University of Bordeaux **Number Theory**
M2 AGNT/ALGANT, 2025–2026

Exercise sheet n°3

Complete fields

Exercice 1 – Let $(K, |\cdot|)$ be an ultrametric field. Show that $|\widehat{K}^*| = |K^*|$ and that the residue fields of K and $\widehat{K}$ are isomorphic.

Exercice 2 – Let K be a field. We equip K with the trivial absolute value and consider $K[[X]]$ as a K-vector space.

1) Let $\rho \in]0,1[$. Set $\|f\|_\rho = \rho^{v_X(f)}$ for all $f \in K[[X]]$. Recall why $\|\cdot\|_\rho$ is a norm on $K[[X]]$.

2) Let $r, s \in \mathbb{R}$ be such that $0 < r < s < 1$. Show that $\|\cdot\|_r$ and $\|\cdot\|_s$ are equivalent but there doesn't exist $c > 0$ such that $\|\cdot\|_s \leq c\|\cdot\|_r$.

Exercice 3 – Consider $\mathbf{Q}[\sqrt{2}]$ as a $\mathbf{Q}$-vector space equipped with the norm $\|\cdot\|$ given by the usual absolute value. Set $\|x + y\sqrt{2}\|' = \max\{|x|_\infty, |y|_\infty\}$ for all $x, y \in \mathbf{Q}$. Show that the norms $\|\cdot\|$ and $\|\cdot\|'$ are not equivalent.

Exercice 4 –

1) Let L/K be a finite separable field extension of degree n. Show that for any $x \in L$

$$N_{L/K}(x) = \prod_{i=1}^{n} \sigma_i(x),$$

where $\sigma_1, \ldots, \sigma_n$ are the K-embeddings of L in the algebraic closure $\overline{K}$ of K. (Hint : first consider the case $L = K[x]$.)

2) Let $K \subset L \subset M$ be a tower of finite separable extensions. Show that

$$N_{M/K}(x) = N_{L/K}(N_{M/L}(x)), \qquad \forall x \in M.$$

(This formula holds without the assumption of the separability.)

Exercice 5 – Let $(K, |\cdot|)$ be a complete ultrametric valued field and let $f(X) = \sum_{k=0}^{n} a_k X^k$ be an irreducible monic polynomial. Show that $|a_k|^n \leq |a_0|^{n-k}$ for all $0 \leq k \leq n$.

Exercice 6 – Let $(K, |\cdot|)$ be a complete ultrametric valued field and let $f(X)$ be a monic polynomial. Prove that if $f(X)$ is irreducible in $O_K[X]$, its image in $k_K[X]$ is the power of an irreducible polynomial.

Exercice 7 – **(Another theorem of Ostrowski)**. Let $(K, |\cdot|)$ be a complete valued field such that $|2| = 2$. Assume that there exists $i \in K$ such that $i^2 = -1$.

1) Construct a morphism $(\mathbf{C}, |\cdot|_\infty) \to (K, |\cdot|)$ of valued fields. We identify $\mathbf{C}$ with its image in K.

2) Fix $a \in K$ and consider the map $f : \mathbf{C} \to \mathbf{R}_+$ defined by $f(z) = |z - a|$. Let $r = \inf_{z \in \mathbf{C}} f(z)$. Show that $f^{-1}(r)$ is closed, bounded and non-empty.

3) Prove that if $r > 0$ and $\gamma_0 \in f^{-1}(r)$, then $B(\gamma_0, r) \subset f^{-1}(r)$. (Hint : if $|\gamma - \gamma_0|_\infty < r$, consider $(\gamma_0 - a)^n - (\gamma_0 - \gamma)^n$.)

4) Conclude that $K = \mathbf{C}$.

Exercice 8 – Prove that a complete archimedean values field is isomorphic to $(\mathbf{R}, |\cdot|_\infty^s)$ or $(\mathbf{C}, |\cdot|_\infty^s)$ for some $s \in]0, 1]$.