UNIVERSITÉ DE BORDEAUX
M2, p-adic Hodge Theory
2024-2025

Solutions to the final exam

Exercise 1.
Part I. Let A be a commutative unitary domain (a commutative ring with unity in which the product of any two nonzero elements is nonzero). Let $\mathcal{G}$ denote the set of formal power series of the form $f(X) = X + \sum_{k=2}^{\infty} a_k X^k$, $a_k \in A$. We equip $\mathcal{G}$ with the binary operation $\circ$ setting:

$$f \circ g(X) = f(g(X)) = g(X) + \sum_{k=2}^{\infty} a_k g(X)^k, \text{ for all } f(X) = X + \sum_{k=2}^{\infty} a_k X^k, \ g(X) = X + \sum_{k=2}^{\infty} b_k X^k.$$

(This operation is often called composition or substitution). It is easy to see (and you can admit) that $(\mathcal{G}, \circ)$ is a group with the identity element X. For any power series $u(X)$ and $v(X)$ we will write $u(X) \equiv v(X) \pmod{X^n}$ if the coefficients of $u(X)$ and $v(X)$ coincide in all degrees $\leqslant n-1$.

1) Let $u(X), v(X) \in \mathcal{G}$. Show that if $u(X) \equiv v(X) \pmod{X^n}$, $(n \geqslant 1)$, then $u(X) = v \circ w(X)$, where $w(X) \equiv X \pmod{X^n}$.

Solution. Since $(\mathcal{G}, \circ)$ is a group, for any $u(X), v(X) \in \mathcal{G}$, there exists a unique $w(X) \in \mathcal{G}$ such that $u(X) = v \circ w(X)$. We only need to show that $w(X) \equiv X \pmod{X^n}$ if $u(X) \equiv v(X) \pmod{X^n}$.

Let $u(X) = X + \sum_{k=2}^{\infty} a_k X^k$, $v(X) = X + \sum_{k=2}^{\infty} b_k X^k$, and $w(X) = X + \sum_{k=2}^{\infty} c_k X^k$. Note that $a_k = b_k$ for $2 \leqslant k \leqslant n-1$. Let m be the smallest integer $\geqslant 2$ such that $c_m \neq 0$, i.e. $w(X) = X + c_m X^m + \dots$. Then

$$v \circ w(X) = w(X) + \sum_{k=2}^{\infty} b_k w(X)^k = X + c_m X^m + \ldots + \sum_{k=2}^{\infty} b_k (X + c_m X^m + \ldots)^k =$$
$$X + b_2 X^2 + \ldots + b_{m-1} X^{m-1} + (b_m + c_m) X^m + \dots.$$

Since $v \circ w(X) = u(X)$, this implies that $m \geqslant n$, i.e. $w(X) \equiv X \pmod{X^n}$.

2) Let $f(X)$ and $g(X)$ be two elements of $\mathcal{G}$ such that $f(X) \equiv X + a_m X^m \pmod{X^{m+1}}$ and $g(X) \equiv X + b_n X^n \pmod{X^{n+1}}$ with some $a_m, b_n \in A$. Compute $f \circ g(X)$ and $g \circ f(X)$ modulo X^{n+m}. Show that for the commutator $[f,g] = f^{-1} \circ g^{-1} f \circ g$ one has

$$[f,g] \equiv X + (m-n) a_m b_n X^{m+n-1} \pmod{X^{m+n}}.$$

(Here f^{-1} and g^{-1} denote the inverse elements of f and g with respect to $\circ$.)

Solution. We have

$$f \circ g(X) = g(X) + a_m (X + b_n X^n + \ldots)^m + a_{m+1} (X + b_n X^n + \ldots)^{m+1}$$
$$\equiv X + \sum_{k=2}^{\infty} b_k X^k + \sum_{k=2}^{\infty} a_k X^k + m a_m b_n X^{m+n-1} \pmod{X^{m+n}}$$

and, analogously,

$$g \circ f(X) \equiv X + \sum_{k=2}^{\infty} b_k X^k + \sum_{k=2}^{\infty} a_k X^k + n a_m b_n X^{m+n-1} \pmod{X^{m+n}}.$$

Set $w(X) := [f,g] = X + \sum_{k=2}^{\infty} c_k X^k$. Then $f \circ g(X) = (g \circ f) \circ w(X)$. Since $f \circ g(X) \equiv g \circ f(X) \equiv \mod X^{n+m-1}$, from question 1) it follows that $w(X) \equiv X \mod X^{n+m-1}$, i.e. $c_k = 0$ for $2 \leqslant k \leqslant n+m-1$. Comparing $(n+m-1)$th coefficients of $f \circ g(X)$ and $(g \circ f) \circ w(X)$, we obtain that

$$ma_m b_n = na_m b_n + c_{n+m-1},$$

and therefore $c_{n+m-1} = (m-n)a_m b_n$. Hence $w(X) \equiv X + (m-n)a_m b_n X^{m+n-1} \pmod{X^{m+n}}$.

Part II. Let L/K be a finite Galois extension of local fields, and let $G = \mathrm{Gal}(L/K)$. We denote by $(G_i)_{i \geqslant -1}$ the ramification filtration on G. Let p denote the characteristic of the residue field k_K of K and π_L be a uniformizer of L.

3) Show that for all $\sigma \in G_i$ and $\tau \in G_j$ $(i, j \geqslant 1)$ one has $[\sigma, \tau] \in G_{i+j}$. Show that

$$[\sigma, \tau](\pi_L) \equiv \pi_L + (i-j)ab\pi_L^{i+j+1} \pmod{\pi_L^{i+j+2}},$$

where a and $b \in O_K$ are such that $\sigma(\pi_L) \equiv \pi_L + a\pi_L^{i+1} \pmod{\pi_L^{i+2}}$ and $\tau(\pi_L) \equiv \pi_L + b\pi_L^{j+1} \pmod{\pi_L^{j+2}}$.

Solution. We can write $\sigma(\pi_L) = f(\pi_L)$ and $\tau(\pi_L) = g(\pi_L)$ for some power series $f(X) = X + aX^{i+1} + \dots$ and $g(X) = X + bX^{j+1} + \dots$. Since σ and τ act trivially on the coefficients of $f(X)$ and $g(X)$, we have

$$\sigma \circ \tau(\pi_L) = \sigma(g(\pi_L)) = g(\sigma(\pi_L)) = g(f(\pi_L)) = (g \circ f)(\pi_L),$$

and

$$[\sigma, \tau](\pi_L) = (g \circ f \circ g^{-1} \circ f^{-1})(\pi_L) = [g^{-1}, f^{-1}](\pi_L) = [f, g]^{-1}(\pi_L).$$

Using the congruence proved in question 2), we obtain that

$$[\sigma, \tau](\pi_L) \equiv \pi_L + (j-i)ab\pi_L^{i+j+1} \pmod{\pi_L^{i+j+2}}.$$

(The sign in the formula should be corrected.)

4) Let c be the biggest integer such that $G_a \neq \{e\}$. Show that if $i \geqslant 1$ is a break of the ramification filtration in the lower numbering (i.e. if $G_i \neq G_{i+1}$), then $i \equiv a \pmod{p}$.

Solution. For any $\sigma \in G_i \setminus G_{i+1}$ and $\tau \in G_a$ we have $\sigma(\pi_L) \equiv \pi_L + u\pi_L^{i+1} \pmod{\pi_L^{i+2}}$ and $\tau(\pi_L) \equiv \pi_L + v\pi_L^{a+1} \pmod{\pi_L^{a+2}}$, where $u, v \not\equiv 0 \pmod{\pi_L}$. Then (see question 3))

$$[\sigma, \tau](\pi_L) \equiv \pi_L + (a-i)uv\pi_L^{i+a+1} \pmod{\pi_L^{i+a+2}}.$$

Therefore $[\sigma, \tau] \in G_{i+a} = \{e\}$. Since $e(\pi_L) = \pi_L$, the above congruence shows that $(a-i)uv$ can not be a unit. Since u and v are units, we conclude that $a - i \equiv 0 \pmod{p}$, i.e. $i \equiv a \pmod{p}$.

5) Show the following strengthening of the property proved in question 3): for all $\sigma \in G_i$ and $\tau \in G_j$ $(i, j \geqslant 1)$ one has $[\sigma, \tau] \in G_{i+j+1}$.

Solution. We can assume that $\sigma \in G_i \setminus G_{i+1}$ and $\tau \in G_j \setminus G_{j+1}$. We have

$$[\sigma, \tau](\pi_L) \equiv \pi_L + (j-i)uv\pi_L^{i+j+1} \pmod{\pi_L^{i+j+2}},$$

where u and v are units. By question 4), $i \equiv a \pmod{p}$ and $j \equiv a \pmod{p}$. Hence p divides $j-i$, and therefore $(j-i)ab\pi_L^{i+j+1} \equiv 0 \pmod{\pi_L^{i+j+2}}$. This implies that $[\sigma, \tau] \in G_{i+j+1}$.

Exercise 2. In this exercise, $\mathbf{F}_p := \mathbf{Z}/p\mathbf{Z}$ is a finite field with p elements and $\mathbf{F}_p((t))$ is the local field of Laurent power series with coefficients in $\mathbf{F}_p$. We denote by v_t the usual discrete valuation on $\mathbf{F}_p((t))$ given by $v_t\left(\sum\limits_{i\gg-\infty} a_it^i\right) = \min\{i \mid a_i \neq 0\}$. For each $n \geqslant 1$, the ring $\mathbf{F}_p((t^{1/p^n}))$ of polynomials in t^{1/p^n} is a purely inseparable extension of $\mathbf{F}_p((t))$ of degree p^n, and we set $\mathbf{F}_p((t^{1/p^\infty})) := \bigcup\limits_{n\geqslant 1}^{\infty} \mathbf{F}_p((t^{1/p^n}))$. The discrete valuation v_t extends to a (non-discrete) valuation on $\mathbf{F}_p((t^{1/p^\infty}))$ with values in $\mathbf{Q} \cup \{+\infty\}$, which we denote again by v_t, namely

$$v_t\left(\sum_{i\gg-\infty} a_it^{i/p^n}\right) = \frac{1}{p^n}\min\{i \mid a_i \neq 0\}.$$

1) Show that $\mathbf{F}_p((t^{1/p^\infty}))$ is not complete for the topology of the valuation v_t.

Solution. Consider the sequence $(S_n)_{n\geq 1}$ defined by $S_1 = t$, and $S_{n+1} = S_n + t^nt^{1/p^n}$ for $n \geqslant 1$. It is clear that $S_n \in \mathbf{F}_p((t^{1/p^n})) \subset \mathbf{F}_p((t^{1/p^\infty}))$ and that $(S_n)_{n\geqslant 1}$ is a Cauchy sequence. Assume that it converges to some $S \in \mathbf{F}_p((t^{1/p^\infty}))$. Then $S \in \mathbf{F}_p((t^{1/p^N}))$ for some $N \geqslant 1$, namely

$$S = \sum_{i\gg-\infty} a_it^{i/p^N}.$$

The convergence implies that for any $M \geqslant 1$ there exists n_0 such that for all $n \geqslant n_0$ the series $S_n \equiv S \pmod{t^M\mathbf{F}_p[[t^{1/p^\infty}]]}$. This leads to a contradiction (take $M > N$ and compare these two series.)

2) Denote by $\mathcal{F} := \mathbf{F}_p\widehat{((t^{1/p^\infty}))}$ the completion of $\mathbf{F}_p((t^{1/p^\infty}))$. Let $O_{\mathcal{F}} := \mathbf{F}_p\widehat{[[t^{1/p^\infty}]]}$ be the ring of integers of $\mathcal{F}$ and (t) the principal ideal of $O_{\mathcal{F}}$ generated by t. For any $a \geqslant 1$, consider the projective limit

$$\varprojlim_{\varphi} \mathbf{F}_p[[t^{1/p^\infty}]]/(t)^a = \varprojlim \left(\mathbf{F}_p[[t^{1/p^\infty}]]/(t)^a \xleftarrow{\varphi} \mathbf{F}_p[[t^{1/p^\infty}]]/(t)^a \xleftarrow{\varphi} \ldots\right)$$

where $\varphi(x) = x^p$ is the Frobenius map. Construct an isomorphism of rings

$$O_{\mathcal{F}} \simeq \varprojlim_{\varphi} \mathbf{F}_p[[t^{1/p^\infty}]]/(t)^a.$$

Solution. We construct a morphism $f : \varprojlim\limits_{\varphi} \mathbf{F}_p[[t^{1/p^\infty}]]/(t)^a \to O_{\mathcal{F}}$. Let $\alpha = (\alpha_n)_{n\geqslant 0} \in \varprojlim\limits_{\varphi} \mathbf{F}_p[[t^{1/p^\infty}]]/(t)^a$. Take any lifts $\widehat{\alpha}_n \in \mathbf{F}_p[[t^{1/p^\infty}]]$ and set

$$f(\alpha) = \lim_{n\to+\infty} \widehat{\alpha}_n^{p^n}.$$

Prove that this limit exists. One has:

$$\varphi(\widehat{\alpha}_{n+1}) = \widehat{\alpha}_{n+1}^p \equiv \widehat{\alpha}_n \pmod{t^a}, \qquad \forall n \geqslant 0.$$

Raising the both sides to p^nth powers and taking into account that the map $x \mapsto x^p$ is a morphism in characteristic p, we obtain that $\widehat{\alpha}_{n+1}^{p^{n+1}} \equiv \widehat{\alpha}_n^{p^n} \pmod{t^{ap^n}}$. This proves that $(\widehat{\alpha}_n^{p^n})_{n\geqslant 0}$ is a Cauchy sequence, and therefore it converges to some element of $O_{\mathcal{F}}$. If $\widetilde{\alpha}_n \in \mathbf{F}_p[[t^{1/p^\infty}]]$ is another system of lifts, then the congruence $\widetilde{\alpha}_n \equiv \widehat{\alpha}_n \pmod{t^a}$ implies that $\widetilde{\alpha}_n^{p^n} \equiv \widehat{\alpha}_n^{p^n} \pmod{t^{ap^n}}$ and $\lim_{n\to+\infty} \widetilde{\alpha}_n^{p^n} = \lim_{n\to+\infty} \widehat{\alpha}_n^{p^n}$. Therefore $f(\alpha)$ doesn't depend on the choice of the lifts.

We construct the inverse map $g : O_{\mathcal{F}} \to \varprojlim_{\varphi} \mathbf{F}_p[[t^{1/p^\infty}]]/(t)^a$. For any $\beta \in O_{\mathcal{F}}$, set $g(\beta) = (\beta_n)_{n\geqslant 0}$, where

$$\beta_n = \varphi^{-n}(\beta) \mod t^a O_{\mathcal{F}} \in O_{\mathcal{F}}/t^a O_{\mathcal{F}} = \mathbf{F}_p[[t^{1/p^\infty}]]/(t)^a.$$

From this definition, it follows easily that $(\beta_n)_{n\geqslant 0} \in \varprojlim \mathbf{F}_p[[t^{1/p^\infty}]]/(t)^a \to O_{\mathcal{F}}$. Hence the map g is well defined. From the definition, it's easy to see that g is a morphism of rings (why?). In addition, $f \circ g$ and $g \circ f$ are the identity maps (please check). To sum up, g is an isomorphism.

In the remainder of this exercise, $K := \bigcup_{n\geqslant 1} \mathbf{Q}_p[\zeta_{p^n}]$, where $(\zeta_{p^n})_{n\geqslant 1}$ is a system of primitive p^nth roots of unity ($\zeta_{p^{n+1}}^p = \zeta_{p^n}$), and O_K denotes the ring of integers of K.

3) Show that O_K/pO_K is isomorphic to the quotient of the ring $\mathbf{F}_p[X_1, X_2, \ldots, X_n, \ldots]$ by the ideal I generated by $X_1^{p-1} + X_1^{p-2} + \cdots + X_1 + 1$ and the polynomials $X_{n+1}^p - X_n$, for all $n \geqslant 1$:

$$I := \left\langle X_1^{p-1} + X_1^{p-2} + \cdots + X_1 + 1, X_2^p - X_1, X_3^p - X_2, \ldots \right\rangle.$$

Deduce *from this description* that the p-adic completion $\widehat{K}$ of K is a perfectoid field.

Solution. a) Since ζ_p is a root of the irreducible polynomial $X_1^{p-1} + X_1^{p-2} + \cdots + X_1 + 1$, we have $\mathbf{Z}_p[\zeta_p] \simeq \mathbf{Z}_p[X_1]/(X_1^{p-1} + X_1^{p-2} + \cdots + X_1 + 1)$. Moreover, since $\zeta_{p^n}^p = \zeta_{p^{n-1}}$, we can write $\mathbf{Z}_p[\zeta_{p^n}] = \mathbf{Z}_p[\zeta_{p^{n-1}}][X_n]/(X_n^p - \zeta_{p^{n-1}})$. Hence, by induction,

$$\mathbf{Z}_p[\zeta_{p^n}] = \mathbf{Z}_p[X_1, X_2, \ldots, X_n]/(X_1^{p-1} + X_1^{p-2} + \cdots + X_1 + 1, X_2^p - X_1, \ldots X_n^p - X_{n-1}).$$

Passing to the direct limit, we obtain that

$$\begin{aligned} O_K &= \bigcup_{n\geqslant 1} \mathbf{Z}_p[\zeta_{p^n}] \\ &= \mathbf{Z}_p[X_1, X_2, \ldots, X_n, \ldots]/(X_1^{p-1} + X_1^{p-2} + \cdots + X_1 + 1, X_2^p - X_1, \ldots X_n^p - X_{n-1}, \ldots). \end{aligned}$$

Therefore

$$O_K/pO_K = \mathbf{F}_p[X_1, X_2, \ldots, X_n, \ldots]/(X_1^{p-1} + X_1^{p-2} + \cdots + X_1 + 1, X_2^p - X_1, \ldots X_n^p - X_{n-1}, \ldots).$$

b) We need to prove that the Frobenius map $\varphi : O_K/pO_K \to O_K/pO_K$ is surjective (other properties of a perfectoid field hold by trivial reasons). For any polynomial $f(X_1, X_2, ...)$, let $\overline{f}(X_1, X_2, ...)$, denote the class of f modulo I. Then $\varphi(\overline{X}_{n+1}) = \overline{X}_n$. Since φ acts trivially on $\mathbf{F}_p$, we have

$$\varphi\left(\overline{f}(X_2, X_3, ...)\right) = \overline{f}(X_1, X_2, ...).$$

Hence φ is surjective.

4) Show that the ring of integers $O_{\widehat{K}}^\flat$ of the tilt of $\widehat{K}$ is isomorphic to $O_{\mathcal{F}}$. (Hint: use a change of variables and question 2)).

Solution. Set $Y_n = X_n - 1$. Then

$$X_1^{p-1} + X_1^{p-2} + \cdots + X_1 + 1 = \frac{X_1^p - 1}{X_1 - 1} = \frac{(Y_1 + 1)^p - 1}{Y_1} \equiv Y_1^{p-1} \pmod p.$$

In addition, $Y_n^p \equiv X_n^p - 1 = X_{n-1} - 1 = Y_{n-1} \pmod p$. Hence

$$O_K/pO_K = \mathbf{F}_p[Y_1, Y_2, \ldots, Y_n, \ldots]/(Y_1^{p-1}, Y_2^p - Y_1, \ldots Y_n^p - Y_{n-1}, \ldots).$$

Setting $t = Y_1$, we see that $Y_n = t^{1/p^{n-1}}$ and

$$O_K/pO_K = \mathbf{F}_p[t^{1/p^\infty}]/(t^{p-1}) = \mathbf{F}_p[[t^{1/p^\infty}]]/(t^{p-1}).$$

Using question 2), one has:

$$O^\flat_{\widehat{K}}=\varprojlim_{\varphi} O_K/pO_K=\varprojlim_{\varphi}\mathbf{F}_p[[t^{1/p^\infty}]]/(t^{p-1})\simeq O_{\mathcal{F}}.$$

5*) Let m be a integer which is coprime with p, and let $\mathcal{E}$ denote the completion of the field $\underset{n\geqslant 1}{\cup}\mathbf{F}_p((t^{m/p^n}))$. Then $\mathcal{E}$ is a perfectoid subfield of $\mathcal{F}$. Show that if, in addition, m is coprime with $p-1$, then $\mathcal{E}$ is not the tilt of a perfectoid subfield of $\widehat{K}$ under the isomorphism $O^\flat_{\widehat{K}}\simeq O_{\mathcal{F}}$ of question 4).

Solution. Proof by contradiction. Assume that $\mathcal{E}=\widehat{E}^\flat$ for some subfield $\mathbf{Q}_p\subset E\subset K$. Then, by the tilting correspondence, $[K:E]=[\mathcal{F}:\mathcal{E}]=m$. Since K is the union of the extensions $\mathbf{Q}_p[\zeta_{p^n}]$ of degree $(p-1)p^{n-1}$ and m is coprime with p, the extension $K/\mathbf{Q}_p$ doesn't have subextensions E such that $[K:E]=m$ (why?). This gives the contradiction.

Exercise 3. Let K be a local field of characteristic 0 with residue field of characteristic p. Let $\mathbf{C}$ denote the completion of the algebraic closure $\overline{K}$ of K, and $\mathfrak{m}_{\mathbf{C}^\flat}$ the maximal ideal of $O^\flat_{\mathbf{C}}$. For any $x\in O^\flat_{\mathbf{C}}$, we denote by $[x]$ the Teichmüller lift of x in the ring $\mathbf{A}_{\mathrm{inf}}:=W(O^\flat_{\mathbf{C}})$. The goal of this exercise is to prove that for any $u\in 1+\mathfrak{m}_{\mathbf{C}^\flat}$, the series

$$\log([u]):=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{([u]-1)^n}{n}$$

converges in the p-adic topology of $\mathbf{B}^+_{\mathrm{cris}}:=\mathbf{A}_{\mathrm{cris}}\otimes_{\mathbf{Z}_p}\mathbf{Q}_p$.

1) Let $u\in 1+\mathfrak{m}_{\mathbf{C}^\flat}$. Show that there exists $n_0\geqslant 1$ such that $\theta(([u]-1)^{n_0})\in pO_{\mathbf{C}}$ (Here θ denotes the usual morphism $\theta:\mathbf{A}_{\mathrm{inf}}\to O_{\mathbf{C}}$ of p-adic Hodge theory.) Deduce that $([u]-1)^{n_0}$ can be written in the form $([u]-1)^{n_0}=px+\xi y$, where ξ is a generator of $\ker(\theta)$ and $x,y\in\mathbf{A}_{\mathrm{inf}}$.

Solution. One has $[u]-1=[a_0]+p[a_1]+\dots$, where $a_0=u-1\in\mathfrak{m}_{\mathbf{C}^\flat}$. Therefore $a_0^{(0)}\in\mathfrak{m}_{\mathbf{C}}$ and

$$\theta([u]-1)=\theta([a_0])+p\theta([a_1])+\ldots=a_0^{(0)}+p\theta([a_1])+\ldots.$$

Hence $c:=\theta([u]-1)\in\mathfrak{m}_{\mathbf{C}}$, i.e. $v_p(c)>0$. Let n_0 be such that $v_p(c^{n_0})=n_0v_p(c)>v_p(p)=1$. Then $\theta(([u]-1)^{n_0})\in pO_{\mathbf{C}}$. Write $\theta(([u]-1)^{n_0})=p\alpha$, where $\alpha\in O_{\mathbf{C}}$. Since the map θ is surjective, there exists $x\in\mathbf{A}_{\mathrm{inf}}$ such that $\theta(x)=\alpha$. Then $\theta(([u]-1)^{n_0}-px)=0$. Since $\ker(\theta)$ is principal, $([u]-1)^{n_0}-px=\xi y$ for some $y\in\mathbf{A}_{\mathrm{inf}}$.

2) Show that $\frac{([u]-1)^{n_0m}}{m!}\in\mathbf{A}_{\mathrm{cris}}$ for all $m\geqslant 1$.

Solution. We have $([u]-1)^{n_0m}=(px+\xi y)^m=\sum\limits_{k=0}^{m}\binom{m}{k}p^{m-k}x^{m-k}y^k\xi^k$. Hence

$$\frac{([u]-1)^{n_0m}}{m!}=\sum_{k=0}^{m}\frac{p^{m-k}x^{m-k}}{(m-k)!}\cdot\frac{y^k\xi^k}{k!}.$$

By the construction of $\mathbf{A}_{\mathrm{cris}}$, for each k one has $\frac{y^k\xi^k}{k!}\in\mathbf{A}_{\mathrm{cris}}$. Since $\frac{p^{m-k}}{(m-k)!}\in\mathbf{Z}_p$, each term of the above sum belongs to $\mathbf{A}_{\mathrm{cris}}$. Therefore $\frac{([u]-1)^{n_0m}}{m!}\in\mathbf{A}_{\mathrm{cris}}$.

3) Show that for any M there exists N such that $\frac{([u]-1)^n}{n}\in p^M\mathbf{A}_{\mathrm{cris}}$ for all $n\geqslant N$. (Hint: write n in the form $n=n_0m+r$.)

Solution. Let $n = n_0 m + r$ with $0 \leqslant r \leqslant n_0 - 1$. Then

$$\frac{([u]-1)^n}{n} = ([u]-1)^r \cdot \frac{m!}{n} \cdot \frac{([u]-1)^{n_0 m}}{m!}.$$

Since $\frac{([u]-1)^{n_0 m}}{m!} \in \mathbf{A}_{\mathrm{cris}}$, we only need to show that $v_p(m!/n) \to +\infty$ when $n \to +\infty$. Writing n in the form $n = p^k n'$ with $(n', p) = 1$, we see that $v_p(n) \leqslant \log_p(n)$ (base p logarithm). On the other hand,

$$v_p(m!) \geqslant \left[\frac{m}{p}\right] \geqslant \frac{m}{p} - 1 \geqslant \frac{n/n_0 - 1}{p} - 1 = \frac{n - n_0}{n_0 p} - 1.$$

Hence $v_p(m!) - v_p(n) \geqslant \frac{n-n_0}{n_0 p} - \log_p(n) - 1 \to +\infty$ when $n \to +\infty$, and we are done.

4) Conclude.

Solution. From question 3) it follows that

$$\log([u]) := \sum_{n=1}^{\infty} (-1)^{n-1} \frac{([u]-1)^n}{n}$$

converges in the p-adic topology of $\mathbf{A}_{\mathrm{cris}}$.