UNIVERSITÉ DE BORDEAUX
M2, p-adic Hodge Theory
2020-2021

Exam May 14th 2021 (Solutions)

Duration: 3 hours

Hard copies of the lecture notes are allowed

You can use all results explicitly stated in the course

Exercise 1. Let K be a local field of characteristic 0 with residue field of characteristic p. Let ψ denote a *non-trivial additive character* of G_K, i.e. a continuous map $\psi : G_K \to \mathbf{Z}_p$ such that

$$\psi(g_1g_2) = \psi(g_1) + \psi(g_2), \qquad \forall g_1, g_2 \in G_K.$$

Since ψ is continuous, $\psi(G_K)$ is a non-trivial closed subgroup of $\mathbf{Z}_p$, hence $\psi(G_K) = p^m\mathbf{Z}_p$ for some $m \geqslant 0$. The kernel $\ker(\psi)$ is a closed subgroup of G_K and $G_K/\ker(\psi) \simeq \mathbf{Z}_p$. Set $K_\infty = \overline{K}^{\ker(\psi)}$. Then K_∞/K is a $\mathbf{Z}_p$-extension.

Assume that K_∞/K is totally ramified. Prove that there does not exist $c \in \mathbf{C}_K$ ($\mathbf{C}_K =$ completion of $\overline{K}$) such that

$$\psi(g) = g(c) - c, \qquad \forall g \in G_K.$$

Hint: use Tate's theorem (Theorem 4.3).

Solution. By Galois theory, $G_{K_\infty} = \ker(\psi)$. Assume that $\psi(g) = g(c) - c$ for all $g \in G_K$. Then $g(c) = c$ for all $g \in G_{K_\infty}$ and therefore $c \in \mathbf{C}_K^{G_{K_\infty}} = \widehat{K}_\infty$. Set $\Gamma = \mathrm{Gal}(K_\infty/K)$ and denote by γ a generator of Γ. We can consider ψ as a character of Γ. Since ψ is non-trivial, the element

$$\psi(\gamma) = \gamma(c) - c \in \mathbf{Z}_p$$

is non-zero. Using the decomposition $\widehat{K}_\infty = K \oplus \widehat{K}_\infty^\circ$, write $c = a + b$ with $a \in K$ and $b \in \widehat{K}_\infty^\circ$. Then

$$\gamma(c) - c = \gamma(a) - a + \gamma(b) - b = \gamma(b) - b.$$

On the other hand, $\gamma(b) - b \in \widehat{K}_\infty^\circ$ and therefore

$$\psi(\gamma) \in K \cap \widehat{K}_\infty^\circ = \{0\}.$$

This gives a contradiction.

Exercise 2. Let K be a local field of characteristic 0 with residue field of characteristic p. Let $\eta : G_K \to \mathbf{Z}_p^*$ be a continuous character. We denote by T_1 a free $\mathbf{Z}_p$-module of rank one equipped with the action of G_K given by

$$g(x) = \eta(g)x, \qquad \forall x \in T_1, g \in G_K.$$

(Note that $T_1 \simeq \mathbf{Z}_p(\eta)$ in the notation of the course). We denote by T_2 a free $\mathbf{Z}_p$-module of rank one equipped with the trivial action of G_K :

$$g(x) = x, \qquad \forall x \in T_2, g \in G_K.$$

Assume that T is a free $\mathbf{Z}_p$-module of rank 2 equipped with a continuous $\mathbf{Z}_p$-linear action of G_K and sitting in an exact sequence of Galois modules

$$0 \to T_1 \to T \to T_2 \to 0. \qquad (1)$$

(thus T_1 is a submodule of T and $T/T_1 \simeq T_2$).

1) Show that there exists a $\mathbf{Z}_p$-basis $\{e_1, e_2\}$ of T such that the action of G_K on this basis is given by

$$g(e_1) = \eta(g)e_1, \qquad g(e_2) = \psi(g)e_1 + e_2, \qquad \forall g \in G_K, \qquad (2)$$

where ψ is some continuous map $\psi : G_K \to \mathbf{Z}_p$. Check that the map ψ satisfies the following property:

$$\psi(g_1g_2) = \eta(g_1)\psi(g_2) + \psi(g_1), \qquad \forall g_1, g_2 \in G_K. \qquad (3)$$

Solution. Let v_1 be an arbitrary basis of T_1. Set $e_1 = v_1$. Then $g(e_1) = \eta(g)e_1$ for all $g \in G_K$. Let v_2 be any basis of V_2 and let e_2 be any element of T such that e_2 maps to v_2 under the map $f : T \to T_2$. It is easy to see that $\{e_1, e_2\}$ is a basis of T over $\mathbf{Z}_p$. Moreover $g(e_2)$ maps to $g(v_2) = v_2$ under f (the sequence (1) is a sequence of G_K-modules) and therefore $g(e_2) - e_2 \in \ker(f) = T_1$. Hence

$$g(e_2) - e_2 = \psi(g)e_1$$

for some $\psi(g) \in \mathbf{Z}_p$. Note that since G_K acts continuously, the map ψ is continuous. This proves the first assertion.

For $g_1, g_2 \in G_K$, one has:

$$g_1g_2(e_2) = g_1(e_2 + \psi(g_2)e_1) = g_1(e_2) + \psi(g_2)g_1(e_1) = e_2 + \psi(g_1)e_1 + \psi(g_2)\eta(g_1)e_1.$$

On the other hand,

$$g_1g_2(e_2) = e_2 + \psi(g_1g_2)e_1.$$

Hence $\psi(g_1g_2) = \psi(g_1) + \psi(g_2)\eta(g_1)$.

2) Conversely, assume that T is a free $\mathbf{Z}_p$-module of rank 2 with a basis $\{e_1, e_2\}$. Let $\psi : G_K \to \mathbf{Z}_p$ be a continuous map such that condition (3) holds. Show that formulas (2) define a continuous linear action of G_K on T and that there exist $T_1 \simeq \mathbf{Z}_p(\eta)$ and $T_2 \simeq \mathbf{Z}_p$ such that $T/T_1 \simeq T_2$.

Solution. Any element of T can be written in a unique way in the form $x = a_1e_1 + a_2e_2$ with $a_1, a_2 \in \mathbf{Z}_p$. Setting

$$g(x) := a_1g(e_1) + a_2g(e_2),$$

we associate to g a $\mathbf{Z}_p$-linear map $T \to T$. To show that it defines a structure on G_K-module on T we only need to check that

a) If $g = e_G$ (neutral element of G_K), then $e_G(x) = x$ for all $x \in T$. We remark that $\eta(e_G) = 1$ and therefore $e_G(e_1) = e_1$. Moreover

$$\psi(e_G) = \psi(e_Ge_G) = \psi(e_G) + \eta(e_G)\psi(e_G) = \psi(e_G) + \psi(e_G).$$

Therefore $\psi(e_G) = 0$ and $e_G(e_2) = e_2$. This implies a).

b) $(g_1g_2)(x) = g_1(g_2(x))$ for all $g_1, g_2 \in G_K$ and $x \in T$. It is sufficient to prove this property for $x = e_1$ and e_2. One has:

$$g_1(g_2(e_1)) = \eta(g_1)(g_2(e_1)) = \eta(g_1)\eta(g_2)e_1 = \eta(g_1g_2)e_1 = (g_1g_2)(e_1).$$

Using property (3), we have:

$$(1) \quad (g_1(g_2(e_2)) = g_1(e_2 + \psi(g_2)e_1) = g_1(e_2) + \psi(g_2)g_1(e_1) =$$
$$= e_2 + \psi(g_1)e_1 + \psi(g_2)\eta(g_1)e_1 = e_2 + \psi(g_1g_2)e_1 = (g_1g_2)(e_2).$$

Set $T_1 = \mathbf{Z}_pe_1$ and $T_2 = T/T_1$. Then $T_1 \simeq \mathbf{Z}_p(\eta)$, and T_2 is a trivial G_K-module of rank one over $\mathbf{Z}_p$.

Set $V = T \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$. Let B be a G_K-regular algebra (see Chapter 5, section 5 of the course). For any p-adic representation U set $\mathbf{D}_B(U) = (U \otimes_{\mathbf{Q}_p} B)^{G_K}$. Recall that the exact sequence (1) induces an exact sequence

$$0 \to \mathbf{D}_B(V_1) \to \mathbf{D}_B(V) \to \mathbf{D}_B(V_2).$$

(the third map is not necessarily surjective).

3) Show that V is B-admissible if and only if there exist an invertible $b \in B$ and $c \in B$ (not necessarily invertible) such that

$$\eta(g) = b/g(b), \qquad \forall g \in G_K$$

and

$$\psi(g) = c - \eta(g) \cdot g(c), \qquad \forall g \in G_K.$$

Hint: First show that V is admissible if and only if V_1 is admissible and the map $\mathbf{D}_B(V) \to \mathbf{D}_B(V_2)$ is surjective.

Solution. Since V_2 is a trivial Galois representation, one has:

$$\mathbf{D}_B(V_2) = (V_2 \otimes_{\mathbf{Q}_p} B)^{G_K} = B^{G_K} =: E.$$

Hence V_2 is B-admissible and $\dim_E \mathbf{D}_B(V_2) = 1$. We remark that

$$\dim_E \mathbf{D}_B(V) \leqslant \dim_E \mathbf{D}_B(V_1) + \dim_E \mathbf{D}_B(V_2) = \dim_E \mathbf{D}_B(V_1) + 1$$

and the equality holds if and only if the map $\mathbf{D}_B(V) \to \mathbf{D}_B(V_2)$ is surjective. Moreover $\dim_E \mathbf{D}_B(V_1) \leqslant 1$ and $\dim_E \mathbf{D}_B(V_1) = 1$ if and only if V_1 is B-admissible. Therefore V is B-admissible if and only if the following conditions hold:

a) V_1 is B-admissible;

b) The map $\mathbf{D}_B(V) \to \mathbf{D}_B(V_2)$ is surjective.

Each element of $V_1 \otimes_{\mathbf{Q}_p} B$ can be written in the form $e_1 \otimes b$ for some $b \in B$. Therefore V_1 is B-admissible if and only if there exists a non-zero b such that

$$g(v_1 \otimes b) = v_1 \otimes b, \qquad \forall g \in G_K.$$

This condition is equivalent to the condition

$$\eta(g)g(b) = b, \qquad \forall g \in G_K.$$

In particular, the line $\mathbf{Q}_p b$ is stable under the action of G_K, and therefore b is invertible. Hence we can write $\eta(g) = b/g(b)$.

The surjectivity of the map $f : \mathbf{D}_B(V) \to \mathbf{D}_B(V_2)$ means that there exists an element $x \in \mathbf{D}_B(V)$ which maps to v_2 under f. Therefore x can be written in the form $x = v_2 + c \otimes v_1$ for some $c \in B$. The condition $g(x) = x$ for all $g \in G_K$ reads:

$$g(x) = g(v_2) + g(c) \otimes g(v_1) = v_2 + \psi(g)v_1 + g(c) \otimes \eta(g)v_1 = v_2 + c \otimes v_1.$$

Therefore it is equivalent to:

$$\psi(g) + \eta(g)g(c) = c, \qquad \forall g \in G_K.$$

In the last question of this exercise, we assume that $\eta = \mathrm{id}$ is the trivial character (and therefore T_1 is trivial as G_K-module). From formula (3) it follows that ψ is an *additive character*, i.e. $\psi(g_1 g_2) = \psi(g_2) + \psi(g_1)$.

4) Set $K_\infty = \overline{K}^{\ker(\psi)}$. Assume that K_∞/K is totally ramified, and ψ is non-trivial. Using Exercise 1, prove that then V is not $\mathbf{C}_K$-admissible. Deduce that V is not Hodge–Tate.

Solution. If $\eta = \mathrm{id}$, V_1 is $\mathbf{C}_K$-admissible. By question 3), V is $\mathbf{C}_K$-admissible if and only if there exists $c \in \mathbf{C}_K$ such that $\psi(g) = c - g(c)$ for all $g \in G_K$. This is impossible by Exercise 1. Hence V is not $\mathbf{C}_K$-admissible.

Assume that V is Hodge–Tate. Then V is $\mathbf{B}_{\mathrm{HT}}$-admissible, and there exists $\alpha = \sum_i c_i t^i$, $c_i \in \mathbf{C}_K$ such that the following identity holds in $\mathbf{B}_{\mathrm{HT}}$.

$$\psi(g) = \alpha - g(\alpha) = \sum_i (c_i - \chi^i(g) g(c_i)) t^i, \qquad \forall g \in G_K.$$

Therefore $\psi(g) = c_0 - g(c_0)$ for all $g \in G_K$. This gives a contradiction.

Exercise 3. For any n we denote by ζ_{p^n} a primitive p^nth root of unity. One can assume (this is not really necessary in this exercise) that $\zeta_{p^n}^p = \zeta_{p^{n-1}}$. Set $K = \mathbf{Q}_p(\zeta_p)$ and $K_\infty = \underset{n\geqslant 1}{\cup} \mathbf{Q}_p(\zeta_{p^n})$. Recall that K_∞/K is a totally ramified $\mathbf{Z}_p$-extension. Let K^{ur}/K denote the maximal unramified extension of K.

1) Show that in the compositum $K^{\mathrm{ur}}K_\infty$ contains infinitely many subfields $K \subset L \subset K^{\mathrm{ur}}K_\infty$ such that L/K are totally ramified $\mathbf{Z}_p$-extensions of K.

Hint: Use Galois theory. What is the Galois group of the compositum of two linearly disjoint Galois extensions?

Solution. If F_1 and F_2 are two Galois extensions of K such that $F_1 \cap F_2 = K$, then $\mathrm{Gal}(F_1F_2/K) = \mathrm{Gal}(F_1/K) \times \mathrm{Gal}(F_2/K)$. Since $\mathrm{Gal}(K^{\mathrm{ur}}/K) \simeq \widehat{\mathbf{Z}} \simeq \prod_\ell \mathbf{Z}_\ell$, there exists an unramified subextension $K^{\mathrm{ur},p}/K$ such that $\mathrm{Gal}(K^{\mathrm{ur},p}/K) \simeq \mathbf{Z}_p$. Take $F_1 = K_\infty$ and $F_2 = K^{\mathrm{ur},p}$, and set $F = F_1F_2$. Then $F \subset K^{\mathrm{ur}}K_\infty$ and $\mathrm{Gal}(F/K) \simeq \mathrm{Gal}(F_1/K) \times \mathrm{Gal}(F_2/K)$, where

$$\mathrm{Gal}(F_1/K) \times \mathrm{Gal}(F_2/K) \simeq \mathbf{Z}_p \times \mathbf{Z}_p.$$

For a subgroup $H \subset \mathrm{Gal}(F_1/K) \times \mathrm{Gal}(F_2/K)$, set $L = F^H$. Since $K^{\mathrm{ur},p} = F^{\mathrm{Gal}(F_1/K)}$, one has

$$L \cap K^{\mathrm{ur},p} = K \Leftrightarrow H \cdot \mathrm{Gal}(F_1/K) = \mathrm{Gal}(F/K).$$

Therefore it is sufficient to show that there exists infinitely many H such that

$$\mathrm{Gal}(F/K)/H \simeq \left(\mathrm{Gal}(F_1/K) \times \mathrm{Gal}(F_2/K)\right)/H \simeq \mathbf{Z}_p$$

and

$$H \cdot \mathrm{Gal}(F_1/K) = \mathrm{Gal}(F_1/K) \times \mathrm{Gal}(F_2/K).$$

To construct such subgroups, it is sufficient to consider the subgroups in $\mathbf{Z}_p \times \mathbf{Z}_p$ topologically generated by $(p^n, 1)$ for different n.

2*) This question is independent on question 1). Moreover, the assertion below holds for any local field of characteristic 0, not only for $K = \mathbf{Q}_p(\zeta_p)$.

Let L_1 and L_2 be two totally ramified $\mathbf{Z}_p$-extensions of K. Show that $\widehat{L}_1^\flat$ and $\widehat{L}_2^\flat$ are isomorphic as non-archimedean fields (i.e. that there exists an isomorphism $\widehat{L}_1^\flat \simeq \widehat{L}_2^\flat$ which is compatible with non-archimedean values on $\widehat{L}_1^\flat$ and $\widehat{L}_2^\flat$).

Solution. By the theory of field of norms, $\mathcal{X}(L_1/K)$ and $\mathcal{X}(L_2/K)$ are both local fields of characteristic p withe the same residue field. Hence $\mathcal{X}(L_1/K) \simeq \mathcal{X}(L_2/K)$. By Theorem 2.7 (Fontaine–Wintenberger)

$$\widehat{L}_1^\flat \simeq \mathcal{X}(L_1/K)^{\mathrm{rad}} \simeq \mathcal{X}(L_2/K)^{\mathrm{rad}} \simeq \widehat{L}_2^\flat.$$

3*) Show that there exist totally ramified $\mathbf{Z}_p$-extensions L_1 and L_2 of K such that $\widehat{L}_1^\flat \simeq \widehat{L}_2^\flat$ but $L_1 \not\simeq L_2$ as abstract fields.

Solution. Use question 1). Take $K = \mathbf{Q}_p(\zeta_p)$ and $K_\infty = \mathbf{Q}_p(\zeta_{p^\infty})$. Take $L_1 = K_\infty$. In the compositum $K_\infty K^{\mathrm{ur}}$ take a totally ramified $\mathbf{Z}_p$-extension L_2 such that $L_2 \neq L_1$. By question 2), $\widehat{L}_1^\flat \simeq \widehat{L}_2^\flat$. To prove that $L_1 \not\simeq L_2$ one can remark that L_2 cannot contain all p^nth roots of unity ζ_{p^n} $(n \geqslant 1)$ since otherwise $L_2 = L_1$.

Exercise 4. Let K be a local field of characteristic 0 with residue field of characteristic p. Fix $u_0 \in U_K$ such that $u_0 \equiv 1 \pmod{pO_K}$. Fix a system of p^nth roots $u_n \in \overline{K}$ of u_0 such that $u_{n+1}^p = u_n$ for all $n \geqslant 0$. We consider the system $u = (u_n)_{n \geqslant 0}$ as an element of $O^\flat_{\mathbf{C}_K}$. It is easy to see that $u_n \equiv 1 \pmod{\mathfrak{m}_{\overline{K}}}$ and therefore $u \equiv 1 \pmod{\mathfrak{m}_{\mathbf{C}_K^\flat}}$. We fix a compatible system $\varepsilon = (\zeta_{p^n})_{n \geqslant 0}$ of p^nth roots of unity and denote by $\chi : G_K \to \mathbf{Z}_p^*$ the cyclotomic character.

1) Show that for any $n \geqslant 1$ there exists a map $\psi_n : G_K \to \mathbf{Z}/p^n\mathbf{Z}$ such that

$$g(u_n) = u_n \zeta_{p^n}^{\psi_n(g)}, \qquad \forall g \in G_K.$$

Show that there exists a continuous map $\psi : G_K \to \mathbf{Z}_p$ such that $\psi_n(g) = \psi(g) \mod p^n$ for all $n \geqslant 1$ and $g \in G_K$.

Solution. If u_n is a fixed root of $X^{p^n} - u_0$, then other roots of this polynomial are of the form $u_n \zeta_{p^n}^a$, where $0 \leqslant a \leqslant p^n - 1$. Since the elements of the Galois group permute the roots, for any $n \geqslant 1$ and $g \in G_K$, there exists $\psi_n(g) \in \mathbf{Z}/p^n\mathbf{Z}$ such that $g(u_n) = u_n \zeta_{p^n}^{\psi_n(g)}$. Moreover,

$$g(u_{n-1}) = g(u_n^p) = g(u_n)^p = u_n^p (\zeta_{p^n}^{\psi_n(g)})^p = u_{n-1} \zeta_{p^{n-1}}^{\psi_n(g)}.$$

This implies that $\psi_{n-1}(g) = \psi_n(g) \pmod{p^{n-1}}$. Therefore $(\psi_n(g))_{n \geqslant 1}$ define an element $\psi(g)$ of $\varprojlim_n \mathbf{Z}/p^n\mathbf{Z} = \mathbf{Z}_p$. Since for each n the set $\{g \in G_K | \psi_n(g) = 0\}$ is open in G_K, the map ψ is continuous.

2) Check that the map ψ satisfies property (3) of Exercise 2 with $\eta = \chi$:

$$\psi(g_1 g_2) = \chi(g_1)\psi(g_2) + \psi(g_1), \qquad g_1, g_2 \in G_K.$$

Solution. For each n one has:

$$g_1 g_2(u_n) = g_1(u_n \zeta_{p^n}^{\psi_n(g_2)}) = g_1(u_n) g_1(\zeta_{p^n}^{\psi_n(g_2)}) = u_n \zeta_{p^n}^{\psi_n(g_1)} \zeta_{p^n}^{\chi(g_1)\psi_n(g_2)} = u_n \zeta_{p^n}^{\psi_n(g_1) + \chi(g_1)\psi_n(g_2)}.$$

Hence $\psi_n(g_1 g_2) = \chi(g_1)\psi_n(g_2) + \psi_n(g_1)$ for all n. Passing to the limit, we obtain the result.

From question 2) of Exercise 2 it follows that there exists an exact sequence of p-adic representations:

$$0 \to V_1 \to V \to V_2 \to 0,$$

where $V_1 \simeq \mathbf{Q}_p(\chi)$ and $V_2 \simeq \mathbf{Q}_p$ (trivial representation).

We denote by $[u] \in \mathbf{A}_{\mathrm{inf}}$ the Teichmüller lift of u in $\mathbf{A}_{\mathrm{inf}} = W(O^\flat_{\mathbf{C}_K})$. The following facts are admitted:

a) The power series

$$\log[u] = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{([u]-1)^n}{n}$$

converges in $\mathbf{B}^+_{\mathrm{cris}}$.

b) If $v = (v_n)_{n \geqslant 0}$ is another system of units satisfying the same properties as u, then $\log[uv] = \log[u] + \log[v]$;

c) The logarithm map commutes with the action of G_K.

3) Show that
$$g(\log[u]) = \log[u] + \psi(g)t, \qquad \forall g \in G_K,$$
where $t = \log[\varepsilon]$.

Solution. One has $g(\log[u]) = \log[g(u)]$. Since $u = (u_n)_{n\geqslant 0}$, by question 1) one has:
$$g(u) = (u_n\zeta_{p^n}^{\psi_n(g)})_{n\geqslant 0} = u\varepsilon^{\psi(g)},$$
where $\varepsilon = (\zeta_{p^n})_{n\geqslant 0}$. Hence
$$g(\log[u]) = \log([u\varepsilon^{\psi(g)}]) = \log([u][\varepsilon]^{\psi(g)}) = \log[u] + \psi(g)\log[\varepsilon] = \log[u] + \psi(g)t.$$

4*) Using the criterion proved in question 3) of Exercise 2 show that V is crystalline.

Solution. $V_1 = \mathbf{Q}_p(\chi)$ is $\mathbf{B}_{\mathrm{cris}}$-admissible. Namely, taking $b = t^{-1}$, we obtain that
$$b/g(b) = g(t)/t = \chi(g).$$
Set $c = -\log[u]/t$. Then
$$c - \chi(g)g(c) = -\log[u]/t + g(\log[u])/t = -\log[u]/t + (\log[u]/t + \psi(g)) = \psi(g).$$
Applying the criterion from Exercise 2, we see that V is crystalline.