gamma(j)\Gamma(j-y+1)}{(-i)^{x-y}2^{x-1}(2\pi)^{2j-y+1}\left\langle \mathbf{f}_x^0,\mathbf{g}_y^0\right\rangle}\cdot L(\mathbf{f}_x^0,\mathbf{g}_y^0,j).$$

In this formula, $\left\langle \mathbf{f}_x^0,\mathbf{g}_y^0\right\rangle$ is the Petersson inner product, $C(\mathbf{f}_x^0)$ is defined in (64), and the Euler-like factor $\mathscr{E}(\mathbf{f}_x^0,\mathbf{g}_y^0,j)$ is given by

$$\mathscr{E}(\mathbf{f}_x^0,\mathbf{g}_y^0,j)=\left(1-\frac{p^{j-1}}{\alpha(\mathbf{f}_x^0)\alpha(\mathbf{g}_y^0)}\right)\left(1-\frac{p^{j-1}}{\alpha(\mathbf{f}_x^0)\beta(\mathbf{g}_y^0)}\right)\left(1-\frac{\beta(\mathbf{f}_x^0)\alpha(\mathbf{g}_l^0)}{p^j}\right)\left(1-\frac{\beta(\mathbf{f}_x^0)\beta(\mathbf{g}_l^0)}{p^j}\right).$$

This theorem was first proved in the ordinary case by Hida [**37**]. In [**58**], Urban introduced the overconvergent projector and sketched a proof in the general non-ordinary case, but his arguments contained a serious gap which was filled only recently in [**59**]. Meanwhile, Loeffler and Zerbes [**47**] gave a complete proof of Theorem 0.3.1 based on the theory of Euler systems and unconditional properties of the overconvergent projector proved in [**58**].

0.4. L-values at near central points. — Assume now that f and g are modular forms of the same weight $k_0=l_0\geqslant 2$. Let $M_{f,g}=M_f\otimes M_g$ be the tensor product of motives associated to f and g. Its p-adic realization is $W_{f,g}=W_f\otimes_E W_g$, where W_f and W_g are the p-adic representations associated to f and g by Deligne [**28**], and E denotes an appropriate finite extension of $\mathbf{Q}_p$. Since $(p,N_fN_g)=1$, the representations W_f, W_g and $W_{f,g}$ are crystalline at p, and we have

$$\mathbf{D}_{\mathrm{cris}}(W_{f,g})=\mathbf{D}_{\mathrm{cris}}(W_f)\otimes_E \mathbf{D}_{\mathrm{cris}}(W_g).$$

The motive $M_{f,g}(k_0)$ is non critical, of motivic weight -2, and its p-adic realization is $V_{f,g}=W_{f,g}(k_0)$. Let $E\eta_f^\alpha$ be the one dimensional eigenspace [(3)] of $\mathbf{D}_{\mathrm{cris}}(W_f)$ associated with the eigenvalue $\alpha(f)$. Set

$$D=\eta_f^\alpha\otimes_E \mathbf{D}_{\mathrm{cris}}(W_g(k_0)).$$

Then D is a φ-submodule of $\mathbf{D}_{\mathrm{cris}}(W_{f,g})$, and an easy computation shows that

$$\mathscr{E}(f,g,k_0)=\mathscr{E}(V_{f,g},D), \tag{4}$$

where the right hand side term is defined by (3).

We define the p-adic L-function $L_{p,\alpha}(f,g,s)$ as the restriction of the three variable p-adic L function from Theorem 0.3.1:

$$L_{p,\alpha}(f,g,s)=L_p(\mathbf{f},\mathbf{g},\omega^{k_0})(k_0,k_0,s).$$

A density argument shows that this function does not depend on the choice of the p-stabilization of g (see Section 7.1).

(3) Here η_f^α denotes the canonical eigenvector associated to $\alpha(f)$. See Section 3.2 below.

Assume that $\varepsilon_f\varepsilon_g \neq \mathrm{id}$. Let $\mathrm{BF}^{[k_0-2]}_{f^*,g^*} \in H^1(\mathbf{Q}, V_{f,g})$ denote the Beilinson–Flach element [(4)] constructed in [**43**]. From the results of Besser [**17**], it follows that the restriction $\mathrm{res}_p\left(\mathrm{BF}^{[k_0-2]}_{f^*,g^*}\right)$ of this element on the decomposition group at p lies in $H^1_f(\mathbf{Q}_p, V_{f,g})$ (see [**43**, Proposition 5.4.1] for detailed arguments). The modular forms f and g^* define canonical bases ω_f of $\mathrm{Fil}^0\mathbf{D}_{\mathrm{cris}}(W_f)$ and ω_{g^*} of $\mathrm{Fil}^0\mathbf{D}_{\mathrm{cris}}(W_{g^*})$. Consider the canonical pairing

$$[\ ,\ ] : \mathbf{D}_{\mathrm{cris}}(W_g) \times \mathbf{D}_{\mathrm{cris}}(W_{g^*}) \to \mathbf{D}_{\mathrm{cris}}(E(1-k_0))$$

Let η_g be any element of $\mathbf{D}_{\mathrm{cris}}(W_g)$ such that

$$[\eta_g, \omega_{g^*}] = e_1^{\otimes(1-k_0)},$$

where e_1 is the canonical basis [(5)] of $\mathbf{D}_{\mathrm{cris}}(E(1))$. Set $b = \omega_f \otimes \eta_g \otimes e_1^{\otimes k_0} \in \mathbf{D}_{\mathrm{cris}}(V_{f,g})$. Then the element

$$\overline{b}_\alpha = b \mod \left(\mathrm{Fil}^0\mathbf{D}_{\mathrm{cris}}(V_{f,g}) + D\right)$$

is a basis of the one dimensional space $\mathbf{D}_{\mathrm{cris}}(V_{f,g})/\left(\mathrm{Fil}^0\mathbf{D}_{\mathrm{cris}}(V_{f,g}) + D\right)$. Therefore the image of $\mathrm{res}_p\left(\mathrm{BF}^{[k_0-2]}_{f^*,g^*}\right)$ under the composition

$$H^1_f(\mathbf{Q}_p, V_{f,g}) \xrightarrow{\log_{V_{f,g}}} t_{V_{f,g}}(\mathbf{Q}_p) \to \mathbf{D}_{\mathrm{cris}}(V_{f,g})/\left(\mathrm{Fil}^0\mathbf{D}_{\mathrm{cris}}(V_{f,g}) + D\right)$$

can be written in a unique way as $\widetilde{R}_p(V_{f,g}, D)\cdot \overline{b}_\alpha$ with $\widetilde{R}_p(V_{f,g}, D) \in E$. We remark that $\widetilde{R}_p(V_{f,g}, D)$ concides with the regulator $R_p(V_{f,g}, D)$ if $H^1_f(\mathbf{Q}, V_{f,g})$ is the one dimensional vector space generated by $\mathrm{BF}^{[k_0-2]}_{f^*,g^*}$. One expect that this always holds [(6)] . We have the following result toward Perrin-Riou's conjecture (2).

Theorem 0.4.1. — *Assume that* $\varepsilon_f\varepsilon_g \neq \mathrm{id}$ *(in particular, this condition implies that $f \neq g^*$). Then the following formula holds:*

$$L_{p,\alpha}(f,g,k_0) = \frac{\varepsilon(f,g,k_0)\cdot \mathscr{E}(V_{f,g},D)}{C(f)\cdot G(\varepsilon_f)\cdot G(\varepsilon_g)\cdot (k_0-2)!}\cdot \widetilde{R}_p(V_{f,g},D).$$

Here $G(\varepsilon_f)$ and $G(\varepsilon_g)$ are Gauss sums associated to ε_f and ε_g, $\varepsilon(f,g,k_0)$ the epsilon constant of the functional equation of the complex L-function, and

$$C(f) = \left(1 - \frac{\beta(f)}{p\alpha(f)}\right)\cdot\left(1 - \frac{\beta(f)}{\alpha(f)}\right).$$

Proof. — This theorem was first proved by Bertolini–Darmon–Rotger [**15**] for modular forms of weight 2. Kings, Loeffler and Zerbes extended the proof to the higher weight case [**43**, Theorem 7.2.6], [**47**, Theorem 7.1.5]. Note that the results proved in [**43**] and [**47**] are in fact more general and include also the case of modular forms of different weights. □

We remark that, combining this formula with the computaiton of the special value of the complex L-function in terms of the Beilinson regulator, one can write this theorem in the form (2) (see [**43**, Theorem 7.2.6]). Also, this result suggests that $L_{p,\alpha}(f,g,s+k_0)$ satisfies the conjectural interpolation properties of $L_p(M_{f,g}(k_0), D, s)$ up to "bad" Euler factors at primes dividing N_fN_g.

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(4) In the weight 2 case, the motivic version of this element was first constructed by Beilinson[**2**]. In [**29**], Flach exploited the p-adic realization of Beilinson's elements in the study of the Selmer group of the symmetric square of an elliptic curve.

(5) More explicitly, $e_1 = \varepsilon \otimes t^{-1}$, where ε is a compatible system of p^nth roots of unity, and $t = \log[\varepsilon]$ the associated element of $\mathbf{B}_{\mathrm{cris}}$.

(6) Beilinson conjectures in the formulation of Bloch and Kato predict that $H^1_f(\mathbf{Q}, V_{f,g})$ has dimension 1.

0.5. The main result. — We keep previous notation and conventions. In this paper, we prove a result toward the extra-zero conjecture for the p-adic L-finction $L_{p,\alpha}(f,g,s)$ as $s=k_0$. In addition to **M1-2)** we assume that the following conditions hold:

M3) ε_f, ε_g and $\varepsilon_f\varepsilon_g$ are primitives modulo N_f, N_g and $\mathrm{lcm}(N_f,N_g)$ respectively.
M4) $\varepsilon_f(p)\varepsilon_g(p)\neq 1$.

Note that the condition **M3)** can be relaxed. We introduce it mainly because in this case the functional equation for the Rankin–Selberg L-function has a simpler form. However, the condition $\varepsilon_f\varepsilon_g\neq \mathrm{id}$ can not be relaxed. In particular, the case $f=g^*$ should be excuded.

Assume that the interpolation factor $\mathscr{E}(f,g,k_0)$ vanishes. Without loss of generality, we can assume that $\alpha(f)\beta(g)=p^{k_0-1}$. Then the φ-module D has a four-step filtration

$$\{0\}\subset D_{-1}\subset D_0\subset D_1\subset \mathbf{D}_{\mathrm{cris}}(V_{f,g}) \tag{5}$$

such that $D_0=D$, D_{-1} is the eigenline of φ associated to the eigenvalue $\alpha(f)\alpha(g)p^{-k_0}$, and D_1 is the unique subspace such that φ acts on D_1/D_0 as the multiplication by $\beta(f)\alpha(g)p^{-k_0}$. Note that φ acts on D_0/D_{-1} as the multiplication by p^{-1}. Taking duals, we have a filtation on $\mathbf{D}_{\mathrm{cris}}(V_{f,g}^*(1))$

$$\{0\}\subset D_{-1}^{\perp}\subset D_0^{\perp}\subset D_1^{\perp}\subset \mathbf{D}_{\mathrm{cris}}(V_{f,g}^*(1)),$$

such that φ acts trivially on $D_1^{\perp}/D_0^{\perp}$. This filtration induces a filtration of the associated (φ,Γ)-modules

$$\{0\}\subset F_0\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_{f,g}^*(1))\subset F_1\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_{f,g}^*(1))\subset \mathbf{D}^{\dagger}_{\mathrm{rig}}(V_{f,g}^*(1)).$$

By [**6**, Proposition 1.5.9], the cohomology of the quotient

$$\mathrm{gr}_1\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_{f,g}^*(1))=F_1\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_{f,g}^*(1))/F_0\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_{f,g}^*(1))$$

has a canonical decomposition

$$H^1\left(\mathrm{gr}_1\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_{f,g}^*(1))\right)=H^1_f\left(\mathrm{gr}_1\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_{f,g}^*(1))\right)\oplus H^1_c\left(\mathrm{gr}_1\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_{f,g}^*(1))\right) \tag{6}$$

into two subspaces of dimension 1, which are both canonically isomorphic to $D_1^{\perp}/D_0^{\perp}$. The interpolation of Beilinson–Flach elements (see [**44**]) provides us with an element $\widetilde{Z}_{f,g}^{[k_0-1]}\in H^1\left(\mathrm{gr}_1\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_{f,g}^*(1))\right)$. Since Beilinson's conjecture and the injectivity of the restriction map $H^1_f(\mathbf{Q},V_{f,g})\to H^1_f(\mathbf{Q}_p,V_{f,g})$ are not known in our case, we can not work with the general definition of the $\mathscr{L}$-invariant proposed in [**8**]. To remeday this problem, we introduce the ad hoc invariant $\widetilde{\mathscr{L}}(V_{f,g},D)$ as the slope of the line generated by $\widetilde{Z}_{f,g}^{[k_0-1]}$ under the decomposition (6). We show that $\widetilde{\mathscr{L}}(V_{f,g},D)$ coincides with the invariant $\mathscr{L}(V_{f,g},D)$ defined in [**8**] if the above mentioned conjectures hold and the regulator $\widetilde{R}_p(V_{f,g},D)$ does not vanish. The main result of this paper is the following theorem (see Theorem 7.3.1).

***Theorem I.* —** *Assume that $\alpha(f)\beta(g)=p^{k_0-1}$. Then*

1) $L_{p,\alpha}(f,g,k_0)=0$.

2) The following conditions are equivalent:

i) $\mathrm{ord}_{s=k_0}L_{p,\alpha}(f,g,s)=1$.

ii) ${}_b\widetilde{Z}_{f,g}^{[k_0-1]}\notin H^1_c\left(\mathrm{gr}_1\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_{f,g}^*(1))\right)$.

3) In addition to the assumption that $\alpha(f)\beta(g)=p^{k_0-1}$, suppose that

$${}_b\widetilde{Z}_{f,g}^{[k_0-1]}\notin H^1_f\left(\mathrm{gr}_1\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_{f,g}^*(1))\right).$$

Then

$$L'_{p,\alpha}(f,g,k_0)=\frac{\varepsilon(f,g,k_0)\cdot\widetilde{\mathscr{L}}(V_{f,g},D)\cdot\mathscr{E}^+(V_{f,g},D)}{C(f)\cdot G(\varepsilon_f)\cdot G(\varepsilon_g)\cdot(k_0-2)!}\cdot\widetilde{R}_p(V_{f,g},D),$$

where

$$\mathscr{E}^+(V_{f,g},D)=\left(1-\frac{p^{k_0-1}}{\alpha(f)\alpha(g)}\right)\left(1-\frac{\beta(f)\alpha(g)}{p^{k_0}}\right)\left(1-\frac{\beta(f)\beta(g)}{p^{k_0}}\right).$$

Remark. — We expect that ${}_b\widetilde{\mathrm{Z}}_{f,g}^{[k_0-1]}$ is in general position with respect to the subspaces $H^1_c\left(\mathrm{gr}_1\mathbf{D}^{\dagger}_{\mathrm{rig}}(V^*_{f,g}(1))\right)$ and $H^1_f\left(\mathrm{gr}_1\mathbf{D}^{\dagger}_{\mathrm{rig}}(V^*_{f,g}(1))\right)$. In this case, $\mathrm{ord}_{s=k_0}L_{p,\alpha}(f,g,s)=1$ and both the p-adic regulator $\widetilde{R}_p(V_{f,g},D)$ and the $\mathscr{L}$-invariant $\widetilde{\mathscr{L}}(V_{f,g},D)$ does not vanish.

It would be interesting to understand the relationship between our approach and the methods of Rivero and Rotger [**54**], where the case $g=f^*$ is studied.

0.6. Outline of the proof. — The proof of Theorem I relies heavily on the theory of Beilinsion–Flach elements initiated by Bertolini, Darmon and Rotger [**15, 16**] and extensively developed by Lei, Kings, Loeffler and Zerbes [**46, 44, 43, 47**]. Note that in the non ordinary case, the overconvergent Shimura isomorphism of Andreatta, Iovita and Stevens [**1**] plays a crucial role in the theory.

Let $\mathbf{f}=\sum\limits_{n=1}^{\infty}\mathbf{a}_nq^n$ and $\mathbf{g}=\sum\limits_{n=1}^{\infty}\mathbf{b}_nq^n$ denote Coleman families passing through the stabilizations f_α and g_α of two forms of weight k_0. Kings, Loeffler and Zerbes [**43**] (in the ordinary case) and Loeffler and Zerbes [**47**] (in the general case) expressed the three variable p-adic L-function as the image of the stabilized three variable Beilinson–Flach element under the large exponential map. Using the semistabilized versions of Beilinson–Flach elements we define, in a neighborhood of k_0, two anaytic p-adic L-functions $L_p^{\mathrm{wc}}(\mathbf{f},\mathbf{g},s)$ and $L_p^{\mathrm{wt}}(\mathbf{f},\mathbf{g},s)$ which can be viewed as "improved" versions of the three variable p-adic L-function. Namely

$$L_p(\mathbf{f},\mathbf{g},\omega^{k_0})(k_0,s,s)=(-1)^{k_0}\left(1-\frac{\mathbf{b}_p(s)}{\varepsilon_g(p)\mathbf{a}_p(k_0)}\right)\left(1-\frac{\varepsilon_g(p)\mathbf{a}_p(k_0)}{p\mathbf{b}_p(s)}\right)^{-1}L_p^{\mathrm{wc}}(\mathbf{f},\mathbf{g},s),$$
$$L_p(\mathbf{f},\mathbf{g},\omega^{k_0-1})(k_0,s,k_0-1)=-\left(1-\frac{p^{k_0-2}}{\mathbf{a}_p(k_0)\mathbf{b}_p(s)}\right)\left(1-\frac{\varepsilon_f(p)\mathbf{b}_p(s)}{\mathbf{a}_p(k_0)}\right)L_p^{\mathrm{wt}}(\mathbf{f},\mathbf{g},s)$$

(see Propositions 6.2.3 and 6.3.3).

The crystalline (φ,Γ)-modules $\mathrm{gr}_1\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_{f,g})$ and $\mathrm{gr}_0\mathbf{D}^{\dagger}_{\mathrm{rig}}(V^*_{f,g}(1))$ are Tate dual to each other, and we denote by

$$[\ ,\ ]:\mathscr{D}_{\mathrm{cris}}\left(\mathrm{gr}_1\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_{f,g})\right)\times\mathscr{D}_{\mathrm{cris}}\left(\mathrm{gr}_0\mathbf{D}^{\dagger}_{\mathrm{rig}}(V^*_{f,g}(1))\right)\to E$$

the resulting duality of Dieudonné modules. Analogously, the (φ,Γ)-modules $\mathrm{gr}_1\mathbf{D}^{\dagger}_{\mathrm{rig}}(V^*_{f,g}(1))$ and $\mathrm{gr}_0\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_{f,g})$ are Tate dual to each other and we denote by

$$\langle\ ,\ \rangle:H^1\left(\mathrm{gr}_1\mathbf{D}^{\dagger}_{\mathrm{rig}}(V^*_{f,g}(1))\right)\times H^1\left(\mathrm{gr}_0\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_{f,g})\right)\to E$$

the induced local duality on cohomology. Let

$$\exp:\mathscr{D}_{\mathrm{cris}}\left(\mathrm{gr}_0\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_{f,g})\right)\to H^1\left(\mathrm{gr}_0\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_{f,g})\right)$$

and

$$\log:H^1\left(\mathrm{gr}_1\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_{f,g})\right)\to\mathscr{D}_{\mathrm{cris}}\left(\mathrm{gr}_1\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_{f,g})\right)$$

denote the Bloch–Kato exponential and logarithm maps for the corresponding (φ,Γ)-modules (see [**8**, Section 2.1.4], [**48**]). The filtered Dieudonné modules $\mathscr{D}_{\mathrm{cris}}\left(\mathrm{gr}_0\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_{f,g})\right)$ and $\mathscr{D}_{\mathrm{cris}}\left(\mathrm{gr}_0\mathbf{D}^{\dagger}_{\mathrm{rig}}(V^*_{f,g}(1))\right)$ have canonical bases which we denote by $d_{\alpha\beta}$ and $n_{\alpha\beta}$ respectively (see Section 6.5). The Beilinson–Flach element $\mathrm{BF}^{[k_0-2]}_{f^*g^*}$ can be "projected" on the subquotient $H^1\left(\mathrm{gr}_1\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_{f,g})\right)$ of $H^1(\mathbf{Q}_p,V_{f,g})$ and we denote by $\mathrm{Z}^{[k_0-2]}_{f^*,g^*}$ its image in $H^1\left(\mathrm{gr}_1\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_{f,g})\right)$ (see

Definition 5.2.5 and Corollary 5.2.6). The functional equation for the improved L-functions has the following interpretation in terms of Beilinson–Flach elements (see Theorem 6.5.3).

Theorem II. — *Assume that $\beta(f)\alpha(g) \neq p^{k_0-1}$. Then the elements $\mathrm{Z}^{[k_0-2]}_{f^*,g^*}$ and $\widetilde{\mathrm{Z}}^{[k_0-1]}_{f,g}$ are related by the equation*

$$\frac{\left\langle \widetilde{\mathrm{Z}}^{[k_0-1]}_{f,g}, \exp(d_{\alpha\beta}) \right\rangle}{G(\varepsilon_f^{-1})G(\varepsilon_g^{-1})} = (-1)^{k_0-1}\varepsilon(f,g,k_0)\cdot \mathscr{E}(V_{f,g},D_{-1})\cdot \frac{\left[\log\left(\mathrm{Z}^{[k_0-2]}_{f^*,g^*}\right), n_{\alpha\beta}\right]}{(k_0-2)!G(\varepsilon_f)G(\varepsilon_g)}, \tag{7}$$

where

$$\mathscr{E}(V_{f,g},D_{-1}) = \det\left(1-p^{-1}\varphi^{-1} \mid D_{-1}\right)\det\left(1-\varphi \mid \mathbf{D}_{\mathrm{cris}}(V_{f,g})/D_{-1}\right).$$

We deduce Theorem I from this theorem. Namely, the machinery developed in [**8**] gives a formula for the derivative of $L_{p,\alpha}(f,g,s)$ at $s=k_0$ in terms of the $\mathscr{L}$-invariant $\widetilde{\mathscr{L}}(V_{f,g},D)$ and the left hand side of equation (7). Using Theorem II, we express it in terms of the right hand side of (7), which is essentially the regulator $\widetilde{R}_p(V_{f,g},D)$.

We hope that our approach could be useful to study some other cases of extra-zeros of non-critical motives.

0.7. The plan of the paper. — The organization of the paper is as follows. In Section 1, we review basic results about the cohomology of (φ,Γ)-modules and the large exponential map. In Sections 2.1-2.2, we review the definition of the $\mathscr{L}$-invariant in the non critical case. Note that in [**8**], the first named author considered only the representations arising from motives of weight -2 because the dual case can be treated using the functional equation. However, to compare this general definition with our ad hoc invariant $\widetilde{\mathscr{L}}(V_{f,g},D)$, it is important to have an intrinsic definition of the $\mathscr{L}$-invariant in the weight 0 case. This is the subject of Section 2.3. In Section 3, for the convenience of the reader, we review the overconvergent étale cohomology of modular curves and its application to Coleman families following [**1**] and [**47**]. In Section 4, we review the construction of Beilinsion–Flach elements following [**43, 46, 44**] and introduce semistabilized Beilinsion–Flach elements, which play a key role in this paper. Local properties of these elements are studied in Section 5. In Section 6, using semistabilized Beilinsion–Flach elements, we define the improved p-adic L-functions $L_p^{\mathrm{wc}}(\mathbf{f},\mathbf{g},s)$ and $L_p^{\mathrm{wt}}(\mathbf{f},\mathbf{g},s)$ and prove Theorem II. In Section 7, we prove Theorem I.

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1. The exponential map

1.1. Notation and conventions. —

1.1.1. — Le p be an odd prime. In this subsection, $\overline{\mathbf{Q}}_p$ denotes a fixed algebraic closure of $\mathbf{Q}_p$ and $\mathbf{C}_p$ the p-adic completion of $\overline{\mathbf{Q}}_p$. For any extension $L/\mathbf{Q}_p$, we set $G_L = \mathrm{Gal}(\overline{\mathbf{Q}}_p/L)$. Fix a system $\varepsilon = (\zeta_{p^n})_{n\geq 0}$ of primitive p^nth roots of unity such that $\zeta_{p^{n+1}}^p = \zeta_{p^n}$ for all $n\geq 0$. Set $K_n = \mathbf{Q}_p(\zeta_{p^n})$, $K_\infty = \underset{n\geq 0}{\cup} K_n$ and $\Gamma = \mathrm{Gal}(K_\infty/\mathbf{Q}_p)$. There is a canonical decomposition

$$\Gamma \simeq \Delta\times\Gamma_1, \qquad \Gamma_1 = \mathrm{Gal}(K_\infty/K_1).$$

We denote by $\chi : \Gamma\to\mathbf{Z}_p^*$ the cyclotomic character and by ω its restriction on $\Delta = \mathrm{Gal}(K_1/\mathbf{Q}_p)$. We also denote by $\langle\chi\rangle$ the composition of χ with the projection $\mathbf{Z}_p^*\to(1+p\mathbf{Z}_p)^*$ induced by the canonical decomposition $\mathbf{Z}_p^*\simeq(\mathbf{Z}/p\mathbf{Z})^*\times(1+p\mathbf{Z}_p)^*$.

1.1.2. — Let $(w_1,\dots,w_d)$ be a finite set of variables. If $E/\mathbf{Q}_p$ is a finite extension, we denote by

$$A = E\,\langle w_1/p^r,\dots,w_d/p^r\rangle \tag{8}$$

the Tate algebra of formal power series

$$F(w_1,\dots,w_d) = \sum_{(m_1,\dots,m_d)\in\mathbf{N}^d} c_{m_1,\dots,m_d}(w_1/p^r)^{m_1}\cdots(w_d/p^r)^{m_d}$$

such that $c_{m_1,\dots,m_d}\to 0$ when $m_1+\cdots+m_d\to+\infty$.

1.1.3. — Fix $k=(k_1,\dots,k_d)\in\mathbf{Z}_p^d$ and consider the closed disk with center k and radius $1/p^{r-1}$ in $\mathbf{Z}_p^d$:

$$D(k,1/p^{r-1}) = k+p^{r-1}\mathbf{Z}_p^d.$$

For each $F\in A$, we define the p-adic analytic function $\mathscr{A}^{\mathrm{wt}}(F)$ on $D(k,1/p^{r-1})$ with values in E by

$$\mathscr{A}^{\mathrm{wt}}(F)(\kappa_1,\dots,\kappa_d) = F\left((1+p)^{\kappa_1-k_1}-1,\dots,(1+p)^{\kappa_d-k_d}-1\right).$$

If M is an A-module, and $x\in\mathrm{Spm}(A)$, we denote by $\mathfrak{m}_x$ the corresponding maximal ideal of A and set $k(x)=A/\mathfrak{m}_x$ and $M_x=M\otimes_A k(x)$. Let

$$\mathrm{sp}_x\,:\,M\to M_x \tag{9}$$

denote the specialization map. If

$$\mathfrak{m}_x = \left((1+w_1)-(1+p)^{\kappa_1-k_1},\dots,(1+w_d)-(1+p)^{\kappa_d-k_d}\right)$$

with $\kappa=(\kappa_1,\dots,\kappa_d)\in D(k,1/p^{r-1})$, we will often write M_κ and sp_κ instead M_x and sp_x respectively.

1.1.4. — Let $\mathscr{H}_E$ denote the ring of power series $f(T)\in E[[T]]$ which converge on the open unit disk. If $\gamma_1\in\Gamma_1$ is a fixed generator of the p-procyclic group Γ_1, then the map $\gamma_1\mapsto T-1$ identifies $\mathscr{H}_E$ with the large Iwasawa algebra $\mathscr{H}_E(\Gamma_1)$. We set $\mathscr{H}_E(\Gamma)=E[\Delta]\otimes_E\mathscr{H}(\Gamma_1)$. Each $h\in\mathscr{H}_E(\Gamma)$ can be written in the form

$$h=\sum_{i=1}^{p-1}\delta_i h_i(\gamma_1-1),\qquad \text{where } \delta_i=\frac{1}{|\Delta|}\sum_{g\in\Delta}\omega(g)^{-i}g.$$

Define

$$\mathscr{A}^{\mathrm{c}}_{\omega^i}(h)(s) = h_i(\chi(\gamma_1)^s-1),\qquad 1\leqslant i\leqslant p-1.$$

Note that the series $\mathscr{A}^{\mathrm{c}}_{\omega^i}(h)(s)$ converge on the open unit disk.

For each $i\in\mathbf{Z}$, we have a Γ-equivariant map

$$\begin{aligned}&\mathrm{sp}^c_m\,:\,\mathscr{H}_E(\Gamma)\to E(\chi^m),\\ &\mathrm{sp}^c_m(f)=\mathscr{A}^c_{\omega^m}(f)(m)\otimes\chi^m.\end{aligned} \tag{10}$$

1.1.5. — If A is a Tate algebra of the form (8), we set $\mathscr{H}_A(\Gamma)=A\widehat{\otimes}_E\mathscr{H}_E(\Gamma)$. For each $F\in\mathscr{H}_A(\Gamma)$ define

$$\mathscr{A}_{\omega^i}(F)(\kappa_1,\dots,\kappa_d,s) = \left(\mathscr{A}^{\mathrm{wt}}\otimes\mathscr{A}^{\mathrm{c}}_{\omega^i}\right)(F),\qquad 1\leqslant i\leqslant p-1. \tag{11}$$

Let $\eta\,:\,\Gamma\to A^*$ be a continuous character. When $\eta|_\Delta=\omega^m$ for some $0\leqslant m\leqslant p-2$. The algebra $\mathscr{H}_A(\Gamma)$ is equipped with the twist operator

$$\mathrm{Tw}_\eta\,:\,\mathscr{H}_A(\Gamma)\to\mathscr{H}_A(\Gamma),\qquad \mathrm{Tw}_\eta\left(F(\gamma_1-1)\delta_i\right)=F(\chi(\gamma_1)^m\gamma_1-1)\delta_{i-m}. \tag{12}$$

If $\eta=\chi^m$ with $m\in\mathbf{Z}$, we write Tw_m instead Tw_{χ^m}. We have

$$\mathrm{Tw}_m\left(F(\gamma_1-1)\delta_i\right)=F(\chi(\gamma_1)^m\gamma_1-1)\delta_{i-m}. \tag{13}$$

The map sp^c_m can be extended by linearity to a map $\mathscr{H}_A(\Gamma)\to A(m)$. Directly from definitions, one has

$$\mathrm{sp}^c_m=\mathrm{sp}^c_0\circ\mathrm{Tw}_m.$$

1.1.6. — Let $A = E\langle w/p^r\rangle$ be the one variable Tate algebra over E. We denote by $\chi : \Gamma \to A^*$ the character defined by

$$\chi(\gamma) = \exp\left(\log_p(1+w)\frac{\log(\langle\langle\chi(\gamma)\rangle\rangle)}{\log(1+p)}\right) \tag{14}$$

Note that $\mathscr{A}^{\mathrm{wt}}(\chi(\gamma))(\kappa) = \langle(\chi(\gamma))\rangle^{\kappa-k}$. The map $\mathrm{Tw}_\chi : \mathscr{H}_A(\Gamma) \to \mathscr{H}_A(\Gamma)$ is explicitly give by

$$\mathrm{Tw}_\chi(F(\gamma_1-1)\delta_i) = F(\chi(\gamma_1)\gamma_1-1)\delta_i. \tag{15}$$

For any $h \in \mathscr{H}_A(\Gamma)$ one has

$$\left(\mathscr{A}^{\mathrm{wt}}\circ \mathrm{Tw}_\chi h\right)\big|_{\kappa=m} = \mathrm{Tw}_{m-k}\circ((\mathscr{A}^{\mathrm{wt}}h)|_{\kappa=m}), \qquad m\equiv k \pmod{(p-1)p^{r-1}}$$

and

$$\mathscr{A}_{\omega^i}(\mathrm{Tw}_\chi h)(\kappa,s) = \mathscr{A}_{\omega^i}(h)(\kappa,s+\kappa-k). \tag{16}$$

1.1.7. — For each $r \in [0,1)$, we denote by $\mathscr{R}_E^{(r)}$ the ring of power series

$$f(X) = \sum_{n\in\mathbf{Z}} a_nX^n, \qquad a_n \in E$$

converging on the open annulus $\mathrm{ann}(r,1) = \{X \in \mathbf{C}_p | r \leqslant |X|_p < 1\}$. These rings are equipped with a canonical Fréchet topology [**11**]. For each affinoid algebra A over E we define

$$\mathscr{R}_A^{(r)} = A\widehat{\otimes}_E\mathscr{R}_E^{(r)}.$$

We define the Robba ring over A the ring $\mathscr{R}_A = \underset{0\leqslant r<1}{\cup} \mathscr{R}_A^{(r)}$. Equip $\mathscr{R}_A$ with a continuous action of Γ and a Frobenius operator φ given by

$$\gamma(f(X)) = f((1+X)^{\chi(\gamma)}-1), \quad \gamma\in\Gamma,$$
$$\varphi(f(X)) = f((1+X)^p-1).$$

In particular,

$$t = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{X^n}{n} \in \mathscr{R}_A,$$

and we have $\varphi(t) = pt$ and $\gamma(t) = \chi(\gamma)t$, $\gamma\in\Gamma$.

1.1.8. — The operator $\varphi : \mathscr{R}_A \to \mathscr{R}_A$ has a left inverse ψ given by

$$\psi(f) = \frac{1}{p}\varphi^{-1}\left(\sum_{\zeta^p=1} f(\zeta(1+X)-1)\right), \qquad f\in\mathscr{R}_A.$$

Set $\mathscr{E}_A = \mathscr{R}_A \cap A[[X]]$. Then $\mathscr{E}_A^{\psi=0}$ is the free $\mathscr{H}_A(\Gamma)$-submodule of $\mathscr{E}_A$ generated by $X+1$.

1.2. Cohomology of (φ,Γ)-modules. —

1.2.1. — In this section, we use freely the theory of (φ,Γ)-modules over relative Robba rings $\mathscr{R}_A$ [**41**]. If $\mathbf{D}$ is a (φ,Γ)-module over $\mathscr{R}_A$, we set $\mathscr{D}_{\mathrm{cris}}(\mathbf{D}) = (\mathbf{D}[1/t])^\Gamma$. Then $\mathscr{D}_{\mathrm{cris}}(\mathbf{D})$ is an A-module equipped with the induced action of φ and a decreasing filtration $(\mathrm{Fil}^i\mathscr{D}_{\mathrm{cris}}(\mathbf{D}))_{i\in\mathbf{Z}}$. For each p-adic representation V of $G_{\mathbf{Q}_p}$ with coefficients in A we denote by $\mathbf{D}^\dagger_{\mathrm{rig},A}(V)$ the associated (φ,Γ)-module For any (φ,Γ)-module $\mathbf{D}$, we denote by $H^i(\mathbf{D})$ the cohomology of the Fontaine–Herr complex

$$C_{\varphi,\gamma_1}(\mathbf{D}) : \mathbf{D}^\Delta \xrightarrow{d_0} \mathbf{D}^\Delta\oplus\mathbf{D}^\Delta \xrightarrow{d_1} \mathbf{D}^\Delta,$$

where γ_1 is a fixed generator of Γ_1, $d_0(x) = ((\varphi-1)x,(\gamma_1-1)x)$ and $d_1(y,z) = (\gamma_1-1)y-(\varphi-1)z$. Let $\mathbf{D}^*(\chi) = \mathrm{Hom}_{\mathscr{R}_A}(\mathbf{D},\mathscr{R}_A(\chi))$ be the Tate dual of $\mathbf{D}$. We have a canonical pairing

$$\langle\ ,\ \rangle_{\mathbf{D}} : H^1(\mathbf{D}^*(\chi))\times H^1(\mathbf{D}) \to H^2(\mathscr{R}_A(\chi)) \simeq A,$$

which generalizes the classical local duality.

The Iwasawa cohomology $H^*_{\mathrm{Iw}}(\mathbf{D})$ of $\mathbf{D}$ is defined as the cohomology of the complex

$$\mathbf{D} \xrightarrow{\psi-1} \mathbf{D}$$

concentrated in degrees 1 and 2. Let $\mathbf{D}\widehat{\otimes}_A \mathscr{H}_A(\Gamma)^\iota$ denote the tensor product $\mathbf{D}\widehat{\otimes}_A \mathscr{H}_A(\Gamma)$ which we consider as a (φ,Γ)-module with the diagonal action of Γ and the additional sructure of $\mathscr{H}(\Gamma)$-module given by

$$\gamma(d\otimes h) = d\otimes h\gamma^{-1}, \qquad d\in \mathbf{D},\ h\in \mathscr{H}(\Gamma),\ \gamma\in\Gamma.$$

There exists a canonical isomorphism of $\mathscr{H}(\Gamma)$-modules

$$H^1_{\mathrm{Iw}}(\mathbf{D}) \simeq H^1\left(\mathbf{D}\widehat{\otimes}_A \mathscr{H}_A(\Gamma)^\iota\right) \tag{17}$$

(see [**41**, Theorem 4.4.8]). We remark that if $\mathbf{D} = \mathbf{D}^\dagger_{\mathrm{rig},A}(V)$ for a p-adic representation V, then $H^i(\mathbf{Q}_p,V)\simeq H^i(\mathbf{D}^\dagger_{\mathrm{rig},A}(V))$ and $H^i_{\mathrm{Iw}}(\mathbf{D}^\dagger_{\mathrm{rig},A}(V))\simeq H^1_{\mathrm{Iw}}(\mathbf{Q}_p,V)$, where $H^1_{\mathrm{Iw}}(\mathbf{Q}_p,V)$ denotes the usual Iwasawa cohomology of V (see [**23**] and [**41**, Corollary 4.4.11]).

For each $m\in\mathbf{Z}$, the map sp^c_{-m} induces a morphism of $\varphi,\Gamma)$-modules

$$\mathrm{sp}^c_{\mathbf{D},m} : \mathbf{D}\widehat{\otimes}_A \mathscr{H}_A(\Gamma)^\iota \to \left(\mathbf{D}\widehat{\otimes}_A \mathscr{H}_A(\Gamma)^\iota\right)\otimes_{\mathscr{H}_A(\Gamma),\mathrm{sp}^c_{-m}} A \simeq \mathbf{D}(\chi^m). \tag{18}$$

Together with the isomorphism (17), it induces homomorphisms on cohomology

$$\mathrm{sp}^c_{\mathbf{D},m} : H^i_{\mathrm{Iw}}(\mathbf{D}) \to H^i(\mathbf{D}(\chi^m)). \tag{19}$$

In the remainder of this paper, we will often omit $\mathbf{D}$ in notation.

1.2.2. — We have a canonical $\mathscr{H}_A(\Gamma)$-linear pairing

$$\{\ ,\ \}_{\mathbf{D}} : H^1_{\mathrm{Iw}}(\mathbf{D}^*(\chi))\times H^1_{\mathrm{Iw}}(\mathbf{D})^\iota \to \mathscr{H}_A(\Gamma) \tag{20}$$

(see [**41**, Definition 4.2.8]). It generalizes the pairing in Iwasawa cohomology of p-adic representations [**51**, Section 3.6].

Lemma 1.2.3. — *i) The pairings $\langle\ ,\ \rangle_{\mathbf{D}}$ and $\{\ ,\ \}_{\mathbf{D}}$ commute with the base change.*
ii) The following diagram commutes

$$\begin{array}{ccc} H^1_{\mathrm{Iw}}(\mathbf{D}^*(\chi))\times H^1_{\mathrm{Iw}}(\mathbf{D})^\iota & \xrightarrow{\{\ ,\ \}_{\mathbf{D}}} & \mathscr{H}_A(\Gamma) \\ \downarrow{\scriptstyle (\mathrm{sp}^c_0,\,\mathrm{sp}^c_0)} & & \downarrow{\scriptstyle \mathrm{sp}^c_0} \\ H^1(\mathbf{D}^*(\chi))\times H^1(\mathbf{D}) & \xrightarrow{\langle\ ,\ \rangle_{\mathbf{D}}} & A. \end{array}$$

Proof. — i) follows immediately from the definition of the pairings and ii) is a particular case of [**41**, Proposition 4.2.9]. □

1.2.4. — Let $\eta : \Gamma\to A^*$ be a continuous character. We denote by

$$\mathrm{Tw}_{\mathbf{D},\eta} : H^i_{\mathrm{Iw}}(\mathbf{D}) \to H^i_{\mathrm{Iw}}(\mathbf{D}(\eta)) \tag{21}$$

the isomorpism given by $d\mapsto d\otimes\eta$. Note that it is not Γ-equivariant. If $\eta=\chi^m$, where χ is the cyclotomic character, we write $\mathrm{Tw}_{\mathbf{D},m}$ instead $\mathrm{Tw}_{\mathbf{D},\chi^m}$. Note that

$$\mathrm{sp}^c_0\circ \mathrm{Tw}_m = \mathrm{sp}^c_m. \tag{22}$$

Lemma 1.2.5. — *Let $\eta : \Gamma\to A^*$ be a continuous character. Then*

$$\left\{\mathrm{Tw}_{\mathbf{D}^*(\chi),\eta}(x), \mathrm{Tw}_{\mathbf{D},\eta^{-1}}(y^\iota)\right\}_{\mathbf{D}(\eta)} = \mathrm{Tw}_{\eta^{-1}}\{x,y^\iota\}_{\mathbf{D}},$$

where the twisting map in the right hand side is defined by (12).

Proof. — By [**41**, Definition 4.2.8 and Lemma 4.2.5]

$$\{x,y^\iota\}_{\mathbf{D}}=\{(\varphi-1)(x),(\varphi-1)(y^\iota)\}^0_{\mathbf{D}},$$

where $\{\ ,\ \}^0_{\mathbf{D}}\,:\,\mathbf{D}^*(\chi)^{\psi=0}\times \mathbf{D}^{\psi=0,\iota}\to \mathscr{R}_A(\Gamma)$ is the unique $\mathscr{R}(\Gamma)$-linear pairing satisfying the following condition: for all $x\in \mathbf{D}^*(\chi)^{\psi=0}$ and $y\in \mathbf{D}^{\psi=0}$ one has

$$\mathrm{res}\left(\{x,y^\iota\}^0_{\mathbf{D}}\cdot \frac{d\gamma_1}{\gamma_1}\right)=\log(\chi(\gamma_1))\cdot \mathrm{res}\left([x,y]_{\mathbf{D}}\cdot \frac{dX}{1+X}\right). \tag{23}$$

Here $[\ ,\ ]_{\mathbf{D}}\,:\,\mathbf{D}^*(\chi)\times \mathbf{D}\to \mathscr{R}_A$ denotes the canonical pairing, and we refer the reader to [**41**, Section 2.1] for any unexplained notation. Let

$$\{\ ,\ \}'\,:\,\mathbf{D}^*(\chi\eta)^{\psi=0}\times \mathbf{D}(\eta^{-1})^{\psi=0,\iota}\to \mathscr{R}_A(\Gamma)$$

denote the map $\{x,y\}'=\mathrm{Tw}_{\eta^{-1}}\left\{\mathrm{Tw}_{\mathbf{D}^*(\chi\eta),\eta^{-1}}(x),\mathrm{Tw}_{\mathbf{D}(\eta^{-1}),\eta}(y^\iota)\right\}^0_{\mathbf{D}}$. An easy computation shows that it is a $\mathscr{R}_A(\Gamma)$-linear pairing which satisfies (23) for the (φ,Γ)-module $\mathbf{D}(\eta^{-1})$. Therefore it coincides with $\{\ ,\ \}^0_{\mathbf{D}(\eta^{-1})}$. This imples the lemma. □

1.3. The large exponential map. —

1.3.1. — Assume that $A=E$. Let $\mathbf{D}$ be a crystalline (φ,Γ)-module over $\mathscr{R}_E$, *i.e.* $\dim_E \mathscr{D}_{\mathrm{cris}}(\mathbf{D})=\mathrm{rk}_{\mathscr{R}_E}\mathbf{D}$. We denote by $H^1_f(\mathbf{D})$ the subgroup of $H^1(\mathbf{D})$ that classifies crystalline extensions of the form $0\to\mathbf{D}\to\mathbf{X}\to\mathscr{R}_E\to 0$ [**6**, Section 1.4]. The equivalence between the category of crystalline (φ,Γ)-modules and that of filtered Dieudonné modules [**13**] induces a canonical homomorphism

$$\exp_{\mathbf{D}}\,:\,\mathscr{D}_{\mathrm{cris}}(\mathbf{D})/\mathrm{Fil}^0\mathscr{D}_{\mathrm{cris}}(\mathbf{D})\to H^1_f(\mathbf{D}),$$

which is a direct generalization of the Bloch–Kato exponential map [**6**, Proposition 1.4.4], [**48**]. Note that $\exp_{\mathbf{D}}$ is an isomorphism if $\mathscr{D}_{\mathrm{cris}}(\mathbf{D})^{\varphi=1}=0$.

If V is a crystalline representation of $G_{\mathbf{Q}_p}$, then $\mathscr{D}_{\mathrm{cris}}(\mathbf{D}^\dagger_{\mathrm{rig},E}(V))\simeq \mathbf{D}_{\mathrm{cris}}(V)$, where $\mathbf{D}_{\mathrm{cris}}$ is classical Fontaine's functor [**31, 32**]. We have a commutative diagram

$$\begin{array}{ccc}
\mathbf{D}_{\mathrm{cris}}(V)/\mathrm{Fil}^0\mathbf{D}_{\mathrm{cris}}(V) & \xrightarrow{\exp_V} & H^1_f(\mathbf{Q}_p,V)\\
\downarrow\simeq & & \downarrow\simeq\\
\mathscr{D}_{\mathrm{cris}}(\mathbf{D})/\mathrm{Fil}^0\mathscr{D}_{\mathrm{cris}}(\mathbf{D}) & \xrightarrow{\exp_{\mathbf{D}}} & H^1_f(\mathbf{D}),
\end{array}$$

where $\exp_V$ is the Bloch–Kato exponential map [**19**].

1.3.2. — Let $\mathbf{D}$ be a (φ,Γ)-module over a Tate algebra A. Assume that for all $x\in \mathrm{Spm}(A)$ the specialization $\mathbf{D}_x=\mathbf{D}\otimes_A A/\mathfrak{m}_x$ of $\mathbf{D}$ at x is a crystalline module. Then $\mathscr{D}_{\mathrm{cris}}(\mathbf{D})$ is a projective A-module of rank $\mathrm{rk}_{\mathscr{R}_A}(\mathbf{D})$ and $\mathscr{D}_{\mathrm{cris}}(\mathbf{D}_x)\simeq \mathscr{D}_{\mathrm{cris}}(\mathbf{D})\otimes_A A/\mathfrak{m}_x$ [**14**, Théorème C]. Moreover, Nakamura [**49**, Section 2] constructed the relative version $\exp_{\mathbf{D}}\,:\,\mathscr{D}_{\mathrm{cris}}(\mathbf{D})/\mathrm{Fil}^0\mathscr{D}_{\mathrm{cris}}(\mathbf{D})\to H^1(\mathbf{D})$ of the exponential map. For any $x\in\mathrm{Spm}(A)$ we have a commtative diagram

$$\begin{array}{ccc}
\mathscr{D}_{\mathrm{cris}}(\mathbf{D})/\mathrm{Fil}^0\mathscr{D}_{\mathrm{cris}}(\mathbf{D}) & \xrightarrow{\exp_{\mathbf{D}}} & H^1(\mathbf{D})\\
\downarrow & & \downarrow\\
\mathscr{D}_{\mathrm{cris}}(\mathbf{D}_x)/\mathrm{Fil}^0\mathscr{D}_{\mathrm{cris}}(\mathbf{D}_x) & \xrightarrow{\exp_{\mathbf{D}_x}} & H^1(\mathbf{D}_x).
\end{array}$$

1.3.3. — Let $\delta\,:\,\mathbf{Q}_p^*\to A^*$ be a continuous character with values in a Tate algebra A. We denote by $\mathscr{R}_A(\delta)$ the (φ,Γ)-module $\mathscr{R}_A\cdot e_\delta$ of rank 1 over $\mathscr{R}_A$ defined by

$$\varphi(e_\delta)=\delta(p)\cdot e_\delta,\qquad \gamma(e_\delta)=\delta(\chi(\gamma))\cdot e_\delta,\quad \gamma\in\Gamma.$$

In Sections 1.3.3-1.3.5, we assume that

$$\delta|_{\mathbb{Z}_p^*}(u)=u^m\qquad \text{for some integer } m\geqslant 1.$$

Then the crystalline module $\mathscr{D}_{\mathrm{cris}}(\mathscr{R}_A(\delta))$ associated to $\mathscr{R}_A(\delta)$ is the free A-module of rank 1 generated by $d_\delta = t^{-m}e_\delta$. The action of φ on d_δ is given by

$$\varphi(d_\delta) = p^{-m}\delta(p)d_\delta.$$

Moreover, $\mathrm{Fil}^0\mathscr{D}_{\mathrm{cris}}(\mathscr{R}_A(\delta)) = 0$, and the exponential map takes the form

$$\exp_{\mathscr{R}_A(\delta)} : \mathscr{D}_{\mathrm{cris}}(\mathscr{R}_A(\delta)) \to H^1(\mathscr{R}_A(\delta)).$$

1.3.4. — We review the construction of the large exponential map for (φ,Γ)-modules of rank one. We refer the reader to [**49**] for general constructions and more detail. Equip the ring $\mathscr{E}_A = \mathscr{R}_A \cap A[[X]]$ with the operator $\partial = (1+X)\dfrac{d}{dX}$. Let $z \in \mathscr{D}_{\mathrm{cris}}(\mathscr{R}_A(\delta)) \otimes_A \mathscr{E}_A^{,\psi=0}$. It may be shown that the equation

$$(\varphi-1)F = z - \frac{\partial^m z(0)}{m!}t^m$$

has a solution in $\mathscr{D}_{\mathrm{cris}}(\mathscr{R}_A(\delta)) \otimes_A \mathscr{R}_A$ and we define

$$\mathrm{Exp}_{\mathscr{R}_A(\delta)}(z) = (-1)^m \frac{\log\chi(\gamma_1)}{p} t^m \partial^m(F).$$

Exactly as in the classical case $A = E$ (see [**12**]), it is not hard to check that $\mathrm{Exp}_{\mathscr{R}_A(\delta)}(z) \in \mathscr{R}_A(\delta)^{\psi=1} \simeq H^1_{\mathrm{Iw}}(\mathscr{R}_A(\delta))$ and we denote by

$$\mathrm{Exp}_{\mathscr{R}(\delta)} : \mathscr{D}_{\mathrm{cris}}(\mathscr{R}_A(\delta)) \otimes_A \mathscr{E}_A^{\psi=0} \to H^1_{\mathrm{Iw}}(\mathscr{R}_A(\delta))$$

the resulting map [**12, 48, 49**]. Let $c \in \Gamma$ denote the unique element such that $\chi(c) = -1$. Set $\mathrm{Exp}^c_{\mathscr{R}(\delta)} = c \circ \mathrm{Exp}_{\mathscr{R}(\delta)}$. For any generator $d \in \mathscr{D}_{\mathrm{cris}}(\mathscr{R}_A(\delta))$ define

$$\begin{aligned}
&\mathfrak{Log}_{\mathscr{R}_A(\delta^{-1}\chi),d} : H^1_{\mathrm{Iw}}(\mathscr{R}_A(\delta^{-1}\chi)) \to \mathscr{H}_A(\Gamma),\\
&\mathfrak{Log}_{\mathscr{R}_A(\delta^{-1}\chi),d}(x) = \left\{x, \mathrm{Exp}^c_{\mathscr{R}_A(\delta)}(d\otimes(1+X)^\iota)\right\}_{\mathscr{R}_A(\delta)}.
\end{aligned}$$

Proposition 1.3.5. — 1) *The maps* $\mathrm{Exp}_{\mathscr{R}_A(\delta)}$ *and* $\mathfrak{Log}_{\mathscr{R}_A(\delta^{-1}\chi),d}$ *commute with the base change.*

2) *Let $A = E$ and let V be a crystalline representation of $G_{\mathbf{Q}_p}$. The choice of a compatible system $\varepsilon = (\zeta_{p^n})_{n\geqslant 0}$ of p^nth roots of unity fixes an isomorphism between $\mathscr{R}_{\mathbf{Q}_p}$ and the ring $\mathbf{B}^\dagger_{\mathrm{rig},\mathbf{Q}_p}$ from* [**11**]. *Assume that $\mathscr{R}_E(\delta)$ is a submodule of $\mathbf{D}^\dagger_{\mathrm{rig},E}(V)$. Then $\mathrm{Exp}_{\mathscr{R}_E(\delta)}$ coincides with the restriction of Perrin-Riou's large exponential map* [**51**]

$$\mathrm{Exp}^\varepsilon_{V,m} : \mathbf{D}_{\mathrm{cris}}(V) \otimes_E \mathscr{E}_E^{\psi=0} \to H^1_{\mathrm{Iw}}(\mathbf{Q}_p, V)$$

on $\mathscr{D}_{\mathrm{cris}}(\mathscr{R}_E(\delta)) \otimes_E \mathscr{E}_E^{\psi=0}$.

3) *Let $k \in \mathbf{Z}$ be an integer such that $k+m \geqslant 1$ and $p^{-k-m}\delta(p)-1$ does not vanish on A. Then*

$$\mathrm{sp}^c_k \circ \mathfrak{Log}_{\mathscr{R}_A(\delta^{-1}\chi),d}(x) = (m+k-1)! \cdot \frac{1-p^{m+k-1}\delta(p)^{-1}}{1-p^{-m-k}\delta(p)} \cdot \left\langle \mathrm{sp}^c_{-k}(x), \exp_{\mathscr{R}_A(\delta\chi^k)}(d[k])\right\rangle_{\mathscr{R}_A(\delta\chi^k)},$$

where we denote by $d[k]$ the image of d under the canonical shift $\mathscr{D}_{\mathrm{cris}}(\mathscr{R}_A(\delta)) \to \mathscr{D}_{\mathrm{cris}}(\mathscr{R}_A(\delta\chi^k))$.

4) *Let $A = E$. Then for any $k \in \mathbf{Z}$ such that $k+m \leqslant 0$ and $p^{-k-m}\delta(p) \neq 1$ one has*

$$\mathrm{sp}^c_k \circ \mathfrak{Log}_{\mathscr{R}_E(\delta^{-1}\chi),d}(x) = \frac{(-1)^{m+k}}{(-m-k)!} \cdot \frac{1-p^{m+k-1}\delta(p)^{-1}}{1-p^{-m-k}\delta(p)} \cdot \left[\log_{\mathscr{R}_E(\delta^{-1}\chi^{1-k})}\left(\mathrm{sp}^c_{-k}(x)\right), d[k]\right]_{\mathscr{R}_E(\delta\chi^k)},$$

where

$$[\ ,\ ]_{\mathscr{R}_E(\delta\chi^k)} : \mathscr{D}_{\mathrm{cris}}\left(\mathscr{R}_E(\delta^{-1}\chi^{1-k})\right) \times \mathscr{D}_{\mathrm{cris}}\left(\mathscr{R}_E(\delta\chi^k)\right) \to E$$

denotes the canonical pairing.

Proof. — Part 1) is clear. Part 2) follows from Berger's construction of the large exponential map [**12**]. Part 3) is essentially the interpolation property of the large exponential map (see [**51**] and [**10**, Corollaire 4.10]. Part 4) is equivalent to Perrin-Riou's explicit reciprocity law. See [**5, 12, 26**] for the proofs in the cassical case of absolutely crystalline representations. The case of (φ,Γ)-modules of rank one over an unramified field is particularly simple and can be treated by the method of Berger [**12**] without additional difficulties. It also can be deduced from the results of Nakamura [**48**], where the approach of Berger was extendend to general de Rham (φ,Γ)-modules. □

Corollary 1.3.6. — *We record the particular cases that will be used in this paper:*

$$\mathrm{sp}_0^c\circ \mathfrak{Log}_{\mathscr{R}_A(\delta^{-1}\chi),d}(x)=(m-1)!\cdot \frac{1-p^{m-1}\delta(p)^{-1}}{1-p^{-m}\delta(p)}\cdot \left\langle \mathrm{sp}_0^c(x),\exp_{\mathscr{R}_A(\delta)}(d)\right\rangle_{\mathscr{R}_A(\delta)},$$
$$\mathrm{sp}_{-m}^c\circ \mathfrak{Log}_{\mathscr{R}_E(\delta^{-1}\chi),d}(x)=\frac{1-p^{-1}\delta(p)^{-1}}{1-\delta(p)}\cdot \left[\log_{\mathscr{R}_E(\delta^{-1}\chi^{m+1})}\left(\mathrm{sp}_m^c(x)\right),d[-m]\right]_{\mathscr{R}_E(\delta\chi^{-m})}.$$

We also need the following technical result.

Proposition 1.3.7. — *Let $m\in \mathbf{Z}$ and let $\delta\,:\,\mathbf{Q}_p^*\to A$ be a continuous character such that*

$$\delta(u)=u^m,\qquad u\in \mathbf{Z}_p^*.$$

Then the following statements hold:

1) If $m\geqslant 1$ or $\delta(p)p^{-m}\neq 1$, then $H^0(\mathscr{R}_A(\delta))=0$.

2a) If A is a principal ideal domain and $m\leqslant 0$, then $H^2(\mathscr{R}_A(\delta))=0$.

2b) If, in addition, $p^{-m}\delta(p)-1$ is invertible in A then $H^1(\mathscr{R}_A(\delta))$ is a free A-module of rank one.

Proof. — 1) If $m\geqslant 1$, then $\mathscr{R}_A(\delta)^\Gamma=0$ and therefore $H^0(\mathscr{R}_A(\delta))=0$. Assume that $m\leqslant 0$. Then $\mathscr{D}_{\mathrm{cris}}(\mathscr{R}_A(\delta))=\mathscr{R}_A(\delta)^\Gamma$ is generated by $d_\delta=t^{-m}e_\delta$, where $\varphi(d_\delta)=p^{-m}\delta(p)d_\delta$. Therefore

$$H^0(\mathscr{R}_A(\delta))=\mathscr{D}_{\mathrm{cris}}(\mathscr{R}_A(\delta))^{\varphi=1}=0.$$

2a) For any bounded complex $C^\bullet$ of A-modules and any A-module M one has a spectral sequence

$$E_2^{i,j}:=\mathrm{Tor}^A_{-i}(H^j(C^\bullet),M)\Rightarrow H^{i+j}(C^\bullet\otimes_A M).$$

In paticular, for any maximal ideal $\mathfrak{m}_x$ of A one has

$$E_2^{i,j}:=\mathrm{Tor}^A_{-i}\left(H^j(\mathscr{R}_A(\delta),k(x)\right)\Rightarrow H^{i+j}\left(\mathscr{R}_{k(x)}(\delta_x)\right), \tag{24}$$

where $k(x)=A/\mathfrak{m}_x$. In degree 2, this gives isomorphisms

$$H^2(\mathscr{R}_A(\delta))\otimes_A k(x)\simeq H^2\left(\mathscr{R}_{k(x)}(\delta_x)\right),\qquad \forall x\in \mathrm{Spm}(A).$$

By local duality and part 1) of the proposition,

$$H^2\left(\mathscr{R}_{k(x)}(\delta_x)\right)=H^0\left(\mathscr{R}_{k(x)}(\delta_x^{-1}\chi)\right)=0,$$

and therefore $H^2(\mathscr{R}_A(\delta))\otimes_A k(x)=0$ for all $x\in \mathrm{Spm}(A)$. Since A is a principal ring, this implies that $H^2(\mathscr{R}_A(\delta))=0$.

2b) The spectral sequence (24) together with 2a) gives

$$H^1(\mathscr{R}_A(\delta))\otimes_A k(x)\simeq H^1\left(\mathscr{R}_{k(x)}(\delta_x)\right),\qquad \forall x\in \mathrm{Spm}(A).$$

From our assumptions, it follows that $H^0\left(\mathscr{R}_{k(x)}(\delta_x)\right)=H^2\left(\mathscr{R}_{k(x)}(\delta_x)\right)=0$ and by the Euler characteristic formula $\dim_{k(x)}H^1\left(\mathscr{R}_{k(x)}(\delta_x)\right)=1$ for all $x\in \mathrm{Spm}(A)$. Thus, $H^1(\mathscr{R}_A(\delta))\otimes_A k(x)$ is a $k(x)$-vector space of dimension 1 for all $x\in \mathrm{Spm}(A)$. Since $H^1(\mathscr{R}_A(\delta))$ is a finitely generated module over the principal ideal domain A, this implies that $H^1(\mathscr{R}_A(\delta))$ is free of rank one over A. The proposition is proved. □

2. Complements on the $\mathscr{L}$-invariant

2.1. Regular submodules. —

2.1.1. — In this section, we first review the definition of the $\mathscr{L}$-invariant of p-adic representations of motivic weight -2 proposed in [**8**]. For the purposes of this paper, it is more convenient to use another construction, which we describe in Section 2.3. We next show that the two definitions are equivalent.

Fix a prime number p and a finite set S of primes of $\mathbf{Q}$ containing p. We denote by $G_{\mathbf{Q},S}$ the Galois group of the maximal algebraic extension of $\mathbf{Q}$ unramified outside $S\cup\{\infty\}$. Let V be a p-adic representation of $G_{\mathbf{Q},S}$ with coefficients in a finite extension E of $\mathbf{Q}_p$. We write $H^*_S(\mathbf{Q},V)$ for the continuous cohomology of $G_{\mathbf{Q},S}$ with coefficients in V. Recall that the Bloch–Kato Selmer group $H^1_f(\mathbf{Q},V)$ is defined as

$$H^1_f(\mathbf{Q},V)=\ker\left(H^1_S(\mathbf{Q},V)\to\bigoplus_{l\in S}\frac{H^1(\mathbf{Q}_l,V)}{H^1_f(\mathbf{Q}_l,V)}\right), \tag{25}$$

where the "local conditions" $H^1_f(\mathbf{Q}_l,V)$ are given by

$$H^1_f(\mathbf{Q}_l,V)=\begin{cases}\ker(H^1(\mathbf{Q}_l,V)\to H^1(I_l,V)) & \text{if } l\neq p,\\ \ker(H^1(\mathbf{Q}_p,V)\to H^1(\mathbf{Q}_p,V\otimes\mathbf{B}_{\mathrm{cris}})) & \text{if } l=p\end{cases} \tag{26}$$

(see [**19**]). Here $\mathbf{B}_{\mathrm{cris}}$ is Fontaine's ring of crystalline periods [**31**]. We also consider the relaxed Selmer group

$$H^1_{f,\{p\}}(\mathbf{Q},V)=\ker\left(H^1_S(\mathbf{Q},V)\to\bigoplus_{l\in S\setminus\{p\}}\frac{H^1(\mathbf{Q}_l,V)}{H^1_f(\mathbf{Q}_l,V)}\right).$$

The Poitou–Tate exact sequence induces an exact sequence

$$0\to H^1_f(\mathbf{Q},V)\to H^1_{f,\{p\}}(\mathbf{Q},V)\to\frac{H^1(\mathbf{Q}_p,V)}{H^1_f(\mathbf{Q}_p,V)}\to H^1_f(\mathbf{Q},V^*(1))^* \tag{27}$$

(see [**33**, Proposition 2.2.1] and [**52**, Lemme 3.3.6]).

2.1.2. — In the rest of this section, we assume that V satisfies the following conditions:

- **C1)** $H^0_S(\mathbf{Q},V)=H^0_S(\mathbf{Q},V^*(1))=0$.
- **C2)** V is crystalline at p and $\mathbf{D}_{\mathrm{cris}}(V)^{\varphi=1}=0$.
- **C3)** $\varphi:\mathbf{D}_{\mathrm{cris}}(V)\to\mathbf{D}_{\mathrm{cris}}(V)$ is semisimple at p^{-1}.
- **C4)** $H^1_f(\mathbf{Q},V^*(1))=0$.
- **C5)** The localization map $\mathrm{res}_p:H^1_f(\mathbf{Q},V)\to H^1_f(\mathbf{Q}_p,V)$ is injective.

We refer the reader to [**9**] for a discussion of these conditions. Here we only remark that if V is the p-adic realization of a pure motive of weight $\leqslant -2$ having good reduction at p, then **C3–5)** are deep conjectures which are known only in some special cases.

From **C2)** it follows that the exponential map $\mathbf{D}_{\mathrm{cris}}(V)/\mathrm{Fil}^0\mathbf{D}_{\mathrm{cris}}(V)\to H^1_f(\mathbf{Q}_p,V)$ is an isomorphism, and we denote by $\log_V$ its inverse. Compositing $\log_V$ with the localization map $H^1_f(\mathbf{Q},V)\to H^1_f(\mathbf{Q}_p,V)$, we obtain a map

$$r_V:H^1_f(\mathbf{Q},V)\to\mathbf{D}_{\mathrm{cris}}(V)/\mathrm{Fil}^0\mathbf{D}_{\mathrm{cris}}(V)$$

which is closely related to the syntomic regulator.

2.1.3. — We introduce the notion of regular submodule, which first appeared in Perrin-Riou's book [**52**] in the context of crystalline representations (see also [**6**]).

Definition 2.1.4 (PERRIN-RIOU). — *Assume that V is a p-adic representation which satisfies the conditions* **C1–5)**.

i) A φ-submodule D of $\mathbf{D}_{\mathrm{cris}}(V)$ is regular if $D\cap \mathrm{Fil}^0\mathbf{D}_{\mathrm{cris}}(V)=0$ and the map

$$r_{V,D}\,:\, H^1_f(\mathbf{Q},V)\to \mathbf{D}_{\mathrm{cris}}(V)/(\mathrm{Fil}^0\mathbf{D}_{\mathrm{cris}}(V)+D),$$

induced by r_V, is an isomorphism.

ii) A φ-submodule D of $\mathbf{D}_{\mathrm{cris}}(V^*(1))$ is regular if

$$D+\mathrm{Fil}^0\mathbf{D}_{\mathrm{cris}}(V^*(1))=\mathbf{D}_{\mathrm{cris}}(V^*(1))$$

and the map

$$D\cap \mathrm{Fil}^0\mathbf{D}_{\mathrm{cris}}(V^*(1))\to H^1_f(\mathbf{Q},V)^*,$$

induced by the dual map $r_V^*\,:\,\mathrm{Fil}^0\mathbf{D}_{\mathrm{cris}}(V^*(1))\to H^1_f(V)^*$, is an isomorphism.

Remark 2.1.5. — *1) Assume that $H^1_f(\mathbf{Q},V)=H^1_f(\mathbf{Q},V^*(1))=0$. Then D is regular if the canonical projection $D\to t_V(\mathbf{Q}_p)$ is an isomorphism of vector spaces, and our definition agrees with the definition given in* [**6**].

2) A φ-submodule D of $\mathbf{D}_{\mathrm{cris}}(V)$ is regular if and only if

$$H^1_f(\mathbf{Q}_p,V)=\mathrm{res}_p\left(H^1_f(\mathbf{Q},V)\right)\oplus H^1_f(\mathbf{D}), \tag{28}$$

where $\mathbf{D}$ is the (φ,Γ)-submodule of $\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)$ associated to D by the theory of Berger [**13**] *(see* [**8**, Section 3.1.3]*).*

2.2. The $\mathscr{L}$-invariant. —

2.2.1. — Let $D\subset \mathbf{D}_{\mathrm{cris}}(V)$ be a regular submodule. From the semisimplicity of φ it follows that, as a φ-module, D decomposes into the direct sum

$$D=D_{-1}\oplus D^{\varphi=p^{-1}},\qquad D_{-1}^{\varphi=p^{-1}}=0,$$

which gives a four step filtration

$$\{0\}\subset D_{-1}\subset D\subset \mathbf{D}_{\mathrm{cris}}(V). \tag{29}$$

Let $F_0\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)$ and $F_{-1}\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)$ denote the (φ,Γ)-submodules of $\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)$ associated to D and D_{-1} by Berger [**13**], thus

$$D=\mathscr{D}_{\mathrm{cris}}\left(F_0\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)\right),\qquad D_{-1}=\mathscr{D}_{\mathrm{cris}}\left(F_{-1}\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)\right).$$

Set $\mathbf{M}=\mathrm{gr}_0\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)$ and $W=D/D_{-1}\simeq \mathscr{D}_{\mathrm{cris}}(\mathbf{M})$. The (φ,Γ)-module $\mathbf{M}$ satisfies

$$\mathrm{Fil}^0\mathscr{D}_{\mathrm{cris}}(\mathbf{M})=0,\qquad \mathscr{D}_{\mathrm{cris}}(\mathbf{M})^{\varphi=p^{-1}}=\mathscr{D}_{\mathrm{cris}}(\mathbf{M}).$$

The cohomology of such modules was studied in detail in [**6**, Proposition 1.5.9 and Section 1.5.10]. In particular, there exists a canonical decomposition of $H^1(\mathbf{M})$ into the direct sum of $H^1_f(\mathbf{M})$ and some canonical space $H^1_c(\mathbf{M})$

$$H^1(\mathbf{M})=H^1_f(\mathbf{M})\oplus H^1_c(\mathbf{M}). \tag{30}$$

Moreover, there are canonical isomorphisms

$$i_{\mathbf{M},f}\,:\,\mathscr{D}_{\mathrm{cris}}(\mathbf{M})\simeq H^1_f(\mathbf{M}),\qquad i_{\mathbf{M},c}\,:\,\mathscr{D}_{\mathrm{cris}}(\mathbf{M})\simeq H^1_c(\mathbf{M}) \tag{31}$$

(see [**6**, Proposition 1.5.9]).

2.2.2. — We have a tautological exact sequence

$$0 \to F_{-1}\mathbf{D}^{\dagger}_{\mathrm{rig}}(V) \to F_0\mathbf{D}^{\dagger}_{\mathrm{rig}}(V) \to \mathbf{M} \to 0$$

which induces exact sequences [**8**, Section 3.1.5]

$$0 \to H^1(F_{-1}\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)) \to H^1(F_0\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)) \to H^1(\mathbf{M}) \to 0,$$
$$0 \to H^1_f(F_{-1}\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)) \to H^1_f(F_0\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)) \to H^1_f(\mathbf{M}) \to 0.$$

Moreover, $H^1_f(F_{-1}\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)) = H^1(F_{-1}\mathbf{D}^{\dagger}_{\mathrm{rig}}(V))$, and we have

$$\frac{H^1(F_0\mathbf{D}^{\dagger}_{\mathrm{rig}}(V))}{H^1_f(F_0\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)} \simeq \frac{H^1(\mathbf{M})}{H^1_f(\mathbf{M})}. \tag{32}$$

2.2.3. — From **C5)** it follows that the localisation map $H^1_{f,\{p\}}(\mathbf{Q},V) \to H^1(\mathbf{Q}_p,V)$ is injective. Let

$$\kappa_D \,:\, H^1_{f,\{p\}}(\mathbf{Q},V) \to \frac{H^1(\mathbf{Q}_p,V)}{H^1_f(F_0\mathbf{D}^{\dagger}_{\mathrm{rig}}(V))}$$

denote the composition of this map with the canonical projection. Then κ_D is an isomorphism [**8**, Lemma 3.1.4]

Let $H^1(D,V)$ denote the inverse image of $H^1(F_0\mathbf{D}^{\dagger}_{\mathrm{rig}}(V))/H^1_f(F_0\mathbf{D}^{\dagger}_{\mathrm{rig}}(V))$ under κ_D. Then κ_D induces an isomorphism

$$H^1(D,V) \simeq \frac{H^1(F_0\mathbf{D}^{\dagger}_{\mathrm{rig}}(V))}{H^1_f(F_0\mathbf{D}^{\dagger}_{\mathrm{rig}}(V))}.$$

Consider the composition map $H^1(D,V) \to H^1(F_0\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)) \to H^1(\mathbf{M})$. From (32) it follows that

$$H^1(D,V) \simeq \frac{H^1(\mathbf{M})}{H^1_f(\mathbf{M})} \tag{33}$$

is an isomorphism. Taking into account (30) and (31) we obtain that $\dim_E H^1(D,V) = \dim \mathscr{D}_{\mathrm{cris}}(\mathbf{M})$. Hence we have a diagram

$$\begin{array}{ccc}
\mathscr{D}_{\mathrm{cris}}(\mathbf{M}) & \xrightarrow[\sim]{i_{\mathbf{M},f}} & H^1_f(\mathbf{M}) \\
\uparrow \rho_{D,f} & & \uparrow p_{\mathbf{M},f} \\
H^1(D,V) & \longrightarrow & H^1(\mathbf{M}) \\
\downarrow \rho_{D,c} & & \downarrow p_{\mathbf{M},c} \\
\mathscr{D}_{\mathrm{cris}}(\mathbf{M}) & \xrightarrow[\sim]{i_{\mathbf{M},c}} & H^1_c(\mathbf{M}),
\end{array}$$

where $\rho_{D,f}$ and $\rho_{D,c}$ are defined as the unique maps making this diagram commute. From (33) it follows that $\rho_{D,c}$ is an isomorphism.

Definition 2.2.4. — *The determinant*

$$\mathscr{L}(V,D) = \det\left(\rho_{D,f} \circ \rho_{D,c}^{-1} \mid \mathscr{D}_{\mathrm{cris}}(\mathbf{M})\right)$$

will be called the $\mathscr{L}$-invariant associated to V and D.

2.3. The dual construction. —

2.3.1. — Let $D^\perp = \mathrm{Hom}_E\left(\mathbf{D}_{\mathrm{cris}}(V)/D, \mathbf{D}_{\mathrm{cris}}(E(1))\right)$. Then $D^\perp$ is a regular submodule of $\mathbf{D}_{\mathrm{cris}}(V^*(1))$. In this section, we define an $\mathscr{L}$-invariant $\mathscr{L}(V^*(1),D^\perp)$ associated to the data $(V^*(1),D^\perp)$.

Let $D_1^\perp$ denote the biggest φ-submodule of $\mathbf{D}_{\mathrm{cris}}(V^*(1))$ such that $(D_1^\perp/D^\perp)^{\varphi=1} = D_1^\perp/D^\perp$. This gives a four step filtration

$$\{0\}\subset D^\perp \subset D_1^\perp \subset \mathbf{D}_{\mathrm{cris}}(V^*(1)). \tag{34}$$

It follows from definition that $D_1^\perp = (D_{-1})^\perp = \mathrm{Hom}_E(\mathbf{D}_{\mathrm{cris}}(V)/D_{-1}, \mathbf{D}_{\mathrm{cris}}(E(1)))$, and therefore (34) is the dual filtration of (29).

We denote by $F_0\mathbf{D}^\dagger_{\mathrm{rig}}(V^*(1))$ and $F_1\mathbf{D}^\dagger_{\mathrm{rig}}(V^*(1))$ the (φ,Γ)-submodules of $\mathbf{D}^\dagger_{\mathrm{rig}}(V^*(1))$ associated to $D^\perp$ and $D_1^\perp$, thus

$$\mathscr{D}_{\mathrm{cris}}\left(F_0\mathbf{D}^\dagger_{\mathrm{rig}}(V^*(1))\right)\simeq D^\perp,\qquad \mathscr{D}_{\mathrm{cris}}\left(F_1\mathbf{D}^\dagger_{\mathrm{rig}}(V^*(1))\right)\simeq D_1^\perp.$$

Then

$$F_0\mathbf{D}^\dagger_{\mathrm{rig}}(V^*(1)) = \mathrm{Hom}_{\mathscr{R}_E}\left(\mathbf{D}^\dagger_{\mathrm{rig}}(V)/F_0\mathbf{D}^\dagger_{\mathrm{rig}}(V), \mathscr{R}_E(\chi)\right),$$
$$F_1\mathbf{D}^\dagger_{\mathrm{rig}}(V^*(1)) = \mathrm{Hom}_{\mathscr{R}_E}\left(\mathbf{D}^\dagger_{\mathrm{rig}}(V)/F_{-1}\mathbf{D}^\dagger_{\mathrm{rig}}(V), \mathscr{R}_E(\chi)\right),$$

and we have an exact sequence

$$0\to F_0\mathbf{D}^\dagger_{\mathrm{rig}}(V^*(1))\to F_1\mathbf{D}^\dagger_{\mathrm{rig}}(V^*(1))\to \mathbf{M}^*(\chi)\to 0, \tag{35}$$

where $\mathbf{M}$ is the (φ,Γ)-module defined in Section 2.2. Note that

$$\mathrm{Fil}^0\mathscr{D}_{\mathrm{cris}}(\mathbf{M}^*(\chi)) = \mathscr{D}_{\mathrm{cris}}(\mathbf{M}^*(\chi)),\qquad \mathscr{D}_{\mathrm{cris}}(\mathbf{M}^*(\chi))^{\varphi=1} = \mathscr{D}_{\mathrm{cris}}(\mathbf{M}^*(\chi)). \tag{36}$$

We refer the reader to [**6**, Proposition 1.5.9 and Section 1.5.10] for the proofs of the following facts.

a) The map

$$\begin{aligned} &i_{\mathbf{M}^*(\chi)}\ :\ \mathscr{D}_{\mathrm{cris}}(\mathbf{M}^*(\chi))\oplus\mathscr{D}_{\mathrm{cris}}(\mathbf{M}^*(\chi))\to H^1(\mathbf{M}^*(\chi)),\\ &i_{\mathbf{M}^*(\chi)}(x,y) = \mathrm{cl}(-x, \frac{\log\chi(\gamma_1)}{p}y)\end{aligned} \tag{37}$$

is an isomorphism.

b) Denote by $i_{\mathbf{M}^*(\chi),f}$ (respectively $i_{\mathbf{M}^*(\chi),c}$) the restriction of $i_{\mathbf{M}^*(\chi)}$ on the first (respectively second) summand. Then $\mathrm{Im}\left(i_{\mathbf{M}^*(\chi),f}\right) = H^1_f(\mathbf{M}^*(\chi))$ and

$$H^1(\mathbf{M}^*(\chi))\simeq H^1_f(\mathbf{M}^*(\chi))\oplus H^1_c(\mathbf{M}^*(\chi)), \tag{38}$$

where $H^1_c(\mathbf{M}^*(\chi)) = \mathrm{Im}\left(i_{\mathbf{M}^*(\chi),c}\right)$.

c) The local duality

$$\langle\ ,\ \rangle_{\mathbf{M}} : H^1(\mathbf{M}^*(\chi))\times H^1(\mathbf{M})\to E$$

has the following explicit description:

$$\left\langle i_{\mathbf{M}^*(\chi)}(\alpha,\beta), i_{\mathbf{M}}(x,y)\right\rangle_{\mathbf{M}} = [\beta,x]_{\mathbf{M}} - [\alpha,y]_{\mathbf{M}}, \tag{39}$$

for all $\alpha,\beta\in\mathscr{D}_{\mathrm{cris}}(\mathbf{M}^*(\chi))$ and $x,y\in\mathscr{D}_{\mathrm{cris}}(\mathbf{M})$.

Lemma 2.3.2. — *i) The following sequences are exact*

$$0\to H^0\left(F_0\mathbf{D}^\dagger_{\mathrm{rig}}(V^*(1))\right)\to H^0\left(F_1\mathbf{D}^\dagger_{\mathrm{rig}}(V^*(1))\right)\to H^0(\mathbf{M}^*(\chi))\to 0,$$
$$0\to H^0\left(F_1\mathbf{D}^\dagger_{\mathrm{rig}}(V^*(1))\right)\to H^0\left(\mathbf{D}^\dagger_{\mathrm{rig}}(V^*(1))\right)\to H^0\left(\mathrm{gr}_2\mathbf{D}^\dagger_{\mathrm{rig}}(V^*(1))\right)\to 0.$$

ii) We have a commutative diagram with exact rows

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & H^1_f\left(F_0\mathbf{D}^{\dagger}_{\mathrm{rig}}(V^*(1))\right) & \longrightarrow & H^1_f\left(F_1\mathbf{D}^{\dagger}_{\mathrm{rig}}(V^*(1))\right) & \longrightarrow & H^1_f\left(\mathbf{M}^*(\chi)\right) & \longrightarrow & 0,\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^1\left(F_0\mathbf{D}^{\dagger}_{\mathrm{rig}}(V^*(1))\right) & \longrightarrow & H^1\left(F_1\mathbf{D}^{\dagger}_{\mathrm{rig}}(V^*(1))\right) & \longrightarrow & H^1\left(\mathbf{M}^*(\chi)\right) & \longrightarrow & 0.
\end{array}
\tag{40}
$$

iii) The natural map $H^1\left(F_1\mathbf{D}^{\dagger}_{\mathrm{rig}}(V^*(1))\right)\to H^1\left(\mathbf{D}^{\dagger}_{\mathrm{rig}}(V^*(1))\right)$ *is injective and*

$$H^1_f\left(F_1\mathbf{D}^{\dagger}_{\mathrm{rig}}(V^*(1))\right)=H^1_f\left(\mathbf{D}^{\dagger}_{\mathrm{rig}}(V^*(1))\right).$$

iv) There is a canonical isomorphism

$$\frac{H^1\left(F_1\mathbf{D}^{\dagger}_{\mathrm{rig}}(V^*(1))\right)}{H^1_f\left(\mathbf{D}^{\dagger}_{\mathrm{rig}}(V^*(1)\right)+H^1\left(F_0\mathbf{D}^{\dagger}_{\mathrm{rig}}(V^*(1))\right)}\simeq\frac{H^1(\mathbf{M}^*(\chi))}{H^1_f(\mathbf{M}^*(\chi))}. \tag{41}$$

Proof. — i) Since the category of crystalline (φ,Γ)-modules is equivalent to the category of filtered Dieudonné modules [**13**], the exact sequence (35) induces an exact sequence

$$0\to\mathrm{Fil}^0\mathscr{D}_{\mathrm{cris}}\left(F_0\mathbf{D}^{\dagger}_{\mathrm{rig}}(V^*(1))\right)\to\mathrm{Fil}^0\mathscr{D}_{\mathrm{cris}}\left(F_1\mathbf{D}^{\dagger}_{\mathrm{rig}}(V^*(1))\right)\to\mathrm{Fil}^0\mathscr{D}_{\mathrm{cris}}\left(\mathbf{M}^*(\chi)\right)\to 0.$$

The semisimplicity of φ implies that the sequence

$$0\to\mathrm{Fil}^0\mathscr{D}_{\mathrm{cris}}\left(F_0\mathbf{D}^{\dagger}_{\mathrm{rig}}(V^*(1))\right)^{\varphi=1}\to\mathrm{Fil}^0\mathscr{D}_{\mathrm{cris}}\left(F_1\mathbf{D}^{\dagger}_{\mathrm{rig}}(V^*(1))\right)^{\varphi=1}\to\mathrm{Fil}^0\mathscr{D}_{\mathrm{cris}}\left(\mathbf{M}^*(\chi)\right)^{\varphi=1}\to 0$$

is exact. Since $H^0(\mathbf{X})=\mathrm{Fil}^0\mathscr{D}_{\mathrm{cris}}(\mathbf{X})^{\varphi=1}$ for any crystalline (φ,Γ)-module $\mathbf{X}$ [**6**, Proposition 1.4.4], the first sequence is exact. The exactness of the second sequence can be proved by the same argument.

ii) We only need to prove that the rows of the diagram (40) are exact. From i) and the long exact cohomology sequence associated to (35) we obtain an exact sequence

$$0\to H^1\left(F_0\mathbf{D}^{\dagger}_{\mathrm{rig}}(V^*(1))\right)\to H^1\left(F_1\mathbf{D}^{\dagger}_{\mathrm{rig}}(V^*(1))\right)\to H^1\left(\mathbf{M}^*(\chi)\right)\to H^2\left(F_0\mathbf{D}^{\dagger}_{\mathrm{rig}}(V^*(1))\right).$$

Condition **C2)** implies that

$$H^0\left(\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)/F_0\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)\right)=\mathrm{Fil}^0\mathscr{D}_{\mathrm{cris}}\left(\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)/F_0\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)\right)^{\varphi=1}=0,$$

and by Poincaré duality for (φ,Γ)-modules, we have

$$H^2\left(F_0\mathbf{D}^{\dagger}_{\mathrm{rig}}(V^*(1))\right)=H^0\left(\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)/F_0\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)\right)^*=0.$$

The exactness of the bottom row is proved. The exactness of the upper row follows from i) and [**6**, Corollary 1.4.6].

iii) The long exact cohomology sequence associated to

$$0\to F_1\mathbf{D}^{\dagger}_{\mathrm{rig}}(V^*(1))\to\mathbf{D}^{\dagger}_{\mathrm{rig}}(V^*(1))\to\mathrm{gr}_2\mathbf{D}^{\dagger}_{\mathrm{rig}}(V^*(1))\to 0$$

together with i) show that the sequence

$$0\to H^1\left(F_1\mathbf{D}^{\dagger}_{\mathrm{rig}}(V^*(1))\right)\to H^1\left(\mathbf{D}^{\dagger}_{\mathrm{rig}}(V^*(1))\right)\to H^1\left(\mathrm{gr}_2\mathbf{D}^{\dagger}_{\mathrm{rig}}(V^*(1))\right)$$

is exact. In particular, the map $H^1\left(F_1\mathbf{D}^{\dagger}_{\mathrm{rig}}(V^*(1))\right)\to H^1\left(\mathbf{D}^{\dagger}_{\mathrm{rig}}(V^*(1))\right)$ is injective. Moreover, since $\mathrm{gr}_2\mathbf{D}^{\dagger}_{\mathrm{rig}}(V^*(1))$ is the Tate dual of $F_{-1}\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)$, from [**6**, Corollary 1.4.10] it follows that

$$\dim_E H^1_f\left(\mathrm{gr}_2\mathbf{D}^{\dagger}_{\mathrm{rig}}(V^*(1)\right)=\dim_E H^1\left(F_{-1}\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)\right)-\dim_E H^1_f\left(F_{-1}\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)\right)=0.$$

Thus, $H^1_f\left(\mathrm{gr}_2\mathbf{D}^\dagger_{\mathrm{rig}}(V^*(1)\right)=0$, and from [**6**, Corollary 1.4.6] it follows that $H^1_f\left(F_1\mathbf{D}^\dagger_{\mathrm{rig}}(V^*(1))\right)=H^1_f\left(\mathbf{D}^\dagger_{\mathrm{rig}}(V^*(1))\right)$.

iv) The last statement follows from ii), iii) and isomorphism theorems.

□

2.3.3. — Consider the exact sequence (27) for the representation $V^*(1)$. Since $H^1_f(\mathbf{Q},V^*(1))=0$, it reads

$$0\to H^1_{f,\{p\}}(\mathbf{Q},V^*(1))\to \frac{H^1(\mathbf{Q}_p,V^*(1))}{H^1_f(\mathbf{Q}_p,V^*(1))}\to H^1_f(\mathbf{Q},V)^*\to 0 \tag{42}$$

(We remark that the surjectivity of the last map follows from **C5)** and the isomorphism $\dfrac{H^1(\mathbf{Q}_p,V^*(1))}{H^1_f(\mathbf{Q}_p,V^*(1))}\simeq H^1_f(\mathbf{Q}_p,V)^*$.) We denote by

$$\kappa_{D^\perp}\ :\ H^1_{f,\{p\}}(\mathbf{Q},V^*(1))\to \frac{H^1(\mathbf{Q}_p,V^*(1))}{H^1_f(\mathbf{Q}_p,V^*(1))+H^1\left(F_0\mathbf{D}^\dagger_{\mathrm{rig}}(V^*(1))\right)}$$

the composition of the second arrow of this exact sequence with the canonical projection.

Lemma 2.3.4. — *The map $\kappa_{D^\perp}$ is an isomorphism.*

Proof. — a) First, we prove the injectivity of $\kappa_{D^\perp}$. Assume that $\kappa_{D^\perp}(x)=0$. Then $\mathrm{res}_p(x)\in H^1_f(\mathbf{Q}_p,V^*(1))+H^1\left(F_0\mathbf{D}^\dagger_{\mathrm{rig}}(V^*(1))\right)$. This implies that $\mathrm{res}_p(x)$ belongs to the orthogonal complement of $H^1_f\left(F_0\mathbf{D}^\dagger_{\mathrm{rig}}(V)\right)$ in $H^1(\mathbf{Q}_p,V^*(1))$. On the other hand, from (42) it follows that $\mathrm{res}_p(x)$ belongs to the orthogonal complement of $\mathrm{res}_p\left(H^1_f(\mathbf{Q},V)\right)$. Then, by (28), we have $\mathrm{res}_p(x)\in \mathrm{res}_p\left(H^1_f(\mathbf{Q},V)\right)\cap H^1_f\left(F_0\mathbf{D}^\dagger_{\mathrm{rig}}(V)\right)=\{0\}$, and therefore $x=0$.

b) From (42) and (28) it follows that

$$\begin{multline}\dim_E H^1_{f,\{p\}}(\mathbf{Q},V^*(1))=\\ =\dim_E H^1_f(\mathbf{Q}_p,V)-\left(\dim_E H^1_f(\mathbf{Q}_p,V)-\dim_E H^1_f\left(F_0\mathbf{D}^\dagger_{\mathrm{rig}}(V)\right)\right)=\dim_E H^1_f\left(F_0\mathbf{D}^\dagger_{\mathrm{rig}}(V)\right).\end{multline} \tag{43}$$

On the other hand, $F_0\mathbf{D}^\dagger_{\mathrm{rig}}(V^*(1))$ is the Tate dual of $\mathrm{gr}_1\mathbf{D}^\dagger_{\mathrm{rig}}(V)$. From the tautological exact sequence

$$0\to F_0\mathbf{D}^\dagger_{\mathrm{rig}}(V)\to \mathbf{D}^\dagger_{\mathrm{rig}}(V)\to \mathrm{gr}_1\mathbf{D}^\dagger_{\mathrm{rig}}(V)\to 0$$

and the semisimplicity of φ we obtain an exact sequence

$$0\to H^1_f\left(F_0\mathbf{D}^\dagger_{\mathrm{rig}}(V)\right)\to H^1_f(\mathbf{Q}_p,V)\to H^1_f\left(\mathrm{gr}_1\mathbf{D}^\dagger_{\mathrm{rig}}V)\right)\to 0.$$

Therefore

$$\begin{multline}\dim_E H^1\left(F_0\mathbf{D}^\dagger_{\mathrm{rig}}(V^*(1))\right)-\dim_E H^1_f\left(F_0\mathbf{D}^\dagger_{\mathrm{rig}}(V^*(1))\right)=\dim_E H^1_f\left(\mathrm{gr}_1\mathbf{D}^\dagger_{\mathrm{rig}}(V)\right)=\\ =\dim_E H^1_f(\mathbf{Q}_p,V)-\dim_E H^1_f\left(F_0\mathbf{D}^\dagger_{\mathrm{rig}}(V)\right).\end{multline}$$

Since $H^1\left(F_0\mathbf{D}^\dagger_{\mathrm{rig}}(V^*(1))\right)\cap H^1_f(\mathbf{Q}_p,V^*(1))=H^1_f\left(F_0\mathbf{D}^\dagger_{\mathrm{rig}}(V^*(1))\right)$, we obtain

$$\dim_E\left(\frac{H^1(\mathbf{Q}_p,V^*(1))}{H^1_f(\mathbf{Q}_p,V^*(1))+H^1(\mathbf{D}^\perp)}\right)=$$

$$\dim_E H^1(\mathbf{Q}_p,V^*(1))-\dim_E H^1_f(\mathbf{Q}_p,V^*(1))-H^1\left(F_0\mathbf{D}^\dagger_{\mathrm{rig}}(V^*(1))\right)+H^1_f\left(F_0\mathbf{D}^\dagger_{\mathrm{rig}}(V^*(1))\right)=$$

$$=\dim_E H^1_f(\mathbf{Q}_p,V)-H^1\left(F_0\mathbf{D}^\dagger_{\mathrm{rig}}(V^*(1))\right)+H^1_f\left(F_0\mathbf{D}^\dagger_{\mathrm{rig}}(V^*(1))\right)=\dim_E H^1_f\left(F_0\mathbf{D}^\dagger_{\mathrm{rig}}(V)\right).$$

Comparing with (43), we see that the source and the target of $\kappa_{D^\perp}$ are of the same dimension. Since $\kappa_{D^\perp}$ is injective, this proves the lemma. □

Remark 2.3.5. — *It is not difficult to prove that there is a canonical isomorphism*

$$\frac{H^1(\mathbf{Q}_p,V^*(1))}{H^1_f(\mathbf{Q}_p,V^*(1))+H^1\left(F_0\mathbf{D}^\dagger_{\mathrm{rig}}(V^*(1))\right)}\simeq\frac{H^1(\mathbf{D}^*(\chi))}{H^1_f(\mathbf{D}^*(\chi))}.$$

2.3.6. — Let $H^1(D^\perp,V^*(1))$ denote the inverse image of

$$H^1\left(F_1\mathbf{D}^\dagger_{\mathrm{rig}}(V^*(1))\right)/\left(H^1_f(\mathbf{Q}_p,V^*(1))+H^1_f\left(F_0\mathbf{D}^\dagger_{\mathrm{rig}}(V^*(1))\right)\right)$$

under $\kappa_{D^\perp}$. Taking into account (41), we see that $\kappa_{D^\perp}$ induces an isomorphism

$$H^1(D^\perp,V^*(1))\simeq\frac{H^1(\mathbf{M}^*(\chi))}{H^1_f(\mathbf{M}^*(\chi))}. \tag{44}$$

Consider the composition map

$$H^1(D^\perp,V^*(1))\to H^1\left(F_1\mathbf{D}^\dagger_{\mathrm{rig}}(V^*(1))\right)\to H^1(\mathbf{M}^*(\chi)).$$

Taking into account (37) and (38), we obtain that $\dim_E H^1(D^\perp,V^*(1))=\dim\mathscr{D}_{\mathrm{cris}}(\mathbf{M})$. Hence we have a diagram

$$\begin{array}{ccc}
\mathscr{D}_{\mathrm{cris}}(\mathbf{M}^*(\chi)) & \xrightarrow[\sim]{i_{\mathbf{M}^*(\chi),f}} & H^1_f(\mathbf{M}^*(\chi))\\
\uparrow{\scriptstyle\rho_{D^\perp,f}} & & \uparrow{\scriptstyle p_{\mathbf{M}^*(\chi),f}}\\
H^1(D^\perp,V^*(1)) & \longrightarrow & H^1(\mathbf{M}^*(\chi))\\
\downarrow{\scriptstyle\rho_{D^\perp,c}} & & \downarrow{\scriptstyle p_{\mathbf{M}^*(\chi),c}}\\
\mathscr{D}_{\mathrm{cris}}(\mathbf{M}^*(\chi)) & \xrightarrow[\sim]{i_{\mathbf{M}^*(\chi),c}} & H^1_c(\mathbf{M}^*(\chi)),
\end{array}$$

where $\rho_{D^\perp,f}$ and $\rho_{D^\perp,c}$ are defined as the unique maps making this diagram commute. From (44) it follows that $\rho_{D^\perp,c}$ is an isomorphism.

Definition 2.3.7. — *The determinant*

$$\mathscr{L}(V^*(1),D^\perp)=(-1)^e\det\left(\rho_{D^\perp,f}\circ\rho^{-1}_{D^\perp,c}\mid\mathscr{D}_{\mathrm{cris}}(\mathbf{M}^*(\chi))\right),$$

where $e=\dim_E\mathscr{D}_{\mathrm{cris}}(\mathbf{M}(\chi))$, will be called the $\mathscr{L}$-invariant associated to $V^*(1)$ and $D^\perp$.

Remark. — The sign $(-1)^e$ corresponds to the sign in the conjectural functional equation for p-adic L-functions. We refer the reader to [**6**, Sections 2.2.6 and 2.3.5] for more detail.

The following proposition is a direct generalization of [**6**, Proposition 2.2.7].

Proposition 2.3.8. — *Assume that D is a regular submodule of $\mathbf{D}_{\mathrm{cris}}(V)$. Then*

$$\mathscr{L}(D^\perp,V^*(1))=(-1)^e\mathscr{L}(D,V).$$

Proof. — The proof repeats *verbatim* the proof of [**6**, Proposition 2.2.7]. We leave the details to the reader. □

3. Modular curves and p-adic representations

3.1. Notation and conventions. —

3.1.1. — Let M and N be two positive integers such that $M+N\geqslant 5$. We denote by $Y_1(M,N)$ the scheme over $\mathbf{Z}[1/MN]$ representing the functor

$$S\mapsto \text{isomorphism classes of } (E,e_1,e_2),$$

where E/S is an elliptic curve, $e_1,e_2\in E(S)$ are such that $Me_1=Ne_2=0$ and the map

$$\begin{cases} \mathbf{Z}/M\mathbf{Z}\times\mathbf{Z}/N\mathbf{Z}\to E,\\ (m,n)\mapsto me_1+ne_2\end{cases}$$

is injective. As usual, we set $Y_1(N)=Y(1,N)$ for $N\geqslant 4$ and write (E,e) instead $(E,0,e_2)$. Recall that

$$Y(M,N)(\mathbf{C})=\Gamma(M,N)\backslash \mathbf{H},$$

where $\mathbf{H}$ is the Poincaré half-plane and

$$\Gamma(M,N)=\left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in \mathrm{SL}_2(\mathbf{Z})\mid a\equiv 1(M), b\equiv 0(M), c\equiv 0(N), d\equiv 1(N)\right\}.$$

In particular, $Y_1(N)(\mathbf{C})=\Gamma_1(N)\backslash\mathbf{H}$, where

$$\Gamma_1(N)=\left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in \mathrm{SL}_2(\mathbf{Z})\mid a\equiv d\equiv 1(N), c\equiv 0(N)\right\}.$$

3.1.2. — Let a be a positive integer. We denote by $\mathrm{Pr}_i\,:\,Y_1(Na)\to Y_1(N)$ $(i=1,2)$ the morphisms defined by

$$\mathrm{Pr}_1\,:\,(E,e)\mapsto (E,ae), \tag{45}$$

$$\mathrm{Pr}_2\,:\,(E,e)\mapsto (E/\langle Ne\rangle, e \mod \langle Ne\rangle), \tag{46}$$

where $\langle ae\rangle$ denotes the cyclic group of order N generated by ae.

3.1.3. — Fix a prime number $p\geqslant 3$ such that $(p,N)=1$. We denote by $Y(N,p)$ the scheme over $\mathbf{Z}[1/Np]$ which represents the functor

$$S\mapsto \text{isomorphism classes of } (E,e,C),$$

where E/S is an elliptic curve, $e\in E(S)$ is a point of order N and $C\subset E$ is a subgroup of order Np such that $e\in C$. We have $Y(N,p)(\mathbf{C})=\Gamma_p(N)\backslash\mathbf{H}$, where $\Gamma_p(N)=\Gamma_1(N)\cap\Gamma_0(p)$ and

$$\Gamma_0(p)=\left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in \mathrm{SL}_2(\mathbf{Z})\mid c\equiv 0(p)\right\}.$$

We have a canonical projection

$$\begin{aligned} &\mathrm{pr}'\,:\,Y_1(Np)\to Y(N,p),\\ &\mathrm{pr}'\,:\,(E,e)\mapsto (E,pe,\langle e\rangle).\end{aligned} \tag{47}$$

We also define projections

$$\mathrm{pr}_i\,:\,Y(N,p)\to Y_1(N),\qquad i=1,2$$

by

$$\begin{aligned} &\mathrm{pr}_1\,:\,(E,e,C)\mapsto (E,e),\\ &\mathrm{pr}_2\,:\,(E,e,C)\mapsto (E/NC,e'),\end{aligned} \tag{48}$$

where $e'\in C$ is the unique element of C such that $pe'=e$.

Note that we have commutative diagrams

(49)
$$\begin{array}{ccc} Y_1(Np) & \xrightarrow{\mathrm{pr}'} & Y(N,p)\,, \\ & \searrow^{\mathrm{Pr}_i} & \downarrow{\scriptstyle \mathrm{pr}_i} \\ & & Y_1(N). \end{array} \qquad i=1,2.$$

3.1.4. — If Y is an unspecified modular curve, we denote by $\lambda : \mathscr{E} \to Y$ the universal elliptic curve over Y and set $\mathscr{F}_n = \mathbf{R}^1\lambda_*\mathbf{Z}/p^n\mathbf{Z}(1)$, $\mathscr{F} = \mathbf{R}^1\lambda_*\mathbf{Z}_p(1)$ and $\mathscr{F}_{\mathbf{Q}_p} = \mathscr{F} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$. Let $\iota_D : D \hookrightarrow \mathscr{E}$ be a subscheme. We assume that D is étale over Y. Consider the diagram

(50)
$$\begin{array}{ccc} \mathscr{E}[p^r]\langle D\rangle & \longrightarrow & \mathscr{E} \\ \downarrow{\scriptstyle \mathrm{p}_{D,r}} & & \downarrow{\scriptstyle [p^r]} \\ D & \xrightarrow{\iota_D} & \mathscr{E} \\ \downarrow{\scriptstyle \lambda_D} & \swarrow_{\lambda} & \\ Y & & \end{array}$$

where $\mathscr{E}[p^r]\langle D\rangle$ is the fiber product of D and $\mathscr{E}$ over $\mathscr{E}$. Define

(51)
$$\Lambda_r(\mathscr{F}_r\langle D\rangle) = \lambda_{D,*}\circ \mathrm{p}_{D,r,*}(\mathbf{Z}/p^r\mathbf{Z}), \qquad \Lambda(\mathscr{F}\langle D\rangle) = (\Lambda_r(\mathscr{F}_r\langle D\rangle))_{r\geqslant 1}\,.$$

We consider $\Lambda(\mathscr{F}\langle D\rangle)$ as an étale pro-sheaf. If $D = Y$ and ι_D is a section $s : Y \to \mathscr{E}$, we use the notation $\Lambda(\mathscr{F}\langle s\rangle)$ to indicate the dependence on s. If s is the zero section, we write $\Lambda(\mathscr{F})$ instead $\Lambda(\mathscr{F}\langle 0\rangle)$. Note that in this case $\mathscr{E}[p^r]\langle D\rangle = \mathscr{E}[p^r]$, and

$$\Lambda_r(\mathscr{F}_r) = \mathbf{Z}/p^r\mathbf{Z}\left[\mathscr{E}[p^r]\right].$$

Therefore, $\Lambda(\mathscr{F})$ can be viewed as the sheaf of Iwasawa algebras associated to the relative Tate module of $\mathscr{E}$.

These sheaves were studied in detail in [**42**] and we refer the reader to *op. cit.* for further information.

3.1.5. — Let $M \mid N$ and let $\mathscr{E} \to Y(M,N)$ be the universal elliptic curve. Let $s_N : Y(M,N) \to \mathscr{E}$ denote the canonical section which sends the class (E,e_1,e_2) to $e_2 \in \mathscr{E}[N]$. For each $r \geqslant 1$, consider the cartesian square

$$\begin{array}{ccc} \mathscr{E}[p^r]\langle s_N\rangle & \longrightarrow & \mathscr{E} \\ \downarrow & & \downarrow{\scriptstyle [p^r]} \\ Y(M,N) & \xrightarrow{s_N} & \mathscr{E}. \end{array}$$

We denote by

(52)
$$\Lambda(\mathscr{F}\langle N\rangle) := \Lambda(\mathscr{F}\langle s_N\rangle)$$

the associated sheaf. From the interpretation of $Y(M,N)$ as moduli space it follows that $Y(M,Np^r) \simeq \mathscr{E}[p^r]\langle s_N\rangle$, and therefore

$$H^1_{\text{ét}}(Y(M,Np^r),\mathbf{Z}/p^r(1)) \simeq H^1_{\text{ét}}(\mathscr{E}[p^r]\langle s_N\rangle,\mathbf{Z}/p^r(1)) \simeq H^1_{\text{ét}}(Y(M,N),\Lambda_r(\mathscr{F}_r\langle N\rangle))$$

Passing to projective limits, we obtain a canonical isomorphism

(53)
$$\varprojlim_r H^1_{\text{ét}}(Y(M,Np^r),\mathbf{Z}/p^r(1)) \simeq H^1_{\text{ét}}(Y(M,N),\Lambda(\mathscr{F}\langle N\rangle)(1)).$$

(see [**44**, Section 4.5]).

3.1.6. — For a module M over a commutative ring A we denote by $\mathrm{S}^k(M)$ (resp. $\mathrm{TS}^k(M)$) the quotient of S_k-coinvariants (resp. the submodule of S_k-invariants) of $M^{\otimes k}$.

3.2. p-adic representations. —

3.2.1. — For each $k \geqslant 2$ we denote by $\mathrm{S}^k(\mathscr{F}_{\mathbf{Q}_p})$ the kth symmetric tensor power of the sheaf $\mathscr{F}_{\mathbf{Q}_p}$. The étale cohomology

$$H^1_{\text{ét},c}\left(Y_1(N)_{\overline{\mathbf{Q}}}, \mathrm{S}^k(\mathscr{F}^\vee_{\mathbf{Q}_p})\right)$$

is equipped with a natural action of the Galois group $G_{\mathbf{Q},S}$ and the Hecke and Atkin–Lehner operators T_l (for primes $(l,N)=1$) and U_l (for primes $l|N$), which commute to each over. Let $f=\sum\limits_{n=1}^{\infty} a_nq^n$, $q=e^{2\pi iz}$ be a normalized cuspidal eigenform of level N_f and weight $k_0=k+2\geqslant 2$. We do not assume that f is a newform. Fix a finite extension $E/\mathbf{Q}_p$ containing a_n for all $n\geqslant 1$. Deligne's p-adic representation associated to f is the maximal subspace

$$W_f=H^1_{\text{ét},c}\left(Y_1(N_f)_{\overline{\mathbf{Q}}}, \mathrm{S}^k(\mathscr{F}^\vee_{\mathbf{Q}_p})\right)_{(f)}$$

of

$$H^1_{\text{ét},c}\left(Y_1(N_f)_{\overline{\mathbf{Q}}}, \mathrm{S}^k(\mathscr{F}^\vee_{\mathbf{Q}_p})\right)\otimes_{\mathbf{Q}_p} E$$

on which the operators T_l (for $(l,N_f)=1$) and U_l (for $l|N_f$) act as multiplication by a_l for all primes l. Then W_f is a two dimensional E-vector space equipped with a continuous action of $G_{\mathbf{Q},S}$, which does not depend on the choice of the level in the following sense. If $N_f|N$ then the canonical projection $\mathrm{Pr}_1\,:\,Y_1(N)\to Y_1(N_f)$ induces a morphism

$$\mathrm{Pr}_{1,*}\,:\,H^1_{\text{ét},c}\left(Y_1(N)_{\overline{\mathbf{Q}}}, \mathrm{S}^k(\mathscr{F}^\vee_{\mathbf{Q}_p})\right)\to H^1_{\text{ét},c}\left(Y_1(N_f)_{\overline{\mathbf{Q}}}, \mathrm{S}^k(\mathscr{F}^\vee_{\mathbf{Q}_p})\right),$$

which is isomorphism on the f-components. Note, that here in our notation we do not distinguish between the sheaves $\mathscr{F}^\vee_{\mathbf{Q}_p}$ on $Y_1(N)$ and $Y_1(N_f)$.

From the Poincaré duality it follows that the dual representation W_f^* can be realized as the quotient

$$W_f^*=H^1_{\text{ét}}\left(Y_1(N_f)_{\overline{\mathbf{Q}}}, \mathrm{TS}^k(\mathscr{F}_{\mathbf{Q}_p})(1)\right)_{[f]} \tag{54}$$

of

$$H^1_{\text{ét}}\left(Y_1(N_f)_{\overline{\mathbf{Q}}}, \mathrm{TS}^k(\mathscr{F}_{\mathbf{Q}_p})(1)\right)\otimes_{\mathbf{Q}_p} E$$

by the submodule generated by the images of T'_l-a_l (for $(l,N_f)=1$) and U'_l-a_l (for $l|N_f$), where T'_l (resp. U'_l) denote the dual Hecke (resp. Atkin–Lehner) operators.

3.2.2. — Let

$$\rho_f\,:\,G_{\mathbf{Q},S}\to \mathrm{GL}(W_f)$$

denote the representation of $G_{\mathbf{Q},S}$ on W_f. The following proposition summarizes some properties of the representation W_f.

Proposition 3.2.3. — *Assume that $f\in S^{\text{new}}_{k_0}(N_f,\varepsilon_f)$ is a newform of level N_f, weight $k_0=k+2\geqslant 2$ and nebentypus ε_f. Let p be a prime such that $(p,N_f)=1$. Then*

i) $\det\rho_f$ is isomorphic to $\varepsilon_f\chi^{1-k_0}$ where χ is the cyclotomic character.

ii) ρ_f is unramified outside the primes $l\mid N_fp$.

iii) If $l\neq p$, then

$$\det(1-\mathrm{Fr}_lX\mid W_f^{I_l})=1-a_lX+\varepsilon_f(l)\,l^{k_0-1}X^2$$

(Deligne–Langlands–Carayol theorem **[28, 45, 22]***).*

iii) The restriction of ρ_f on the decomposition group at p is crystalline with Hodge–Tate weights $(0,k_0-1)$. Moreover

$$\det(1-\varphi X\mid \mathbf{D}_{\mathrm{cris}}(W_f))=1-a_pX+\varepsilon_f(p)\,p^{k_0-1}X^2$$

(Saito's theorem **[56]***).*

3.2.4. — We retain previous notation. Let $f(q)=\sum\limits_{n=1}^{\infty} a_nq^n$ be a newform of weight $k_0\geqslant 2$, level N_f and nebentypus ε_f. Fix a prime number p such that $(p,N_f)=1$ and denote by $\alpha(f)$ and $\beta(f)$ the roots of the Hecke polynomial $X^2-a_pX+\varepsilon_f(p)p^{k_0-1}$. Till the end of this chapter we assume that the following conditions hold:

1) $\alpha(f)\neq\beta(f)$.

2) If $v_p(\alpha)=k_0-1$ then $\rho_f\big|_{G_{\mathbf{Q}_p}}$ does not split.

We remark that **1)** conjecturally always holds but is known to be true only in the weight 2 case [**25**]. One expects (see for example [**34**]) that **2)** does not hold (*i.e.* ρ_f locally splits) only if f is a CM form with p split.

3.2.5. — The p-stabilization $f_\alpha(q)=f(q)-\beta(f)\cdot f(q^p)$ is an oldform with respect to the subgroup $\Gamma_p(N_f)$. The map pr', defined in (47), induces an isomorphism

$$W_{f_\alpha}^*=H^1_{\text{ét}}\left(Y_1(N_fp)_{\overline{\mathbf{Q}}},\mathrm{TS}^k(\mathscr{F}_{\mathbf{Q}_p})(1)\right)_{[f\alpha]}\xrightarrow[\sim]{\mathrm{pr}'_*}H^1_{\text{ét}}\left(Y(N_f,p)_{\overline{\mathbf{Q}}},\mathrm{TS}^k(\mathscr{F}_{\mathbf{Q}_p})(1)\right)_{[f\alpha]}.$$

Moreover, the map

$$\mathrm{Pr}^{\alpha}_*\;:\;H^1_{\text{ét}}\left(Y_1(N_fp)_{\overline{\mathbf{Q}}},\mathrm{TS}^k(\mathscr{F}_{\mathbf{Q}_p})(1)\right)\to H^1_{\text{ét}}\left(Y_1(N_f)_{\overline{\mathbf{Q}}},\mathrm{TS}^k(\mathscr{F}_{\mathbf{Q}_p})(1)\right)$$

defined by

$$\mathrm{Pr}^{\alpha}_*=\mathrm{Pr}_{1,*}-\frac{\beta(f)}{p^{k_0-1}}\cdot\mathrm{Pr}_{2,*}\tag{55}$$

factorizes through appropriate quotients and induces an isomorphism

$$\mathrm{Pr}^{\alpha}_*\;:\;W^*_{f_\alpha}\simeq W^*_f\tag{56}$$

(see [**44**, Proposition 2.4.5]). Taking into account the diagram (49), we can be summarize this information in the following commutative diagram

$$\begin{array}{ccc} W^*_{f_\alpha} & \xrightarrow[\sim]{\mathrm{pr}'_*} & H^1_{\text{ét}}\left(Y(N_f,p)_{\overline{\mathbf{Q}}},\mathrm{TS}^k(\mathscr{F}_{\mathbf{Q}_p})(1)\right)_{[f\alpha]} \\ & \searrow^{\mathrm{Pr}^{\alpha}_*} & \downarrow{\scriptstyle \mathrm{pr}^{\alpha}_*} \\ & & W^*_f. \end{array}\tag{57}$$

Here

$$\mathrm{pr}^{\alpha}_*=\mathrm{pr}_{1,*}-\frac{\beta(f)}{p^{k_0-1}}\cdot\mathrm{pr}_{2,*}.$$

We denote by

$$\mathrm{Pr}^*_{\alpha}\;:\;W_f\simeq W_{f_\alpha}\tag{58}$$

the dual isomorphism.

3.2.6. — The newform f defines a canonical generator ω_f of the one-dimensional E-vector space $\mathrm{Fil}^{k_0-1}\mathbf{D}_{\mathrm{cris}}(W_f)$ (see [**40**, Section 11.3]). Note that $\mathrm{Pr}^{\alpha,*}(\omega_f)=\omega_{f_\alpha}$, where $\mathrm{Pr}^{\alpha,*}\,:\,\mathbf{D}_{\mathrm{cris}}(W_f)\to\mathbf{D}_{\mathrm{cris}}(W_{f_\alpha})$ denotes the isomorphism induced by (58).

Let $f^*=\sum\limits_{n=1}^{\infty}\overline{a}_nq^n$ denote the complex conjugate of f. The Atkin–Lehner operator w_{N_f} acts on f by

$$w_{N_f}(f)=\lambda_{N_f}(f)f^*,$$

where $\lambda_{N_f}(f)$ is called the pseudo-eigenvalue of f. The canonical pairing $W_f\times W_{f^*}\to E(1-k_0)$ induces a pairing

$$[\,,\,]\;:\;\mathbf{D}_{\mathrm{cris}}(W_f)\times\mathbf{D}_{\mathrm{cris}}(W_{f^*})\to\mathbf{D}_{\mathrm{cris}}(E(1-k_0)).$$

The filtered module $\mathbf{D}_{\text{cris}}(E(1-k_0))$ has the canonical generator $e_{1-k_0}=(\varepsilon\otimes t)^{\otimes(1-k_0)}$, where $\varepsilon=(\zeta_{p^n})_{n\geqslant 0}$ and $t=\log[\varepsilon]\in\mathbf{B}_{\text{dR}}$ is the associated uniformizer of the field of de Rham periods (note that e_{1-k_0} does not depend on the choice of ε). Since $\alpha(f)\neq\beta(f)$, we have $\mathbf{D}_{\text{cris}}(W_f)=\mathbf{D}_{\text{cris}}(W_f)^{\varphi=\alpha(f)}\oplus\mathbf{D}_{\text{cris}}(W_f)^{\varphi=\beta(f)}$. From **2)**, it follows that ω_{f^*} is not an eigenvector of φ, and we denote by η_f^{α} the unique element of $\mathbf{D}_{\text{cris}}(W_f)^{\varphi=\alpha(f)}$ such that

$$\left[\eta_f^{\alpha},\omega_{f^*}\right]=e_{1-k_0}.$$

We also denote by ω_f^{β} the unique element of $\mathbf{D}_{\text{cris}}(W_f)^{\varphi=\beta(f)}$ such that

$$\omega_f^{\beta}\equiv\omega_f \mod \mathbf{D}_{\text{cris}}(W_f)^{\varphi=\alpha(f)}.$$

Note that $\{\eta_f^{\alpha},\omega_f^{\beta}\}$ is a basis of $\mathbf{D}_{\text{cris}}(W_f)$.

3.2.7. — Set

$$\alpha(f^*)=\frac{p^{k_0-1}}{\beta(f)},\qquad \beta(f^*)=\frac{p^{k_0-1}}{\alpha(f)}.$$

Then $\alpha(f^*)$ and $\beta(f^*)$ are the roots of the Hecke polynomial of f^* at p. We denote by $\{\eta_{f^*}^{\alpha},\omega_{f^*}^{\beta}\}$ the corresponding basis of $\mathbf{D}_{\text{cris}}(W_{f^*})$. It is easy to check that it is dual to the basis $\{\eta_f^{\alpha},\omega_f^{\beta}\}$.

3.3. Overconvergent étale cohomology. —

3.3.1. — In this section, we review the construction of p-adic representations associated to Coleman families [**36, 47**]. It relies heavily on the overconvergent Eichler–Shimura isomorphism of Andreatta, Iovita and Stevens [**1**]. Let $\mathscr{W}=\mathrm{Hom}_{\text{cont}}(\mathbf{Z}_p^*,\mathbf{G}_m)$ denote the weight space. As usual, we consider $\mathscr{W}$ as a rigid analytic space over some fixed finite extension E of $\mathbf{Q}_p$. Namely, since $\mathbf{Z}_p^*\simeq\mu_{p-1}\times(1+p\mathbf{Z}_p)^*$, each continuous character $\eta\,:\,\mathbf{Z}_p^*\to L^*$ is completely determined by its restriction on μ_{p-1} and its value at $1+p$. This identifies $\mathscr{W}$ with the union of $p-1$ rigid open balls of radius 1. Let $\mathscr{W}^*$ denote the subspace of weights κ such that $v_p(\kappa(z)^{p-1}-1)\geqslant\frac{1}{p-1}$. Note that $\mathscr{W}^*$ contains all characters κ of the form $\kappa(z)=z^m$, $m\in\mathbf{Z}$. If U is an open disk, we denote by $\mathscr{O}_U^0$ the ring of rigid analytic functions on U bounded by 1 and set $\mathscr{O}_U=\mathscr{O}_U^0[1/p]$. We remark that $\mathscr{O}_U^0=O_E[[u]]$ for some u and denote by $\mathfrak{m}_U$ the maximal ideal of $\mathscr{O}_U^0$. The inclusion $U\subset\mathscr{W}$ fixes a character

$$\kappa_U\in\mathscr{W}(U)=\mathrm{Hom}_{\text{cont}}(\mathbf{Z}_p^*,\mathscr{O}_U^*).$$

If $x\in U(L)$, we can consider x as a homomorphism $x\,:\,\mathscr{O}_U\to L$. Let $\kappa_x\,:\,\mathbf{Z}_p^*\to L^*$ denote the character parametrized by x. Then we have

$$\kappa_x=x\circ\kappa_U.$$

3.3.2. — Consider the following compact subsets of $\mathbf{Z}_p^2$:

$$T_0=\mathbf{Z}_p^*\times\mathbf{Z}_p,\qquad T_0'=p\mathbf{Z}_p\times\mathbf{Z}_p^*.$$

For any weight $\kappa\in\mathscr{W}^*(L)$ we denote by $A_\kappa^0(T_0)$ (respectively $A_\kappa^0(T_0')$) the module of functions $f\,:\,T_0\to O_L$ (respectively $f\,:\,T_0'\to O_L$) such that:

- f is homogeneous of weight κ *i.e.*

$$f(ax,ay)=\kappa(a)f(x,y),\qquad \forall a\in\mathbf{Z}_p.$$

- $f(1,z)=\sum\limits_{n=0}^{\infty}c_nz^n$, $c_n\in O_L$, where $(c_n)_{n\geqslant 0}$ converges to 0.

Analogously, for any open disk $U\subset\mathscr{W}^*$ we denote by $A_U^0(T_0)$ (respectively $A_U^0(T_0')$) the module of functions $f\,:\,T_0\to\mathscr{O}_U^0$ (respectively $f\,:\,T_0'\to\mathscr{O}_U^0$) such that:

- f is homogeneous of weight κ_U, *i.e.*

$$f(ax,ay)=\kappa_U(a)f(x,y), \qquad \forall a\in \mathbf{Z}_p.$$

- $f(1,z)=\sum_{n=0}^{\infty} c_nz^n$, $c_n\in \mathscr{O}_U^0$, where $(c_n)_{n\geqslant 0}$ converges to 0 in the $\mathfrak{m}_U$-adic topology.

Define

$$D_\kappa^0=\mathrm{Hom}_{\mathrm{cont}}(A_\kappa^0(T_0),O_L), \qquad D_U^0=\mathrm{Hom}_{\mathrm{cont}}(A_U^0(T_0),\mathscr{O}_U^0)$$

and

$$D_\kappa=D_\kappa^0[1/p], \qquad D_U=D_U^0[1/p].$$

We have the specialization map

$$\begin{aligned}&\mathrm{sp}_\kappa\,:\,D_U\to D_\kappa,\\ &\mathrm{sp}_\kappa(\mu_U)(f)=\kappa(\mu_U(f_U)), \qquad \text{where } f_U(x,y)=f(1,y/x)\kappa_U(x)\end{aligned} \tag{59}$$

(see [**1**, Section 3.1]).

For each positive integer k, we denote by P_k^0 the O_E-module of homogeneous polynomials of degree k with coefficients in O_E. We remark that there exists a canonical isomorphism

$$\mathrm{Hom}_{O_E}(P_k^0,O_E)\simeq \mathrm{TS}^k(O_E^2). \tag{60}$$

The restriction of distributions on P_k^0 provides us with a map $D_k^0\to \mathrm{TS}^k(O_E^2)$. Composing this map with (59), we obtain a map

$$\theta_k\,:\,D_U^0\to \mathrm{TS}^k(O_E^2).$$

3.3.3. — Let $\Lambda(T_0)$, $\Lambda(T_0')$ and $\Lambda(\mathbf{Z}_p^2)$ denote the modules of p-adic measures with values in O_E on T_0, T_0' and $\mathbf{Z}_p^2$ respectively. We remark that $\Lambda(\mathbf{Z}_p^2)$ is canonically isomorphic to the Iwasawa algebra of $\mathbf{Z}_p^2$ and for each integer $k\geqslant 0$ we have the moment map

$$\begin{aligned}&\mathrm{m}^k\,:\,\Lambda(\mathbf{Z}_p^2)\to \mathrm{TS}^k(O_E^2),\\ &\mathrm{m}^k(h)=h^{\otimes k}, \qquad h\in \mathbf{Z}_p^2.\end{aligned} \tag{61}$$

(see [**42**, Section 2]). For $T=T_0,T_0'$, we have a commutative diagram

$$\begin{array}{ccc}\Lambda(T) & \longrightarrow & D_U^0(T)\\ \big\downarrow & & \big\downarrow{\scriptstyle \theta_k}\\ \Lambda(\mathbf{Z}_p^2) & \xrightarrow{\mathrm{m}^k} & \mathrm{TS}^k(O_E^2).\end{array},$$

where $k\in U\cap \mathbf{Z}$ (see [**47**, Proposition 4.2.10]).

3.3.4. — Let $N\geqslant 4$ and let p be an odd prime such that $(p,N)=1$. The fundamental group of $Y(N,p)(\mathbf{C})$ is $\Gamma_p(N)=\Gamma_1(N)\cap\Gamma_0(p)$. Its p-adic completion $\widehat{\Gamma_p(N)}$ is isomorphic to the p-Iwahori subgroup $U_0(p)$ and it acts on the pro-p-covering $Y_1(p^\infty,Np^\infty)$ of $Y(N,p)$. This defines a morphism $\pi_1^{\text{ét}}(Y(N,p))\to U_0(p)$. Thus the natural action of $U_0(p)$ on $D_U^0(T_0)$ and $D_U^0(T_0')$ provides these objects with an action of $\pi_1^{\text{ét}}(Y(N,p))$. Therefore, $D_U^0(T_0)$ and $D_U^0(T_0')$ define pro-étale sheaves on $Y(N,p)$, which we will denote by $\mathfrak{D}_U^0(\mathscr{F})$ and $\mathfrak{D}_U^0(\mathscr{F}')$ respectively.

3.3.5. — Let $\mathscr{E}\to Y(N,p)$ denote the universal elliptic curve over $Y(N,p)$. Let $C\subset \mathscr{E}[p]$ denote the canonical subgroup of order p of $\mathscr{E}[p]$. Set $D=\mathscr{E}[p]-C$ and $D'=C-\{0\}$ and consider the pro-sheaves $\Lambda(\mathscr{F}\langle D\rangle)$ and $\Lambda(\mathscr{F}\langle D'\rangle)$ defined by (50- 51).

Proposition 3.3.6. — *i) The sheaves $\Lambda(\mathscr{F}\langle D\rangle)$, $\Lambda(\mathscr{F}\langle D'\rangle)$ and $\Lambda(\mathscr{F})$ are induces by the modules $\Lambda(T_0)$, $\Lambda(T_0')$ and $\Lambda(\mathbf{Z}_p^2)$ equipped with the natural action of $\pi_1^{\text{ét}}(Y(N,p))$.*

ii) The natural inclusions $\Lambda(T_0)\to D_U(T_0)$ and $\Lambda(T_0')\to D_U(T_0')$ induce morphisms of sheaves $\Lambda(\mathscr{F}\langle D\rangle)\to \mathfrak{D}_U^0(\mathscr{F})$ and $\Lambda(\mathscr{F}\langle D'\rangle)\to \mathfrak{D}_U^0(\mathscr{F}')$.

iii) For any $k \in U \cap \mathbf{Z}$, we have commutative diagrams

$$\begin{array}{ccc} \Lambda(\mathscr{F}\langle D\rangle) & \longrightarrow & \mathfrak{D}_U(\mathscr{F}) \\ \downarrow [p]_* & & \downarrow \theta_k \\ \Lambda(\mathscr{F}) & \xrightarrow{\mathrm{m}^k} & \mathrm{TS}^k(\mathscr{F}_{\mathbf{Q}_p}), \end{array} \qquad \begin{array}{ccc} \Lambda(\mathscr{F}\langle D'\rangle) & \longrightarrow & \mathfrak{D}_U(\mathscr{F}') \\ \downarrow [p]_* & & \downarrow \theta_k \\ \Lambda(\mathscr{F}) & \xrightarrow{\mathrm{m}^k} & \mathrm{TS}^k(\mathscr{F}_{\mathbf{Q}_p}), \end{array}$$

where m^k is the moment map on sheaves induced by (61).

Proof. — See [**47**, Propositions 4.4.2 and 4.4.5]. □

3.3.7. — In [**1**], Andreatta, Iovita and Stevens defined the étale cohomology with coefficients in the sheaves $\mathfrak{D}_U(\mathscr{F})$ and $\mathfrak{D}_U(\mathscr{F}')$. Set

$$W(U)^0 = H^1_{\text{ét}}(Y(N,p)_{\overline{\mathbf{Q}}}, \mathfrak{D}_U(\mathscr{F}))(-\kappa_U), \qquad W'(U)^0 = H^1_{\text{ét}}(Y(N,p)_{\overline{\mathbf{Q}}}, \mathfrak{D}_U(\mathscr{F}')(1))$$

and

$$W(U) = W(U)^0 \otimes_{O_E} E, \qquad W'(U) = W'(U)^0 \otimes_{O_E} E.$$

We remark that $W(U)$ and $W'(U)$ are $\mathscr{O}_U$-modules equipped with a continuous linear action of the Galois group $G_{\mathbf{Q},S}$ and an action of Hecke operators which commute to each other.

3.3.8. — Assume that $k \in U$. The map $x \mapsto x+k$ defines a canonical bijection between $U-k$ and U and, therefore, an isomorphism $t_k : \mathscr{O}^0_{U-k} \simeq \mathscr{O}^0_U$. If $F \in A^0_{U-k}(T'_0)$ and $G \in P^0_k$ is a homogeneous polynomial of degree k, then $t_k \circ (FG) \in A^0_U(T'_0)$, and we have a well defined map $A^0_{U-k}(T'_0) \otimes P^0_k \to A^0_U(T'_0)$. Passing to the duals and using the isomorphism (60) we obtain a map

$$\beta^*_k : D^0_U(T'_0) \to D^0_{U-k}(T'_0) \otimes \mathrm{TS}^k(O^2_E).$$

We use the same notation for the induced map of sheaves

$$\beta^*_k : \mathfrak{D}^0_U(\mathscr{F}') \to \mathfrak{D}^0_{U-k}(\mathscr{F}') \otimes \mathrm{TS}^k(\mathscr{F}). \tag{62}$$

Let

$$\delta_k : A^0_U(T'_0) \to A_{U-k}(T'_0) \otimes P^0_k$$

be the map defined by

$$\delta_k(F) = \frac{1}{k!} \sum_{i+j=k} \frac{\partial^k F(x,y)}{\partial x^i \partial y^j} \otimes x^i y^j.$$

Passing to the duals we obtain a map

$$\delta^*_k : D^0_U(T'_0) \to D_{U-k}(T'_0) \otimes \mathrm{TS}^k(O^2_E).$$

We use the same notation for the induced map of sheaves

$$\delta^*_k : \mathfrak{D}^0_{U-k}(\mathscr{F}') \otimes \mathrm{TS}^k(\mathscr{F}) \to \mathfrak{D}_U(\mathscr{F}'). \tag{63}$$

Set $\ell = \dfrac{\log_p \kappa_U(1+p)}{\log_p(1+p)}$. Then $\ell \in \mathscr{O}_U$, and $\ell(x) = x$ for each $x \in U \cap \mathbf{Z}$. Let

$$\binom{\ell}{k} = \frac{\ell(\ell-1)\cdots(\ell-k+1)}{k!}.$$

Then

$$\delta^*_k \circ \beta^*_k = \binom{\ell}{k}$$

(see [**47**, Proposition 5.2.1]).

3.4. Coleman families. —

3.4.1. — Let $f(q)=\sum_{n=1}^{\infty} a_nq^n$ be a newform of weight $k_0\geqslant 2$, level N_f and nebentypus ε_f. We assume that the conditions **C1-2)** of Section 3.2 hold for some fixed odd prime $p \nmid N_f$. Define

$$I_f=\{x\in \mathbf{Z}\mid x\geqslant 2,\quad x\equiv k_0 \mod (p-1)\}.$$

We identify I_f with a subset of $\mathscr{W}^*$. Let $U\subset \mathscr{W}^*$ be an open disk centered in k_0. For any $F\in \mathscr{O}_U$ and $x\in I_f$, we denote by F_x the value of F at x. For any sufficiently large $r\geqslant 1$, we consider $E\langle w/p^r\rangle$ as the ring of analytic functions on the closed disk $D(k_0,p^{-r})\subset U$. Recall that for each $F(w)\in E\langle w/p^r\rangle$ we set $\mathscr{A}^{\mathrm{wt}}(F)(x)=F((1+p)^{x-k_0}-1)$ (see Section 1.1.3). Then $F_x=\mathscr{A}^{\mathrm{wt}}(F)(x)$. The following proposition summarizes the main properties of Coleman families we need in this paper.

Proposition 3.4.2. — *Assume that $v_p(\alpha(f))<k_0-1$. Then for a sufficiently small open disk U centered in k_0 the following conditions hold:*

1) There exists a unique formal power series

$$\mathbf{f}=\sum_{n=1}^{\infty}\mathbf{a}_nq^n\in \mathscr{O}_U[[q]]$$

with coefficients $\mathscr{O}_U$ such that

1a) For each $x\in I_f\cap U$ such that $v_p(\alpha(f))\neq x/2-1$, the specialization $\mathbf{f}_x$ at x is a p-stabilization of a newform f_x^0 of weight x and level N_f.

1b) $\mathbf{f}_{k_0}=f_\alpha$.

2) Fix $D(k_0,p^{-r})\subset U$ and denote by $A_{\mathbf{f}}$ its E-affinoid algebra. Let

$$W_{\mathbf{f}}=W(U)_{(\mathbf{f})}\otimes_{\mathscr{O}_U}A_{\mathbf{f}},$$

where $W(U)_{(\mathbf{f})}$ is the maximal submodule of the $\mathscr{O}_U$-module $W(U)$ on which the operators T_l (for $(l,N_f)=1$) and U_l (for $l|N_f$) act as multiplication by $\mathbf{a}_l$ for all primes l. Then

2a) $W_{\mathbf{f}}$ is a free $A_{\mathbf{f}}$-module of rank 2 equipped with a continuous linear action of $G_{\mathbf{Q},S}$.

2b) The specialization of $W_{\mathbf{f}}$ at each integer $x\geqslant 2$ is isomorphic to Deligne's representation associated to $\mathbf{f}_x$.

2c) The (φ,Γ)-module $\mathbf{D}_{\mathbf{f}}=\mathbf{D}^{\dagger}_{\mathrm{rig},A_{\mathbf{f}}}(W_{\mathbf{f}})$ has a triangulation of the form

$$0\to F^+\mathbf{D}_{\mathbf{f}}\to \mathbf{D}_{\mathbf{f}}\to F^-\mathbf{D}_{\mathbf{f}}\to 0,$$

where

$$F^+\mathbf{D}_{\mathbf{f}}=\mathscr{R}_{A_{\mathbf{f}}}(\delta_{\mathbf{f}}^+),\quad \delta_{\mathbf{f}}^+(p)=\mathbf{a}_p,\qquad \delta_{\mathbf{f}}^+|_{\mathbf{Z}_p^*}=1;$$
$$F^-\mathbf{D}_{\mathbf{f}}=\mathscr{R}_{A_{\mathbf{f}}}(\delta_{\mathbf{f}}^-),\quad \delta_{\mathbf{f}}^-(p)=\varepsilon(p)\mathbf{a}_p^{-1},\qquad \delta_{\mathbf{f}}^-|_{\mathbf{Z}_p^*}=\chi_{\mathbf{f}}^{-1},$$

and

$$\chi_{\mathbf{f}}(\gamma)=\chi(\gamma)^{k_0-1}\exp\left(\log_p(1+w)\frac{\log(\langle(\chi(\gamma))\rangle}{\log(1+p)}\right)$$

denotes the character $\chi^{k_0-1}\chi$ for the algebra $A_{\mathbf{f}}$.

3) Let $W'_{\mathbf{f}}=W'(U)_{[\mathbf{f}]}\otimes_{\mathscr{O}_U}A_{\mathbf{f}}$, where $W'(U)_{[\mathbf{f}]}$ denotes the maximal quotient of the $\mathscr{O}_U$-module $W'(U)$ by the submodule generated by the images of $T'_l-\mathbf{a}_l$ (for $(l,N_f)=1$) and $U'_l-\mathbf{a}_l$ (for $l|N_f$). There exists a pairing

$$W'_{\mathbf{f}}\times W_{\mathbf{f}}\to A_{\mathbf{f}},$$

which induces a canonical isomorphism

$$W'_{\mathbf{f}}\simeq W^*_{\mathbf{f}}:=\mathrm{Hom}_{A_{\mathbf{f}}}(W_{\mathbf{f}},A_{\mathbf{f}}).$$

Proof. — 1) This is the central result of Coleman's theory [**24**] together with [**4**, Lemma 2.7].

The statements 2a) and 2b) and 3) follow from the theory of Andreatta, Iovita and Stevens. See [**47**, Theorem 4.6.6] and the unpublished preprint [**36**] for comments and more detail.

The statement 2c) is a theorem of Liu [**53**]. □

3.4.3. — We say that $x \in I_f$ is *classical* if $v_p(\alpha) \neq x/2-1$ and denote by f_x^0 the newform of level N_f whose p-stabilization is $\mathbf{f}_x$. For each classical weight x we have isomorphisms

$$\mathbf{D_f} \otimes_{A_\mathbf{f}} (A_\mathbf{f}/\mathfrak{m}_x) \simeq \mathbf{D}^\dagger_{\mathrm{rig}}(W_{\mathbf{f}_x}) \simeq \mathbf{D}^\dagger_{\mathrm{rig}}(W_{f_x^0}),$$

where the second isomorphism is induced by (58) for f_x^0.

The (φ,Γ)-module $F^+\mathbf{D_f}$ is crystalline of Hodge–Tate weight 0 and the operator φ acts on $\mathscr{D}_{\mathrm{cris}}(F^+\mathbf{D_f})$ as multiplication by $\mathbf{a}_p$. The (φ,Γ)-module $F^-\mathbf{D_f}(\chi_\mathbf{f})$ is crystalline of Hodge–Tate weight -1 and φ acts on $\mathscr{D}_{\mathrm{cris}}(F^-\mathbf{D_f}(\chi_\mathbf{f}))$ as multiplication by $\varepsilon_f(p)p^{-1}\mathbf{a}_p^{-1}$.

3.4.4. — Define

$$C(f)=\left(1-\frac{\beta(f)}{p\alpha(f)}\right)\cdot\left(1-\frac{\beta(f)}{\alpha(f)}\right). \tag{64}$$

Proposition 3.4.5. — *Let r be a sufficiently large integer. Then*

1) There exists an element $\eta_\mathbf{f} \in \mathscr{D}_{\mathrm{cris}}(F^+\mathbf{D_f})$ such that for all classical $x \in I_f$ the specialization $\eta_\mathbf{f}(x) := \mathscr{A}^{\mathrm{wt}}(\eta_\mathbf{f})(x)$ of $\eta_\mathbf{f}$ at weight x satisfies

$$\eta_\mathbf{f}(x)=\lambda_{N_f}^{-1}(f_x^0)C(f_x^0)^{-1}\mathrm{Pr}_\alpha^*(\eta_{f_x^0}),$$

where the map Pr_α^* is defined in (58) and $C(f_x^0)$ is (64) for the form f_x^0.

2) There exists an element $\xi_\mathbf{f} \in \mathscr{D}_{\mathrm{cris}}(F^-\mathbf{D_f}(\chi_\mathbf{f}))$ such that for all classical $x \in I_f$ the specialization $\xi_\mathbf{f}(x) := \mathscr{A}^{\mathrm{wt}}(\xi_\mathbf{f})(x)$ of $\xi_\mathbf{f}$ at x satisfies

$$\xi_\mathbf{f}(x)=\mathrm{Pr}_\alpha^*(\omega_{f_x^0})\otimes e_{x-1} \mod \mathscr{D}_{\mathrm{cris}}(F^+\mathbf{D}_{\mathbf{f}_x}(\chi^{x-1}))$$

where e_{x-1} is the canonical generator of $\mathscr{D}_{\mathrm{cris}}(\mathscr{R}_E(\chi^{x-1}))\simeq \mathbf{D}_{\mathrm{cris}}(E(x-1))$.

Proof. — This is Theorem 6.4.1 and Corollary 6.4.3 of [**47**]. □

3.4.6. — We review the construction of the overconvergent projector defined by Loeffler and Zerbes in [**47**, Section 5.2]. For any integer $j \geqslant 1$, the maps β_j^* and δ_j^*, defined by (62) and (63), induce morphisms

$$\beta_j^*\,:\,W'(U)\to H^1_{\text{ét}}\left(Y(N_f,p)_{\overline{\mathbf{Q}}},\mathfrak{D}_{U-j}(\mathscr{F}')\otimes \mathrm{TS}^j(\mathscr{F})(1)\right),$$
$$\delta_j^*\,:\,H^1_{\text{ét}}\left(Y(N_f,p)_{\overline{\mathbf{Q}}},\mathfrak{D}_{U-j}(\mathscr{F}')\otimes \mathrm{TS}^j(\mathscr{F})(1)\right)\to W'(U)$$

such that $\delta_j^*\circ\beta_j^*=\binom{\ell}{j}$.

Let $\mathbf{f}$ be the Coleman family passing through f_α as in Proposition 3.4.2 and let

$$\pi_{\mathbf{f},U}\,:\,W'(U)\to W'(U)_{[\mathbf{f}]}$$

denote the canonical projection.

Proposition 3.4.7. — *Assume that the open set U satisfies assumptions from Proposition 3.4.2. Then*

i) The image of the composition

$$\pi_{\mathbf{f},U}\circ\delta_j^*\,:\,H^1_{\text{ét}}\left(Y(N_f,p)_{\overline{\mathbf{Q}}},\mathfrak{D}_{U-j}(\mathscr{F}')\otimes \mathrm{TS}^j(\mathscr{F})(1)\right)\to W'(U)_{[\mathbf{f}]}$$

is contained in $\binom{\ell}{j}W'(U)_{[\mathbf{f}]}$.

ii) There exists a unique map

$$\pi_{\mathbf{f},U}^{[j]}\,:\,H^1_{\text{ét}}\left(Y(N_f,p)_{\overline{\mathbf{Q}}},\mathfrak{D}_{U-j}(\mathscr{F}')\otimes \mathrm{TS}^j(\mathscr{F})(1)\right)\to W'(U)_{[\mathbf{f}]}$$

such that $=\binom{\ell}{j}\pi_{\mathbf{f},U}^{[j]}=\pi_{\mathbf{f},U}\circ\delta_j^*$.

iii) We have a commutative diagram

$$\begin{array}{ccc} W'(U) & & \\ \downarrow \beta_j^* & \searrow^{\pi_{\mathbf{f},U}} & \\ H^1_{\text{ét}}\left(Y(N_f,p)_{\overline{\mathbf{Q}}}, \mathfrak{D}_{U-j}(\mathscr{F}') \otimes \mathrm{TS}^j(\mathscr{F})(1)\right) & \xrightarrow[\pi^{[j]}_{\mathbf{f},U}]{} & W'(U)_{[\mathbf{f}]}. \end{array}$$

Proof. — See [**47**, Proposition 5.2.5]. □

Remark 4.4.8. — If U contains none of the integers $\{0,1,\dots,j-1\}$, then the function $\binom{\ell}{j}$ is invertible on U.

If $A_{\mathbf{f}}$ is the affinoid algebra of a closed disk centered in k as in Proposition 3.4.2, we denote by

$$\pi^{[j]}_{\mathbf{f}} : H^1_{\text{ét}}\left(Y(N_f,p)_{\overline{\mathbf{Q}}}, \mathfrak{D}_{U-j}(\mathscr{F}') \otimes \mathrm{TS}^j(\mathscr{F})(1)\right) \to W^*_{\mathbf{f}} \qquad (65)$$

the composition of $\pi^{[j]}_{\mathbf{f},U}$ with the natural map $W'(U)_{[\mathbf{f}]} \to W^*_{\mathbf{f}} \simeq W'(U)_{[\mathbf{f}]} \otimes A_{\mathbf{f}}$.

4. Beilinson–Flach elements

4.1. Eisenstein classes. —

4.1.1. — In this section, we review the theory of Beilinson–Flach elements introduced first by Beilinson [**2**] and extensively studied the last years by Bertolini, Darmon, Rotger [**15, 16**], Lei, Loeffler, Zerbes [**46**] and Kings, Loeffler and Zerbes [**43, 44**]. We follow [**44, 43**] closely. We maintain notation of Section 3. Let $N \geqslant 4$ be a fixed integer. We denote by

$$\mathrm{Eis}^k_{b,N} \in H^1_{\text{ét}}\left(Y_1(N), \mathrm{TS}^k(\mathscr{F}_{\mathbf{Q}_p})(1)\right), \qquad k \geqslant 0, \quad b \in \mathbf{Z}/N\mathbf{Z}$$

the étale realization of the Beilinson–Levin motivic Eisenstein elements [(7)] constructed in [**3**]. Note that for $k=0$, we have

$$b^2 \mathrm{Eis}^0_{1,N} - \mathrm{Eis}^0_{b,N} = \partial({}_b g_{0,1/N}),$$

where $\partial : \mathscr{O}(Y_1(N))^* \to H^1_{\text{ét}}(Y_1(N), \mathbf{Q}_p(1))$ denotes the Kummer map and ${}_b g_{0,1/N}$ is the Siegel unit as defined in [**40**].

4.1.2. — Set

$$H^i_{\text{ét}}\left(Y_1(Np^\infty), \mathrm{TS}^k(\mathscr{F})(1)\right) = \varprojlim_n H^i_{\text{ét}}\left(Y_1(Np^n), \mathrm{TS}^k(\mathscr{F}_n)(1)\right),$$

where the projective limit is taken with respect to the trace map. The Siegel units $({}_b g_{0,1/Np^n})_{n\geqslant 0}$ form a norm compatible system [**40**] and therefore we have a well defined element

$${}_b\mathbf{Eis}_N := (\partial({}_b g_{0,1/Np^n}))_{n\geqslant 0} \in H^1_{\text{ét}}(Y_1(Np^\infty), \mathbf{Z}_p(1)) \simeq H^1_{\text{ét}}(Y_1(N), \Lambda(\mathscr{F}\langle N\rangle)(1)),$$

where $\Lambda(\mathscr{F}\langle N\rangle)$ denotes the Iwasawa sheaf (52) associated to the canonical section $s_N : Y_1(N) \to \mathscr{E}[N]$. (Here we use the isomorphism (53).) Consider the map

$$\mathrm{m}^k_{\langle N\rangle} : H^1_{\text{ét}}(Y_1(N), \Lambda(\mathscr{F}\langle N\rangle)(1)) \xrightarrow{[N]} H^1_{\text{ét}}(Y_1(N), \Lambda(\mathscr{F})(1)) \xrightarrow{\mathrm{m}^k} H^1_{\text{ét}}\left(Y_1(N), \mathrm{TS}^k(\mathscr{F}_{\mathbf{Q}_p})(1)\right),$$

where the first arrow is induced by the multiplication by N on the universal elliptic curve and the second one is induced by the moment map from Proposition 3.3.6, iii).

(7) We normalize this element as in [**44**].

The main property of the elements ${}_b\mathbf{Eis}_N$ is that they interpolate Eisenstein elements, namely

$$\mathrm{m}^k_{\langle N\rangle}({}_b\mathbf{Eis}_N)=b^2\mathrm{Eis}^k_{1,N}-b^{-k}\mathrm{Eis}^k_{b,N}.$$

We refere the reader to [**44**, Theorem 4.5.1] for the proof and further detail.

4.2. Rankin-Eisenstein classes. —

4.2.1. — Let $Y_1(N)^2=Y_1(N)\times Y_1(N)$. We denote by $\mathrm{p}_i : Y_1(N)^2\to Y_1(N)$ $(i=1,2)$ the projections onto the first and second copy of $Y_1(N)$ respectively and by $\Delta : Y_1(N)\to Y_1(N)^2$ the diagonal map. For any integers $k,l\geqslant 0$, we consider the sheaf on $Y_1(N)^2$ defined by

$$\mathrm{TS}^{[k,l]}(\mathscr{F}_\star)=\mathrm{p}_1^*\left(\mathrm{TS}^k(\mathscr{F}_\star)\right)\otimes\mathrm{p}_2^*\left(\mathrm{TS}^l(\mathscr{F}_\star)\right),\qquad \star\in\{n,\emptyset,\mathbf{Q}_p\}.$$

Note that $\Delta^*(\mathrm{TS}^{[k,l]}(\mathscr{F}_\star))=\mathrm{TS}^k(\mathscr{F}_\star)\otimes\mathrm{TS}^l(\mathscr{F}_\star)$. Let j be an integer such that

$$0\leqslant j\leqslant\min\{k,l\}$$

In this situation, Kings, Loeffler and Zerbes [**43**, Section 5] defined a map

$$\mathrm{CG}^{[k,l,j]} : \mathrm{TS}^{k+l-2j}\left(\mathscr{F}_{\mathbf{Q}_p}\right)\to\mathrm{TS}^k\left(\mathscr{F}_{\mathbf{Q}_p}\right)\otimes\mathrm{TS}^l\left(\mathscr{F}_{\mathbf{Q}_p}\right)(-j),$$

called the Clebsch–Gordan map in *op. cit.*. We will use the same notation for the induced map on cohomology

$$H^1_{\text{ét}}\left((Y_1(N),\mathrm{TS}^{k+l-2j}\left(\mathscr{F}_{\mathbf{Q}_p}\right)(1)\right)\to H^1_{\text{ét}}\left((Y_1(N),\mathrm{TS}^k\left(\mathscr{F}_{\mathbf{Q}_p}\right)\otimes\mathrm{TS}^l\left(\mathscr{F}_{\mathbf{Q}_p}\right)(1-j)\right).$$

Taking the composition of this map with the Gysin map

$$H^1_{\text{ét}}\left(Y_1(N),\Delta^*(\mathrm{TS}^{[k,l]}(\mathscr{F}_{\mathbf{Q}_p}))(1-j)\right)\to H^3_{\text{ét}}\left(Y_1(N)^2,\mathrm{TS}^{[k,l]}(\mathscr{F}_{\mathbf{Q}_p})(2-j)\right),$$

induced by the diagonal embedding Δ, we get a map

$$H^1_{\text{ét}}\left(Y_1(N),\mathrm{TS}^{k+l-2j}(\mathscr{F}_{\mathbf{Q}_p})(1)\right)\to H^3_{\text{ét}}\left(Y_1(N)^2,\mathrm{TS}^{[k,l]}(\mathscr{F}_{\mathbf{Q}_p})(2-j)\right).\tag{66}$$

The spectral sequence

$$H^r_S\left(\mathbf{Q},H^s_{\text{ét}}(Y_1(N)^2_{\overline{\mathbf{Q}}},\mathrm{TS}^{[k,k']}(\mathscr{F}_{\mathbf{Q}_p})(2-j)\right)\Rightarrow H^{r+s}_{\text{ét}}\left(Y_1(N)^2,\mathrm{TS}^{[k,k']}(\mathscr{F}_{\mathbf{Q}_p})(2-j)\right)$$

gives rise to a map

$$H^3_{\text{ét}}\left(Y_1(N)^2,\mathrm{TS}^{[k,l]}(\mathscr{F}_{\mathbf{Q}_p})(2-j)\right)\to H^1_S\left(\mathbf{Q},H^2_{\text{ét}}(Y_1(N)^2_{\overline{\mathbf{Q}}},\mathrm{TS}^{[k,l]}(\mathscr{F}_{\mathbf{Q}_p})(2-j))\right).$$

Composing this map with (66), we get a map

$$\Delta^{[k,l,j]} : H^1_{\text{ét}}\left(Y_1(N),\mathrm{TS}^{k+l-2j}(\mathscr{F}_{\mathbf{Q}_p})(1)\right)\longrightarrow H^1_S\left(\mathbf{Q},H^2_{\text{ét}}\left(Y_1(N)^2_{\overline{\mathbf{Q}}},\mathrm{TS}^{[k,l]}(\mathscr{F}_{\mathbf{Q}_p})(2-j)\right)\right).\tag{67}$$

Definition 4.2.2. — *The elements*

$$\mathrm{Eis}^{[k,l,j]}_{b,N}=\Delta^{[k,l,j]}\left(\mathrm{Eis}^{k+l-2j}_{b,N}\right),\qquad 0\leqslant j\leqslant\min\{k,l\},\quad b\in\mathbf{Z}/N\mathbf{Z},$$

are called Rankin–Eisenstein classes.

4.2.3. — We define Rankin–Iwasawa classes following [**44**, Section 5]. Set

$$\Lambda(\mathscr{F}\langle N\rangle)^{[j]} = \Lambda(\mathscr{F}\langle N\rangle)\otimes \mathrm{TS}^j(\mathscr{F}).$$

From the definition of the sheaves $\Lambda_r(\mathscr{F}_r\langle N\rangle)$ (see Sections 3.1.4 and 3.1.5) it is clear that the diagonal embeddings $\mathscr{E}[p^r]\langle s_N\rangle \to \mathscr{E}[p^r]\langle s_N\rangle \times_{Y_1(Np^n)} \mathscr{E}[p^r]\langle s_N\rangle$ induce morphisms of sheaves

$$\Lambda_r(\mathscr{F}_r\langle N\rangle) \to \Lambda_r(\mathscr{F}_r\langle N\rangle)\otimes \Lambda_r(\mathscr{F}_r\langle N\rangle).$$

Tensoring this map with the Clebsch–Gordan map

$$\mathrm{CG}^{[j,j,j]}: \mathbf{Z}_p \to \mathrm{TS}^j(\mathscr{F}_{\mathbf{Q}_p})\otimes \mathrm{TS}^j(\mathscr{F}_{\mathbf{Q}_p})(-j),$$

and passing to inverse limits, we get a map

$$\Lambda(\mathscr{F}\langle N\rangle) \to \Lambda(\mathscr{F}\langle N\rangle)^{[j]}\widehat{\otimes}\Lambda(\mathscr{F}\langle N\rangle)^{[j]}(-j).$$

This induces a map on cohomology

$$H^1_{\text{ét}}(Y_1(N),\Lambda(\mathscr{F}\langle N\rangle)(1)) \to H^1_{\text{ét}}\left(Y_1(N),\Lambda(\mathscr{F}\langle N\rangle)^{[j]}\widehat{\otimes}\Lambda(\mathscr{F}\langle N\rangle)^{[j]}(1-j)\right). \tag{68}$$

Define

$$\Lambda(\mathscr{F}\langle N\rangle)^{[j,j]} = \mathrm{p}_1^*\left(\Lambda(\mathscr{F}\langle N\rangle)^{[j]}\right)\otimes \mathrm{p}_2^*\left(\Lambda(\mathscr{F}\langle N\rangle)^{[j]}\right).$$

Then the diagonal embedding induces the Gysin map

$$H^1_{\text{ét}}\left(Y_1(N),\Lambda(\mathscr{F}\langle N\rangle)^{[j]}\widehat{\otimes}\Lambda(\mathscr{F}\langle N\rangle)^{[j]}(1-j)\right) \to H^3_{\text{ét}}\left(Y_1(N)^2,\Lambda(\mathscr{F}\langle N\rangle)^{[j,j]}(2-j)\right).$$

Taking the composition of this map with (68), we obtain a map

$$H^1_{\text{ét}}(Y_1(N),\Lambda(\mathscr{F}\langle N\rangle)(1)) \to H^3_{\text{ét}}\left(Y_1(N)^2,\Lambda(\mathscr{F}\langle N\rangle)^{[j,j]}(2-j)\right). \tag{69}$$

Composing this map with the map

$$H^3_{\text{ét}}\left(Y_1(N)^2,\Lambda(\mathscr{F}\langle N\rangle)^{[j,j]}(2-j)\right) \to H^1_S\left(\mathbf{Q},H^2_{\text{ét}}\left(Y_1(N)^2_{\overline{\mathbf{Q}}},\Lambda(\mathscr{F}\langle N\rangle)^{[j,j]}(2-j)\right)\right)$$

induced by the Grothendick spectral sequence, we obtain an Iwasawa theoretic analog of the map (67)

$$\Delta^{[j]}_{\Lambda}: H^1_{\text{ét}}(Y_1(N),\Lambda(\mathscr{F}\langle N\rangle)(1)) \to H^1_S\left(\mathbf{Q},H^2_{\text{ét}}\left(Y_1(N)^2_{\overline{\mathbf{Q}}},\Lambda(\mathscr{F}\langle N\rangle)^{[j,j]}(2-j)\right)\right).$$

Definition 4.2.4. — *The elements*

$$_b\mathbf{RI}^{[j]}_N = \Delta^{[j]}_{\Lambda}({_b}\mathbf{Eis}_N) \in H^1_S\left(\mathbf{Q},H^2_{\text{ét}}\left(Y_1(N)^2_{\overline{\mathbf{Q}}},\Lambda(\mathscr{F}\langle N\rangle)^{[j,j]}(2-j)\right)\right), \qquad j\geqslant 0$$

are called Rankin–Iwasawa classes.

We remark that these classes interpolate p-adically the elements $\mathrm{Eis}^{[k,l,j]}_{1,N}$ (see [**44**, Proposition 5.2.3]) and refer the reader to *op. cit.* for the proof and further results.

4.2.5. — In this subsection, we assume that $N\geqslant 4$ and $(p,N)=1$. We have a commutative diagram

$$\begin{array}{ccc} \mathscr{E}_{Np} & \longrightarrow & Y_1(Np) \\ \downarrow & & \downarrow \mathrm{pr}' \\ \mathscr{E}_{N,p} & \longrightarrow & Y(N,p), \end{array}$$

where $\mathscr{E}_*$ denotes the relevant universal elliptic curve and pr' is the map defined in Section 3.1.3. Recall that $\mathscr{E}_{Np}$ is equipped with a canonical subscheme D_{Np} of points of order Np together with the canonical section $s_{Np}: Y_1(Np)\to D_{Np}$. The universal curve $\mathscr{E}_{N,p}$ is equipped with a canonical subscheme D' of points of degree p (see Section 3.3.5). The map pr' together with multiplication by N induce finite morphisms

$$\mathscr{E}_{Np}[p^r]\langle s_{Np}\rangle \to \mathscr{E}_{N,p}[p^r]\langle D'\rangle, \qquad r\geqslant 1,$$

and therefore we have a map

$$\mathrm{tr}'_* : H^2_{\text{ét}}\left(Y_1(Np)^2_{\overline{\mathbf{Q}}}, \Lambda(\mathscr{F}\langle Np\rangle)^{[j,j]}\right) \to H^2_{\text{ét}}\left(Y(N,p)^2_{\overline{\mathbf{Q}}}, \Lambda(\mathscr{F}\langle D'\rangle)^{[j,j]}\right). \tag{70}$$

Analogously, the map $\mathrm{pr}_1 : Y(N,p) \to Y_1(N)$ (see (48)) together with multiplication by p induce finite morphisms

$$\mathscr{E}_{N,p}[p^r]\langle D'\rangle \to \mathscr{E}_N[p^r], \qquad r \geqslant 1.$$

This gives us a map

$$\mathrm{pr}_{1,*} : H^2_{\text{ét}}\left(Y_1(N,p)^2_{\overline{\mathbf{Q}}}, \Lambda(\mathscr{F}\langle D'\rangle)^{[j,j]}\right) \to H^2_{\text{ét}}\left(Y(N)^2_{\overline{\mathbf{Q}}}, \Lambda(\mathscr{F})^{[j,j]}\right).$$

***Definition 4.2.6.* —** *We denote by*

$${}_b\mathbf{RI}^{[j]}_{N(p)} = \mathrm{tr}'_*\left({}_b\mathbf{RI}^{[j]}_{1,Np}\right) \in H^1_S\left(\mathbf{Q}, H^2_{\text{ét}}\left(Y_1(N,p)^2_{\overline{\mathbf{Q}}}, \Lambda(\mathscr{F}\langle D'\rangle)^{[j,j]}(2-j)\right)\right) \tag{71}$$

the image of the Beilinson–Flach element ${}_b\mathbf{RI}^{[j]}_{Np}$ under the map pr'_* induced by pr'.

Note that

$$\mathrm{pr}_{1,*}\left({}_b\mathbf{RI}^{[j]}_{N(p)}\right) = {}_b\mathbf{RI}^{[j]}_{N}.$$

4.3. Beilinson–Flach elements. —

4.3.1. — Let $f = \sum_{n=1}^{\infty} a_n q^n$ and $g = \sum_{n=1}^{\infty} b_n q^n$ be two eigenforms of weights $k_0 = k+2$ and $l_0 = l+2$ with $k, l \geqslant 0$ and levels N_f, N_g respectively. By (54), we have canonical projections

$$\pi_f : H^1_{\text{ét}}\left(Y_1(N_f)_{\overline{\mathbf{Q}}}, \mathrm{TS}^k(\mathscr{F})(1)\right) \otimes_{\mathbf{Z}_p} E \to W^*_f,$$
$$\pi_g : H^1_{\text{ét}}\left(Y_1(N_g)_{\overline{\mathbf{Q}}}, \mathrm{TS}^l(\mathscr{F})(1)\right) \otimes_{\mathbf{Z}_p} E \to W^*_g.$$

Let N be any positive integer divisible by N_f and N_g and such that N and N_fN_g have the same prime divisors. Without loss of generality, assume that W_f and W_g are defined over the same field E. Set $W_{f,g} = W_f \otimes_E W_g$. Künneth theorem gives an isomorphism

$$H^2_{\text{ét}}\left(Y_1(N)^2_{\overline{\mathbf{Q}}}, \mathrm{TS}^{[k,l]}(\mathscr{F}_{\mathbf{Q}_p})(2)\right) \simeq H^1_{\text{ét}}\left(Y_1(N)_{\overline{\mathbf{Q}}}, \mathrm{TS}^k(\mathscr{F}_{\mathbf{Q}_p})(1)\right) \otimes H^1_{\text{ét}}\left(Y_1(N)_{\overline{\mathbf{Q}}}, \mathrm{TS}^l(\mathscr{F}_{\mathbf{Q}_p})(1)\right).$$

We also have the maps induced on cohomology by the projections (45):

$$H^1_{\text{ét}}\left(Y_1(N)_{\overline{\mathbf{Q}}}, \mathrm{TS}^k(\mathscr{F}_{\mathbf{Q}_p})(1)\right) \to H^1_{\text{ét}}\left(Y_1(N_f)_{\overline{\mathbf{Q}}}, \mathrm{TS}^k(\mathscr{F}_{\mathbf{Q}_p})(1)\right),$$
$$H^1_{\text{ét}}\left(Y_1(N)_{\overline{\mathbf{Q}}}, \mathrm{TS}^l(\mathscr{F}_{\mathbf{Q}_p})(1)\right) \to H^1_{\text{ét}}\left(Y_1(N_g)_{\overline{\mathbf{Q}}}, \mathrm{TS}^l(\mathscr{F}_{\mathbf{Q}_p})(1)\right).$$

Composing Künneth decomposition with these projections and $\pi_f \otimes \pi_g$, we obtain a map

$$\mathrm{pr}^{[j]}_{f,g} : H^2_{\text{ét}}\left(Y_1(N)^2_{\overline{\mathbf{Q}}}, \mathrm{TS}^{[k,l]}\mathscr{F}(2-j)\right) \otimes_{\mathbf{Z}_p} E \to W^*_{f,g}(-j).$$

***Definition 4.3.2.* —** *The elements*

$$\mathrm{BF}^{[j]}_{f,g} = \mathrm{pr}^{[j]}_{f,g}\left(\mathrm{Eis}^{[k,l,j]}_{1,N}\right) \in H^1_S(\mathbf{Q}, W^*_{f,g}(-j)), \qquad 0 \leqslant j \leqslant \min\{k,l\}$$

are called Belinson–Flach elements associated to the forms f and g.

One can prove that the definition of $\mathrm{BF}^{[j]}_{f,g}$ does not depend on the choice of N.

4.3.3. — In this subsection, we assume that f and g are newforms of nebentypus ε_f and ε_g and p is an odd prime such that $(p,N_fN_g)=1$. We denote by $\alpha(f)$ and $\beta(f)$ (respectively by $\alpha(g)$ and $\beta(g)$) the roots of the Hecke polynomial of f (respectively g) at p. We assume that $\alpha(f)\neq\beta(f)$ and $\alpha(g)\neq\beta(g)$. As before, f_α and f_β (resp. g_α and g_β) denote the stabilizations of f (respectively g). Recall the isomorphisms (56) for f and g

$$\mathrm{Pr}^{\alpha}_{*}\,:\,W^*_{f_\alpha}\simeq W^*_f,\qquad \mathrm{Pr}^{\alpha}_{*}\,:\,W^*_{g_\alpha}\simeq W^*_g,$$

which we denote by the same symbol to simplify notation. These isomorphisms induce isomorphisms on Galois cohomology

$$\begin{aligned}(\mathrm{Pr}^{\alpha}_{*},\mathrm{Pr}^{\alpha}_{*})\,&:\,H^1_S(\mathbf{Q},W^*_{f_\alpha,g_\alpha})\to H^1_S(\mathbf{Q},W^*_{f,g}),\\ (\mathrm{id},\mathrm{Pr}^{\alpha}_{*})\,&:\,H^1_S(\mathbf{Q},W^*_{f,g_\alpha})\to H^1_S(\mathbf{Q},W^*_{f,g}).\end{aligned}$$

Proposition 4.3.4. — *For any* $0\leqslant j\leqslant \min\{k,l\}$ *we have*

i) $$(\mathrm{Pr}^{\alpha}_{*},\mathrm{Pr}^{\alpha}_{*})\left(\mathrm{BF}^{[j]}_{f_\alpha,g_\alpha}\right)=\left(1-\frac{\alpha(f)\beta(g)}{p^{j+1}}\right)\left(1-\frac{\beta(f)\alpha(g)}{p^{j+1}}\right)\left(1-\frac{\beta(f)\beta(g)}{p^{j+1}}\right)\mathrm{BF}^{[j]}_{f,g}.$$

ii) $$(\mathrm{id},\mathrm{Pr}^{\alpha}_{*})\left(\mathrm{BF}^{[j]}_{f,g_\alpha}\right)=\left(1-\frac{\alpha(f)\beta(g)}{p^{j+1}}\right)\left(1-\frac{\beta(f)\beta(g)}{p^{j+1}}\right)\mathrm{BF}^{[j]}_{f,g}.$$

Proof. — The first formula is proved in [**44**, Theorem 5.7.6]. The second formula is stated in Remark 7.7.7 of *op. cit.* For convenience of the reader, we give a short proof here.

Let $N=\mathrm{lcm}(N_f,N_g)$. Consider the commutative diagram

(72)
$$\begin{array}{ccc}
H^1\left(Y_1(Np)_{\overline{\mathbf{Q}}},\mathrm{TS}^k(\mathscr{F}_{\mathbf{Q}_p})(1)\right) & & \\
\Big\downarrow \widetilde{\mathrm{Pr}}_{1,*} & \searrow^{\widetilde{\pi}_{f_\alpha}} & \\
H^1\left(Y_1(N_fp)_{\overline{\mathbf{Q}}},\mathrm{TS}^k(\mathscr{F}_{\mathbf{Q}_p})(1)\right) & \xrightarrow{\ \pi_{f_\alpha}\ } & W^*_{f_\alpha}\\
\Big\downarrow \mathrm{Pr}^{\alpha}_{*} & & \Big\downarrow \mathrm{Pr}^{\alpha}_{*}\\
H^1\left(Y_1(N_f)_{\overline{\mathbf{Q}}},\mathrm{TS}^k(\mathscr{F}_{\mathbf{Q}_p})(1)\right) & \xrightarrow{\ \pi_f\ } & W^*_f
\end{array}$$

and the analogous commutative diagram with f_β instead f_α. Here we denote by $\widetilde{\mathrm{Pr}}_1$ the map (45) for $Y_1(Np)$ over $Y_1(N)$ to distinguish it from the map (45) for $Y_1(N_fp)$ over $Y_1(N_f)$, which we denote simply by Pr_1. Set

$$\mathrm{pr}_f=\pi_f\circ\mathrm{Pr}_{1,*}\circ\widetilde{\mathrm{Pr}}_{1,*}\,:\,H^1\left(Y_1(Np)_{\overline{\mathbf{Q}}},\mathrm{TS}^k(\mathscr{F}_{\mathbf{Q}_p})(1)\right)\to W^*_f.$$

By definition, we have

(73) $$\mathrm{BF}^{[j]}_{f,g_\alpha}=\mathrm{pr}^{[j]}_{f,g_\alpha}\left(\mathrm{Eis}^{[k,l,j]}_{1,Np}\right),$$

where the map $\mathrm{pr}^{[j]}_{f,g_\alpha}$ is induced on Galois cohomology by the projection $(\mathrm{pr}_f,\mathrm{pr}_{g_\alpha})$ twisted by the $(-j)$th power of the cyclotomic character.

From (55), it follows that

$$\mathrm{Pr}_{1,*}=\frac{\alpha(f)\cdot\mathrm{Pr}^{\alpha}_{*}-\beta(f)\cdot\mathrm{Pr}^{\beta}_{*}}{\alpha(f)-\beta(f)}.$$

This formula together with the commutativity of (72) show that

$$\mathrm{pr}_f=\pi_f\circ\left(\frac{\alpha(f)\cdot \mathrm{Pr}_*^{\alpha}-\beta(f)\cdot \mathrm{Pr}_*^{\beta}}{\alpha(f)-\beta(f)}\right)\circ \widetilde{\mathrm{Pr}}_{1,*}=\frac{\alpha(f)\cdot(\mathrm{Pr}_*^{\alpha}\circ\pi_{f_\alpha})-\beta(f)\cdot\left(\mathrm{Pr}_*^{\beta}\circ\pi_{f_\beta}\right)}{\alpha(f)-\beta(f)}. \tag{74}$$

From (73) and (74), we obtain that

$$\mathrm{BF}^{[j]}_{f,g_\alpha}=\frac{1}{\alpha(f)-\beta(f)}\left(\alpha(f)\cdot(\mathrm{Pr}_*^{\alpha},\mathrm{id})\left(\mathrm{BF}^{[j]}_{f_\alpha,g_\alpha}\right)-\beta(f)\cdot\left(\mathrm{Pr}_*^{\beta},\mathrm{id}\right)\left(\mathrm{BF}^{[j]}_{f_\beta,g_\alpha}\right)\right),$$

and therefore

$$(\mathrm{id},\mathrm{Pr}_*^{\alpha})\left(\mathrm{BF}^{[j]}_{f,g_\alpha}\right)=$$
$$=\frac{1}{\alpha(f)-\beta(f)}\left(\alpha(f)\cdot(\mathrm{Pr}_*^{\alpha},\mathrm{Pr}_*^{\alpha})\left(\mathrm{BF}^{[j]}_{f_\alpha,g_\alpha}\right)-\beta(f)\cdot\left(\mathrm{Pr}_*^{\beta},\mathrm{Pr}_*^{\alpha}\right)\left(\mathrm{BF}^{[j]}_{f_\beta\cdot g_\alpha}\right)\right).$$

Applying part i) to compute $(\mathrm{Pr}_*^{\alpha},\mathrm{Pr}_*^{\alpha})\left(\mathrm{BF}^{[j]}_{f_\alpha,g_\alpha}\right)$ and $\left(\mathrm{Pr}_*^{\beta},\mathrm{Pr}_*^{\alpha}\right)\left(\mathrm{BF}^{[j]}_{f_\beta,g_\alpha}\right)$, we obtain ii).

□

4.3.5. — We maintain previous assumptions. Let f and g be two newforms satisfying conditions **M1-3)**. Consider the composition

$$\mathrm{m}^{[j],i}_{\langle N\rangle}\ :\ H^1\left(Y_1(N)_{\overline{\mathbf{Q}}},\Lambda(\mathscr{F}\langle N\rangle)^{[j]}\right)\xrightarrow{\mathrm{m}^{i-j}_{\langle N\rangle}\otimes\mathrm{id}} H^1\left(Y_1(N)_{\overline{\mathbf{Q}}},\mathrm{TS}^{i-j}(\mathscr{F}_{\mathbf{Q}_p})\otimes\mathrm{TS}^{j}(\mathscr{F}_{\mathbf{Q}_p})\right)\to H^1\left(Y_1(N)_{\overline{\mathbf{Q}}},\mathrm{TS}^{i}(\mathscr{F}_{\mathbf{Q}_p})\right), \tag{75}$$

where the last map is induced by the natural map $\mathrm{TS}^{i-j}(\mathscr{F}_{\mathbf{Q}_p})\otimes\mathrm{TS}^{j}(\mathscr{F}_{\mathbf{Q}_p})\to\mathrm{TS}^{i}(\mathscr{F}_{\mathbf{Q}_p})$. For all $0\leqslant j\leqslant\min\{k,l\}$ we have a map

$$H^1_S\left(\mathbf{Q},H^2_{\text{ét}}\left(Y_1(N)_{\overline{\mathbf{Q}}},\Lambda(\mathscr{F}\langle N\rangle)^{[j,j]}(2-j)\right)\right)\xrightarrow{\left(\mathrm{m}^{[j],k}_{\langle N\rangle},\mathrm{m}^{[j],l}_{\langle N\rangle}\right)} H^1_S\left(\mathbf{Q},H^2_{\text{ét}}\left(Y_1(N)_{\overline{\mathbf{Q}}},\mathrm{TS}^{[k,l]}(\mathscr{F}_{\mathbf{Q}_p})(2-j)\right)\right)\xrightarrow{\mathrm{pr}^{[j]}_{f,g}} H^1_S(\mathbf{Q},W^*_{f,g}(-j)). \tag{76}$$

Definition 4.3.6. — *For any integer $0\leqslant j\leqslant\min\{k,l\}$, we denote by ${}_b\mathrm{BF}^{[j]}_{f,g}$ the image of the element ${}_b\mathbf{RI}^{[j]}_N$ under the composition (76).*

We have

$${}_b\mathrm{BF}^{[j]}_{f,g}=(b^2-b^{2j-k-l}\varepsilon_f^{-1}(b)\varepsilon_g^{-1}(b))\cdot\mathrm{BF}^{[j]}_{f,g},\qquad (b,N_fN_g)=1. \tag{77}$$

(see [**44**, Proposition 5.2.3]).

4.4. Stabilized Beilinson–Flach families. —

4.4.1. — Let f and g be two newforms. Denote by $\alpha(f)$ and $\beta(f)$ (resp. by $\alpha(g)$ and $\beta(g)$) the roots of the Hecke polyunomial of f (resp. g) at p, $(p,N_fN_g)=1$. We will always assume that the following conditions hold:

M1) $\alpha(f)\neq\beta(f)$ and $\alpha(g)\neq\beta(g)$;
M2) $v_p(\alpha(f))<k_0-1$ and $v_p(\alpha(g))<l_0-1$.

As before, f_α and g_α denote the stabilizations of f and g with respect to $\alpha(f)$ and $\alpha(g)$ respectively. Let $\mathbf{f}=\sum\limits_{n=1}^{\infty}\mathbf{a}_nq^n\in A_{\mathbf{f}}[[q]]$ and $\mathbf{g}=\sum\limits_{n=1}^{\infty}\mathbf{b}_nq^n\in A_{\mathbf{g}}[[q]]$ denote the Coleman families passing through f_α and g_α. We fix open disks U_f and U_g and affinoid algebras $A_{\mathbf{f}}=E\langle w_1/p^r\rangle$ and $A_{\mathbf{g}}=E\langle w_2/p^r\rangle$ such that the conditions of Propositions 3.4.2 and 3.4.5 hold for f and g. Then

$$W_{\mathbf{f},\mathbf{g}}=W_{\mathbf{f}}\widehat{\otimes}_EW_{\mathbf{g}}$$

is a p-adic Galois representation of rank 4 with coefficients in $A = A_{\mathbf{f}} \widehat{\otimes}_E A_{\mathbf{g}} \simeq E\left\langle w_1/p^r, w_2/p^r \right\rangle$. Let $N = \operatorname{lcm}(N_f, N_g)$.

4.4.2. — By Proposition 3.3.6, ii) there exists a natural morphism of sheaves $\Lambda(\mathscr{F}\left\langle D' \right\rangle) \to \mathfrak{D}_{U_f - j}(\mathscr{F}')$. It induces a morphism $\Lambda(\mathscr{F}\left\langle D' \right\rangle)^{[j]} \to \mathfrak{D}_{U_f - j}(\mathscr{F}') \otimes \mathrm{TS}^j(\mathscr{F})$. Consider the composition

$$(78) \qquad H^1\left(Y(N,p)_{\overline{\mathbf{Q}}}, \Lambda(\mathscr{F}\left\langle D' \right\rangle)^{[j]}(1)\right) \xrightarrow{\mathrm{Pr}^{(N,p)}_{(N_g,p)}} H^1\left(Y(N_g,p)_{\overline{\mathbf{Q}}}, \Lambda(\mathscr{F}\left\langle D' \right\rangle)^{[j]}(1)\right) \xrightarrow{\kappa_g} H^1\left(Y(N_g,p)_{\overline{\mathbf{Q}}}, \mathfrak{D}_{U_g - j}(\mathscr{F}') \otimes \mathrm{TS}^j(\mathscr{F})(1)\right) \xrightarrow{\pi_{\mathbf{g}}^{[j]}} W_{\mathbf{g}}^*,$$

where the first map is induced by the projection $Y(N,p) \to Y(N_g,p)$ and the last map is defined by (65). We also have the analogous morphism for the family $\mathbf{f}$. Composing these maps with Künneth's isomorphism

$$(79) \qquad H^2\left(Y(N,p)_{\overline{\mathbf{Q}}}, \Lambda(\mathscr{F}\left\langle D' \right\rangle)^{[j,j]}(2)\right) \simeq H^1\left(Y(N,p)_{\overline{\mathbf{Q}}}, \Lambda(\mathscr{F}\left\langle D' \right\rangle)^{[j]}(1)\right)^{\otimes 2}$$

we obtain a map

$$(80) \qquad H^2\left(Y(N,p)_{\overline{\mathbf{Q}}}, \Lambda(\mathscr{F}\left\langle D' \right\rangle)^{[j,j]}(2)\right) \to W_{\mathbf{f},\mathbf{g}}^*.$$

This map induces a map on Galois cohomology

$$(81) \qquad \mathrm{pr}_{\mathbf{f},\mathbf{g}}^{[j]} : H^1_S\left(\mathbf{Q}, H^2\left(Y(N,p)_{\overline{\mathbf{Q}}}, \Lambda(\mathscr{F}\left\langle D' \right\rangle)^{[j,j]}(2-j)\right)\right) \to H^1_S\left(\mathbf{Q}, W_{\mathbf{f},\mathbf{g}}^*(-j)\right).$$

Definition 4.4.3. — *We define stabilized Beilinson–Flach classes associated to* $\mathbf{f}$ *and* $\mathbf{g}$ *by*

$$_b\mathbf{BF}_{\mathbf{f},\mathbf{g}}^{[j]} = \mathrm{pr}_{\mathbf{f},\mathbf{g}}^{[j]}\left({}_b\mathbf{RI}_{N(p)}^{[j]}\right),$$

where $_b\mathbf{RI}_{N(p)}^{[j]}$ *is the Rankin–Iwasawa element defined by (71).*

We denote again by $\mathrm{sp}_{x,y}^{\mathbf{f},\mathbf{g}} : H^1_S\left(\mathbf{Q}, W_{\mathbf{f},\mathbf{g}}^*(-j)\right) \to H^1_S\left(\mathbf{Q}, W_{\mathbf{f}_x,\mathbf{g}_y}^*(-j)\right)$ the morphism induced by the specialization map $W_{\mathbf{f},\mathbf{g}}^* \to W_{\mathbf{f}_x,\mathbf{g}_y}^*$.

Proposition 4.4.4. — *i) For all integers* x, y *such that* $0 \leqslant j \leqslant \min\{x,y\} - 2$ *one has*

$$\binom{x-2}{j} \cdot \binom{y-2}{j} \cdot \mathrm{sp}_{x,y}^{\mathbf{f},\mathbf{g}}\left({}_b\mathbf{BF}_{\mathbf{f},\mathbf{g}}^{[j]}\right) = {}_b\mathrm{BF}_{\mathbf{f}_x,\mathbf{g}_y}^{[j]}.$$

ii) Let $\lambda = v_p(\alpha(f)) + v_p(\alpha(g))$. *There exists a unique element*

$$_b\mathbf{BF}_{\mathbf{f},\mathbf{g}}^{\mathrm{Iw}} \in H^1_{\mathrm{Iw},S}(\mathbf{Q}, W_{\mathbf{f},\mathbf{g}}^*) \otimes_\Lambda \mathscr{H}_E^{[\lambda]}(\Gamma)$$

such that for any integer $j \geqslant 0$ *one has*

$$\mathrm{sp}^c_{-j}\left({}_b\mathbf{BF}_{\mathbf{f},\mathbf{g}}^{\mathrm{Iw}}\right) = \frac{(-1)^j}{j!}\left(1 - \frac{p^j}{\mathbf{a}_p \mathbf{b}_p}\right) {}_b\mathbf{BF}_{\mathbf{f},\mathbf{g}}^{[j]}.$$

Proof. — i) The first statement follows directly from the definition of the maps $\pi_{\mathbf{f}}^{[j]}$ and $\pi_{\mathbf{g}}^{[j]}$ (see [**47**, Proposition 5.3.4]).

ii) The second statement is proved in [**47**, Proposition 2.3.3 and the proof of Theorem 5.4.2]. □

4.5. Semistabilized Beilinson–Flach elements. —

4.5.1. — Define
$$W_{f,\mathbf{g}}=W_f\otimes_E W_{\mathbf{g}}.$$
This is a p-adic representation of $G_{\mathbf{Q},S}$ with coefficients in $A_{\mathbf{g}}$. For any $0\leqslant j\leqslant k$, consider the composition of maps

$$(82)\quad H^1\left(Y(N,p)_{\overline{\mathbf{Q}}},\Lambda(\mathscr{F}\langle D'\rangle)^{[j]}(1)\right)\xrightarrow{\mathrm{m}_{\langle p\rangle}^{k-j}\otimes \mathrm{id}} H^1\left(Y(N,p)_{\overline{\mathbf{Q}}},\mathrm{TS}^k(\mathscr{F}_{\mathbf{Q}_p})(1)\right)$$
$$\xrightarrow{\mathrm{Pr}_{N_f}^{(N,p)}} H^1\left(Y(N_f)_{\overline{\mathbf{Q}}},\mathrm{TS}^k(\mathscr{F}_{\mathbf{Q}_p})(1)\right)\xrightarrow{\pi_f} W_f^*.$$

Composing Künneth's isomorphism (79) with (78) and (82), we obtain a map
$$(83)\qquad H^2\left(Y(N,p)_{\overline{\mathbf{Q}}},\Lambda(\mathscr{F}\langle D'\rangle)^{[j,j]}(2)\right)\to W_{f,\mathbf{g}}^*.$$
We denote by
$$\mathrm{pr}_{f,\mathbf{g}}^{[j]}:H^1_S\left(\mathbf{Q},H^2\left(Y(N,p)_{\overline{\mathbf{Q}}},\Lambda(\mathscr{F}\langle D'\rangle)^{[j,j]}(2-j)\right)\right)\to H^1_S\left(\mathbf{Q},W_{f,\mathbf{g}}^*(-j)\right)$$
the induced map on Galois cohomology.

Definition 4.5.2. — *Assume that* $0\leqslant j\leqslant k$. *The elements*
$$(84)\qquad {}_b\mathfrak{BF}_{f,\mathbf{g}}^{[j]}=\mathrm{pr}_{f,\mathbf{g}}^{[j]}\left({}_b\mathbf{RI}_{N(p)}^{[j]}\right),$$
will be called semistabilized Beilinson–Flach elements.

4.5.3. — For each $y\in\mathrm{Spm}(A_{\mathbf{g}})$, we denote again by
$$\mathrm{sp}_y^{\mathbf{g}}:H^1_S(\mathbf{Q},W_{f,\mathbf{g}}^*(-j))\to H^1_S(\mathbf{Q},W_{f,\mathbf{g}_y}^*(-j))$$
the morphism induced by the specialization map $\mathrm{sp}_y^{\mathbf{g}}:W_{\mathbf{g}}\to W_{\mathbf{g}_y}$. Recall that
$$I_g=\{y\in\mathbf{Z}\cap\mathrm{Spm}(A_{\mathbf{g}})\mid y\geqslant 2,\quad y\equiv l_0\mod (p-1)\}.$$

Proposition 4.5.4. — *i) For each* $y\in I_g$ *such that* $y\geqslant j+2$ *we have*
$${}_b\mathrm{BF}_{f,\mathbf{g}_y}^{[j]}=\binom{y-2}{j}\cdot\mathrm{sp}_y^{\mathbf{g}}\left({}_b\mathfrak{BF}_{f,\mathbf{g}}^{[j]}\right).$$
ii) In particular,
$$\binom{y-2}{j}\cdot(\mathrm{id},\mathrm{Pr}_*^{\alpha})\circ\mathrm{sp}_y^{\mathbf{g}}\left({}_b\mathfrak{BF}_{f,\mathbf{g}}^{[j]}\right)=\left(1-\frac{\alpha(f)\cdot\beta(g_y^0)}{p^{j+1}}\right)\left(1-\frac{\beta(f)\cdot\beta(g_y^0)}{p^{j+1}}\right){}_b\mathrm{BF}_{f,g_y^0}^{[j]}.$$

Proof. — i) Since the moment maps commute with the traces (see [**42**, Proposition 2.2.2]), we have a commutative diagram

$$\begin{array}{ccc}
H^2_{\text{ét}}\left(Y_1(Np)_{\overline{\mathbf{Q}}},\Lambda(\mathscr{F}\langle Np\rangle)^{[j,j]}\right) & \xrightarrow{\left(\mathrm{m}_{\langle Np\rangle}^{[j],k},\mathrm{m}_{\langle Np\rangle}^{[j],y-2}\right)} & H^2_{\text{ét}}\left(Y_1(Np)_{\overline{\mathbf{Q}}},\mathrm{TS}^{[k,y-2]}(\mathscr{F}_{\mathbf{Q}_p})\right)\\
\downarrow \mathrm{tr}'_* & & \downarrow \mathrm{pr}'_*\\
H^2_{\text{ét}}\left(Y_1(N,p)_{\overline{\mathbf{Q}}},\Lambda(\mathscr{F}\langle D'\rangle)^{[j,j]}\right) & \xrightarrow{\left(\mathrm{m}_{\langle p\rangle}^{[j],k},\mathrm{m}_{\langle p\rangle}^{[j],y-2}\right)} & H^2_{\text{ét}}\left(Y_1(N,p)_{\overline{\mathbf{Q}}},\mathrm{TS}^{[k,y-2]}(\mathscr{F}_{\mathbf{Q}_p})\right)
\end{array}$$

Here the maps tr'_* and $\mathrm{m}_{\langle *\rangle}^{[j],*}$ are defined by (70) and (75) respectively. Taking into account (71) and the definition of ${}_b\mathrm{BF}_{f,\mathbf{g}_y}^{[j]}$, we obtain that
$$(85)\qquad {}_b\mathrm{BF}_{f,\mathbf{g}_y}^{[j]}=(\pi_f,\pi_{\mathbf{g}_y})\circ\left(\mathrm{Pr}_{N_f}^{(N,p)},\mathrm{Pr}_{(N_g,p)}^{(N,p)}\right)\circ\left(\mathrm{m}_{\langle p\rangle}^{[j],k},\mathrm{m}_{\langle p\rangle}^{[j],y-2}\right)\left(\mathbf{BF}_{1,N(p)}^{[j]}\right).$$
The elements ${}_b\mathrm{BF}_{f,\mathbf{g}_y}^{[j]}$ and ${}_b\mathfrak{BF}_{f,\mathbf{g}}^{[j]}$ are defined via the maps (76) and (83) respectively.

Consider the following diagram

$$
\begin{array}{ccc}
H^1_{\text{ét}}(Y(N_g,p)_{\overline{\mathbf{Q}}},\Lambda(\mathscr{F}\langle D'\rangle)^{[j]}(1)) & \xrightarrow{\mathrm{m}^{[j],y-2}_{\langle p\rangle}} & H^1_{\text{ét}}(Y(N_g,p)_{\overline{\mathbf{Q}}},\mathrm{TS}^{y-2}(\mathscr{F}_{\mathbf{Q}_p})(1)) \\
\downarrow \alpha & \nearrow \theta_{y-j-2}\otimes \mathrm{id} & \downarrow \pi_{\mathbf{g}_y} \\
H^1_{\text{ét}}\left(Y(N_g,p)_{\overline{\mathbf{Q}}},\mathfrak{D}_{U_g-j}(\mathscr{F}')\otimes \mathrm{TS}^j(\mathscr{F})(1)\right) & & W^*_{\mathbf{g}_y} \\
\downarrow \delta^*_j & \searrow \binom{\ell}{j}\pi^{[j]}_{\mathbf{g}} & \uparrow \mathrm{sp}_y \\
W'(U) & \xrightarrow{\pi_{\mathbf{g}}} & W^*_{\mathbf{g}}
\end{array}
\tag{86}
$$

Compairing (85) with (78) and (82), it is easy to see that we only need to prove the formula

$$\pi_{\mathbf{g}_y}\circ \mathrm{m}^{[j],y-2}_{\langle p\rangle}=\binom{y-2}{j}\mathrm{sp}^{\mathbf{g}}_y\circ \pi^{[j]}_{\mathbf{g}}\circ \alpha, \tag{87}$$

which in turn follows from the commutativity of the diagram (86). The commutativity of the upper triangle follows from Proposition 3.3.6, iii). The commutativity of the lower triangle follows from Proposition 3.4.7. Directly from the definition of the maps θ_{y-j-2}, δ^*_j and $\pi_{\mathbf{g}}$ it follows that $\pi'_{\mathbf{g}_y}\circ(\theta_{y-j-2},\mathrm{id})=\mathrm{sp}^{\mathbf{g}}_y\circ \pi_{\mathbf{g}}\circ \delta^*_j$. Therefore, the diagram (86) commutes, and (87) is proved.

ii) The second statement follows from i), Proposition 4.3.4, ii) and (77). □

In the following proposition, we compare Beilinson–Flach Euler systems ${}_b\mathbf{BF}^{\mathrm{Iw}}_{[\mathbf{f},\mathbf{g}]}$ with semistabilized Beilinson–Flach elements. This result plays a key role in the proof of the main theorem of this paper.

Proposition 4.5.5. — *Let*

$${}_b\mathfrak{BF}^{\mathrm{Iw}}_{f,\mathbf{g}}=(\mathrm{Pr}^{\alpha}_*,\mathrm{id})\circ \mathrm{sp}^{\mathbf{f}}_{k_0}\left({}_b\mathbf{BF}^{\mathrm{Iw}}_{\mathbf{f},\mathbf{g}}\right)\in H^1_{\mathrm{Iw},S}\left(\mathbf{Q},W^*_{f,\mathbf{g}}\right)\otimes_{\Lambda}\mathscr{H}^{[\lambda]}_E(\Gamma).$$

Then for any integer $0\leqslant j\leqslant \min\{k,l\}$ *there exists a neighborhood* $U_g=\mathrm{Spm}(A_{\mathbf{g}})$ *such that*

$$\mathrm{sp}^c_{-j}\left({}_b\mathfrak{BF}^{\mathrm{Iw}}_{f,\mathbf{g}}\right)=\frac{(-1)^j}{j!}\cdot\binom{k}{j}\cdot\left(1-\frac{p^j}{\alpha(f)\cdot \mathbf{b}_p}\right)\cdot\left(1-\frac{\beta(f)\cdot \mathbf{b}_p}{p^{j+1}}\right)\cdot {}_b\mathfrak{BF}^{[j]}_{f,\mathbf{g}}$$

Proof. — Shrinking, if necessary, the neighborhood $U_{\mathbf{g}}$, we can assume that, as an $A_{\mathbf{g}}$-module, $H^1_S(\mathbf{Q},W_{f,\mathbf{g}})\simeq A^r_{\mathbf{g}}\oplus T$, where T is a $\mathfrak{m}_{l_0}$-primary torsion module. Let $y\in \mathrm{Spm}(A_{\mathbf{g}})$ be an integral weight such that $y\geqslant l_0$. From Propositions 4.3.4 ii), 4.4.4 and 4.5.4 it follows that the map $(\mathrm{id},\mathrm{Pr}^{\alpha}_*)\circ \mathrm{sp}^{\mathbf{g}}_y$ sends the both sides of the formula to

$$\frac{(-1)^j}{j!}\cdot\binom{k}{j}\cdot\binom{y-2}{j}\cdot\left(1-\frac{p^j}{\alpha(f)\alpha(g^0_y)}\right)\cdot\left(1-\frac{\alpha(f)\beta(g^0_y)}{p^{j+1}}\right)\times$$
$$\times\left(1-\frac{\beta(f)\alpha(g^0_y)}{p^{j+1}}\right)\cdot\left(1-\frac{\beta(f)\beta(g^0_y)}{p^{j+1}}\right){}_b\mathbf{BF}^{[j]}_{f,g^0_y}.$$

Since Pr^{α}_* is an isomorphism, this shows that the specializations of the both sides coincide at infinitely many points, including l_0. This proves the proposition. □

5. Triangulations

5.1. Triangulations. —

5.1.1. — Let $f=\sum\limits_{n=1}^{\infty} a_nq^n$ and $g=\sum\limits_{n=1}^{\infty} b_nq^n$ be two eigenforms of weights $k_0=k+2$ and $l_0=l+2$, levels N_f, N_g and nebentypus ε_f and ε_g respectively. Fix an odd prime p such that $(p,N_fN_g)=1$ and denote by $\alpha(f)$ and $\beta(f)$ (respectively by $\alpha(g)$ and $\beta(g)$) the roots of the Hecke polynomial of f (respectively g) at p. We will always assume that the conditions **M1-2)** from Section 4.4 hold. Let f_α and g_α denote the p-stabilizations of f and g with respect to $\alpha(f)$ and $\alpha(g)$ respectively. Denote by $\mathbf{f}=\sum\limits_{n=1}^{\infty}\mathbf{a}_nq^n\in A_{\mathbf{f}}[[q]]$ and $\mathbf{g}=\sum\limits_{n=1}^{\infty}\mathbf{b}_nq^n\in A_{\mathbf{g}}[[q]]$ Coleman families passing through f_α and g_α. Shrinking the neighborhoods of k_0 and l_0 in the weight space, we can assume that the affinoid algebras $A_{\mathbf{f}}=E\left<w_1/p^r\right>$ and $A_{\mathbf{g}}=E\left<w_2/p^r\right>$ satisfy the conditions of Propositions 3.4.2 and 3.4.5. Then

$$W_{\mathbf{f},\mathbf{g}}=W_{\mathbf{f}}\widehat{\otimes}_E W_{\mathbf{g}}$$

is a p-adic Galois representation of rank 4 with coefficients in $A=A_{\mathbf{f}}\widehat{\otimes}_E A_{\mathbf{g}}\simeq E\left<w_1/p^r,w_2/p^r\right>$. Let $\mathbf{D}_{\mathbf{f}}=\mathbf{D}^{\dagger}_{\mathrm{rig},A_{\mathbf{f}}}(W_{\mathbf{f}})$ and $\mathbf{D}_{\mathbf{g}}=\mathbf{D}^{\dagger}_{\mathrm{rig},A_{\mathbf{g}}}(W_{\mathbf{g}})$ and let

$$\begin{aligned}&0\to F^+\mathbf{D}_{\mathbf{f}}\to\mathbf{D}_{\mathbf{f}}\to F^-\mathbf{D}_{\mathbf{f}}\to 0,\\ &0\to F^+\mathbf{D}_{\mathbf{g}}\to\mathbf{D}_{\mathbf{g}}\to F^-\mathbf{D}_{\mathbf{g}}\to 0\end{aligned}\tag{88}$$

be the canonical triangulations of $\mathbf{D}_{\mathbf{f}}$ and $\mathbf{D}_{\mathbf{g}}$ respectively (see Proposition 3.4.2, 2c)). We denote by $\boldsymbol{\eta}_{\mathbf{f}}$ and $\boldsymbol{\xi}_{\mathbf{f}}$ (respectively by $\boldsymbol{\eta}_{\mathbf{g}}$ and $\boldsymbol{\xi}_{\mathbf{g}}$) the elements constructed in Proposition 3.4.5. Set $\mathbf{D}_{\mathbf{f},\mathbf{g}}=\mathbf{D}_{\mathbf{f}}\widehat{\otimes}_{\mathscr{R}_E}\mathbf{D}_{\mathbf{g}}$. Then $\mathbf{D}_{\mathbf{f},\mathbf{g}}=\mathbf{D}^{\dagger}_{\mathrm{rig},A}(W_{\mathbf{f},\mathbf{g}})$. We denote by $(F_i\mathbf{D}_{\mathbf{f},\mathbf{g}})_{i=-2}^2$ the triangulation

$$\{0\}\subset F_{-1}\mathbf{D}_{\mathbf{f},\mathbf{g}}\subset F_0\mathbf{D}_{\mathbf{f},\mathbf{g}}\subset F_1\mathbf{D}_{\mathbf{f},\mathbf{g}}\subset F_2\mathbf{D}_{\mathbf{f},\mathbf{g}}$$

given by

$$F_i\mathbf{D}_{\mathbf{f},\mathbf{g}}=\begin{cases}\{0\}, & \text{if } i=-2,\\ F^+\mathbf{D}_{\mathbf{f}}\widehat{\otimes}_{\mathscr{R}_E}F^+\mathbf{D}_{\mathbf{g}}, & \text{if } i=-1,\\ F^+\mathbf{D}_{\mathbf{f}}\widehat{\otimes}_{\mathscr{R}_E}\mathbf{D}_{\mathbf{g}}, & \text{if } i=0,\\ (F^+\mathbf{D}_{\mathbf{f}}\widehat{\otimes}_{\mathscr{R}_E}\mathbf{D}_{\mathbf{g}})+(\mathbf{D}_{\mathbf{f}}\widehat{\otimes}_{\mathscr{R}_E}F^+\mathbf{D}_{\mathbf{g}}), & \text{if } i=1,\\ \mathbf{D}_{\mathbf{f},\mathbf{g}}, & \text{if } i=2.\end{cases}\tag{89}$$

We will denote by $(\mathrm{gr}_i\mathbf{D}_{\mathbf{f},\mathbf{g}})_{i=-2}^2$ the associated graded module. In particular,

$$\mathrm{gr}_0\mathbf{D}_{\mathbf{f},\mathbf{g}}=F^+\mathbf{D}_{\mathbf{f}}\widehat{\otimes}_{\mathscr{R}_E}F^-\mathbf{D}_{\mathbf{g}},\qquad \mathrm{gr}_1\mathbf{D}_{\mathbf{f},\mathbf{g}}=F^-\mathbf{D}_{\mathbf{f}}\widehat{\otimes}_{\mathscr{R}_E}F^+\mathbf{D}_{\mathbf{g}}.$$

Note that

$$\mathrm{gr}_0\mathbf{D}_{\mathbf{f},\mathbf{g}}\simeq\mathscr{R}_A(\boldsymbol{\delta}_{\mathbf{f},\mathbf{g}}\boldsymbol{\chi}_{\mathbf{g}}^{-1}),\qquad \boldsymbol{\delta}_{\mathbf{f},\mathbf{g}}(p)=\varepsilon_g(p)\mathbf{a}_p\mathbf{b}_p^{-1},\qquad \boldsymbol{\delta}_{\mathbf{f},\mathbf{g}}|_{\mathbf{Z}_p^*}=1,\tag{90}$$

5.1.2. — We denote by $(F_i\mathbf{D}^*_{\mathbf{f},\mathbf{g}})_{i=-2}^2$ the dual filtration on $\mathbf{D}^*_{\mathbf{f},\mathbf{g}}$

$$F_i\mathbf{D}^*_{\mathbf{f},\mathbf{g}}=\mathrm{Hom}_{\mathscr{R}_A}\left(\mathbf{D}_{\mathbf{f},\mathbf{g}}/F_{-i}\mathbf{D}_{\mathbf{f},\mathbf{g}},\mathscr{R}_A\right).$$

Lemma 5.1.3. — *Let $\alpha(f^*)=p^{k_0-1}/\beta(f)$ and $\alpha(g^*)=p^{l_0-1}/\beta(g)$ and let $\mathbf{f}^*$ and $\mathbf{g}^*$ denote the Coleman families passing through the stabilizations of f^* and g^* with respect to $\alpha(f^*)$ and $\alpha(g^*)$. Then the filtrations $(F_i\mathbf{D}^*_{\mathbf{f},\mathbf{g}})_{i=-2}^2$ and $(F_i\mathbf{D}_{\mathbf{f}^*,\mathbf{g}^*})_{i=-2}^2$ are compatible with the duality $\mathbf{D}_{\mathbf{f}^*,\mathbf{g}^*}\times\mathbf{D}_{\mathbf{f},\mathbf{g}}\to\mathscr{R}_A(\boldsymbol{\chi}_{\mathbf{f}}^{-1}\boldsymbol{\chi}_{\mathbf{g}}^{-1})$. Namely*

$$F_i\mathbf{D}^*_{\mathbf{f},\mathbf{g}}\simeq F_i\mathbf{D}_{\mathbf{f}^*,\mathbf{g}^*}(\boldsymbol{\chi}_{\mathbf{f}}\boldsymbol{\chi}_{\mathbf{g}}).\tag{91}$$

Proof. — The proof is straightforward and left to the reader. □

5.1.4. — Set

$$\mathbf{M}_{\mathbf{f},\mathbf{g}} = \mathrm{gr}_0\mathbf{D}_{\mathbf{f},\mathbf{g}}(\chi\chi_{\mathbf{g}}) \simeq \mathscr{R}_A(\delta_{\mathbf{f},\mathbf{g}}\chi). \tag{92}$$

Then $\mathbf{M}_{\mathbf{f},\mathbf{g}}$ is a crystalline (φ,Γ)-module of Hodge–Tate weight 1 [(8)] and $\mathscr{D}_{\mathrm{cris}}(\mathbf{M}_{\mathbf{f},\mathbf{g}})$ is a free A-module of rank one generated by

$$m_{\mathbf{f},\mathbf{g}} := \eta_{\mathbf{f}} \otimes \xi_{\mathbf{g}} \otimes e_1, \tag{93}$$

where $\eta_{\mathbf{f}}$ and $\xi_{\mathbf{g}}$ are defined in Proposition 3.4.5. From Lemma 5.1.3 it follows that $\mathrm{gr}_1\mathbf{D}_{\mathbf{f}^*,\mathbf{g}^*}(\chi_{\mathbf{f}})$ is the Tate dual of $\mathbf{M}_{\mathbf{f},\mathbf{g}}$:

$$\mathbf{M}^*_{\mathbf{f},\mathbf{g}}(\chi) = \mathrm{gr}_1\mathbf{D}^*_{\mathbf{f},\mathbf{g}}(\chi_{\mathbf{g}}^{-1}) \simeq \mathrm{gr}_1\mathbf{D}_{\mathbf{f}^*,\mathbf{g}^*}(\chi_{\mathbf{f}}) \simeq \mathscr{R}_A(\delta_{\mathbf{f},\mathbf{g}}^{-1}). \tag{94}$$

5.1.5. — Let

$$\mathbf{D}_{f,\mathbf{g}} = \mathbf{D}^{\dagger}_{\mathrm{rig},A_{\mathbf{g}}}(W_{f,\mathbf{g}}) \simeq \mathbf{D}^{\dagger}_{\mathrm{rig}}(W_f) \otimes_{\mathscr{R}_E} \mathbf{D}^{\dagger}_{\mathrm{rig},A_{\mathbf{g}}}(W_{\mathbf{g}}).$$

The isomorphism (58) identifies $\mathbf{D}_{f,\mathbf{g}}$ with the specialization of the (φ,Γ)-module $\mathbf{D}_{\mathbf{f},\mathbf{g}}$ at f_α. In particular, for each $j \in \mathbf{Z}$ we have a tautological short exact sequence

$$0 \to \mathrm{gr}_1\mathbf{D}^*_{f,\mathbf{g}}(\chi^j) \to \mathbf{D}^*_{f,\mathbf{g}}/F_0\mathbf{D}^*_{f,\mathbf{g}}(\chi^j) \to \mathrm{gr}_2\mathbf{D}^*_{f,\mathbf{g}}(\chi^j) \to 0.$$

Lemma 5.1.6. — *1) For each $j \in \mathbf{Z}$ the induced sequence*

$$0 \to H^1\left(\mathrm{gr}_1\mathbf{D}^*_{f,\mathbf{g}}(\chi^j)\right) \to H^1\left(\mathbf{D}^*_{f,\mathbf{g}}/F_0\mathbf{D}^*_{f,\mathbf{g}}(\chi^j)\right) \to H^1\left(\mathrm{gr}_2\mathbf{D}^*_{f,\mathbf{g}}(\chi^j)\right)$$

is exact.

2) Assume that $j \neq 1 - \dfrac{k_0+l_0}{2}$. Then for a sufficiently small neighborhood U_g the $A_{\mathbf{g}}$-module $H^1\left(\mathrm{gr}_2\mathbf{D}^*_{f,\mathbf{g}}(\chi^j)\right)$ is free of rank one.

3) Assume that $j \leqslant 0$ and $y \in I_g$ is such that

$$\frac{k_0+y}{2} \notin \{1-j, 2-j\}.$$

Then $H^1_g\left(\mathrm{gr}_2\mathbf{D}^*_{f,\mathbf{g}_y}(\chi^j)\right) = 0$.

Proof. — 1) We only need to prove that $H^0\left(\mathrm{gr}_2\mathbf{D}^*_{f,\mathbf{g}}(\chi^j)\right) = 0$. From Proposition 3.4.2 it follows that $\mathrm{gr}_2\mathbf{D}^*_{f,\mathbf{g}} \simeq \mathscr{R}_{A_{\mathbf{g}}}(\psi_{f,\mathbf{g}})$, where $\psi_{f,\mathbf{g}}(p) = \alpha(f)^{-1}\mathbf{b}_p^{-1}$ and $\psi_{f,\mathbf{g}}\big|_{\mathbf{Z}_p^*} = 1$. By Proposition 1.3.7, it is sufficient to check that $\psi_{f,\mathbf{g}}(p)p^{-j} \neq 1$, but this follows from the fact that $|\mathbf{b}_p(y)| = p^{\frac{y-1}{2}}$ for any $y \in I_g$.

2) We have

$$\left(\psi_{f,\mathbf{g}}(p)p^{-j}\right)(l_0) = \alpha(f)^{-1}\alpha(g)^{-1}p^{-j}.$$

If $j \neq 1 - \dfrac{k_0+l_0}{2}$, then

$$\left|\alpha(f)^{-1}\alpha(g)^{-1}p^{-j}\right| = p^{-j-\frac{k_0-1}{2}-\frac{l_0-1}{2}} = p^{-j+1-\frac{k_0+l_0}{2}} \neq 1.$$

Therefore $\psi_{f,\mathbf{g}}(p)p^{-j} - 1 \in A_{\mathbf{g}}$ does not vanish at l_0, and 2) follows from Proposition 1.3.7, 2b).

3) By [**6**, Corollary 1.4.5], $H^1_g\left(\mathrm{gr}_2\mathbf{D}^*_{f,\mathbf{g}_y}(\chi^j)\right) = 0$ if the following conditions hold:

a) $j \leqslant 0$.

b) $H^0\left(\mathrm{gr}_2\mathbf{D}^*_{f,\mathbf{g}_y}(\chi^j)\right) = 0$.

c) $\mathscr{D}_{\mathrm{cris}}\left(\mathrm{gr}_2\mathbf{D}^*_{f,\mathbf{g}_y}(\chi^j)\right)^{\varphi=p^{-1}} = 0$.

(8) We call Hodge–Tate weights the jumps of the Hodge–Tate filtration on the associated filtered module. In particular, the Hodge–Tate weight of $\mathbf{Q}_p(1)$ is -1.

Since φ acts on $\mathscr{D}_{\mathrm{cris}}\left(\mathrm{gr}_2\mathbf{D}^*_{f,\mathbf{g}_y}(\chi^j)\right)$ as the multiplication by $\alpha(f)^{-1}\mathbf{b}(y)^{-1}p^{-j}$, the same argument as for 2) applies and shows that b) and c) hold. □

5.2. Local properties of Beilinson–Flach elements. —

5.2.1. — We maintain previous notation and conventions. Fix $0 \leqslant j \leqslant k$. Consider the diagram

(95)
$$\begin{array}{ccccccc} & & & & H^1(\mathbf{Q}_p, W^*_{f,\mathbf{g}}(-j)) & & \\ & & & & \downarrow & & \\ 0 \longrightarrow & H^1\left(\mathrm{gr}_1\mathbf{D}^*_{f,\mathbf{g}}(\chi^{-j})\right) & \longrightarrow & & H^1\left(\mathbf{D}^*_{f,\mathbf{g}}/F_0\mathbf{D}^*_{f,\mathbf{g}}(\chi^{-j})\right) & \longrightarrow & H^1\left(\mathrm{gr}_2\mathbf{D}^*_{f,\mathbf{g}}(\chi^{-j})\right). \end{array}$$

Recall that the bottom row is exact by Lemma 5.1.6. Let $\mathrm{res}_p\left({}_b\mathfrak{BF}^{[j]}_{f,\mathbf{g}}\right) \in H^1(\mathbf{Q}_p, W_{f,\mathbf{g}}(-j))$ denote the image of the semistabilized Beilinson–Flach element under the localization map.

Definition 5.2.2. — *We denote by* ${}_b\mathfrak{Z}^{[j]}_{f,\mathbf{g}}$ *the image of* $\mathrm{res}_p\left({}_b\mathfrak{BF}^{[j]}_{f,\mathbf{g}}\right)$ *under the map*

$$H^1(\mathbf{Q}_p, W^*_{f,\mathbf{g}}(-j)) \to H^1\left(\mathbf{D}^*_{f,\mathbf{g}}/F_0\mathbf{D}^*_{f,\mathbf{g}}(\chi^{-j})\right).$$

Proposition 5.2.3. — *Assume that* $j \neq \dfrac{k_0+l_0}{2} - 1$. *Then for a sufficiently small neighborhood* U_g *of* l_0 *the image of* ${}_b\mathfrak{Z}^{[j]}_{f,\mathbf{g}}$ *under the map* $H^1\left(\mathbf{D}^*_{f,\mathbf{g}}/F_0\mathbf{D}^*_{f,\mathbf{g}}(\chi^{-j})\right) \to H^1\left(\mathrm{gr}_2\mathbf{D}^*_{f,\mathbf{g}}(\chi^{-j})\right)$ *is zero, and therefore*

$${}_b\mathfrak{Z}^{[j]}_{f,\mathbf{g}} \in H^1\left(\mathrm{gr}_1\mathbf{D}^*_{f,\mathbf{g}}(\chi^{-j})\right).$$

Proof. — Let z denote the image of ${}_b\mathfrak{Z}^{[j]}_{f,\mathbf{g}}$ in $H^1\left(\mathrm{gr}_2\mathbf{D}^*_{f,\mathbf{g}}(\chi^{-j})\right)$ under the natural projection. From Lemma 5.1.6, 3) it follows that, for a sufficiently small U_g, the cohomology $H^1\left(\mathrm{gr}_2\mathbf{D}^*_{f,\mathbf{g}}(\chi^{-j})\right)$ is a free module of rank one over the principal ideal domain $A_{\mathbf{g}}$.

On the other hand, by [**44**, Proposition 5.4.1]

$$\mathrm{res}_p\left({}_b\mathrm{BF}^{[j]}_{f,g}\right) \in H^1_f(\mathbf{Q}_p, W^*_{f,g}(-j)).$$

Taking into account Proposition 4.5.4, we obtain that

$$\mathrm{sp}^{\mathbf{g}}_y\left(\mathrm{res}_p\left({}_b\mathfrak{BF}^{[j]}_{f,\mathbf{g}}\right)\right) \in H^1_g(\mathbf{Q}_p, W^*_{f,\mathbf{g}_y}(-j)), \qquad \forall y \in I_g \quad \text{such that } y \geqslant j+2.$$

Hence

$$\mathrm{sp}^{\mathbf{g}}_y(z) \in H^1_g\left(\mathrm{gr}_2\mathbf{D}^*_{f,\mathbf{g}_y}(\chi^{-j})\right), \qquad \forall y \in I_g \quad \text{such that } y \geqslant j+2.$$

From Lemma 5.1.6, 3) it follows that $H^1_g\left(\mathrm{gr}_2\mathbf{D}^*_{f,\mathbf{g}_y}(\chi^{-j})\right) = 0$ for infinitely many values of $y \in I_g$. Therefore $\mathrm{sp}^{\mathbf{g}}_y(z) = 0$ at infinitely many values of $y \in I_g$. Since $H^1\left(\mathrm{gr}_2\mathbf{D}^*_{f,\mathbf{g}}(\chi^{-j})\right)$ is free by Lemma 5.1.6, 3), this implies that $z = 0$. □

5.2.4. — In this subsection, we record a corollary of Proposition 5.2.3. Assume that f and g are of the same weight k_0. Since $W_{f_\alpha,g_\alpha} \simeq W_{f,g}$, the specialization of the triangulation $\left(F_i\mathbf{D}_{\mathbf{f},\mathbf{g}}\right)_{i=-2}^2$ at (k_0,k_0) defines a triangulation $(F_i\mathbf{D}_{f,g})_{i=-2}^2$ of $\mathbf{D}_{f,g}$. It is clear, that this triangulation can be defined directly in terms of $W_{f,g}$ by formulas analogous to (89).

Consider the diagram

$$
\begin{array}{ccccccc}
 & & & H^1(\mathbf{Q}_p, W_{f,g}(k_0)) & & & \\
 & & & \downarrow & & & \\
0 \longrightarrow & H^1\left(\mathrm{gr}_1\mathbf{D}_{f,g}(\chi^{k_0})\right) & \longrightarrow & H^1\left(\mathbf{D}_{f,g}/F_0\mathbf{D}_{f,g}(\chi^{k_0})\right) & \longrightarrow & H^1\left(\mathrm{gr}_2\mathbf{D}_{f,g}(\chi^{k_0})\right). &
\end{array} \tag{96}
$$

Using the canonical isomorphism $W^*_{f^*,g^*}(2-k_0) \simeq W_{f,g}(k_0)$, we can consider $\mathrm{BF}^{[k_0-2]}_{f^*,g^*}$ as an element

$$\mathrm{BF}^{[k_0-2]}_{f^*,g^*} \in H^1_S(\mathbf{Q}, W_{f,g}(k_0)).$$

Definition 5.2.5. — *We denote by* $\mathrm{Z}^{[k_0-2]}_{f^*,g^*}$ *the image of* $\mathrm{res}_p\left(\mathrm{BF}^{[k_0-2]}_{f^*,g^*}\right) \in H^1(\mathbf{Q}_p, W_{f,g}(k_0))$ *in* $H^1\left(\mathbf{D}_{f,g}/F_0\mathbf{D}_{f,g}(\chi^{k_0})\right)$.

Corollary 5.2.6. — *Assume that* $\alpha(f)\alpha(g) \neq p^{k_0-1}$ *and* $\beta(f)\alpha(g) \neq p^{k_0-1}$. *Then*
1) $H^1_f\left(\mathrm{gr}_1\mathbf{D}_{f,g}(\chi^{k_0})\right) = H^1\left(\mathrm{gr}_1\mathbf{D}_{f,g}(\chi^{k_0})\right)$;
2) $\mathrm{Z}^{[k_0-2]}_{f^*,g^*} \in H^1\left(\mathrm{gr}_1\mathbf{D}_{f,g}(\chi^{k_0})\right)$.

Proof. — 1) The (φ,Γ)-module $\mathrm{gr}_1\mathbf{D}_{f,g}(\chi^{k_0})$ is of Hodge–Tate weight -1, and φ acts on $\mathscr{D}_{\mathrm{cris}}(\mathrm{gr}_1\mathbf{D}_{f,g}(\chi^{k_0}))$ as multiplication by $\beta(f)\alpha(g)/p^{k_0}$. In particular, $\mathscr{D}_{\mathrm{cris}}(\mathrm{gr}_1\mathbf{D}_{f,g}(\chi^{k_0}))^{\varphi=1} = \mathscr{D}_{\mathrm{cris}}(\mathrm{gr}_1\mathbf{D}_{f,g}(\chi^{k_0}))^{\varphi=p^{-1}} = 0$. This implies 1).

2) The (φ,Γ)-module $\mathrm{gr}_2\mathbf{D}_{f,g}(\chi^{k_0})$ is of Hodge–Tate weight $k_0-2 \geqslant 0$, and φ acts on $\mathscr{D}_{\mathrm{cris}}(\mathrm{gr}_2\mathbf{D}_{f,g}(\chi^{k_0}))$ as multiplication by $\beta(f)\beta(g)/p^{k_0}$. If $\beta(f)\beta(g) \neq p^{k_0-1}$ this implies that $\mathscr{D}_{\mathrm{cris}}(\mathrm{gr}_2\mathbf{D}_{f,g}(\chi^{k_0}))^{\varphi=1} = \mathscr{D}_{\mathrm{cris}}(\mathrm{gr}_2\mathbf{D}_{f,g}(\chi^{k_0}))^{\varphi=p^{-1}} = 0$. Therefore $H^1_g(\mathrm{gr}_2\mathbf{D}_{f,g}(\chi^{k_0})) = 0$, and by the same argument as in the proof of Proposition 5.2.3, we conclude that $\mathrm{Z}^{[k_0-2]}_{f^*,g^*} \in H^1(\mathrm{gr}_1\mathbf{D}_{f,g}(\chi^{k_0}))$ in this case.

In the general case, the proof is slightly different. Consider the diagram (95) for the forms f^* and g^* instead f and g and $j = k_0-2$. Then the canonical isomorphism $W^*_{f^*,g^*_\alpha}(2-k_0) \simeq W_{f,g}(k_0)$ identifies the specialization of this diagram at weight k_0 with the diagram (96). From Proposition 5.2.3 we have

$$\mathrm{sp}^{\mathbf{g}^*}_{k_0}\left({}_b\mathfrak{Z}^{[k_0-2]}_{f^*,\mathbf{g}^*}\right) \in H^1\left(\mathrm{gr}_1\mathbf{D}^*_{f^*,g^*_\alpha}(\chi^{2-k_0})\right).$$

On the other hand, Proposition 4.5.4 and (77) imply that

$$(\mathrm{id}, \mathrm{Pr}^{\alpha}_*) \circ \mathrm{sp}^{\mathbf{g}^*}_{k_0}\left({}_b\mathfrak{Z}^{[k_0-2]}_{f^*,\mathbf{g}^*}\right) = \left(1 - \frac{p^{k_0-1}}{\beta(f)\alpha(g)}\right)\left(1 - \frac{p^{k_0-1}}{\alpha(f)\alpha(g)}\right) {}_b\mathrm{Z}^{[k_0-2]}_{f^*,g^*},$$

where ${}_b\mathrm{Z}^{[k_0-2]}_{f^*,g^*} = (b^2 - \varepsilon_f^{-1}(b)\varepsilon_g^{-1}(b))\mathrm{Z}^{[k_0-2]}_{f^*,g^*}$. Since the terms in brackets do not vanish, $\mathrm{Z}^{[k_0-2]}_{f^*,g^*} \in H^1(\mathrm{gr}_1\mathbf{D}_{f,g}(\chi^{k_0}))$, and 2) is proved. □

5.2.7. — Consider the diagram

$$
\begin{array}{ccccccc}
 & & & H^1_{\mathrm{Iw}}(\mathbf{D}^*_{\mathbf{f},\mathbf{g}}) & & & \\
 & & & \downarrow & & & \\
0 \longrightarrow & H^1_{\mathrm{Iw}}\left(\mathrm{gr}_1\mathbf{D}^*_{\mathbf{f},\mathbf{g}}\right) & \longrightarrow & H^1_{\mathrm{Iw}}\left(\mathbf{D}^*_{\mathbf{f},\mathbf{g}}/F_0\mathbf{D}^*_{\mathbf{f},\mathbf{g}}\right) & \longrightarrow & H^1_{\mathrm{Iw}}\left(\mathrm{gr}_2\mathbf{D}^*_{\mathbf{f},\mathbf{g}}\right). &
\end{array}
$$

Let

$$\mathrm{res}_p\left({}_b\mathbf{BF}^{\mathrm{Iw}}_{\mathbf{f},\mathbf{g}}\right) \in H^1_{\mathrm{Iw}}(\mathbf{Q}_p, W^*_{\mathbf{f},\mathbf{g}}) \otimes_\Lambda \mathscr{H}_E(\Gamma) \simeq H^1_{\mathrm{Iw}}\left(\mathbf{D}^*_{\mathbf{f},\mathbf{g}}\right)$$

denote the localization of the Beilinson–Flach Iwasawa class ${}_b\mathbf{BF}^{\mathrm{Iw}}_{\mathbf{f},\mathbf{g}}$.

Definition 5.2.8. — *We denote by* ${}_b\mathbf{Z}^{\mathrm{Iw}}_{\mathbf{f},\mathbf{g}}$ *the image of* $\mathrm{res}_p\left({}_b\mathbf{BF}^{\mathrm{Iw}}_{\mathbf{f},\mathbf{g}}\right)$ *under the map* $H^1_{\mathrm{Iw}}(\mathbf{D}^*_{\mathbf{f},\mathbf{g}}) \to H^1_{\mathrm{Iw}}\left(\mathbf{D}^*_{\mathbf{f},\mathbf{g}}/F_0\mathbf{D}^*_{\mathbf{f},\mathbf{g}}\right)$.

We have the following analog of Proposition 5.2.3.

Proposition 5.2.9. — *The image of* ${}_b\mathbf{Z}^{\mathrm{Iw}}_{\mathbf{f},\mathbf{g}}$ *under the map* $H^1_{\mathrm{Iw}}\left(\mathbf{D}^*_{\mathbf{f},\mathbf{g}}/F_0\mathbf{D}^*_{\mathbf{f},\mathbf{g}}\right) \to H^1_{\mathrm{Iw}}\left(\mathrm{gr}_2\mathbf{D}^*_{\mathbf{f},\mathbf{g}}\right)$ *is zero, and therefore*

$$ {}_b\mathbf{Z}^{\mathrm{Iw}}_{\mathbf{f},\mathbf{g}} \in H^1_{\mathrm{Iw}}\left(\mathrm{gr}_1\mathbf{D}^*_{\mathbf{f},\mathbf{g}}\right). $$

Proof. — See [**47**, Theorem 7.1.2]. □

We record the following corollary of Proposition 4.5.5.

Proposition 5.2.10. — *Let*

$$ {}_b\mathfrak{Z}^{\mathrm{Iw}}_{f,\mathbf{g}} = (\mathrm{Pr}^{\alpha}_*, \mathrm{id}) \circ \mathrm{sp}^{\mathbf{f}}_{k_0}\left({}_b\mathbf{Z}^{\mathrm{Iw}}_{\mathbf{f},\mathbf{g}}\right) \in H^1_{\mathrm{Iw}}(\mathrm{gr}_1\mathbf{D}^*_{f,\mathbf{g}}). $$

Then for any integer $0 \leqslant j \leqslant \min\{k,l\}$ *there exists a neighborhood* $U_g = \mathrm{Spm}(A_{\mathbf{g}})$ *such that*

$$ \mathrm{sp}^c_{-j}\left({}_b\mathfrak{Z}^{\mathrm{Iw}}_{f,\mathbf{g}}\right) = \frac{(-1)^j}{j!}\cdot\binom{k}{j}\cdot\left(1-\frac{p^j}{\alpha(f)\cdot\mathbf{b}_p}\right)\cdot\left(1-\frac{\beta(f)\cdot\mathbf{b}_p}{p^{j+1}}\right)\cdot {}_b\mathfrak{Z}^{[j]}_{f,\mathbf{g}}. $$

6. p-adic L-functions

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6.1. Three-variable p-adic L-function. —

6.1.1. — We maintain notation and assumptions of Section 5. In particular, we assume that the forms f and g satisfy conditions **M1-2)** of Section 4.4. We will also assume that $\varepsilon_f\varepsilon_g \neq 1$. Let $\mathbf{M}_{\mathbf{f},\mathbf{g}}$ be the (φ,Γ)-modules defined by (92). Recall that $\mathbf{M}^*_{\mathbf{f},\mathbf{g}}(\chi\chi_{\mathbf{g}}) = \mathrm{gr}_1\mathbf{D}^*_{\mathbf{f},\mathbf{g}}$ by (94). We have pairings on Iwasawa cohomology

$$ \begin{aligned} &\{\cdot,\cdot\}_{\mathbf{M}_{\mathbf{f},\mathbf{g}}} : H^1_{\mathrm{Iw}}(\mathbf{M}^*_{\mathbf{f},\mathbf{g}}(\chi)) \times H^1_{\mathrm{Iw}}(\mathbf{M}_{\mathbf{f},\mathbf{g}})^{\iota} \to \mathscr{H}_A(\Gamma),\\ &\{\cdot,\cdot\}_{\mathbf{M}_{\mathbf{f},\mathbf{g}}(\chi_{\mathbf{g}}^{-1})} : H^1_{\mathrm{Iw}}(\mathbf{M}^*_{\mathbf{f},\mathbf{g}}(\chi\chi_{\mathbf{g}})) \times H^1_{\mathrm{Iw}}(\mathbf{M}_{\mathbf{f},\mathbf{g}}(\chi_{\mathbf{g}}^{-1}))^{\iota} \to \mathscr{H}_A(\Gamma). \end{aligned} $$

Let $m_{\mathbf{f},\mathbf{g}}$ denote the canonical generator of $\mathbf{M}_{\mathbf{f},\mathbf{g}}$ defined by (93). Set

$$ \widetilde{m}_{\mathbf{f},\mathbf{g}} = m_{\mathbf{f},\mathbf{g}} \otimes (1+X). $$

Recall that for an unspecified large exponential map Exp we set $\mathrm{Exp}^c = c\circ\mathrm{Exp}$, where $c\in\Gamma$ is the unique element of order 2 (see Section 1.3.4). For any $\mathbf{z}\in H^1_{\mathrm{Iw}}(\mathbf{M}^*_{\mathbf{f},\mathbf{g}}(\chi\chi_{\mathbf{g}}))$ define

$$ \mathfrak{L}_p(\mathbf{z},\omega^a,x,y,s) = \mathscr{A}_{\omega^a}\left(\left\{\mathbf{z}, \mathrm{Tw}_{\chi_{\mathbf{g}}^{-1}}\circ\mathrm{Exp}^c_{\mathbf{M}_{\mathbf{f},\mathbf{g}}}(\widetilde{m}_{\mathbf{f},\mathbf{g}})^{\iota}\right\}_{\mathbf{M}_{\mathbf{f},\mathbf{g}}(\chi_{\mathbf{g}}^{-1})}\right)(x,y,s), $$

where the transform $\mathscr{A}_{\omega^a}$ was defined in (11), and

$$ x = k_0 + \log(1+w_1)/\log(1+p), \qquad y = l_0 + \log(1+w_2)/\log(1+p). $$

Lemma 6.1.2. — *We have*

$$ \mathfrak{L}_p(\mathbf{z},\omega^a,x,y,s) = \mathscr{A}_{\omega^{a-l_0+1}}\left(\mathfrak{Log}_{\mathbf{M}_{\mathbf{f},\mathbf{g}},m_{\mathbf{f},\mathbf{g}}}\left(\mathrm{Tw}_{\chi_{\mathbf{g}}^{-1}}(\mathbf{z})\right)\right)(x,y,s-y+1), $$

where $\mathfrak{Log}$ *is the map defined in Section 1.3.4.*

Proof. — By Lemma 1.2.5, we have

$$\mathfrak{L}_p(\mathbf{z},\omega^a,x,y,s)=\mathscr{A}_{\omega^a}\left(\mathrm{Tw}_{\chi_{\mathbf{g}}^{-1}}\circ\left\{\mathrm{Tw}_{\chi_{\mathbf{g}}^{-1}}(\mathbf{z}),\mathrm{Exp}^c_{\mathbf{M}_{\mathbf{f},\mathbf{g}}}(\widetilde{m}_{\mathbf{f},\mathbf{g}})^{\iota}\right\}_{\mathbf{M}_{\mathbf{f},\mathbf{g}}}\right).$$

Writing $\mathrm{Tw}_{\chi_{\mathbf{g}}}=\mathrm{Tw}_{\chi}\circ\mathrm{Tw}_{l_0-1}$ and taking into account (13) and (16) we get

$$\mathscr{A}_{\omega^a}\left(\mathrm{Tw}_{\chi_{\mathbf{g}}^{-1}}\circ\left\{\mathrm{Tw}_{\chi_{\mathbf{g}}^{-1}}(\mathbf{z}),\mathrm{Exp}^c_{\mathbf{M}_{\mathbf{f},\mathbf{g}}}(\widetilde{m}_{\mathbf{f},\mathbf{g}})^{\iota}\right\}_{\mathbf{M}_{\mathbf{f},\mathbf{g}}}\right)(x,y,s)=$$
$$=\mathscr{A}_{\omega^{a-l_0+1}}\left(\mathrm{Tw}_{\chi^{-1}}\circ\left\{\mathrm{Tw}_{\chi_{\mathbf{g}}^{-1}}(\mathbf{z}),\mathrm{Exp}^c_{\mathbf{M}_{\mathbf{f},\mathbf{g}}}(\widetilde{m}_{\mathbf{f},\mathbf{g}})^{\iota}\right\}_{\mathbf{M}_{\mathbf{f},\mathbf{g}}}\right)(x,y,s-l_0+1)=$$
$$=\mathscr{A}_{\omega^{a-l_0+1}}\left(\left\{\mathrm{Tw}_{\chi_{\mathbf{g}}^{-1}}(\mathbf{z}),\mathrm{Exp}^c_{\mathbf{M}_{\mathbf{f},\mathbf{g}}}(\widetilde{m}_{\mathbf{f},\mathbf{g}})^{\iota}\right\}_{\mathbf{M}_{\mathbf{f},\mathbf{g}}}\right)(x,y,s-y+1),$$

and the lemma is proved. □

6.1.3. — Let $L_p(\mathbf{f},\mathbf{g},\omega^a)(x,y,s)$ denote the three-variable p-adic L-function. Recall the element

$${}_b\mathbf{Z}^{\mathrm{Iw}}_{\mathbf{f},\mathbf{g}}\in H^1\left(\mathrm{gr}_1\mathbf{D}^*_{\mathbf{f},\mathbf{g}}\right)$$

defined in Section 5.2.

Theorem 6.1.4 (KINGS–LOEFFLER–ZERBES). — *One has*

$$B_b(\omega^a,x,y,s)L_p(\mathbf{f},\mathbf{g},\omega^a)(x,y,s)=(-1)^a\mathfrak{L}_p({}_b\mathbf{Z}^{\mathrm{Iw}}_{\mathbf{f},\mathbf{g}},\omega^{a-1},x,y,s-1),$$

where

$$B_b(\omega^a,x,y,s)=\frac{G(\varepsilon_f^{-1})G(\varepsilon_g^{-1})}{\lambda_{N_f}(\mathbf{f})(x)}\left(b^2-\omega(b)^{2a-k_0-l_0+2}\langle b\rangle^{2s-x-y+2}\varepsilon_f^{-1}(b)\varepsilon_g^{-1}(b)\right)\tag{97}$$

and $G(-)$ denotes the corresponding Gauss sum.

Proof. — The theorem was first proved in the ordinary case in [**44**, Theorem 10.2.2]. The non-ordinary case is treated in [**47**, Theorem 7.1.5]. □

6.2. The first improved p-adic L-function. — In this section we assume that $k_0=l_0$. Recall that $\mathbf{M}^*_{\mathbf{f},\mathbf{g}}(\chi)=F^{-+}\mathbf{D}^*_{\mathbf{f},\mathbf{g}}(\chi_{\mathbf{g}}^{-1})$ (see (94)). Let $\mathrm{Tw}_{\chi_{\mathbf{g}}^{-1}}\left({}_b\mathbf{Z}^{\mathrm{Iw}}_{\mathbf{f},\mathbf{g}}\right)$ denote the image of ${}_b\mathbf{Z}^{\mathrm{Iw}}_{\mathbf{f},\mathbf{g}}$ in $H^1_{\mathrm{Iw}}\left(\mathbf{M}^*_{\mathbf{f},\mathbf{g}}(\chi)\right)$ under the canonical map $m\mapsto m\otimes\chi_{\mathbf{g}}^{-1}$.

Definition 6.2.1. — *We denote by*

$${}_b\mathbf{Z}_{\mathbf{f},\mathbf{g}}\in H^1(\mathbf{M}^*_{\mathbf{f},\mathbf{g}}(\chi)).$$

the image of $\mathrm{Tw}_{\chi_{\mathbf{g}}^{-1}}\left({}_b\mathbf{Z}^{\mathrm{Iw}}_{\mathbf{f},\mathbf{g}}\right)$ *under the canonical projection* $H^1_{\mathrm{Iw}}(\mathbf{M}^*_{\mathbf{f},\mathbf{g}}(\chi))\to H^1(\mathbf{M}^*_{\mathbf{f},\mathbf{g}}(\chi))$.

We have

$$B_b(\omega^{k_0},k_0,s,s)=\frac{G(\varepsilon_f^{-1})G(\varepsilon_g^{-1})}{\lambda_{N_f}(\mathbf{f})(x)}\left(b^2-\omega(b)^2\langle b\rangle^{s-k_0+2}\varepsilon_f^{-1}(b)\varepsilon_g^{-1}(b)\right).$$

Since $\varepsilon_f\varepsilon_g\neq 1$, we can and will choose b such that $B_b(\omega^{k_0},k_0,k_0,k_0)\neq 0$.

Recall that $\mathbf{M}_{\mathbf{f},\mathbf{g}}$ is a crystalline module of Hodge–Tate weight 1. We denote by

$$\exp_{\mathbf{M}_{\mathbf{f},\mathbf{g}}}:\mathscr{D}_{\mathrm{cris}}(\mathbf{M}_{\mathbf{f},\mathbf{g}})\to H^1(\mathbf{M}_{\mathbf{f},\mathbf{g}}).$$

the Bloch–Kato exponential map for $\mathbf{M}_{\mathbf{f},\mathbf{g}}$.

***Definition 6.2.2.* —** *We define the first improved p-adic L-function as the analytic function given by*

$$L_p^{\mathrm{wc}}(\mathbf{f},\mathbf{g},s)=B_b(\omega^{k_0},k_0,s,s)^{-1}\cdot\mathscr{A}^{\mathrm{wt}}\left(\left\langle {}_b\mathbf{Z}_{\mathbf{f},\mathbf{g}},\exp_{\mathbf{M}_{\mathbf{f},\mathbf{g}}}(m_{\mathbf{f},\mathbf{g}})\right\rangle_{\mathbf{M}_{\mathbf{f},\mathbf{g}}}\right)(k_0,s),$$

where $\langle\ ,\ \rangle_{\mathbf{M}_{\mathbf{f},\mathbf{g}}}: H^1(\mathbf{M}^*_{\mathbf{f},\mathbf{g}}(\chi))\times H^1(\mathbf{M}_{\mathbf{f},\mathbf{g}})\to A$ *is the local duality.*

***Proposition 6.2.3.* —** *Assume that* $k_0=l_0$. *Then in a sufficiently small neighborhood of* k_0, *one has*

$$L_p(\mathbf{f},\mathbf{g},\omega^{k_0})(k_0,s,s)=(-1)^{k_0}\left(1-\frac{\mathbf{b}_p(s)}{\varepsilon_g(p)\mathbf{a}_p(k_0)}\right)\left(1-\frac{\varepsilon_g(p)\mathbf{a}_p(k_0)}{p\mathbf{b}_p(s)}\right)^{-1}L_p^{\mathrm{wc}}(\mathbf{f},\mathbf{g},s).$$

In particular, $L_p^{\mathrm{wc}}(\mathbf{f},\mathbf{g},s)$ *does not depend on the choice of b.*

Proof. — By Lemma 6.1.2, we have

$$\mathfrak{L}_p({}_b\mathbf{Z}^{\mathrm{Iw}}_{\mathbf{f},\mathbf{g}},\omega^{k_0-1},k_0,s,s-1)=\mathscr{A}_{\omega^0}\left(\left\{\mathrm{Tw}_{\chi_{\mathbf{g}}^{-1}}\left({}_b\mathbf{Z}^{\mathrm{Iw}}_{\mathbf{f},\mathbf{g}}\right),\mathrm{Exp}^c_{\mathbf{M}_{\mathbf{f},\mathbf{g}}}(\widetilde{m}_{\mathbf{f},\mathbf{g}})^{\iota}\right\}_{\mathbf{M}_{\mathbf{f},\mathbf{g}}}\right)(k_0,s,0).\tag{98}$$

By (90), the action of φ on $\mathbf{M}_{\mathbf{f},\mathbf{g}}$ is given by $\varphi(m_{\mathbf{f},\mathbf{g}})=\left(p^{-1}\varepsilon_g(p)\mathbf{a}_p\mathbf{b}_p^{-1}\right)m_{\mathbf{f},\mathbf{g}}$. Applying the first formula of Corollary 1.3.6, we obtain

$$\begin{gathered}\mathscr{A}_{\omega^0}\left(\left\{\mathrm{Tw}_{\chi_{\mathbf{g}}^{-1}}\left({}_b\mathbf{Z}^{\mathrm{Iw}}_{\mathbf{f},\mathbf{g}}\right),\mathrm{Exp}^c_{\mathbf{M}_{\mathbf{f},\mathbf{g}}}(\widetilde{m}_{\mathbf{f},\mathbf{g}})^{\iota}\right\}_{\mathbf{M}_{\mathbf{f},\mathbf{g}}}\right)(k_0,s,0)=\\ \left(1-\frac{\mathbf{b}_p(s)}{\varepsilon_g(p)\mathbf{a}_p(k_0)}\right)\left(1-\frac{\varepsilon_g(p)\mathbf{a}_p(k_0)}{p\mathbf{b}_p(s)}\right)^{-1}\mathscr{A}^{\mathrm{wt}}\left(\left\langle {}_b\mathbf{Z}_{\mathbf{f},\mathbf{g}},\exp_{\mathbf{M}_{\mathbf{f},\mathbf{g}}}(m_{\mathbf{f},\mathbf{g}})\right\rangle_{\mathbf{M}_{\mathbf{f},\mathbf{g}}}\right)(k_0,s).\end{gathered}\tag{99}$$

The proposition follows from formulas (98), (99) and Theorem 6.1.4. □

6.3. The second improved p-adic L-function. —

6.3.1. — We continue to assume that $k_0=l_0$. Let

$$\mathbf{M}_{f,\mathbf{g}}=\mathrm{gr}_0\mathbf{D}_{f,\mathbf{g}}(\chi\chi_{\mathbf{g}}),\qquad \mathbf{N}_{f,\mathbf{g}}=\mathrm{gr}_0\mathbf{D}_{f,\mathbf{g}}(\chi^{k_0-1})=\mathbf{M}_{f,\mathbf{g}}(\chi^{-1}\chi^{-1}).$$

Define

$$\mathfrak{m}_{f,\mathbf{g}}=\frac{1}{C(f)\cdot\lambda_{N_f}(f)}\eta_f\otimes\boldsymbol{\xi}_{\mathbf{g}}\otimes e_1\in\mathscr{D}_{\mathrm{cris}}(\mathbf{M}_{f,\mathbf{g}}).\tag{100}$$

Let $(\mathrm{Pr}^*_{\alpha},\mathrm{id}):\mathscr{D}_{\mathrm{cris}}(\mathbf{M}_{f,\mathbf{g}})\to\mathscr{D}_{\mathrm{cris}}(\mathbf{M}_{f_\alpha,\mathbf{g}})$ denote the map induced by the map (58). By Proposition 3.4.5,

$$\mathrm{sp}^{\mathbf{f}}_{k_0}(m_{\mathbf{f},\mathbf{g}})=(\mathrm{Pr}^*_{\alpha},\mathrm{id})(\mathfrak{m}_{f,\mathbf{g}}).\tag{101}$$

Consider the composition

$$H^1_{\mathrm{Iw}}(\mathbf{M}_{f,\mathbf{g}})\xrightarrow{\mathrm{Tw}_{\chi_{\mathbf{g}}^{-1}}}H^1_{\mathrm{Iw}}(\mathbf{M}_{f,\mathbf{g}}(\chi_{\mathbf{g}}^{-1}))\xrightarrow{\mathrm{sp}^c_k}H^1(\mathbf{N}_{f,\mathbf{g}}),$$

where $k=k_0-2$ and sp^c_k is the map (19). We have

$$B_b(\omega^{k_0-1},k_0,s,k_0-1)=\frac{G(\varepsilon_f^{-1})G(\varepsilon_g^{-1})}{\lambda_{N_f}(\mathbf{f})(k_0)}\left(b^2-\langle b\rangle^{k_0-s}\varepsilon_f^{-1}(b)\varepsilon_g^{-1}(b)\right).$$

Therefore $B_b(\omega^{k_0-1},k_0,k_0,k_0-1)\neq 0$ for any $b\neq 1$.

***Definition 6.3.2.* —** *We define the second improved p-adic L-function as the analytic function given by*

$$\begin{aligned}L_p^{\mathrm{wt}}(\mathbf{f},\mathbf{g},s)=\Gamma(k_0-1)^{-1}\cdot B_b(\omega^{k_0-1},k_0,s,k_0-1)^{-1}\times\\ \times\mathscr{A}^{\mathrm{wt}}\left(\left\langle {}_b\mathfrak{Z}^{[k_0-2]}_{f,\mathbf{g}},\mathrm{sp}^c_{k_0-2}\circ\mathrm{Tw}_{\chi_{\mathbf{g}}^{-1}}\circ\mathrm{Exp}^c_{\mathbf{M}_{f,\mathbf{g}}}(\widetilde{\mathfrak{m}}_{f,\mathbf{g}})^{\iota}\right\rangle_{\mathbf{N}_{f,\mathbf{g}}}\right)(s),\end{aligned}$$

where $\langle\ ,\ \rangle_{\mathbf{N}_{f,\mathbf{g}}}$ is the local duality pairing and ${}_b\mathfrak{Z}^{[k]}_{f,\mathbf{g}}$ is the element constructed in Section 5.2.

Proposition 6.3.3. — *Assume that $k_0 = l_0$. Then on a sufficiently small neighborhood $U_g = \mathrm{Spm}(A_{\mathbf{g}})$ of k_0 one has*

$$L_p(\mathbf{f},\mathbf{g},\omega^{k_0-1})(k_0,s,k_0-1) = -\left(1-\frac{p^{k_0-2}}{\mathbf{a}_p(k_0)\mathbf{b}_p(s)}\right)\left(1-\frac{\varepsilon_f(p)\mathbf{b}_p(s)}{\mathbf{a}_p(k_0)}\right)L_p^{\mathrm{wt}}(\mathbf{f},\mathbf{g},s).$$

Proof. — Let $\mathbf{N}_{\mathbf{f},\mathbf{g}} = \mathrm{gr}_0\mathbf{D}_{\mathbf{f},\mathbf{g}}(\chi^{k_0-1})$. By Theorem 6.1.4 and Lemma 1.2.5 one has

$$\begin{aligned}(102)\quad & G(f,g)B_b(k_0,s,k_0-1)\cdot L_p(\mathbf{f},\mathbf{g},\omega^{k_0-1})(k_0,s,k_0-1) = \\ & (-1)^{k_0-1}\mathscr{A}_{\omega^{k_0-2}}\left(\left\{{}_b\mathbf{Z}^{\mathrm{Iw}}_{\mathbf{f},\mathbf{g}}, \mathrm{Tw}_{\chi_{\mathbf{g}}^{-1}}\circ \mathrm{Exp}^c_{\mathbf{M}_{\mathbf{f},\mathbf{g}}}(\widetilde{m}_{\mathbf{f},\mathbf{g}})^{\iota}\right\}_{\mathbf{M}_{\mathbf{f},\mathbf{g}}(\chi_{\mathbf{g}}^{-1})}\right)(k_0,s,k_0-2) = \\ & (-1)^{k_0-1}\mathscr{A}_{\omega^0}\left(\left\{\mathrm{Tw}_{2-k_0}\left({}_b\mathbf{Z}^{\mathrm{Iw}}_{\mathbf{f},\mathbf{g}}\right), \mathrm{Tw}_{k_0-2}\circ\mathrm{Tw}_{\chi_{\mathbf{g}}^{-1}}\circ \mathrm{Exp}^c_{\mathbf{M}_{\mathbf{f},\mathbf{g}}}(\widetilde{m}_{\mathbf{f},\mathbf{g}})^{\iota}\right\}_{\mathbf{N}_{\mathbf{f},\mathbf{g}}}\right)(k_0,s,0) = \\ & (-1)^{k_0-1}\mathscr{A}^{\mathrm{wt}}\left(\left\langle \mathrm{sp}^c_{2-k_0}\left({}_b\mathbf{Z}^{\mathrm{Iw}}_{\mathbf{f},\mathbf{g}}\right), \mathrm{sp}^c_{k_0-2}\circ\mathrm{Tw}_{\chi_{\mathbf{g}}^{-1}}\circ \mathrm{Exp}^c_{\mathbf{M}_{\mathbf{f},\mathbf{g}}}(\widetilde{m}_{\mathbf{f},\mathbf{g}})^{\iota}\right\rangle_{\mathbf{N}_{\mathbf{f},\mathbf{g}}}\right)(k_0,s).\end{aligned}$$

Let ${}_b\mathfrak{Z}^{\mathrm{Iw}}_{f,\mathbf{g}} = (\mathrm{Pr}^{\alpha}_*,\mathrm{id})\circ \mathrm{sp}^{\mathbf{f}}_{k_0}\left({}_b\mathbf{Z}^{\mathrm{Iw}}_{\mathbf{f},\mathbf{g}}\right)$. Since the maps Pr^{α}_* and Pr^*_{α} are dual to each other, from (101) we obtain that

$$\begin{aligned}(103)\quad & \mathscr{A}^{\mathrm{wt}}\left(\left\langle \mathrm{sp}^c_{2-k_0}\left({}_b\mathbf{Z}^{\mathrm{Iw}}_{\mathbf{f},\mathbf{g}}\right), \mathrm{sp}^c_{k_0-2}\circ\mathrm{Tw}_{\chi_{\mathbf{g}}^{-1}}\circ \mathrm{Exp}^c_{\mathbf{M}_{\mathbf{f},\mathbf{g}}}(\widetilde{m}_{\mathbf{f},\mathbf{g}})^{\iota}\right\rangle_{\mathbf{N}_{\mathbf{f},\mathbf{g}}}\right)(k_0,s) = \\ & \mathscr{A}^{\mathrm{wt}}\left(\left\langle \mathrm{sp}^c_{2-k_0}\left({}_b\mathfrak{Z}^{\mathrm{Iw}}_{f,\mathbf{g}}\right), \mathrm{sp}^c_{k_0-2}\circ\mathrm{Tw}_{\chi_{\mathbf{g}}^{-1}}\circ \mathrm{Exp}^c_{\mathbf{M}_{f,\mathbf{g}}}(\widetilde{\mathfrak{m}}_{f,\mathbf{g}})^{\iota}\right\rangle_{\mathbf{N}_{f,\mathbf{g}}}\right)(s).\end{aligned}$$

By Proposition 5.2.10,

$$\mathrm{sp}^c_{2-k_0}\left({}_b\mathfrak{Z}^{\mathrm{Iw}}_{f,\mathbf{g}}\right) = \frac{(-1)^{k_0}}{(k_0-2)!}\cdot\left(1-\frac{p^{k_0-2}}{\alpha(f)\cdot\mathbf{b}_p}\right)\cdot\left(1-\frac{\beta(f)\cdot\mathbf{b}_p}{p^{k_0-1}}\right)\cdot {}_b\mathfrak{Z}^{[k_0-2]}_{f,\mathbf{g}}.$$

Combining this formula with (102-103) and taking into account that $\alpha(f)\beta(f) = \varepsilon_f(p)p^{k_0-1}$, we obtain the wanted formula. □

6.4. The functional equation. —

6.4.1. — In this subsection, we establish a functional equation for our improved p-adic L-functions. In addition to **M1-2)**, we assume that the following conditions hold:

> **M3)** The characters ε_f, ε_g and $\varepsilon_f\varepsilon_g$ are primitive modulo N_f, N_g and $\mathrm{lcm}(N_f,N_g)$ respectively.

We remark that **M3)** implies that $\varepsilon_f\varepsilon_g \neq 1$. In particular, the case $f \neq g^*$ is excluded.

Write $\lambda_{N_f}(f) = p^{k_0/2}w(f)$, $\lambda_{N_g}(g) = p^{l_0/2}w(g)$, $w(\varepsilon_f\varepsilon_g) = G(\varepsilon_f\varepsilon_g)N^{-1/2}$ and set

$$w(f,g) = (-1)^{l_0}\cdot w(f)\cdot w(g)\cdot w(\varepsilon_f\varepsilon_g)\cdot\frac{\mathbf{a}^c_{d_g}(k_0)\cdot\mathbf{b}^c_{d_f}(y)}{d_g^{(k_0-1)/2}\cdot d_f^{(l_0-1)/2}},$$

where c denotes the complex multiplication. The complete Rankin–Selberg L-function $\Lambda(f,g,s) = \Gamma(s)\Gamma(s-l_0+1)(2\pi)^{l_0-1-2s}L(f,g,s)$ has a holomorphic continuation to all $\mathbf{C}$ and satisfies the functional equation

$$\Lambda(f,g,s) = \varepsilon(f,g,s)\cdot\Lambda(f^*,g^*,k_0+l_0-1-s)$$

where $\varepsilon(f,g,s) = w(f,g)\cdot(NN_fN_g)^{\frac{k_0+l_0-1}{2}-s}$ (see, for example, [**38**, Section 9.5]).

Denote by f^*_α and g^*_α the p-stabilizations of f^* and g^* with respect to the roots $\alpha(f^*) = p^{k_0-1}/\beta(f)$ and $\alpha(g^*) = p^{k_0-1}/\beta(g)$ respectively and by $\mathbf{f}^*$ and $\mathbf{g}^*$ the Coleman families passing through f^*_α and g^*_α.

Proposition 6.4.2. — *The three variable p-adic function $L_p(\mathbf{f},\mathbf{g},\omega^a)(x,y,s)$ satisfies the functional equation*

$$L_p(\mathbf{f},\mathbf{g},\omega^a)(x,y,s)=\varepsilon_p^{[\mathbf{f},\mathbf{g},a]}(x,y,s)\cdot L_p(\mathbf{f}^*,\mathbf{g}^*,\omega^{a^*})(x,y,x+y-s-1),$$

where $a^*=k_0+l_0-a-1$ and

$$\varepsilon_p^{[\mathbf{f},\mathbf{g},a]}(x,y,s)=w(f,g)\cdot(NN_fN_g)^{\frac{k_0+l_0-1}{2}-a}\cdot\left\langle NN_fN_g\right\rangle^{a-s}\cdot\left\langle N_f\right\rangle^{\frac{y-l_0}{2}}\cdot\langle N_g\rangle^{\frac{x-k_0}{2}}.$$

Proof. — This proposition follows from the interpolation properties of $L_p(\mathbf{f},\mathbf{g},\omega^a)(x,y,s)$ and the functional equation for the complex Rankin–Selberg L-function. We leave the details to the reader. □

Corollary 6.4.3. — *The improved L-functions are related by the functional equation*

$$L_p^{\mathrm{wc}}(\mathbf{f},\mathbf{g},s)=A_p^{[\mathbf{f},\mathbf{g}]}(s)\cdot\left(1-\frac{\varepsilon_f(p)\varepsilon_g(p)p^{k_0-2}}{\mathbf{a}_p(k_0)\mathbf{b}_p(s)}\right)\cdot\left(1-\frac{\varepsilon_g(p)\mathbf{a}_p(k_0)}{p\mathbf{b}_p(s)}\right)\cdot L_p^{\mathrm{wt}}(\mathbf{f}^*,\mathbf{g}^*,s),$$

where

$$A_p^{[\mathbf{f},\mathbf{g}]}(s)=(-1)^{k_0-1}w(f,g)\cdot(NN_fN_g)^{-1/2}\cdot\left\langle NN_fN_g\right\rangle^{k_0-s}\cdot\left\langle N_f\right\rangle^{\frac{s-k_0}{2}}.$$

Proof. — Set $\mathbf{f}^*=\sum\limits_{n=1}^{\infty}\mathbf{a}_n^*q^n$ and $\mathbf{g}^*=\sum\limits_{n=1}^{\infty}\mathbf{b}_n^*q^n$. The functional equation gives

$$L_p(\mathbf{f},\mathbf{g},\omega^{k_0})(k_0,s,s)=\varepsilon_p^{[\mathbf{f},\mathbf{g},k_0]}(k_0,s,s)\cdot L_p(\mathbf{f}^*,\mathbf{g}^*,\omega^{k_0-1})(k_0,s,k_0-1).$$

Applying Propositions 6.2.3 and 6.3.3 and taking into account that $\mathbf{a}^*=\varepsilon_f^{-1}(p)\mathbf{a}$ and $\mathbf{b}^*=\varepsilon_g^{-1}(p)\mathbf{b}$, we get

$$\left(1-\frac{\mathbf{b}_p(s)}{\varepsilon_g(p)\mathbf{a}_p(k_0)}\right)\left(1-\frac{\varepsilon_g(p)\mathbf{a}_p(k_0)}{p\mathbf{b}_p(s)}\right)^{-1}L_p^{\mathrm{wc}}(\mathbf{f},\mathbf{g},s)=$$
$$=A_p^{[\mathbf{f},\mathbf{g}]}(s)\left(1-\frac{\varepsilon_f(p)\varepsilon_g(p)p^{k_0-2}}{\mathbf{a}_p(k_0)\mathbf{b}_p(s)}\right)\left(1-\frac{\mathbf{b}_p(s)}{\varepsilon_g(p)\mathbf{a}_p(k_0)}\right)L_p^{\mathrm{wt}}(\mathbf{f}^*,\mathbf{g}^*,s).$$

Since the function $\left(1-\dfrac{\mathbf{b}_p(s)}{\varepsilon_g(p)\mathbf{a}_p(k_0)}\right)$ is not identically zero, we can cancel it in this equation. This gives us the wanted formula. □

Remark 6.4.4. — *One has $A_p^{[\mathbf{f},\mathbf{g}]}(k_0)=(-1)^{k_0-1}\varepsilon(f,g,k_0)$.*

6.5. Functional equation for zeta elements. —

6.5.1. — In this section, we interpret the functional equation for p-adic L-functions in terms of Beilinson–Flach elements and prove Theorem II. We assume that f and g are newforms of the same weight $k_0\geqslant 2$ which satisfy conditions **M1-3)**. Set $V_g=W_g(k_0)$ and $V_{f,g}=W_{f,g}(k_0)$. We consider the canonical basis of $\mathbf{D}_{\mathrm{cris}}(V_{f,g})$ formed by the eigenvectors

$$\begin{aligned}&d_{\alpha\alpha}=\eta_f^{\alpha}\otimes\eta_g^{\alpha}\otimes e_{k_0},\quad d_{\alpha\beta}=\eta_f^{\alpha}\otimes\omega_g^{\beta}\otimes e_{k_0},\\&d_{\beta\alpha}=\omega_f^{\beta}\otimes\eta_g^{\alpha}\otimes e_{k_0},\quad d_{\beta\beta}=\omega_f^{\beta}\otimes\omega_g^{\beta}\otimes e_{k_0}.\end{aligned}\tag{104}$$

Let $D=\eta_f^{\alpha}\otimes\mathbf{D}_{\mathrm{cris}}(V_g)$. We associate to D the filtration $(D_i)_{i=-2}^{2}$ on $\mathbf{D}_{\mathrm{cris}}(V_{f,g})$ defined by

$$D_i=\begin{cases}0, & \text{if } i=-2,\\ E\cdot d_{\alpha\alpha}, & \text{if } i=-1,\\ E\cdot d_{\alpha\alpha}+E\cdot d_{\alpha\beta}, & \text{if } i=0,\\ E\cdot d_{\alpha\alpha}+E\cdot d_{\alpha\beta}+E\cdot d_{\beta\alpha}, & \text{if } i=1,\\ \mathbf{D}_{\mathrm{cris}}(V_{f,g}), & \text{if } i=2.\end{cases}\tag{105}$$

Note that $D_0 = D$. This filtration defines a unique triangulation $\left(F_i\mathbf{D}^\dagger_{\mathrm{rig}}(V_{f,g})\right)_{i=-2}^{2}$ of $\mathbf{D}^\dagger_{\mathrm{rig}}(V_{f,g})$ such that $D_i = \mathscr{D}_{\mathrm{cris}}(F_i\mathbf{D}^\dagger_{\mathrm{rig}}(V_{f,g}))$ for all $-2 \leqslant i \leqslant 2$. From definition it follows that the isomorphism

$$(\mathrm{Pr}^*_\alpha, \mathrm{Pr}^*_\alpha) \,:\, W_{f,g} \simeq \mathrm{sp}^{\mathbf{f},\mathbf{g}}_{k_0,k_0}\left(W_{\mathbf{f},\mathbf{g}}\right),$$

identifies $\left(F_i\mathbf{D}^\dagger_{\mathrm{rig}}(V_{f,g})\right)_{i=-2}^{2}$ with the specialization at (f_α, g_α) of the triangulation $\left(F_i\mathbf{D}_{\mathbf{f},\mathbf{g}}(\chi^{k_0})\right)_{i=-2}^{2}$ constructed in Section 5.1.

To simplify notation, set

$$\mathbf{M}_{f,g} = \mathrm{gr}_0\mathbf{D}^\dagger_{\mathrm{rig}}(V_{f,g}), \qquad \mathbf{N}_{f^*,g^*} = \mathrm{gr}_0\mathbf{D}^\dagger_{\mathrm{rig}}(V^*_{f,g}(1)).$$

Since $\mathrm{sp}^{\mathbf{f},\mathbf{g}}_{k_0,k_0}(\mathbf{M}_{\mathbf{f},\mathbf{g}}) = \mathrm{gr}_0\mathbf{D}_{f_\alpha,g_\alpha}(\chi^{k_0})$ and

$$\mathrm{sp}^{\mathbf{f},\mathbf{g}}_{k_0,k_0}(\mathbf{N}_{\mathbf{f}^*,\mathbf{g}^*}) = \mathbf{M}_{f^*_\alpha,g^*_\alpha}(\chi^{-1}) = \mathrm{gr}_0\mathbf{D}_{f^*_\alpha,g^*_\alpha}(\chi^{k_0-1}),$$

we have canonical isomorphisms

$$\mathbf{M}_{f,g} \simeq \mathrm{sp}^{\mathbf{f},\mathbf{g}}_{k_0,k_0}(\mathbf{M}_{\mathbf{f},\mathbf{g}}), \qquad \mathbf{N}_{f^*,g^*} \simeq \mathrm{sp}^{\mathbf{f}^*,\mathbf{g}^*}_{k_0,k_0}(\mathbf{N}_{f^*,\mathbf{g}^*}).$$

Thus our notation agrees with that of Sections 5.1 and 6.3.

By (94), we have

$$\mathbf{M}^*_{f,g}(\chi) \simeq \mathrm{gr}_1\mathbf{D}^\dagger_{\mathrm{rig}}(V^*_{f,g}(1)), \qquad V^*_{f,g}(1) \simeq W_{f^*,g^*}(k_0-1). \tag{106}$$

Fix canonical generators

$$\begin{aligned} &d_{\alpha\beta} \in \mathscr{D}_{\mathrm{cris}}(\mathbf{M}_{f,g}),\\ &n_{\alpha\beta} = \eta^\alpha_{f^*} \otimes \omega^\beta_{g^*} \otimes e_{k_0-1} \in \mathscr{D}_{\mathrm{cris}}(\mathbf{N}_{f^*,g^*}). \end{aligned}$$

Note that by Proposition 3.4.5

$$(\mathrm{Pr}^*_\alpha, \mathrm{Pr}^*_\alpha)(d_{\alpha\beta}) = \lambda_{N_f}(f)\cdot C(f)\cdot \mathrm{sp}^{\mathbf{f},\mathbf{g}}_{k_0,k_0}(m_{\mathbf{f},\mathbf{g}}). \tag{107}$$

We denote by

$$\exp\,:\, \mathscr{D}_{\mathrm{cris}}(\mathbf{M}_{f,g}) \to H^1(\mathbf{M}_{f,g})$$

and

$$\log\,:\, H^1\left(\mathrm{gr}_1\mathbf{D}^\dagger_{\mathrm{rig}}(V_{f,g})\right) \to \mathscr{D}_{\mathrm{cris}}\left(\mathrm{gr}_1\mathbf{D}^\dagger_{\mathrm{rig}}(V_{f,g})\right)$$

the Bloch–Kato exponential and logarithm maps respectively.

6.5.2. — Let

$$\mathrm{Z}^{[k_0-2]}_{f^*,g^*} \in H^1\left(\mathrm{gr}_1\mathbf{D}^\dagger_{\mathrm{rig}}(V_{f,g})\right) \tag{108}$$

denote the element constructed in Definition 5.2.5. Choose b such that $\varepsilon_f(b)\varepsilon_g(b) \neq 1$ and set

$$\begin{aligned} {}_b\widetilde{\mathrm{Z}}^{[k_0-1]}_{f_\alpha,g_\alpha} &= \mathrm{sp}^{\mathbf{f},\mathbf{g},c}_{k_0,k_0,1-k_0}\left({}_b\mathbf{Z}^{\mathrm{Iw}}_{\mathbf{f},\mathbf{g}}\right) \in H^1(\mathbf{M}^*_{f_\alpha,g_\alpha}(\chi)),\\ {}_b\widetilde{\mathrm{Z}}^{[k_0-1]}_{f,g} &= (\mathrm{Pr}^\alpha_*, \mathrm{Pr}^\alpha_*)\left({}_b\widetilde{\mathrm{Z}}^{[k_0-1]}_{f_\alpha,g_\alpha}\right) \in H^1\left(\mathrm{gr}_1\mathbf{D}^\dagger_{\mathrm{rig}}(V^*_{f,g}(1))\right). \end{aligned} \tag{109}$$

(here we use the isomorphism (106)). Note that ${}_b\widetilde{\mathrm{Z}}^{[k_0-1]}_{f_\alpha,g_\alpha} = \mathrm{sp}^{\mathbf{f},\mathbf{g}}_{k_0,k_0}({}_b\mathbf{Z}_{\mathbf{f},\mathbf{g}})$, where ${}_b\mathbf{Z}_{\mathbf{f},\mathbf{g}}$ is the element introduced in Definition 6.2.1. Set

$$\widetilde{\mathrm{Z}}^{[k_0-1]}_{f,g} = b^{-2}(1-\varepsilon_f(b)\varepsilon_g(b))^{-1}\,{}_b\widetilde{\mathrm{Z}}^{[k_0-1]}_{f,g}.$$

We remark that $\mathrm{Z}^{[k_0-2]}_{f^*,g^*}$ is constructed directly from the Beilinson–Flach element $\mathrm{BF}^{[k_0-2]}_{f^*,g^*}$ whereas the construction of $\widetilde{\mathrm{Z}}^{[k_0-1]}_{f,g}$ relies on Proposition 4.4.4 and involves p-adic interpolation and Iwasawa twist.

Theorem 6.5.3. — *Assume that $\beta(f)\alpha(g) \neq p^{k_0-1}$. Then the elements $\mathrm{Z}^{[k_0-2]}_{f^*,g^*}$ and $\widetilde{\mathrm{Z}}^{[k_0-1]}_{f,g}$ are related by the equation*

$$\frac{\left\langle \widetilde{\mathrm{Z}}^{[k_0-1]}_{f,g}, \exp(d_{\alpha\beta}) \right\rangle_{\mathbf{M}_{f,g}}}{G(\varepsilon_f^{-1})G(\varepsilon_g^{-1})} = (-1)^{k_0-1}\varepsilon(f,g,k_0)\cdot \mathscr{E}(V_{f,g},D_{-1})\cdot \frac{\left[\log\left(\mathrm{Z}^{[k_0-2]}_{f^*,g^*}\right), n_{\alpha\beta}\right]_{\mathbf{N}_{f^*,g^*}}}{(k_0-2)!G(\varepsilon_f)G(\varepsilon_g)},$$

where

$$\mathscr{E}(V_{f,g},D_{-1}) = \det\left(1-p^{-1}\varphi^{-1}\mid D_{-1}\right)\det\left(1-\varphi\mid \mathbf{D}_{\mathrm{cris}}(V_{f,g})/D_{-1}\right).$$

Proof. — From Definition 6.2.2 we have

$$L_p^{\mathrm{wc}}(\mathbf{f},\mathbf{g},k_0) = B_b(\omega^{k_0},k_0,k_0,k_0)^{-1}\left\langle {}_b\widetilde{\mathrm{Z}}^{[k_0-1]}_{f_\alpha,g_\alpha}, \exp_{\mathbf{M}_{f_\alpha,g_\alpha}}(\mathrm{sp}^{\mathbf{f},\mathbf{g}}_{k_0,k_0}(m_{\mathbf{f},\mathbf{g}}))\right\rangle_{\mathbf{M}_{f_\alpha,g_\alpha}},$$

where

$$B_b(\omega^{k_0},k_0,k_0,k_0) = \frac{b^2G(\varepsilon_f^{-1})G(\varepsilon_g^{-1})}{\lambda_{N_f}(f)}\cdot(1-\varepsilon_f^{-1}(b)\varepsilon_g^{-1}(b))\neq 0.$$

Taking into account (107) and (109), we obtain that

$$L_p^{\mathrm{wc}}(\mathbf{f},\mathbf{g},k_0) = \frac{\left\langle \widetilde{\mathrm{Z}}^{[k_0-1]}_{f,g}, \exp(d_{\alpha\beta})\right\rangle_{\mathbf{M}_{f,g}}}{C(f)G(\varepsilon_f^{-1})G(\varepsilon_g^{-1})}. \tag{110}$$

On the other hand,

$$B_b(\omega^{k_0-1},k_0,k_0,k_0) = \frac{G(\varepsilon_f)G(\varepsilon_g)}{\lambda_{N_f}(f^*)}\cdot(b^2-\varepsilon_f(b)\varepsilon_g(b)). \tag{111}$$

The Frobenius φ acts on $\mathscr{D}_{\mathrm{cris}}(\mathbf{M}_{f^*,g^*_\alpha})$ as multiplication by

$$\frac{\alpha(f^*)\beta(g^*)}{p^{k_0}} = \frac{p^{k_0-2}}{\beta(f)\alpha(g)}.$$

By Proposition 4.5.4, i), one has

$$\mathrm{sp}^{\mathbf{g}}_{k_0}\left({}_b\mathfrak{Z}^{[k_0-2]}_{f^*,\mathbf{g}^*}\right) = {}_b\mathrm{Z}^{[k_0-2]}_{f^*,g^*_\alpha}.$$

From Definition 6.3.2 and Proposition 1.3.5 it follows that

$$L_p^{\mathrm{wt}}(\mathbf{f}^*,\mathbf{g}^*,k_0) = \Gamma(k_0-1)^{-1}B_b(\omega^{k_0-1},k_0,k_0,k_0)^{-1}\left(1-\frac{\beta(f)\alpha(g)}{p^{k_0}}\right)\times$$
$$\times\left(1-\frac{p^{k_0-1}}{\beta(f)\alpha(g)}\right)^{-1}\cdot\left[\log\left({}_b\mathrm{Z}^{[k_0-2]}_{f^*,g^*_\alpha}\right), \mathrm{sp}^{\mathbf{g}}_{k_0}(\mathfrak{m}_{f^*,\mathbf{g}^*}\otimes e_{-1}))\right]_{\mathbf{N}_{f^*,g^*_\alpha}}. \tag{112}$$

Since $C(f^*) = C(f)$, from (100) we have that

$$\mathrm{sp}^{\mathbf{g}}_{k_0}(\mathfrak{m}_{f^*,\mathbf{g}^*}\otimes e_{-1}) = \frac{1}{C(f)\lambda_{N_f}(f^*)}n_{\alpha\beta}. \tag{113}$$

Proposition 4.3.4 and (77) give

$$(\mathrm{id},\mathrm{Pr}^\alpha_*)\left({}_b\mathrm{Z}^{[k_0-2]}_{f^*,g^*_\alpha}\right) = (b^2-\varepsilon_f(b)\varepsilon_g(b))\cdot\left(1-\frac{p^{k_0-1}}{\beta(f)\alpha(g)}\right)\cdot\left(1-\frac{p^{k_0-1}}{\alpha(f)\alpha(g)}\right)\mathrm{Z}^{[k_0-2]}_{f^*,g^*}. \tag{114}$$

Taking into account (111), (113) and (114), we can write (112) in the form

$$L_p^{\mathrm{wt}}(\mathbf{f}^*,\mathbf{g}^*,k_0) = \frac{1}{(k_0-2)!\cdot C(f)G(\varepsilon_f)G(\varepsilon_g)}\left(1-\frac{\beta(f)\alpha(g)}{p^{k_0}}\right)\times$$
$$\times\left(1-\frac{p^{k_0-1}}{\alpha(f)\alpha(g)}\right)\left[\log\left(\mathrm{Z}^{[k_0-2]}_{f^*,g^*}\right), n_{\alpha\beta}\right]_{\mathbf{N}_{f^*,g^*}}. \tag{115}$$

Now the theorem follows from (110), (115) and Corollary 6.4.3. □

***Remark 6.5.4.* —** The explicit form of the Euler-like factor $\mathscr{E}(V_{f,g},D_{-1})$ is

$$\mathscr{E}(V_{f,g},D_{-1})=\left(1-\frac{p^{k_0-1}}{\alpha(f)\alpha(g)}\right)\left(1-\frac{\alpha(f)\beta(g)}{p^{k_0}}\right)\left(1-\frac{\beta(f)\alpha(g)}{p^{k_0}}\right)\left(1-\frac{\beta(f)\beta(g)}{p^{k_0}}\right).$$

7. Extra zeros of Rankin–Selberg L-functions

7.1. The p-adic regulator. —

7.1.1. — In this section, we prove the main result of the paper. Let f and g be two newforms of the same weight $k_0\geqslant 2$, levels N_f and N_g and nebentypus ε_f and ε_g respectively. Fix a prime number $p\geqslant 5$ such that $(p,N_fN_g)=1$. As before, we denote by $\alpha(f)$ and $\beta(f)$ (respectively by $\alpha(g)$ and $\beta(g)$) the roots of the Hecke polynomial of f (respectively g) at p. We will always assume that conditions **M1-4)** hold, namely

M1) $\alpha(f)\neq\beta(f)$ and $\alpha(g)\neq\beta(g)$.
M2) $v_p(\alpha(f))<k_0-1$ and $v_p(\alpha(g))<k_0-1$.
M3) The characters ε_f, ε_g and $\varepsilon_f\varepsilon_g$ are primitive modulo N_f, N_g and $\mathrm{lcm}(N_f,N_g)$ respectively.

We make also the following additional assumption which will allow us to apply Theorem 6.5.3:

M4) $\varepsilon_f(p)\varepsilon_g(p)\neq 1$.

We maintain the notation of Section 6.5. Let $V_g=W_g(k_0)$ and $V_{f,g}=W_{f,g}(k_0)$. The two-dimensional E-subspace

$$D=E\eta_f^{\alpha}\otimes\mathbf{D}_{\mathrm{cris}}(V_g)\subset\mathbf{D}_{\mathrm{cris}}(V_{f,g})$$

is stable under the action of φ. Let $\mathbf{f}$ and $\mathbf{g}$ be Coleman families passing through f_α anf g_α and let $L_p(\mathbf{f},\mathbf{g},\omega^{k_0})(x,y,s)$ denote the three-variable p-adic L-function.

***Definition 7.1.2.* —** *We define the one-variable p-adic L-function $L_{p,\alpha}(f,g,s)$ by*

$$L_{p,\alpha}(f,g,s)=L_p(\mathbf{f},\mathbf{g},\omega^{k_0})(k_0,k_0,s).$$

Note that if $v_p(\beta(g))<k_0-1$ and $\tilde{\mathbf{g}}$ denotes the Coleman family passing through g_β, then the density argument shows that

$$L_p(\mathbf{f},\mathbf{g},\omega^{k_0})(x,k_0,s)=L_p(\mathbf{f},\tilde{\mathbf{g}},\omega^{k_0})(x,k_0,s),$$

and therefore our definition does not depend on the choice of the stabilization of g (see [**20**, Proposition 3.6.3]).

The Euler-like factor (3) takes the form

$$\mathscr{E}(V_{f,g},D)=\left(1-\frac{p^{k_0-1}}{\alpha(f)\alpha(g)}\right)\cdot\left(1-\frac{p^{k_0-1}}{\alpha(f)\beta(g)}\right)\cdot\left(1-\frac{\beta(f)\alpha(g)}{p^{k_0}}\right)\cdot\left(1-\frac{\beta(f)\beta(g)}{p^{k_0}}\right).$$

The weight argument shows that only the first two factors of this product can vanish and that they can not vanish simultaneously. Exchanging $\alpha(g)$ and $\beta(g)$ if necessary, without loss of generality we can assume that

M5) $\alpha(f)\alpha(g)\neq p^{k_0-1}$.

7.1.3. — Let $\mathrm{BF}^{[k_0-2]}_{f^*,g^*}\in H^1_S(\mathbf{Q},W^*_{f^*,g^*}(2-k_0))$ denote the element of Beilinson–Flach associated to the forms f^*, g^* (see Definition 4.3.2). Using the canonical isomorphism $W^*_{f^*,g^*}(2-k_0)\simeq V_{f,g}$ we can consider it as an element

$$\mathrm{BF}^{[k_0-2]}_{f^*,g^*}\in H^1_S(\mathbf{Q},V_{f,g}).$$

For any prime l, we denote by $\mathrm{res}_l\left(\mathrm{BF}^{[k_0-2]}_{f^*,g^*}\right)\in H^1(\mathbf{Q}_l,V_{f,g})$ the localization of this element at l.

Lemma 7.1.4. — *The following holds true:*

1) $\mathrm{res}_p\left(\mathrm{BF}^{[k_0-2]}_{f^*,g^*}\right) \in H^1_f(\mathbf{Q}_p, V_{f,g})$.

2) Assume that for each prime divisor $l \mid N_f N_g$ *the factorization of* N_f *or* N_g *contains* l *with multiplicity* 1. *Then*

$$\mathrm{BF}^{[k_0-2]}_{f^*,g^*} \in H^1_{f,\{p\}}(\mathbf{Q}, V_{f,g}).$$

Proof. — The first statement is proved in [**43**, Proposition 5.4.1] and was already mentioned in Section 5.2. The second statement follows from the fact that $H^1_f(\mathbf{Q}_l, V_{f,g}) = H^1(\mathbf{Q}_l, V_{f,g})$ if and only if $H^0(\mathbf{Q}_l, V^*_{f,g}(1)) = 0$ and the monodromy-weight conjecture for modular forms [**57**]. □

7.1.5. — Recall that $\mathbf{D}_{\mathrm{cris}}(V_{f,g})$ is equipped with the filtration (105). We denote by $\left(F_i\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_{f,g})\right)_{i=-2}^{2}$ the associated triangulation of $\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_{f,g})$. The eigenvector $d_{\beta\alpha} = \omega_f^{\beta} \otimes \eta_g^{\alpha} \otimes e_{k_0}$ defined in (104) is a canonical basis of $\mathscr{D}_{\mathrm{cris}}\left(\mathrm{gr}_1\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_{f,g})\right)$. As in Section 6.5, we denote by

$$\log : H^1\left(\mathrm{gr}_1\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_{f,g})\right) \to \mathscr{D}_{\mathrm{cris}}\left(\mathrm{gr}_1\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_{f,g})\right)$$

the logarithm map of Bloch and Kato. Let

$$\mathrm{Z}^{[k_0-2]}_{f^*,g^*} \in H^1\left(\mathrm{gr}_1\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_{f,g})\right)$$

be the image of $\mathrm{res}_p\left(\mathrm{BF}^{[k_0-2]}_{f^*,g^*}\right)$ under the canonical projection

$$H^1(\mathbf{Q}_p, V_{f,g}) \to H^1\left(\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_{f,g})/F_0\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_{f,g})\right)$$

(see Section 5.2.4). Denote by $\widetilde{R}_p(V_{f,g}, D) \in E$ the unique element of E such that

$$\log\left(\mathrm{Z}^{[k_0-2]}_{f^*,g^*}\right) = \widetilde{R}_p(V_{f,g}, D) \cdot d_{\beta\alpha}.$$

Since $d_{\beta\alpha} \in \mathscr{D}_{\mathrm{cris}}\left(\mathrm{gr}_1\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_{f,g})\right)$ is the dual basis of

$$n_{\alpha\beta} \in \mathscr{D}_{\mathrm{cris}}(\mathbf{N}_{f^*,g^*}) \simeq \mathscr{D}_{\mathrm{cris}}\left(\mathrm{gr}_0\mathbf{D}^{\dagger}_{\mathrm{rig}}(V^*_{f,g}(1))\right)$$

(see Section 6.5), we have

$$\widetilde{R}_p(V_{f,g}, D) = \left[\log\left(\mathrm{Z}^{[k_0-2]}_{f^*,g^*}\right\rangle, n_{\alpha\beta}\right]_{\mathbf{N}_{f^*,g^*}}. \tag{116}$$

Let $\eta_g \in \mathbf{D}_{\mathrm{cris}}(W_g)$ be any vector such that $[\eta_g, \omega_{g^*}] = e_{1-k_0}$. Set $b = \omega_f \otimes \eta_g \otimes e_{k_0} \in \mathbf{D}_{\mathrm{cris}}(V_{f,g})$. Then the class

$$\overline{b}_{\alpha} = b \mod \left(\mathrm{Fil}^0\mathbf{D}_{\mathrm{cris}}(V_{f,g}) + D\right)$$

is nonzero, does not depend on the choice of η_g and therefore gives a canonical basis of the one-dimensional vecor space $\mathbf{D}_{\mathrm{cris}}(V_{f,g})/\left(\mathrm{Fil}^0\mathbf{D}_{\mathrm{cris}}(V_{f,g}) + D\right)$.

Proposition 7.1.6. — *1) The representation* $V_{f,g}$ *satisfies conditions* **C1-3)** *of Section 2.1.*

2) Assume that the representation $V_{f,g}$ *satisfies conditions* **C4-5)**, *namely that* $H^1_f(\mathbf{Q}, V^*_{f,g}(1)) = 0$ *and the localization map* $H^1_f(\mathbf{Q}, V_{f,g}) \to H^1_f(\mathbf{Q}_p, V_{f,g})$ *is injective. Then*

i) D *is a regular submodule if and only if* $\mathrm{Z}^{[k_0-2]}_{f^*,g^*} \neq 0$.

ii) If that is the case, then $\widetilde{R}_p(V_{f,g}, D)$ *coincides with the determinant of the regulator map*

$$H^1_f(\mathbf{Q}, V_{f,g}) \to \mathbf{D}_{\mathrm{cris}}(V_{f,g})/\left(\mathrm{Fil}^0\mathbf{D}_{\mathrm{cris}}(V_{f,g}) + D\right)$$

computed in the bases $\mathrm{BF}^{[k_0-2]}_{f^*,g^*} \in H^1_f(\mathbf{Q}, V_{f,g})$ *and* $\overline{b}_{\alpha} \in \mathbf{D}_{\mathrm{cris}}(V_{f,g})/(\mathrm{Fil}^0\mathbf{D}_{\mathrm{cris}}(V_{f,g}) + D)$.

Proof. — 1) The weight argument shows that $\mathbf{D}_{\mathrm{cris}}(V_{f,g})^{\varphi=1}=0$ and $H^0(\mathbf{Q},V_{f,g})=0$. Since $V_{f,g}^*(1)=\mathrm{Hom}_E(W_f,W_{g^*})$ and $f\neq g^*$, we obtain that $H^0(\mathbf{Q},V_{f,g})=\mathrm{Hom}_{E[G_{\mathbf{Q}_p}]}(W_f,W_{g^*})=0$. The semisimplicity of φ follows from **M1)**.

2) From the congruences $\eta_g\equiv\eta_g^{\alpha} \pmod{\mathrm{Fil}^{k_0-1}\mathbf{D}_{\mathrm{cris}}(W_g)}$ and $\omega_f\equiv\omega_f^{\beta} \mod \mathbf{D}_{\mathrm{cris}}(W_f)^{\varphi=\alpha(f)}$ it is easy to see that $\overline{b}_{\alpha}=\overline{d}_{\beta\alpha}$. Now the second statement follows directly from the definition of the regulator map. □

7.2. The $\mathscr{L}$-invariant. —

7.2.1. — Set

$$ {}_b\widetilde{\mathrm{BF}}_{f,g}^{[k_0-1]}=(\mathrm{Pr}_*^{\alpha},\mathrm{Pr}_*^{\alpha})\circ \mathrm{sp}_{k_0,k_0,1-k_0}^{\mathbf{f},\mathbf{g},c}\left({}_b\mathbf{BF}_{\mathbf{f},\mathbf{g}}^{\mathrm{Iw}}\right)\in H^1_S(\mathbf{Q},V_{f,g}^*(1)), $$

where ${}_b\mathbf{BF}_{\mathbf{f},\mathbf{g}}^{\mathrm{Iw}}$ is the class in Iwasawa cohomology constructed in Proposition 4.4.4. Note that, unlike ${}_b\mathrm{BF}_{f^*,g^*}^{[k_0-2]}$, the element ${}_b\widetilde{\mathrm{BF}}_{f,g}^{[k_0-1]}$ is not a proper Beilinson–Flach class and its construction involves p-adic interpolation.

Lemma 7.2.2. — *For all primes $l\neq p$ we have*

$$ \mathrm{res}_l\left({}_b\widetilde{\mathrm{BF}}_{f,g}^{[k_0-1]}\right)\in H^1_f(\mathbf{Q}_l,V_{f,g}^*(1)), $$

and therefore

$$ {}_b\widetilde{\mathrm{BF}}_{f,g}^{[k_0-1]}\in H^1_{f,\{p\}}(\mathbf{Q},V_{f,g}^*(1)). $$

Proof. — By [**50**, Section 2.1.7], the image of the projection map

$$ H^1_{\mathrm{Iw}}(\mathbf{Q}_l,V_{f,g}^*(1))\to H^1(\mathbf{Q}_l,V_{f,g}^*(1)) $$

is contained in $H^1_f(\mathbf{Q}_l,V_{f,g}^*(1))$. This implies the lemma. □

Choose b such that $\varepsilon_f(b)\varepsilon_g(b)\neq 1$. The element ${}_b\widetilde{\mathrm{Z}}_{f,g}^{[k_0-1]}$ constructed in Section 6.5 is the image of ${}_b\widetilde{\mathrm{BF}}_{f,g}^{[k_0-1]}$ under the composition

$$ H^1_S(\mathbf{Q},V_{f,g}^*(1))\xrightarrow{\mathrm{res}_p}H^1(\mathbf{Q}_p,V_{f,g}^*(1))\to H^1\left(\mathbf{D}_{\mathrm{rig}}^{\dagger}(V_{f,g}^*(1))/F_0\mathbf{D}_{\mathrm{rig}}^{\dagger}(V_{f,g}^*(1))\right). $$

From Proposition 5.2.9 it follows that

$$ {}_b\widetilde{\mathrm{Z}}_{f,g}^{[k_0-1]}\in H^1\left(\mathrm{gr}_1\mathbf{D}_{\mathrm{rig}}^{\dagger}(V_{f,g}^*(1))\right). $$

Note that φ acts on $\mathscr{D}_{\mathrm{cris}}\left(\mathrm{gr}_1\mathbf{D}_{\mathrm{rig}}^{\dagger}(V_{f,g}^*(1))\right)$ as multiplication by $\dfrac{p^{k_0-1}}{\alpha(f)\beta(g)}$.

7.2.3. — To simplify notation, we set $\mathbf{M}_{f,g}=\mathrm{gr}_0\mathbf{D}_{\mathrm{rig}}^{\dagger}(V_{f,g})$ (see Section 6.5). Then $\mathbf{M}_{f,g}^*(\chi)\simeq \mathrm{gr}_1\mathbf{D}_{\mathrm{rig}}^{\dagger}(V_{f,g}^*(1))$. Assume that $\alpha(f)\beta(g)=p^{k_0-1}$. Then formulas (90) show that $\mathbf{M}_{f,g}\simeq\mathscr{R}_E(\chi)$ and, dually, $\mathbf{M}_{f,g}^*(\chi)=\mathscr{R}_E$. Therefore $\mathbf{M}_{f,g}^*(\chi)$ is the (φ,Γ)-module associated to the trivial representation, and we have canonical isomorphisms $\mathscr{D}_{\mathrm{cris}}(\mathbf{M}_{f,g}^*(\chi))\simeq E$ and $H^1(\mathbf{M}_{f,g}^*(\chi))\simeq H^1(\mathbf{Q}_p,E)$. Clearly, $\mathbf{M}_{f,g}^*(\chi)$ satisfies condition (36) and the decomposition (37) has the following interpretation in terms of Galois cohomology. Let ord : $\mathrm{Gal}(\mathbf{Q}_p^{\mathrm{ur}}/\mathbf{Q}_p)\to\mathbf{Z}_p$ denote the unramified character defined by $\mathrm{ord}(\mathrm{Fr}_p)=-1$, where Fr_p is the geometric Frobenius. Denote by $\log\chi$ the logarithm of the cyclotomic character viewed as an additive character of the whole Galois group.

We have a commuative diagram

$$\begin{array}{ccc} \mathscr{D}_{\mathrm{cris}}(\mathbf{M}^*_{f,g}(\chi)) \oplus \mathscr{D}_{\mathrm{cris}}(\mathbf{M}^*_{f,g}(\chi)) & \xrightarrow{i_{\mathbf{M}^*_{f,g}(\chi)}} & H^1(\mathbf{M}^*_{f,g}(\chi)) \\ \downarrow = & & \downarrow = \\ E \oplus E & \longrightarrow & H^1(\mathbf{Q}_p, E), \end{array}$$

where the bottom horizontal arrow is given by $(x,y) \mapsto x \cdot \mathrm{ord} + y \cdot \log \chi$. This follows from the explicit description of the Galois cohomology in terms of (φ,Γ)-modules (see [**5**, Proposition 1.3.2] or [**23**, Proposition I.4.1]). Under the right vertical map, the subspaces $H^1_f(\mathbf{M}^*_{f,g}(\chi))$ and $H^1_c(\mathbf{M}^*_{f,g}(\chi))$ are mapped onto the subspaces generated by the characters ord and $\log \chi$ respectively. We refer the reader to [**6**, Section 1.5] for further comments.

Definition 7.2.4. — *Assume that $\alpha(f)\beta(g) = p^{k_0-1}$ and ${}_b\widetilde{\mathrm{Z}}^{[k_0-1]}_{f,g} \notin H^1_f(\mathbf{M}^*_{f,g}(\chi))$. Then*

$${}_b\widetilde{\mathrm{Z}}^{[k_0-1]}_{f,g} = A \cdot \mathrm{ord}_p + B \cdot \log \chi$$

for unique $A, B \in E$ such that $B \neq 0$, and we define

$$\widetilde{\mathscr{L}}(V_{f,g}, D) = A/B.$$

Proposition 7.2.5. — *Assume that the following holds:*

a) $\alpha(f)\beta(g) = p^{k_0-1}$.

b) The representation $V_{f,g}$ satisfies conditions **C4-5)** *of Section 2.1.*

c) $Z^{[k_0-2]}_{f^*,g^*} \neq 0$.

Then $\widetilde{\mathscr{L}}(V_{f,g}, D) = \mathscr{L}(V_{f,g}, D)$, where $\mathscr{L}(V_{f,g}, D)$ is the invariant defined in Section 2.2.

Proof. — From Proposition 7.1.6 it follows that D is regular. By Proposition 5.2.9, the image of the element $\widetilde{\mathrm{BF}}^{[k_0-1]}_{f,g} \in H^1_{f,\{p\}}(\mathbf{Q}, V^*_{f,g}(1))$ under the map

$$H^1_{f,\{p\}}(\mathbf{Q}, V^*_{f,g}(1)) \to \frac{H^1(\mathbf{Q}_p, V^*(1))}{H^1\left(F_0\mathbf{D}^{\dagger}_{\mathrm{rig}}(V^*(1))\right)}$$

is $\widetilde{Z}^{[k_0-1]}_{f,g} \in H^1(\mathbf{M}^*_{f,g}(\chi))$. Since $\varepsilon_f(p)\varepsilon_g(p) \neq 1$, the condition a) implies that $\beta(f)\alpha(g) \neq p^{k_0-1}$, and we can apply Theorem 6.5.3. By assumption, $Z^{[k_0-2]}_{f^*,g^*} \neq 0$. Therefore $\left\langle \widetilde{Z}^{[k_0-1]}_{f,g}, \exp(d_{\alpha\beta}) \right\rangle_{\mathbf{M}_{f,g}} \neq 0$ and $\widetilde{Z}^{[k_0-1]}_{f,g} \notin H^1_f(\mathbf{M}^*_{f,g}(\chi))$. Comparing this with the isomorphism (44), we see that

$$\widetilde{\mathrm{BF}}^{[k_0-1]}_{f,g} \in H^1(D^{\perp}, V^*_{f,g}(1)).$$

Now it is easy to see that our construction of the ad hoc invariant concides (up to the sign) with the invariant defined in Section 2.3:

$$\widetilde{\mathscr{L}}(V_{f,g}, D) = -\mathscr{L}(V^*_{f,g}(1), D^{\perp}).$$

Hence, the proposition follows from Proposition 2.3.8. □

7.3. The main theorem. — In this section, we prove Theorem I. We keep previous notation and conventions.

Theorem 7.3.1. — *Assume that $\alpha(f)\beta(g) = p^{k_0-1}$. Then*

1) $L_{p,\alpha}(f,g,k_0) = 0$.

2) The following conditions are equivalent:

i) $\mathrm{ord}_{s=0} L_{p,\alpha}(f,g,s) = 1$.

ii) ${}_b\widetilde{Z}^{[k_0-1]}_{f,g} \notin H^1_c\left(\mathrm{gr}_1 \mathbf{D}^{\dagger}_{\mathrm{rig}}(V^*_{f,g}(1))\right)$.

3) In addition to the assumption that $\alpha(f)\beta(g)=p^{k_0-1}$, suppose that

$$_b\widetilde{\mathrm{Z}}_{f,g}^{[k_0-1]} \notin H^1_f\left(\mathrm{gr}_1\mathbf{D}^{\dagger}_{\mathrm{rig}}(V^*_{f,g}(1))\right).$$

Then

$$L'_{p,\alpha}(f,g,k_0)=\frac{\varepsilon(f,g,k_0)\cdot\widetilde{\mathscr{L}}(V_{f,g},D)\cdot\mathscr{E}^+(V_{f,g},D)}{C(f)\cdot G(\varepsilon_f)\cdot G(\varepsilon_g)\cdot(k_0-2)!}\cdot\widetilde{R}_p(V_{f,g},D),$$

where

$$\mathscr{E}^+(V_{f,g},D)=\left(1-\frac{p^{k_0-1}}{\alpha(f)\alpha(g)}\right)\left(1-\frac{\beta(f)\alpha(g)}{p^{k_0}}\right)\left(1-\frac{\beta(f)\beta(g)}{p^{k_0}}\right).$$

Proof. — 1) From Theorem 6.1.4, Lemma 6.1.2 and the identity (107) it follows that

(117)

$$B_b(\omega^{k_0},k_0,k_0,k_0+s)L_{p,\alpha}(f,g,s)=(-1)^{k_0}\mathscr{A}_{\omega^0}\left(\mathfrak{Log}_{\mathbf{M}_{\mathbf{f},\mathbf{g}},m_{\mathbf{f},\mathbf{g}}}\left(\mathrm{Tw}_{\boldsymbol{\chi}_{\mathbf{g}}^{-1}}({}_b\mathbf{Z}^{\mathrm{Iw}}_{\mathbf{f},\mathbf{g}})\right)\right)(k_0,k_0,s)=$$
$$=\frac{(-1)^{k_0}}{C(f)\lambda_{N_f}(f)}\cdot\mathscr{A}_{\omega^0}\left(\mathfrak{Log}_{\mathbf{M}_{f,g},d_{\alpha\beta}}\left(\mathrm{Tw}_{1-k_0}({}_b\mathrm{Z}^{\mathrm{Iw}}_{f,g})\right)\right)(s).$$

By Proposition 1.3.5 we have

$$B_b(\omega^{k_0},k_0,k_0,k_0)L_p(f,g,k_0)=$$
$$=\frac{(-1)^{k_0}}{C(f)\lambda_{N_f}(f)}\cdot\left(1-\frac{p^{k_0-1}}{\alpha(f)\beta(g)}\right)\cdot\left(1-\frac{\alpha(f)\beta(g)}{p^{k_0}}\right)\cdot\left\langle {}_b\widetilde{\mathrm{Z}}_{f,g}^{[k_0-1]},\exp_{\mathbf{M}_{f,g}}(d_{\alpha\beta})\right\rangle_{\mathbf{M}_{f,g}}.$$

This proves 1).

2) The derivative of the large logarithm map in presence of trivial zeros is computed in [**7**, Propositions 1.3.6 and 2.2.2][(9)]. Applying it to our case (see especially formulas (24) and (25) of op. cit.), we obtain

(118)

$$\frac{d}{ds}\mathscr{A}_{\omega^0}\left(\mathfrak{Log}_{\mathbf{M}_{f,g},d_{\alpha\beta}}\left(\mathrm{Tw}_{1-k_0}({}_b\mathrm{Z}^{\mathrm{Iw}}_{f,g})\right)\right)(s)\Big|_{s=0}=-\left(1-\frac{1}{p}\right)^{-1}\cdot\left\langle {}_b\widetilde{\mathrm{Z}}_{f,g}^{[k_0-1]},i_{\mathbf{M}_{f,g},c}(d_{\alpha\beta})\right\rangle_{\mathbf{M}_{f,g}}.$$

Write ${}_b\widetilde{\mathrm{Z}}_{f,g}^{[k_0-1]}=A\cdot\mathrm{ord}_p+B\cdot\log\chi$. Then

$$\left\langle {}_b\widetilde{\mathrm{Z}}_{f,g}^{[k_0-1]},i_{\mathbf{M}_{f,g},c}(d_{\alpha\beta})\right\rangle_{\mathbf{M}_{f,g}}=A\left\langle\mathrm{ord}_p,i_{\mathbf{M}_{f,g},c}(d_{\alpha\beta})\right\rangle_{\mathbf{M}_{f,g}}=-A. \qquad (119)$$

This implies 2).

3) By [**6**, Theorem 1.5.7], $\exp_{\mathbf{M}_{f,g}}(d_{\alpha\beta})=i_{\mathbf{M}_{f,g},f}(d_{\alpha\beta})$, and therefore

$$\left\langle {}_b\widetilde{\mathrm{Z}}_{f,g}^{[k_0-1]},\exp_{\mathbf{M}_{f,g}}(d_{\alpha\beta})\right\rangle_{\mathbf{M}_{f,g}}=\left\langle {}_b\widetilde{\mathrm{Z}}_{f,g}^{[k_0-1]},i_{\mathbf{M}_{f,g},f}(d_{\alpha\beta})\right\rangle_{\mathbf{M}_{f,g}}=B.$$

Taking into account Definition 7.2.4 and (119), we obtain that

$$\left\langle {}_b\widetilde{\mathrm{Z}}_{f,g}^{[k_0-1]},i_{\mathbf{M}_{f,g},c}(d_{\alpha\beta})\right\rangle_{\mathbf{M}_{f,g}}=-\widetilde{\mathscr{L}}(V_{f,g},D)\cdot B= \qquad (120)$$
$$=-\widetilde{\mathscr{L}}(V_{f,g},D)\cdot\left\langle {}_b\widetilde{\mathrm{Z}}_{f,g}^{[k_0-1]},\exp_{\mathbf{M}_{f,g}}(d_{\alpha\beta})\right\rangle_{\mathbf{M}_{f,g}}.$$

Formulas (117), (118) and (120) give

$$B_b(\omega^{k_0},k_0,k_0,k_0)\cdot L'_{p,\alpha}(f,g,k_0)=$$
$$=\frac{(-1)^{k_0-1}}{C(f)\lambda_{N_f}(f)}\left(1-\frac{1}{p}\right)^{-1}\cdot\widetilde{\mathscr{L}}(V_{f,g},D)\cdot\left\langle {}_b\widetilde{\mathrm{Z}}_{f,g}^{[k_0-1]},\exp_{\mathbf{M}_{f,g}}(d_{\alpha\beta})\right\rangle_{\mathbf{M}_{f,g}}.$$

(9) In [**7**], only the case of p-adic representations is considered, but for (φ,Γ)-modules the proof is exactly the same.

Since $\alpha(f)\beta(g)=p^{k_0-1}$, the condition **M4)** implies that $\beta(f)\alpha(g)\neq p^{k_0-1}$ and we can apply Theorem 6.5.3. Taking into account Remark 6.5.4, (97) and (108), we obtain that

$$L'_{p,\alpha}(f,g,k_0)=\frac{\varepsilon(f,g,k_0)\cdot\widetilde{\mathscr{L}}(V_{f,g},D)\cdot\mathscr{E}^+(V_{f,g},D)}{C(f)\cdot G(\varepsilon_f)\cdot G(\varepsilon_g)\cdot(k_0-2)!}\cdot\left[\log\left(\mathrm{Z}^{[k_0-2]}_{f^*,g^*}\right),n_{\alpha\beta}\right]_{\mathbf{N}_{f^*,g^*}}.$$

Using (116), we can replace $\left[\log\left(\mathrm{Z}^{[k_0-2]}_{f^*,g^*}\right),n_{\alpha\beta}\right]_{\mathbf{N}_{f^*,g^*}}$ by $\widetilde{R}_p(V_{f,g},D)$. The theorem is proved. □

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