University of Bordeaux **Number Theory**
M2 AGNT/ALGANT, 2025–2026

Exercise sheet n°6
Ramification groups

Exercice 1 – Let K be the splitting field of the polynomial $f(X) = X^p - p \in \mathbb{Q}_p[X]$ (*i.e.* K is generated over $\mathbb{Q}_p$ by the roots of $f(X)$). Set $G = \mathrm{Gal}(K/\mathbb{Q}_p)$.

1) Show that $K = \mathbb{Q}_p(\alpha, \zeta_p)$ where α is a root of $f(X)$ and ζ_p is a primitive pth root of unity.

2) Show that $[K : \mathbb{Q}_p] = p(p-1)$ and that $H = \mathrm{Gal}(K/\mathbb{Q}_p(\zeta_p))$ is a normal subgroup of $G = \mathrm{Gal}(K/\mathbb{Q}_p)$ of index $(p-1)$.

3) Show that $K/\mathbb{Q}_p$ is a totally ramified extension and give an uniformizer of K.

4) Describe the ramification subgroups G_i of G.

Exercice 2 – Let $K = \mathbf{F}_p((t))$, thus K is a local field of characteristic p. Set $f(X) = X^p - X - \dfrac{1}{t} \in K[X]$.

1) Show that $f(X)$ has no roots in K.

2) Let $L = K(\alpha)$, where α is a root of $f(X)$. Express the roots of $f(X)$ in terms of α. Show that L is a splitting field of $f(X)$ *i.e.* that $f(X)$ decomposes over L into linear factors.

3) Show that L/K is a Galois extension and that the map

$$\begin{cases} \varphi : \mathrm{Gal}(L/K) \to \mathbf{F}_p, \\ \varphi(g) = g(\alpha) - \alpha \end{cases}$$

is an injective homomorphism. Deduce that $[L : K] = p$.

4) Show that L/K is totally ramified and give an uniformizer of L.

5) Describe the ramification subgroups of $G = \mathrm{Gal}(L/K)$.

Exercice 3 – Let L/K be a Galois extension of local fields and let $G = \mathrm{Gal}(L/K)$. Let v_L denote the discrete valuation on L such that $v_L(L^*) = \mathbf{Z}$.

1) Recall why there exists $\alpha \in L$ such that $O_L = O_K[\alpha]$.

2) For each $\sigma \in G$, set $i_G(\sigma) = v_L(\sigma(\alpha) - \alpha)$. (In particular, $i_G(e) = +\infty$.) Show that $G_i = \{\sigma \in G \mid i_G(\sigma) \geq i + 1\}$ for all $i \geq -1$.

3) Let $f(X) \in O_K[X]$ denote the minimal polynomial of α. Show that

$$v_L(f'(\alpha)) = \sum_{\sigma \neq e} i_G(\sigma).$$

4) Let $\mathcal{D}_{L/K}$ denote the different ideal of L/K and let $v_L(\mathcal{D}_{L/K}) = \min\{v_L(x) \mid x \in \mathcal{D}_{L/K}\}$. Show that

$$v_L(\mathcal{D}_{L/K}) = \sum_{i=0}^{\infty} (|G_i| - 1).$$

5) Let e denote the ramification index of L/K. Show that $v_L(\mathcal{D}_{L/K}) = e - 1$ if and only in L/K is tamely ramified.