University of Bordeaux **Number Theory**
M2 AGNT/ALGANT, 2026–2027

Exercise sheet n°5

Extension of absolute values

Exercise 1 – Let $(K, |\cdot|_K)$ be a complete non-Archimedean field and let L be a finite Galois extension of K. Let $|\cdot|_L$ denote the extension of $|\cdot|_K$ to L. Show that $|g(x)|_L = |x|_L$ for all $x \in L$ and $g \in \mathrm{Gal}(L/K)$.

Exercise 2 – Let K be a complete non-Archimedean field and let K^{sep} denote the separable closure of K. Show that $G_K = \mathrm{Gal}(K^{\mathrm{sep}}/K)$ acts continuously on K^{sep}.

Exercise 3 – Let K be a complete non-Archimedean field. We denote by $|\cdot|$ the extension of the absolute value on K to its algebraic closure $\overline{K}$. Let $f(X) \in K[X]$ be an irreducible polynomial. Show that for any two roots α and β of $f(X)$, one has $|\alpha| = |\beta|$.

Exercise 4 – Let K be field complete for a discrete valuation v_K. Let L be a finite extension of K. Show that there exists a unique extension of v_K to a discrete valuation on L.

Exercise 5 – **(Continuity of roots of polynomials)** Let $(K, |\cdot|)$ be a complete non-Archimedean field. Fix an algebraic closure $\overline{K}$ of K. The absolute value $|\cdot|$ extends uniquely to an absolute value on $\overline{K}$, which we denote again by $|\cdot|$. Let $P \in K[X]$ be a monic polynomial of degree n.

1) Show that $|\alpha| \leq |P|$, where $|P|$ is the Gauss norm of P.

2) Let $Q \in K[X]$ be another monic polynomial of degree n. Show that for any root α of P one has

$$|Q(\alpha)| \leq |P - Q| \cdot |P|^n.$$

3) Show that for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any monic $Q \in K[X]$ of degree n and such that $|P - Q| < \delta$ and any root α of P there exists a root β of Q such that $|\alpha - \beta| < \varepsilon$.

Exercise 6 – **(Krasner's lemma)** We denote by $K^{\mathrm{sep}} \subset \overline{K}$ the separable closure of K.

1) Let $\alpha \in K^{\mathrm{sep}}$ and let $\alpha_1 := \alpha, \alpha_2, \cdots, \alpha_n$ denote the conjugates of α over K. Let $\beta \in \overline{K}$ be such that $|\alpha - \beta| < |\alpha_1 - \alpha_i|$ for all $2 \leq i \leq n$. Prove that $K[\alpha] \subset K[\beta]$.

2) Let p be a prime number and let ζ_p denote a primitive pth root of unity. Show that $\mathbf{Q}_p[\zeta_p] = \mathbf{Q}_p[\sqrt[p-1]{-p}]$.

Exercise 7 – **(Completion of the algebraic closure)** Let $(K, |\cdot|)$ be a complete non-Archimedean field and let $\mathbb{C}_K$ denote the completion of $\overline{K}$.

1) Show that $\mathbb{C}_K$ is algebraically closed.

2) Show that $\overline{\mathbb{Q}}_p$ is not complete.

Exercise 8 – Let $\mathbb{C}$ denote the completion of an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$. Let $v : \mathbb{C} \to \mathbb{R} \cup \{+\infty\}$ denote the extension to $\mathbb{C}$ of the standard p-adic valuation on $\mathbb{Q}_p$; hence $v(p) = 1$. We write $O_{\mathbb{C}}$ for its ring of integers and $\mathfrak{m}_{\mathbb{C}}$ for the maximal ideal of $O_{\mathbb{C}}$. Consider the formal power series

$$\exp(X) = \sum_{n=0}^{\infty} \frac{X^n}{n!}, \qquad \log(1+X) = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{X^n}{n}.$$

We assume that the following formulas hold true in $\mathbb{Q}[[X, Y]]$:

$$\exp(X+Y) = \exp(X)\exp(Y), \quad \log((1+X)(1+Y)) = \log(1+X)+\log(1+Y).$$

(They can be proved by a direct computation involving binomial coefficients.)

1) Show that $\log(1+X)$ converges on $\mathfrak{m}_{\mathbb{C}}$.

2) Let $U^1_C = \{x \in O_{\mathbb{C}} \mid x \equiv 1 \pmod{\mathfrak{m}_{\mathbb{C}}}\}$. Show that the rule $x \mapsto \log(x)$ defines a morphism $\log : (U^1_{\mathbb{C}}, *) \to (\mathbb{C}, +)$.

3) Show that the series $\exp(X)$ converges on the set

$$\left\{ x \in O_{\mathbb{C}} \mid v(x) > \frac{1}{p-1} \right\}.$$

4) Show that the multiplicative group $(1 + p^2 O_{\mathbb{C}})^*$ is isomorphic to the additive group $p^2 O_{\mathbb{C}}$.

5) Let $\mu_{p^\infty} = \bigcup_n^{\infty} \mu_{p^n}$, where μ_{p^n} denotes the group of p^nth roots of unity. Show that $\ker(\log) = \mu_{p^\infty}$.

Exercise 9 – Let K be a complete discrete valued field. Fix an algebraic closure $\overline{K}/K$. Let k/k_K be a Galois extension.

1) Recall why there exists a unique subfield $L \subset \overline{K}$ such that L/K is unramified with residue field k.

2) Show that L/K is a Galois extension and $\mathrm{Gal}(L/K) \simeq \mathrm{Gal}(k/k_K)$.

Exercise 10 – Let K be a complete discrete valued field. Let $L, M \subset \overline{K}$ be finite extensions of K. Assume that L/K is unramified. Show that LM/M is unramified.

Exercise 11 – Let K be the splitting field of the polynomial $f(X) = X^p - p \in \mathbb{Q}_p[X]$ (*i.e.* K is generated over $\mathbb{Q}_p$ by the roots of $f(X)$). Set $G = \mathrm{Gal}(K/\mathbb{Q}_p)$.

1) Show that $K = \mathbb{Q}_p(\alpha, \zeta_p)$ where α is a root of $f(X)$ and ζ_p is a primitive pth root of unity.

2) Show that $[K : \mathbb{Q}_p] = p(p-1)$ and that $H = \mathrm{Gal}(K/\mathbb{Q}_p(\zeta_p))$ is a normal subgroup of $G = \mathrm{Gal}(K/\mathbb{Q}_p)$ of index $(p-1)$.

3) Show that $K/\mathbb{Q}_p$ is a totally ramified extension and give an uniformizer of K.

Exercise 12 – Let $K = \mathbf{F}_p((t))$, thus K is a local field of characteristic p. Set $f(X) = X^p - X - \dfrac{1}{t} \in K[X]$.

1) Show that $f(X)$ has no roots in K.

2) Let $L = K(\alpha)$, where α is a root of $f(X)$. Express the roots of $f(X)$ in terms of α. Show that L is a splitting field of $f(X)$ *i.e.* that $f(X)$ decomposes over L into linear factors.

3) Show that L/K is a Galois extension and that the map

$$\begin{cases} \varphi : \mathrm{Gal}(L/K) \to \mathbf{F}_p, \\ \varphi(g) = g(\alpha) - \alpha \end{cases}$$

is an injective homomorphism. Deduce that $[L : K] = p$.

4) Show that L/K is totally ramified and give an uniformizer of L.