University of Bordeaux **Number Theory**
M2 AGNT/ALGANT, 2025–2026

Exercise sheet n°2

Absolute values, valuations

Exercice 1 – Let K be a field.

1) Prove that if K is finite, then the only absolute value on K is the trivial one.

2) Does the converse of the above statement hold ?

Exercice 2 –

1) Let $(K, |\cdot|)$ be a valued field. Show that $|\cdot|^s$ is an absolute value on K for all $s \in]0;1]$.

2) Find all $s > 0$ such that $|\cdot|_\infty^s$ is an absolute value on $\mathbf{Q}$.

Exercice 3 –

1) Let $(K, |\cdot|)$ be a valued field. Show that if $|\cdot|$ is ultrametric on a subfield of K, then it is ultrametric on K.

2) Prove that every absolute value on a field of positive characteristic is ultrametric.

Exercice 4 – Let $(K, |\cdot|)$ be a valued field. Show that $|\cdot|$ is ultrametric if and only if $B(0,1) \cap B(1,1) = \emptyset$.

Exercice 5 – Let $(K, |\cdot|)$ be an ultrametric field.

1) Show that for each $r \in]0,1]$ the open ball $B(1,r)$ is a subgroup of K^*.

2) Prove that each open ball in K is closed.

3) Deduce that K is totally disconnected (the only connected subsets are singletons.)

Exercice 6 – Let K be a field and v be a non-trivial valuation on K. Prove that the following properties are equivalent :
(a) O_v is a principal ideal domain ;
(b) O_v is noetherian ;
(c) the ideal $\mathfrak{m}_v$ is principal ;
(d) $v(K^*)$ is a discrete subgroup of $\mathbf{R}$.

Exercice 7 – **(Ostrowski theorem for rational functions.)** Let k be a finite field, and let $k(t)$ denote the field of fractions of the polynomial ring $k[t]$. For each irreducible polynomial $P \in k[t]$, we denote by v_P the discrete valuation on $k(t)$ attached to P, namely

$$v_P(f) = k, \text{if } f(t) = \frac{r(t)}{s(t)} P(t)^k, \text{ where } r(t) \text{ and } s(t) \text{ are coprime with } P(t).$$

1) For each $f(t) = a(t)/b(t)$ with $a(t), b(t) \in k[t]$, $b(t) \neq 0$, define :

$$v_\infty(f) = \deg(b) - \deg(a).$$

(Recall that $\deg(0) = -\infty$.) Check that $v_\infty(f)$ does not depend on the choice of $a(t)$ and $b(t)$ and that the above formula defines a discrete valuation on $k(t)$.

Let now $|\cdot|$ be a nontrivial absolute value on K.

2) Assume that $|t| > 1$ and set $c = 1/|t|$. Show that

$$|a(t)| = c^{-\deg(f)}, \qquad \forall a(t) \in k[t],$$

and deduce that $|f(t)| = c^{v_\infty(f)}$ for all $f \in k(t)$.

3) Assume that $|t| \leq 1$.

a) Show that there exists a nonzero polynomial $g \in k[t]$ such that $|g| < 1$.

b) Let $P \in k[t]$ denote a unitary polynomial of smallest degree such that $|P| \leq 1$. Prove that P is irreducible.

c) Show that for any nonzero polynomial $h \in k[t]$, coprime with P, one has $|h| = 1$.

d) Show that for any $f(t) \in k[t]$, one has $|f(t)| = c^{v_P(f(t))}$, where $c = |P|$. Deduce that the above formula holds for all $f(t) \in k(t)$.

Exercice 8 – (Weak approximation theorem of Artin-Whaples.) Let K be a field equipped with pairwise inequivalent nontrivial absolute values $|\cdot|_1, \ldots, |\cdot|_m$, $m \geq 2$.

1) Show that there exists $a \in K$ such that $|a|_1 > 1$ and $|a|_2 < 1$.

2) Prove by induction on m that there exists $a \in K$ such that $|a|_1 > 1$ and $|a|_i < 1$ for all $2 \leq i \leq m$. (Hint : assume that $|b|_1 > 1$ and $|b|_i < 1$ for all $2 \leq i \leq m-1$ and take c such that $|c|_1 > 1$ and $|c|_n < 1$. Consider the sequences $a_k = cb^k$ and $\frac{b^k}{b^k+1}c$.)

3) Prove that given any $\alpha_i \in K$ $(1 \leq i \leq m)$ and $\varepsilon > 0$, there exists $\beta \in K$ such that simultaneously

$$|\beta - \alpha_i|_i < \varepsilon, \qquad \text{for all } 1 \leq i \leq m.$$

4) Compare the above result with the Chinese Remainder Theorem.