University of Bordeaux **Number Theory**
M2 AGNT/ALGANT, 2025–2026

Exercise sheet n°1

Dedekind rings, discrete valuations

Exercice 1 – Let A be an integral domain and $\alpha \in A \setminus \{0\}$. Assume that $A[\alpha^{-1}]$ is integrally closed and that $A/\alpha A$ is reduced. Show that A is integrally closed.

Exercice 2 – Let A be a Dedekind ring. Show that for each nonzero prime ideal $\mathfrak{p}$ of A, one has $A/\mathfrak{p}^n \simeq A_{\mathfrak{p}}/\mathfrak{p}^n A_{\mathfrak{p}}$.

Exercice 3 – Show that a semilocal Dedekind domain (Dedekind domain with a finite number of prime ideals) is principal.

Exercice 4 – Let $\mathfrak{a}$ be a nonzero ideal of a Dedekind ring A. Show that $A/\mathfrak{a}$ is a principal ideal ring.

Exercice 5 – Every ideal of a Dedekind ring can be generated by two elements.

Exercice 6 – Decide which of the following rings are discrete valuation rings :

$$\mathbf{Z}; \quad k[[X]] \text{ where } k \text{ is a field}; \quad k[[X^2, X^3]].$$

Exercice 7 – Let A be a local noetherian ring such that its maximal ideal $\mathfrak{m}$ is generated by a non-nilpotent element π. The goal of this exercise is to prove that A is a principal ideal domain.

1) Verify that $I = \{x \in A \mid \exists m \in \mathbf{N} : \pi^m x = 0\}$ is an ideal of A and that there exists $N \in \mathbf{N}$ such that $\pi^N I = \{0\}$.

2) Show that $\bigcap_{n \geq 1} \mathfrak{m}^n = \{0\}$. Hint : if $y = \pi^n x_n$ for every $n \in \mathbf{N}$, then the chain of ideals $(I + x_n A)_{n \in \mathbf{N}}$ is ascending.

3) Prove that any element $y \in A \setminus \{0\}$ can be written in a unique way in the form $y = \pi^{v(y)} u$ with $v(y) \in \mathbf{N}$ and $u \in A^*$, and that A is an integral domain.