

Exercise sheet n°2
Absolute values, valuations

Exercise 1 – Prove that the rings $\mathbb{Z}_p/p^n\mathbb{Z}_p$ and $\mathbb{Z}/p^n\mathbb{Z}$ are isomorphic. In particular, $\mathbb{Z}_p/p\mathbb{Z}_p$ is isomorphic to the field $\mathbb{F}_p$.

Exercise 2 – Let K be a field.

- 1) Prove that if K is finite, then the only absolute value on K is the trivial one.
- 2) Does the converse of the above statement hold?

Exercise 3 –

- 1) Let $(K, |\cdot|)$ be a valued field. Show that $|\cdot|^s$ is an absolute value on K for all $s \in]0; 1]$.
- 2) Find all $s > 0$ such that $|\cdot|_\infty^s$ is an absolute value on $\mathbb{Q}$.

Exercise 4 –

- 1) Let $(K, |\cdot|)$ be a valued field. Show that if $|\cdot|$ is non-Archimedean on a subfield of K , then it is non-Archimedean on K .
- 2) Prove that every absolute value on a field of positive characteristic is non-Archimedean.

Exercise 5 – Let $(K, |\cdot|)$ be a valued field. Show that if $|\cdot|$ is non-Archimedean, then $B(0, 1) \cap B(1, 1) = \emptyset$.

Exercise 6 – Let $(K, |\cdot|)$ be a non-Archimedean field.

- 1) Show that for each $r \in]0, 1]$ the open ball $B(1, r)$ is a subgroup of K^* .
- 2) Prove that each open ball in K is closed.
- 3) Deduce that K is totally disconnected (the only connected subsets are singletons.)

Exercise 7 – Let K be a field and v be a non-trivial valuation on K . Prove that the following properties are equivalent :

- (a) $\mathcal{O}_v$ is a principal ideal domain ;
- (b) $\mathcal{O}_v$ is noetherian ;
- (c) the ideal $\mathfrak{m}_v$ is principal ;
- (d) $v(K^*)$ is a discrete subgroup of $\mathbb{R}$.

Exercise 8 – (**Ostrowski theorem for rational functions.**) Let k be a finite field, and let $k(t)$ denote the field of fractions of the polynomial ring $k[t]$. For each irreducible polynomial $P \in k[t]$, we denote by v_P the discrete valuation on $k(t)$ attached to P , namely

$$v_P(f) = k, \text{ if } f(t) = \frac{r(t)}{s(t)} P(t)^k, \text{ where } r(t) \text{ and } s(t) \text{ are coprime with } P(t).$$

- 1) For each $f(t) = a(t)/b(t)$ with $a(t), b(t) \in k[t]$, $b(t) \neq 0$, define :

$$v_\infty(f) = \deg(b) - \deg(a).$$

(Recall that $\deg(0) = -\infty$.) Check that $v_\infty(f)$ does not depend on the choice of $a(t)$ and $b(t)$ and that the above formula defines a discrete valuation on $k(t)$.

Let now $|\cdot|$ be a nontrivial absolute value on K .

- 2) Assume that $|t| > 1$ and set $c = 1/|t|$. Show that

$$|a(t)| = c^{-\deg(f)}, \quad \forall a(t) \in k[t],$$

and deduce that $|f(t)| = c^{v_\infty(f)}$ for all $f \in k(t)$.

- 3) Assume that $|t| \leq 1$.

a) Show that there exists a nonzero monic polynomial $g \in k[t]$ of degree > 0 such that $|g| < 1$.

b) Let $P \in k[t]$ denote a monic polynomial of smallest positive degree such that $|P| < 1$. Prove that P is irreducible.

c) Show that for any nonzero polynomial $h \in k[t]$, coprime with P , one has $|h| = 1$.

d) Show that for any $f(t) \in k(t)$, one has $|f(t)| = c^{v_P(f(t))}$, where $c = |P|$. Deduce that the above formula holds for all $f(t) \in k(t)$.

Exercice 9 – (Weak approximation theorem of Artin-Whaples.)

Let K be a field equipped with pairwise inequivalent nontrivial absolute values $|\cdot|_1, \dots, |\cdot|_m$, $m \geq 2$.

- 1) Show that there exists $a \in K$ such that $|a|_1 > 1$ and $|a|_2 < 1$.

- 2) Prove by induction on m that there exists $a \in K$ such that $|a|_1 > 1$ and $|a|_i < 1$ for all $2 \leq i \leq m$. (Hint : assume that $|b|_1 > 1$ and $|b|_i < 1$ for all $2 \leq i \leq m-1$ and take c such that $|c|_1 > 1$ and $|c|_m < 1$. Consider the sequences $a_k = cb^k$ and $\frac{b^k}{b^k+1}c$.)

- 3) Prove that given any $\alpha_i \in K$ ($1 \leq i \leq m$) and $\varepsilon > 0$, there exists $\beta \in K$ such that simultaneously

$$|\beta - \alpha_i|_i < \varepsilon, \quad \text{for all } 1 \leq i \leq m.$$

- 4) Compare the above result with the Chinese Remainder Theorem.