

Exercise sheet n°4

Extension of valuations. Complete fields II.

Exercise 1 – Let $(K, |\cdot|)$ be a complete ultrametric valued field, and L a finite extension of K . Let $|\cdot|_L$ denote the unique extension of $|\cdot|_K$ to L . Prove that $|x|_L = \lim_{n \rightarrow +\infty} \|x^n\|^{1/n}$, where $\|\cdot\|$ is any norm on the K -vector space L .

Exercise 2 – Let p be a prime number such that $p \equiv 3 \pmod{4}$. Let $\overline{\mathbf{Q}}_p$ be an algebraic closure of $\mathbf{Q}_p$.

- 1) Show that $X^2 + 1$ is irreducible in $\mathbf{Q}_p[X]$.
- 2) Let $i \in \overline{\mathbf{Q}}_p$ be a root of $X^2 + 1$, $K = \mathbf{Q}_p[i]$, and $|\cdot|_K$ be the extension of $|\cdot|_p$ to K . For $a, b \in \mathbf{Q}_p$ and $x = a + bi$, show that $|x|_K = \max\{|a|_p, |b|_p\}$.
- 3) Determine the value group and the residue field of $(K, |\cdot|_K)$.

Exercise 3 – Let K be a number field with r_1 real embeddings and $2r_2$ complex embeddings. Show that usual absolute value $|\cdot|_\infty$ on $\mathbf{Q}$ admits exactly r_1 extensions to K with completion isomorphic to $\mathbf{R}$ and r_2 extensions to K with completion isomorphic to $\mathbf{C}$.

Exercise 4 – (**Newton polygons**) Let K be a field complete for a non-trivial valuation v . Recall that v has a unique extension to the algebraic closure $\overline{K}$ of K , which we denote again by v . Let $f(X) = \sum_{k=0}^n a_k X^k \in K[X]$ be a polynomial. Assume that $a_0, a_n \neq 0$. To each nonzero term $a_k X^k$ we attach the point $(k, v(a_k)) \in \mathbf{R}^2$, and denote by D the set of such points. The Newton polygon of f is defined as the lower convex envelope of D . It consists of line segments joining two points of D . Assume that the segment S joining the points $(j, v(a_j))$ and $(i, v(a_i))$ with $j > i$ appears in the Newton polygon, and call the slope of S the number

$$s = \frac{v(a_j) - v(a_i)}{j - i}.$$

Show that f has exactly $j - i$ roots of value $-s$.

Exercise 5 – (**Krasner's lemma**) Let $(K, |\cdot|)$ be a complete ultrametric valued field. Fix an algebraic closure $\overline{K}$ of K . We denote by $K^{\text{sep}} \subset \overline{K}$ the separable closure of K . The absolute value $|\cdot|$ extends uniquely to an absolute value on $\overline{K}$, which we denote again by $|\cdot|$.

- 1) Let $\alpha \in K^{\text{sep}}$ and let $\alpha_1 := \alpha, \alpha_2, \dots, \alpha_n$ denote the conjugates of α over K . Let $\beta \in \overline{K}$ be such that $|\alpha - \beta| < |\alpha_1 - \alpha_i|$ for all $2 \leq i \leq n$. Prove that $K[\alpha] \subset K[\beta]$.

- 2) Let $P, Q \in K[X]$ be two monic polynomials of the same degree. Assume that P is separable and irreducible. Show that if $|P - Q|_G$ is small enough (here $|\cdot|_G$ denotes the Gauss norm), then Q is also separable and irreducible, and there exists a root $\beta \in \overline{K}$ such that $K[\alpha] \subset K[\beta]$.
- 3) Let $\mathbf{C}$ denote the completion of $\overline{K}$ with respect to $|\cdot|$. Using Krasner's lemma, show that $\mathbf{C}$ is algebraically closed and K^{sep} is dense in $\mathbf{C}$.
- 4) Let p be a prime number and let ζ_p denote a primitive p th root of unity. Show that $\mathbf{Q}_p[\zeta_p] = \mathbf{Q}_p[\sqrt[p-1]{-p}]$.