AN INTRODUCTION TO p-ADIC HODGE THEORY

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Abstract. These notes provide an introduction to p-adic Hodge theory. They are based on the series of lectures given by the author at the International Center of Theoretical Sciences of Tata Institute in 2019.

Contents

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Introduction

0.1. These notes grew out of author's lectures at the International Center of Theoretical Sciences of Tata Institute in Bangalore in September, 2019. Their aim is to provide a self-contained introduction to p-adic Hodge theory with minimal prerequisties. The reader should be familiar with valuations, complete fields and basic results in the theory of local fields, including ramification theory as, for example, the first four chapters of Serre's book [142]. In Sections 3 and 4, we use the language of continuous cohomology. Sections 15 and 16 require the knowledge of Galois cohomology and local class field theory, as in [142] or [140].

0.2. Section 1 is utilitarian. For the convenience of the reader, it assembles basic definitions and results from the theory of local fields repeatedly used in the text. In Section 2, we discuss the structure of the absolute Galois group of a local field. Although only a portion of this material is used in the remainder of the text, we think that it is important in its own right. In Section 3, we illustrate the ramification theory by the example of $\mathbf{Z}_p$-extensions. Following Tate, we define the normalized trace map and compute continuous cohomology of Galois groups of such extensions.

Krasner [100] was probably the first to remark that local fields of caracteristic p appear as "limits" of totally ramified local fields of characteristic 0 [1]. In Sections 4-6, we study three important manifestations of this phenomenon. In Section 4, we introduce Tate's method of almost étale extensions. We consider deeply ramified extensions of local fields and prove that finite extensions of a deeply ramified field are almost étale. The main reference here is the paper of Coates and Greenberg [37]. The book of Gabber and Ramero [78] provides a new conceptual approach to this theory in a very general setting, but uses the tools which are beyond the scope of these notes. As an application, we compute continuous Galois cohomology of the absolute Galois group of a local field.

In Section 5, we study perfectoid fields following Scholze [130] and Fargues–Fontaine [60]. The connection of this notion with the theory of deeply ramified extensions is given by a theorem of Gabber–Ramero. Again, we limit our study to the arithmetic case and refer the interested reader to [130] for the general treatment. In Section 6, we review the theory of field of norms of Fontaine–Wintenberger and discuss its relation with perfectoid fields.

Sections 7-13 are devoted to the general theory of p-adic representations. In Section 7, we introduce basic notions and examples and discuss Grothendieck's ℓ-adic monodromy theorem. Next we turn to the case $\ell = p$. Section 8 gives an

[1] See [52] for a modern exposition of Krasner's results.

introduction to Fontaine's theory of (φ,Γ)-modules [69]. Here we classify p-adic representations of local fields using the link between the fields of characteristic 0 and p studied in Sections 5-6. In Sections 9-13, we introduce and study special classes of p-adic representations. The general formalism of admissible representations is reviewed in Section 9. In Section 10, we discuss the notion of a Hodge–Tate representation and put it in the frame of Sen's theory of $\mathbf{C}_K$-representations. Here the computation of the continuous Galois cohomology from Section 4 plays a fundamental role. In Section 11-13, we define the rings of p-adic periods $\mathbf{B}_{\mathrm{dR}}$, $\mathbf{B}_{\mathrm{cris}}$ and $\mathbf{B}_{\mathrm{st}}$ and introduce Fontaine's hierarchy of p-adic representations. Its relation with p-adic comparison isomorphisms is quickly discussed at the end of Section 13.

In the remainder of the text, we study p-adic representations arising from formal groups. In this case, the main constructions of the theory have an explicit description, and p-adic representations can be studied without an extensive use of algebraic geometry. In Section 14, we review the p-adic integration on formal groups following Colmez [38]. A completely satisfactory exposition of this material should cover the general case of p-divisible groups, which we decided not to include in these notes. For this material, we refer the reader to [64], [66], [39], [30]. In Sections 15-16, we illustrate the p-adic Hodge theory of formal groups by two applications: complex multiplication of abelian varieties and Hilbert pairings on formal groups. In Section 17, we prove the theorem "weakly admissible $\Rightarrow$ admissible" in the case of dimension one by the method of Laffaille [102]. This implies the surjectivity of the Gross–Hopkins period map. Finally, we apply the theory of formal groups to the study of the spaces $(\mathbf{B}_{\mathrm{cris}}^{+})^{\varphi^h=p}$, which play an important role in the theory of Fargues–Fontaine. For further detail and applications of these results, we refer the reader to [60].

0.3. These notes should not be viewed as a survey paper. Several important aspects of p-adic Hodge theory are not even mentioned. As a partial substitute, we propose some references for further reading in the body of the text.

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1. Local fields. Preliminaries

1.1. **Non-archimedean fields.**

1.1.1. We recall basic definitions and facts about non-archimedean fields.

Definition. *A non-archimedean field is a field K equipped a non-archimedean absolute value that is, an absolute value $|\cdot|_K$ satisfying the ultrametric triangle inequality:*

$$|x+y|_K \leqslant \max\{|x|_K, |y|_K\}, \qquad \forall x,y\in K.$$

We will say that K is complete if it is complete for the topology induced by $|\cdot|_K$.

To any non-archimedean field K, one associates its ring of integers

$$O_K = \{x \in K \mid |x|_K \leqslant 1\}.$$

The ring O_K is local, with the maximal ideal

$$\mathfrak{m}_K = \{x \in K \mid |x|_K < 1\}.$$

The group of units of O_K is

$$U_K = \{x \in K \mid |x|_K = 1\}.$$

The residue field of K is defined as

$$k_K = O_K/\mathfrak{m}_K.$$

Theorem 1.1.2. *Let K be a complete non-archimedean field and let L/K be a finite extension of degree $n = [L : K]$. Then the absolute value $|\cdot|_K$ has a unique continuation $|\cdot|_L$ to L, which is given by*

$$|x|_L = \left|N_{L/K}(x)\right|_K^{1/n},$$

where $N_{L/K}$ is the norm map.

Proof. See, for example, [10, Chapter 2, Theorem 7]. □

1.1.3. We fix an algebraic closure $\overline{K}$ of K and denote by K^{sep} the separable closure of K in $\overline{K}$. If $\mathrm{char}(K) = p > 0$, we denote by $K^{\mathrm{rad}} := K^{1/p^\infty}$ the purely inseparable closure of K. Thus $\overline{K} = K^{\mathrm{sep}}$ if $\mathrm{char}(K) = 0$, and $\overline{K} = (K^{\mathrm{rad}})^{\mathrm{sep}}$ if $\mathrm{char}(K) = p > 0$. Theorem 1.1.2 allows to extend $|\cdot|_K$ to $\overline{K}$. To simplify notation, we denote again by $|\cdot|_K$ the extension of $|\cdot|_K$ to $\overline{K}$.

Proposition 1.1.4 (Krasner's lemma)**.** *Let K be a complete non-archimedean field. Let $\alpha \in K^{\mathrm{sep}}$ and let $\alpha_1 = \alpha, \alpha_2, \dots, \alpha_n$ denote the conjugates of α over K. Set*

$$d_\alpha = \min\{|\alpha - \alpha_i|_K \mid 2 \leqslant i \leqslant n\}.$$

If $\beta \in K^{\mathrm{sep}}$ is such that $|\alpha - \beta| < d_\alpha$, then $K(\alpha) \subset K(\beta)$.

Proof. We recall the proof (see, for example, [119, Proposition 8.1.6]). Assume that $\alpha \notin K(\beta)$. Then $K(\alpha,\beta)/K(\beta)$ is a non-trivial extension, and there exists an embedding $\sigma : K(\alpha,\beta)/K(\beta) \to \overline{K}/K(\beta)$ such that $\alpha_i := \sigma(\alpha) \neq \alpha$. Hence

$$|\beta - \alpha_i|_K = |\sigma(\beta - \alpha)|_K = |\beta - \alpha|_K < d_\alpha,$$

and

$$|\alpha - \alpha_i|_K = |(\alpha - \beta) + (\beta - \alpha_i)|_K \leqslant \max\{|\alpha - \beta|_K, |\beta - \alpha_i|_K\} < d_\alpha.$$

This gives a contradiction. □

Proposition 1.1.5 (Hensel's lemma)**.** *Let K be a complete non-archimedean field. Let $f(X) \in O_K[X]$ be a monic polynomial such that:*

a) the reduction $\bar{f}(X) \in k_K[X]$ of $f(X)$ modulo $\mathfrak{m}_K$ has a root $\bar{\alpha} \in k_K$;

b) $\bar{f}'(\bar{\alpha}) \neq 0$.

Then there exists a unique $\alpha \in O_K$ such that $f(\alpha) = 0$ and $\bar{\alpha} = \alpha \pmod{\mathfrak{m}_K}$.

Proof. See, for example, [106, Chapter 2, §2]. □

1.1.6. Recall that a valuation on K is a function $v_K : K \to \mathbf{R}\cup\{+\infty\}$ satisfying the following properties:

1) $v_K(xy) = v_K(x)+v_K(y), \qquad \forall x,y\in K^*$;
2) $v_K(x+y) \geqslant \min\{v_K(x),v_K(y)\}, \qquad \forall x,y\in K^*$;
3) $v_K(x)=\infty \Leftrightarrow x=0$.

For any $\rho\in]0,1[$, the function $|x|_\rho = \rho^{v_K(x)}$ defines an ultrametric absolute value on K. Conversely, if $|\cdot|_K$ is an ultrametric absolute value, then for any $\rho\in]0,1[$ the function $v_\rho(x) = \log_\rho |x|_K$ is a valuation on K. This establishes a one to one correspondence between equivalence classes of non-archimedean absolute values and equivalence classes of valuations on K.

Definition. *A discrete valuation field is a field K equipped with a valuation v_K such that $v_K(K^*)$ is a discrete subgroup of $\mathbf{R}$. Equivalently, K is a discrete valuation field if it is equipped with an absolute value $|\cdot|_K$ such that $|K^*|_K\subset \mathbf{R}_+$ is discrete.*

Let K be a discrete valuation field. In the equivalence class of discrete valuations on K, we can choose the unique valuation v_K such that $v_K(K^*)=\mathbf{Z}$. An element $\pi_K\in K$ such that $v_K(\pi_K)=1$ is called a uniformizer of K. Every $x\in K^*$ can be written in the form $x=\pi_K^{v_K(x)}u$ with $u\in U_K$, and one has:

$$K^*\simeq \langle \pi_K\rangle\times U_K, \qquad \mathfrak{m}_K=(\pi_K).$$

1.1.7. Let K be a complete non-archimedean field. We finish this section by discussing the Galois action on the completion $\mathbf{C}_K$ of $\overline{K}$.

Theorem 1.1.8 (Ax–Sen–Tate)**.** *Let K be a complete non-archimedean field. The the following statements hold true:*

i) The completion $\mathbf{C}_K$ of $\overline{K}$ is an algebraically closed field, and K^{sep} is dense in $\mathbf{C}_K$.

ii) The absolute Galois group $G_K=\mathrm{Gal}(K^{\mathrm{sep}}/K)$ acts continuously on $\mathbf{C}_K$, and this action identifies G_K with the group of all continuous automorphisms of $\mathbf{C}_K$ that act trivially on K.

iii) For any closed subgroup $H\subset G_K$, the field $\mathbf{C}_K^H$ coincides with the completion of the purely inseparable closure of $(K^{\mathrm{sep}})^H$ in $\overline{K}$.

Proof. The statement i) follows easily from Krasner's lemma, and ii) is an immediate consequence of continuity of the Galois action. The last statement was first proved by Tate [151] for local fields of characteristic 0. In full generality, the theorem was proved by Ax [11]. Tate's proof is based on the ramification theory and leads to the notion of an almost étale extension, which is fundamental for p-adic Hodge theory. We review it in Section 4. □

1.2. **Local fields.**

1.2.1. In these notes, we adopt the following convention.

Definition 1.2.2. *A local field is a complete discrete valuation field K whose residue field k_K is finite.*

Note that many (but not all) results and constructions of the theory are valid under the weaker assumption that the residue field k_K is perfect.

We will always assume that the discrete valuation

$$v_K : K \to \mathbf{Z} \cup \{+\infty\}$$

is surjective. Let $p = \mathrm{char}(k_K)$. The following well-known classification of local fields can be easily proved using Ostrowski's theorem:

- If $\mathrm{char}(K) = p$, then K is isomorphic to the field $k_K((x))$ of Laurent power series, where k_K is the residue field of K and x is transcendental over k. The discrete valuation on K is given by

$$v_K(f(x)) = \mathrm{ord}_x f(x).$$

  Note that x is a uniformizer of K and $O_K \simeq k_K[[x]]$.
- If $\mathrm{char}(K) = 0$, then K is isomorphic to a finite extension of the field of p-adic numbers $\mathbf{Q}_p$. The absolute value on K is the extension of the p-adic absolute value

$$\left| \frac{a}{b} p^k \right|_p = p^{-k}, \qquad p \nmid a, b.$$

In all cases, set $f_K = [k_K : \mathbf{F}_p]$ and denote by $q_K = p^{f_K}$ the cardinality of k_K. The group of units U_K is equipped with the exhaustive descending filtration:

$$U_K^{(n)} = 1 + \pi_K^n O_K, \qquad n \geqslant 0.$$

For the factors of this filtration, one has:

$$\text{(1)} \qquad U_K / U_K^{(1)} \simeq k_K^*, \qquad\qquad U_K^{(n)} / U_K^{(n+1)} \simeq \mathfrak{m}_K^n / \mathfrak{m}_K^{n+1}. \qquad \text{if } n \geqslant 1.$$

1.2.3. If L/K is a finite extension of local fields, the ramification index $e(L/K)$ and the inertia degree $f(L/K)$ of L/K are defined as follows:

$$e(L/K) = v_L(\pi_K), \qquad f(L/K) = [k_L : k_K].$$

Recall the fundamental formula:

$$f(L/K)\, e(L/K) = [L : K]$$

(see, for example, [10, Chapter 3, Theorem 6]).

Definition 1.2.4. *One says that L/K is*
i) unramified if $e(L/K) = 1$ (and therefore $f(L/K) = [L : K]$);
ii) totally ramified if $e(L/K) = [L : K]$ (and therefore $f(L/K) = 1$).

The following useful proposition follows easily from Krasner's lemma.

Proposition 1.2.5. *Let K be a local field of characteristic* 0. *For any $n \geqslant 1$ there exists only a finite number of extensions of K of degree $\leqslant n$.*

Proof. See [106, Chapter 2, Proposition 14]. □

We remark that, looking at Artin–Schreier extensions, it's easy to see that a local field of characteristic p has infinitely many separable extensions of degree p.

1.2.6. The unramified extensions can be described entirely in terms of the residue field k_K. Namely, there exists a one-to-one correspondence

$$\{\text{finite extensions of } k_K\} \longleftrightarrow \{\text{finite unramified extensions of } K\},$$

which can be explicitly described as follows. Let k/k_K be a finite extension of k_K. Write $k = k_K(\alpha)$ and denote by $f(X) \in k_K[X]$ the minimal polynomial of α. Let $\widehat{f}(X) \in O_K[X]$ denote any lift of $f(X)$. Then we associate to k the extension $L = K(\widehat{\alpha})$, where $\widehat{\alpha}$ is the unique root of $\widehat{f}(X)$ whose reduction modulo $\mathfrak{m}_L$ is α. An easy argument using Hensel's lemma shows that L doesn't depend on the choice of the lift $\widehat{f}(X)$.

Unramified extensions form a distinguished class of extensions in the sense of [104]. In particular, for any finite extension L/K, one can define its maximal unramified subextension L_{ur} as the compositum of all its unramified subextensions. Then

$$f(L/K) = [L_{\mathrm{ur}} : K], \qquad e(L/K) = [L : L_{\mathrm{ur}}].$$

The extension L/L_{ur} is totally ramified.

1.2.7. Assume that L/K is totally ramified of degree n. Let π_L be any uniformizer of L, and let

$$f(X) = X^n + a_{n-1}X^{n-1} + \cdots + a_1X + a_0 \in O_K[X]$$

be the minimal polynomial of π_L. Then $f(X)$ is an Eisenstein polynomial, namely

$$v_K(a_i) \geqslant 1 \qquad \text{for } 0 \leqslant i \leqslant n-1, \text{ and } v_K(a_0) = 1.$$

Conversely, if α is a root of an Eisenstein polynomial of degree n over K, then $K(\alpha)/K$ is totally ramified of degree n, and α is an uniformizer of $K(\alpha)$.

Definition 1.2.8. *One says that an extension L/K is*

i) tamely ramified if $e(L/K)$ is coprime to p.

ii) totally tamely ramified if it is totally ramified and $e(L/K)$ is coprime to p.

Using Krasner's lemma, it is easy to give an explicit description of totally tamely ramified extensions.

Proposition 1.2.9. *If L/K is totally tamely ramified of degree n, then there exists a uniformizer $\pi_K \in K$ such that*

$$L = K(\pi_L), \qquad \pi_L^n = \pi_K.$$

Proof. Assume that L/K is totally tamely ramified of degree n. Let Π be a uniformizer of L and $f(X) = X^n + \cdots + a_1X + a_0$ its minimal polynomial. Then $f(X)$ is Eisenstein, and $\pi_K := -a_0$ is a uniformizer of K. Let $\alpha_i \in \overline{K}$ ($1 \leqslant i \leqslant n$) denote the roots of $g(X) := X^n + a_0$. Then

$$|g(\Pi)|_K = |g(\Pi) - f(\Pi)|_K \leqslant \max_{1\leqslant i\leqslant n-1} |a_i\Pi^i|_K < |\pi_K|_K$$

Since $|g(\Pi)|_K = \prod_{i=1}^{n}(\Pi - \alpha_i)$, and $\Pi = (-1)^n \prod_{i=1}^{n}\alpha_i$, we have:

$$\prod_{i=1}^{n} |\Pi - \alpha_i|_K < \prod_{i=1}^{n} |\alpha_i|_K.$$

Therefore there exists i_0 such that

$$|\Pi-\alpha_{i_0}|_K<|\alpha_{i_0}|_K. \tag{2}$$

Set $\pi_L=\alpha_{i_0}$. Then

$$\prod_{i\neq i_0}(\pi_L-\alpha_i)=g'(\pi_L)=n\pi_L^{n-1}.$$

Since $(n,p)=1$ and $|\pi_L-\alpha_i|_K\leqslant|\pi_L|_K$, the previous equality implies that

$$d:=\min_{i\neq i_0}|\pi_L-\alpha_i|_K=|\pi_L|_K.$$

Together with (2), this gives:

$$|\Pi-\alpha_{i_0}|_K<d.$$

Applying Krasner's lemma, we find that $K(\pi_L)\subset L$. Since $[L:K]=[K(\pi_L):K]=n$, we obtain that $L=K(\pi_L)$, and the proposition is proved.

□

1.2.10. Let L/K be a finite separable extension of local fields. Consider the bilinear non-degenerate form

$$t_{L/K}\,:\,L\times L\to K,\qquad t_{L/K}(x,y)=\mathrm{Tr}_{L/K}(xy), \tag{3}$$

where $\mathrm{Tr}_{L/K}$ is the trace map. The set

$$O_L':=\{x\in L\mid t_{L/K}(x,y)\in O_K,\quad \forall y\in O_L\}$$

is a fractional ideal, and

$$\mathfrak{D}_{L/K}:=O_L^{-1}:=\{x\in L\mid xO_L'\subset O_L\}$$

is an ideal of O_L.

Definition. *The ideal $\mathfrak{D}_{L/K}$ is called the different of L/K.*

If $K\subset L\subset M$ is a tower of separable extensions, then

$$\mathfrak{D}_{M/K}=\mathfrak{D}_{M/L}\mathfrak{D}_{L/K}. \tag{4}$$

(see, for example, [106, Chapter 3, Proposition 5]).

Set

$$v_L(\mathfrak{D}_{L/K})=\inf\{v_L(x)\mid x\in\mathfrak{D}_{L/K}\}.$$

Proposition 1.2.11. *Let L/K be a finite separable extension of local fields and $e=e(L/K)$ the ramification index. The following assertions hold true:*

i) If $O_L=O_K[\alpha]$, and $f(X)\in O_K[X]$ is the minimal polynomial of α, then $\mathfrak{D}_{L/K}=(f'(\alpha))$.

ii) $\mathfrak{D}_{L/K}=O_L$ if and only if L/K is unramified.

iii) $v_L(\mathfrak{D}_{L/K})\geqslant e-1$.

iv) $v_L(\mathfrak{D}_{L/K})=e-1$ if and only if L/K is tamely ramified.

Proof. The first statement holds in the more general setting of Dedekind rings (see, for example, [106, Chapter 3, Proposition 2]). We prove ii-iv) for reader's convenience (see also [106, Chapter 3, Proposition 8]).

a) Let L/K be an unramified extension of degree n. Write $k_L = k_K(\bar{\alpha})$ for some $\bar{\alpha} \in k_L$. Let $f(X) \in k_K[X]$ denote the minimal polynomial of $\bar{\alpha}$. Then $\deg(\bar{f}) = n$. Take any lift $f(X) \in O_K[X]$ of $\bar{f}(X)$ of degree n. By Proposition 1.1.5 (Hensel's lemma) there exists a unique root $\alpha \in O_L$ of $f(X)$ such that $\bar{\alpha} = \alpha \pmod{\mathfrak{m}_K}$. It's easy to see that $O_L = O_K[\alpha]$. Since $\bar{f}(X)$ is separable, $\bar{f}'(\bar{\alpha}) \neq 0$, and therefore $f'(\alpha) \in U_L$. Applying i), we obtain:

$$\mathfrak{D}_{L/K} = (f'(\alpha)) = O_L.$$

Therefore $\mathfrak{D}_{L/K} = O_L$ if L/K is unramified.

b) Assume that L/K is totally ramified. Then $O_L = O_K[\pi_L]$, where π_L is any uniformizer of O_L. Let $f(X) = X^e + a_{e-1}X^{e-1} + \cdots + a_1X + a_0$ be the minimal polynomial of π_L. Then

$$f'(\pi_L) = e\pi_L^{e-1} + (e-1)a_{e-1}\pi_L^{e-2} + \cdots + a_1.$$

Since $f(X)$ is Eisenstein, $v_L(a_i) \geqslant e$, and an easy estimation shows that $v_L(f'(\pi_L)) \geqslant e-1$. Thus

$$v_L(\mathfrak{D}_{L/K}) = v_L(f'(\alpha)) \geqslant e-1.$$

This proves iii). Moreover, $v_L(f'(\alpha)) = e-1$ if and only if $(e,p) = 1$, i.e. if and only if L/K is tamely ramified. This proves iv).

c) Assume that $\mathfrak{D}_{L/K} = O_L$. Then $v_L(\mathfrak{D}_{L/K}) = 0$. Let L_{ur} denote the maximal unramified subextension of L/K. By (4), a) and b) we have:

$$v_L(\mathfrak{D}_{L/K}) = v_L(\mathfrak{D}_{L/L_{\mathrm{ur}}}) \geqslant e-1.$$

Thus $e = 1$, and we showed that each extension L/K such that $\mathfrak{D}_{L/K} = O_L$ is unramified. Together with a), this proves i). □

1.3. **Ramification filtration.**

1.3.1. Let L/K be a finite Galois extension of local fields. Set $G = \mathrm{Gal}(L/K)$. For any integer $i \geqslant -1$ define

$$G_i = \{g \in G \mid v_L(g(x) - x) \geqslant i+1, \quad \forall x \in O_L\}.$$

Then G_i are normal subgroups of G, called ramification subgroups. We have a descending chain

$$G = G_{-1} \supset G_0 \supset G_1 \supset \cdots \supset G_m = \{1\}$$

called the ramification filtration on G (in low numbering). From definition, it easily follows that

$$G_0 = \mathrm{Gal}(L/L_{\mathrm{ur}}), \qquad G/G_0 \simeq \mathrm{Gal}(k_L/k_K).$$

Below, we summarize some basic results about the factors of the ramification filtration. First remark that for each $i \geqslant 0$, one has:

$$G_i = \left\{ g \in G_0 \mid v_L\left(1 - \frac{g(\pi_L)}{\pi_L}\right) \geqslant i \right\}.$$

Proposition 1.3.2. *i) For all $i \geqslant 0$, the map*

$$s_i : G_i/G_{i+1} \to U_L^{(i)}/U_L^{(i+1)}, \tag{5}$$

which sends $\bar{g} = g \mod G_{i+1}$ to $s_i(\bar{g}) = \dfrac{g(\pi_L)}{\pi_L} \pmod{U_L^{(i+1)}}$, is a well defined monomorphism which doesn't depend on the choice of the uniformizer π_L of L.

ii) The composition of s_i with the maps (1) gives monomorphisms:

$$\delta_0 : G_0/G_1 \to k^*, \qquad \delta_i : G_i/G_{i+1} \to \mathfrak{m}_K^i/\mathfrak{m}_K^{i+1}, \quad \textit{for all } i \geqslant 1. \tag{6}$$

Proof. The proof is straightforward. See [142, Chapitre IV, Propositions 5-7]. □

An important corollary of this proposition is that the Galois group G is solvable for any Galois extension. Also, since $\mathrm{char}(k_K) = p$, the order of G_0/G_1 is coprime to p, and the order of G_1 is a power of p. Therefore $L_{\mathrm{tr}} = L^{G_1}$ is the maximal tamely ramified subextension of L. From this, one can easily deduce that the class of tamely ramified extensions is distinguished. To sup up, we have the tower of extensions:

(7)
$$\begin{tikzcd}
L \arrow[d, dash, "G_1"] \arrow[dd, dash, bend right, "G_0"'] \\
L_{\mathrm{tr}} \arrow[d, dash, "G_0/G_1"] \\
L_{\mathrm{ur}} \arrow[d, dash, "G/G_0"] \\
K
\end{tikzcd}$$

Definition 1.3.3. *The groups $I_{L/K} := G_0$ and $P_{L/K} := G_1$ are called the inertia subgroup and the wild inertia subgroup respectively.*

1.3.4. The different $\mathfrak{D}_{L/K}$ of a finite Galois extension can be computed in terms of the ramification subgroups.

Proposition 1.3.5. *Let L/K be a finite Galois extension of local fields. Then*

$$v_L(\mathfrak{D}_{L/K}) = \sum_{i=0}^{\infty} (|G_i| - 1). \tag{8}$$

Proof. Let $O_L = O_K[\alpha]$, and let $f(X)$ be the minimal polynomial of α. For any $g \in G$, set $i_{L/K}(g) = v_L(g(\alpha) - \alpha)$. From the definition of ramification subgroups it follows that $g \in G_i$ if and only if $i_{L/K}(g) \geqslant i + 1$. Since

$$f'(\alpha) = \prod_{g \neq 1} (\alpha - g(\alpha)),$$

we have:

$$v_L(\mathfrak{D}_{L/K}) = v_L(f'(\alpha)) = \sum_{g\neq 1} v_L(\alpha - g(\alpha)) = \sum_{g\neq 1} i_{L/K}(g) = \sum_{i=0}^{\infty} (i+1)(|G_i| - |G_{i+1}|)$$

$$= \sum_{i=0}^{\infty} (i+1)((|G_i|-1) - (|G_{i+1}|-1)) = \sum_{i=0}^{\infty} (|G_i|-1).$$

□

1.3.6. We review Hasse–Herbrand's theory of upper ramification. It is convenient to define G_u for all *real* $u \geqslant -1$ setting

$$G_t = G_i, \quad \text{where } i \text{ is the smallest integer} \geqslant u.$$

For any finite Galois extension the Hasse–Herbrand functions are defined as follows:

$$\begin{aligned} &\varphi_{L/K}(u) = \int_0^u \frac{dt}{(G_0 : G_t)}, \\ &\psi_{L/K}(v) = \varphi_{L/K}^{-1}(v) \qquad \text{(the inverse of } \varphi_{L/K}). \end{aligned} \tag{9}$$

Proposition 1.3.7. *Let $K \subset F \subset L$ be a tower of finite Galois extensions. Set $G = \mathrm{Gal}(L/K)$ and $H = \mathrm{Gal}(L/F)$. Then the following holds true:*

i) $\varphi_{L/K} = \varphi_{F/K} \circ \varphi_{L/F}$ *and* $\psi_{L/K} = \psi_{L/F} \circ \psi_{F/K}$.

ii) (Herbrand's theorem) *For any* $u \geqslant 0$,

$$G_u H/H \simeq (G/H)_{\varphi_{M/L}(u)}.$$

Proof. See [142, Chapter IV, §3]. □

Definition. *The ramification subgroups in upper numbering $G^{(v)}$ are defined by*

$$G^{(v)} = G_{\psi_{L/K}(v)},$$

or, equivalently, by $G^{(\varphi_{L/K}(u))} = G_u$.

Therefore Herbrand's theorem can be stated as follows:

$$(G/H)^{(v)} = G^{(v)}/G^{(v)} \cap H, \qquad \forall v \geqslant 0. \tag{10}$$

The Hasse–Herbrand function $\psi_{L/K}$ can be written as

$$\psi_{L/K}(v) = \int_0^v (G^{(0)} : G^{(t)}) dt.$$

1.3.8. Hebrand's theorem allows to define the ramification filtration for infinite Galois extensions. Namely, for any (finite or infinite) Galois extension of local fields L/K define

$$\mathrm{Gal}(L/K)^{(v)} = \varprojlim_F \mathrm{Gal}(F/K)^{(v)},$$

where F runs through finite Galois subextensions of L/K. In particular, we can consider the ramification filtration on the absolute Galois group G_K of K. This filtration contains fundamental information about the field K. We discuss it in more detail in Section 2.3.

Definition. *A real number $v \geqslant 0$ is a ramification jump of a Galois extension L/K if*

$$\mathrm{Gal}(L/K)^{(v+\varepsilon)} \neq \mathrm{Gal}(L/K)^{(v)} \qquad \text{for any } \varepsilon > 0.$$

1.3.9. Formula (8) can be written in terms of upper ramification subgroups:

$$v_K(\mathfrak{D}_{L/K}) = \int_{-1}^{\infty} \left(1 - \frac{1}{|G^{(v)}|}\right) dv.$$

In this form, it can be generalized to arbitrary finite extensions as follows. For any $v \geqslant 0$ define

$$\overline{K}^{(v)} = \overline{K}^{G_K^{(v)}}.$$

Then for any finite extension L/K one has:

$$v_K(\mathfrak{D}_{L/K}) = \int_{-1}^{\infty} \left(1 - \frac{1}{[L : L \cap \overline{K}^{(v)}]}\right) dv \tag{11}$$

(see [37, Lemma 2.1]).

1.3.10. The description of the ramification filtration for general Galois extensions is a difficult problem (see Section 2.3) below). Is is completely solved for abelian extensions (see Section 2.2). In particular, the ramification jumps of an abelian extension are rational integers (theorem of Hasse–Arf). For non-abelian extensions we have the following result.

Theorem 1.3.11 (Sen). *Let K_∞/K be an infinite totally ramified Galois extension whose Galois group $G = \mathrm{Gal}(K_\infty/K)$ is a p-adic Lie group. Fix a Lie filtration $(G(n))_{n \geqslant 0}$ on G. Then there exists a constant $c \geqslant 0$ such that*

$$G^{(ne_K+c)} \subset G(n) \subset G^{(ne_K-c)}, \qquad \forall n \geqslant 0.$$

In particular, $(G : G^{(v)}) < +\infty$ for all $v \geqslant 0$.

Proof. This is the main result of [134]. □

1.4. **Norms and traces.**

1.4.1. The results proved in this section are technical by the nature, but they play a crucial role in our discussion of deeply ramified extensions and the field of norms functor. Assume that L/K is a finite extension of local fields of characteristic 0.

Lemma 1.4.2. *One has:*

$$\mathrm{Tr}_{L/K}(\mathfrak{m}_L^n) = \mathfrak{m}_K^r,$$

where $r = \left[\frac{v_L(\mathfrak{D}_{L/K})+n}{e(L/K)}\right]$.

Proof. From the definition of the different if follows immediately that

$$\mathrm{Tr}_{L/K}(\mathfrak{D}_{L/K}^{-1}) = O_K.$$

Set $\delta = v_L(\mathfrak{D}_{L/K})$ and $e = e(L/K)$. Then:

$$\mathfrak{m}_K^r = \mathrm{Tr}_{L/K}\left(\mathfrak{m}_K^r \mathfrak{D}_{L/K}^{-1}\right) = \mathrm{Tr}_{L/K}\left(\mathfrak{m}_L^{re-\delta}\right) \subset \mathrm{Tr}_{L/K}\left(\mathfrak{m}_L^{(\delta+n)-\delta}\right) = \mathrm{Tr}_{L/K}\left(\mathfrak{m}_L^n\right).$$

Conversely, one has:

$$\mathrm{Tr}_{L/K}(\mathfrak{m}_L^n\mathfrak{m}_K^{-r})=\mathrm{Tr}_{L/K}(\mathfrak{m}_L^n\mathfrak{m}_L^{-er})\subset \mathrm{Tr}_{L/K}(\mathfrak{m}_L^{n-(\delta+n)})=\mathrm{Tr}_{L/K}(\mathfrak{D}_{L/K}^{-1})=O_K,$$

Therefore $\mathrm{Tr}_{L/K}(\mathfrak{m}_L^n)\subset \mathfrak{m}_K^r$, and the lemma is proved. □

1.4.3. Assume that L/K is a totally ramified Galois extension of degree p. Set $G=\mathrm{Gal}(L/K)$ and denote by t the maximal natural number such that $G_t=G$ (and therefore $G_{t+1}=\{1\}$). Formula (8) reads:

$$v_L(\mathfrak{D}_{L/K})=(p-1)(t+1). \tag{12}$$

Lemma 1.4.4. *For any $x\in\mathfrak{m}_L^n$,*

$$N_{L/K}(1+x)\equiv 1+N_{L/K}(x)+\mathrm{Tr}_{L/K}(x)\pmod{\mathfrak{m}_K^s},$$

where $s=\left[\frac{(p-1)(t+1)+2n}{p}\right]$.

Proof. Set $G=\mathrm{Gal}(L/K)$, and for each $1\leqslant n\leqslant p$ denote by C_n the set of all n-subsets $\{g_1,\dots,g_n\}$ of G (note that $g_i\neq g_j$ if $i\neq j$). Then:

$$\begin{aligned}N_{L/K}(1+x)=\prod_{g\in G}(1+g(x))&=1+N_{L/K}(x)+\mathrm{Tr}_{L/K}(x)\\ &+\sum_{\{g_1,g_2\}\in C_2}g_1(x)g_2(x)+\dots+\sum_{\{g_1,\dots g_{p-1}\}\in C_{p-1}}g_1(x)\cdots g_{p-1}(x).\end{aligned}$$

It's clear that the rule

$$g\star\{g_1,\dots,g_n\}=\{gg_1,\dots,gg_n\}$$

defines an action of G on C_n. Moreover, from the fact that $|G|=p$ is a prime number, it follows that all stabilizers are trivial, and therefore each orbit has p elements. This implies that each sum

$$\sum_{\{g_1,\dots g_n\}\in C_n}g_1(x)\cdots g_n(x),\qquad 2\leqslant n\leqslant p-1$$

can be written as the trace $\mathrm{Tr}_{L/K}(x_n)$ of some $x_n\in\mathfrak{m}_L^{2n}$. From (12) and Lemma 1.4.2 it follows that $\mathrm{Tr}_{L/K}(x_n)\in\mathfrak{m}_K^s$. The lemma is proved. □

Corollary 1.4.5. *Let L/K is a totally ramified Galois extension of degree p. Then*

$$v_K(N_{L/K}(1+x)-1-N_{L/K}(x))\geqslant\frac{t(p-1)}{p}.$$

Proof. From Lemmas 1.4.2 and 1.4.4 if follows that

$$v_K(N_{L/K}(1+x)-1-N_{L/K}(x))\geqslant\left[\frac{(p-1)(t+1)}{p}\right],$$

Since

$$\left[\frac{(p-1)(t+1)}{p}\right]=\left[\frac{(p-1)t}{p}+1-\frac{1}{p}\right]\geqslant\frac{t(p-1)}{p},$$

the corollary is proved. □

1.5. **Witt vectors.**

1.5.1. In this subsection, we review the theory of Witt vectors. Consider the sequence of polynomials $w_0(x_0), w_1(x_0,x_1), \dots$ defined by

$$\begin{aligned}
&w_0(x_0) = x_0,\\
&w_1(x_0,x_1) = x_0^p + px_1,\\
&w_2(x_0,x_1,x_2) = x_0^{p^2} + px_1^p + p^2x_2,\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&w_n(x_0,x_1,\dots x_n) = x_0^{p^n} + px_1^{p^{n-1}} + p^2x_2^{p^{n-2}} + \dots + p^nx_n,\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots
\end{aligned}$$

Proposition 1.5.2. *Let $F(x,y) \in \mathbf{Z}[x,y]$ be a polynomial with coefficients in $\mathbf{Z}$ such that $F(0,0)=0$. Then there exists a unique sequence of polynomials*

$$\begin{aligned}
&\Phi_0(x_0,y_0) \in \mathbf{Z}[x_0,y_0],\\
&\Phi_1(x_0,y_0,x_1,y_1) \in \mathbf{Z}[x_0,y_0,x_1,y_1],\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&\Phi_n(x_0,y_0,x_1,y_1,\dots,x_n,y_n) \in \mathbf{Z}[x_0,y_0,x_1,y_1,\dots,x_n,y_n],\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots
\end{aligned}$$

such that

(13)
$$w_n(\Phi_0,\Phi_1,\dots,\Phi_n) = F(w_n(x_0,x_1,\dots,x_n), w_n(y_0,y_1,\dots,y_n)), \qquad \textit{for all } n \geqslant 0.$$

To prove this proposition, we need the following elementary lemma.

Lemma 1.5.3. *Let $f \in \mathbf{Z}[x_0,\dots,x_n]$. Then*

$$f^{p^m}(x_0,\dots,x_n) \equiv f^{p^{m-1}}(x_0^p,\dots,x_n^p) \pmod{p^m}, \qquad \textit{for all } m \geqslant 1.$$

Proof. The proof is left to the reader. □

1.5.4. *Proof of Proposition 1.5.2.* We prove the proposition by induction on n. For $n=0$ we have $\Phi_0(x_0,y_0) = F(x_0,y_0)$. Assume that $\Phi_0,\Phi_1,\dots,\Phi_{n-1}$ are constructed. From (13) it follows that

$$\Phi_n = \frac{1}{p^n}\Big(F(w_n(x_0,x_1,\dots,x_n), w_n(y_0,y_1,\dots,y_n)) - (\Phi_0^{p^n} + \dots + p^{n-1}\Phi_{n-1}^p)\Big). \tag{14}$$

This proves the uniqueness. It remains to prove that Φ_n has coefficients in $\mathbf{Z}$. Since

$$w_n(x_0,\dots,x_{n-1},x_n) \equiv w_{n-1}(x_0^p,\dots,x_{n-1}^p) \pmod{p^n},$$

we have:

$$\begin{aligned}
&F(w_n(x_0,\dots,x_{n-1},x_n), w_n(y_0,\dots,y_{n-1},y_n))\\
&\qquad\qquad \equiv F(w_{n-1}(x_0^p,\dots,x_{n-1}^p), w_{n-1}(y_0^p,\dots,y_{n-1}^p)) \pmod{p^n}.
\end{aligned} \tag{15}$$

On the other hand, applying Lemma 1.5.3 and the induction hypothesis, we have:

$$\begin{aligned}
\Phi_0^{p^n} + \dots + p^{n-1}\Phi_{n-1}^p &\equiv w_{n-1}\Big(\Phi_0(x_0^p,y_0^p),\dots,\Phi_{n-1}(x_0^p,y_0^p,\dots,x_{n-1}^p,y_{n-1}^p)\Big)\\
&\equiv F(w_{n-1}(x_0^p,\dots,x_{n-1}^p), w_{n-1}(y_0^p,\dots,y_{n-1}^p)) \pmod{p^n}.
\end{aligned} \tag{16}$$

From (15) and (16) we obtain that

$$F(w_n(x_0,\dots,x_{n-1},x_n),w_n(y_0,\dots,y_{n-1},y_n))\equiv \Phi_0^{p^n}+\cdots+p^{n-1}\Phi_{n-1}^p \pmod{p^n}.$$

Together with (14), this whows that Φ_n has coeffiients in $\mathbf{Z}$. The proposition is proved.

1.5.5. Let $(S_n)_{n\geqslant 0}$ denote the polynomials $(\Phi_n)_{n\geqslant 0}$ for $F(x,y)=x+y$ and $(P_n)_{n\geqslant 0}$ denote the polynomials $(\Phi_n)_{n\geqslant 0}$ for $F(x,y)=xy$. In particular,

$$S_0(x_0,y_0)=x_0+y_0,\quad S_1(x_0,y_0,x_1,y_1)=x_1+y_1+\frac{x_0^p+y_0^p-(x_0+y_0)^p}{p},$$
$$P_0(x_0,y_0)=x_0y_0,\qquad P_1(x_0,y_0,x_1,y_1)=x_0^py_1+x_1y_0^p+px_1y_1.$$

1.5.6. For any commutative ring A, we denote by $W(A)$ the set of infinite vectors $a=(a_0,a_1,\dots)\in A^{\mathbf{N}}$ equipped with the addition and multiplication defined by the formulas:

$$a+b=(S_0(a_0,b_0),S_1(a_0,b_0,a_1,b_1),\dots),$$
$$a\cdot b=(P_0(a_0,b_0),P_1(a_0,b_0,a_1,b_1),\dots).$$

Theorem 1.5.7 (Witt)**.** *With addition and multiplication defined as above, $W(A)$ is a commutative unitary ring with the identity element*

$$1=(1,0,0,\dots).$$

Proof. a) We show the associativity of addition. From construction it is clear that there exist polynomials $(u_n)_{n\geqslant 0}$, and $(v_n)_{n\geqslant 0}$ with integer coefficients such that $u_n,v_n\in\mathbf{Z}[x_0,y_0,z_0,\dots,x_n,y_n,z_n]$ and for any $a,b,c\in W(A)$

$$(a+b)+c=(u_0(a_0,b_0,c_0),\dots,u_n(a_0,b_0,c_0,\dots,a_n,b_n,c_n),\dots),$$
$$a+(b+c)=(v_0(a_0,b_0,c_0),\dots,v_n(a_0,b_0,c_0,\dots,a_n,b_n,c_n),\dots).$$

Moreover,

$$\begin{aligned}w_n(u_0,\dots,u_n)&=w_n(f_0(x_0,y_0),f_1(x_0,y_0,x_1,y_1),\dots)+w_n(z_0,\dots,z_n)\\&=w_n(x_0,\dots,x_n)+w_n(y_0,\dots,y_n)+w_n(z_0,\dots,z_n),\end{aligned}$$

and

$$\begin{aligned}w_n(v_0,\dots,v_n)&=w_n(x_0,\dots,x_n)+w_n(f_0(y_0,z_0),f_1(y_0,z_0,y_1,z_1),\dots)\\&=w_n(x_0,\dots,x_n)+w_n(y_0,\dots,y_n)+w_n(z_0,\dots,z_n).\end{aligned}$$

Therefore

$$w_n(u_0,\dots,u_n)=w_n(v_0,\dots,v_n),\qquad \forall n\geqslant 0,$$

and an easy induction shows that $u_n=v_n$ for all n. This proves the associativity of addition.

b) We will show the formula:

$$(x_0,x_1,x_2,\dots)\cdot(y_0,0,0,\dots)=(x_0y_0,x_1y_0^p,x_1y_0^{p^2},\dots). \tag{17}$$

In particular, it implies that $1=(1,0,0,\dots)$ is the identity element of $W(A)$. We have:

$$(x_0,x_1,x_2,\dots)\cdot(y_0,0,0,\dots)=(h_0,h_1,\dots),$$

where $h_0, h_1, \dots$ are some polynomials in $y_0, x_0, x_1 \dots$. We prove by induction that $h_n = x_n y_0^n$. For $n = 0$, we have $h_0 = g_0(x_0, y_0) = x_0 y_0$. Assume that the formula is proved for all $i \leqslant n-1$. We have:

$$w_n(h_0, h_1, \dots, h_n) = w_n(x_0, x_1, \dots, x_n) w_n(y_0, 0, \dots, 0).$$

Hence:

$$h_0^{p^n} + p h_1^{p^{n-1}} + \cdots + p^{n-1} h_1 + p^n h_n = (x_0^{p^n} + p x_1^{p^{n-1}} + \cdots + p^{n-1} x_1 + p^n x_n) y_0^{p^n}.$$

By induction hypothesis, $h_i = x_i y_0^{p^i}$ for $0 \leqslant i \leqslant n-1$. Then $h_n = x_n y_0^{p^n}$, and the statement is proved.

Other properties can be proved by the same method. □

1.5.8. Below, we assemble some properties of the ring $W(A)$:

1) For any homomorphism $\psi : A \to B$, the map

$$W(A) \to W(B), \qquad \psi(a_0, a_1, \dots) = (\psi(a_0), \psi(a_1), \dots)$$

is an homomorphism.

2) If p is invertible in A, then there exists an isomorphism of rings $W(A) \simeq A^{\mathbf{N}}$.

Proof. The map

$$w : W(A) \to A^{\mathbf{N}}, \qquad w(a_0, a_1, \dots) = (w_0(a_0), w_1(a_0, a_1), w_2(a_0, a_1, a_2), \dots)$$

is an homomorphism by the definition of the addition and multiplication in $W(A)$. If p is invertible, then for any $(b_0, b_1, b_2, \dots)$, the system of equations

$$w_0(x_0) = b_0, \quad w_1(x_0, x_1) = b_1, \quad w_2(x_0, x_1, x_2) = b_2, \dots$$

has a unique solution in A. Therefore w is an isomorphism. □

3) For any $a \in A$, define its Teichmüller lift $[a] \in W(A)$ by

$$[a] = (a, 0, 0, \dots).$$

Then $[ab] = [a][b]$ for all $a, b \in A$. This follows from (17).

4) The shift map (Verschiebung)

$$V : W(A) \to W(A), \qquad (a_0, a_1, 0, \dots) \mapsto (0, a_0, a_1, \dots)$$

is additive, i.e. $V(a+b) = V(a) + V(b)$. This can be proved by the same method as for Theorem 1.5.7.

5) For any $n \geqslant 0$, define:

$$I_n(A) = \{(a_0, a_1, \dots) \in W(A) \mid a_i = 0 \text{ for all } 0 \leqslant i \leqslant n\}.$$

Then $(I_n(A))_{n \geqslant 0}$ is a descending chain of ideals, which defines a separable filtration on $W(A)$. Set:

$$W_n(A) := W(A)/I_n(A).$$

Then

$$W(A) = \varprojlim W(A)/I_n(A).$$

We equip $W(A)/I_n(A)$ with the discrete topology and define the *standard topology* on $W(A)$ as the topology of the projective limit. It is clearly Hausdorff. This topology coincides with the topology of the direct product on $W(A)$:

$$W(A) = A \times A \times A \times \cdots,$$

where each copy of A is equipped with the discrete topology. The ideals $I_n(A)$ form a base of neighborhoods of 0 (each open neighborhood of 0 contains $I_n(A)$ for some n).

6) For any $a = (a_0, a_1, \dots) \in W(A)$, one has:

$$(a_0, a_1, a_2, \dots) = \sum_{n=0}^{\infty} V^n[a_n].$$

This can be proved by the same method as for Theorem 1.5.7.

7) If A is a ring of characteristic p, then the map

$$\varphi : W(A) \to W(A), \qquad (a_0, a_1, \dots) \mapsto (a_0^p, a_1^p, \dots),$$

is a ring endomomorphism. In addition,

$$\varphi V = V\varphi = p.$$

Proof. The map φ is induced by the absolute Frobenius

$$\varphi : A \to A, \qquad \varphi(x) = x^p.$$

We should show that

$$p(a_0, a_1, \dots) = (0, a_0^p, a_1^p, \dots).$$

By definition of Witt vectors, the multiplication by p is given by

$$p(a_0, a_1, \dots) = (\bar{h}_0(a_0), \bar{h}_1(a_0, a_1), \dots),$$

where $\bar{h}_n(x_0, x_1, \dots, x_n)$ is the reduction $\mod p$ of the polynomials defined by the relations:

$$w_n(h_0, h_1, \dots, h_n) = pw_n(x_0, x_1, \dots, x_n), \qquad n \geqslant 0.$$

An easy induction shows that $h_n \equiv x_{n-1}^p \pmod{p}$, and the formula is proved. □

Definition. *Let A be a ring of charactersitic p. We say that A is perfect if φ is an isomorphism. We will say that A is semiperfect if φ is surjective.*

Proposition 1.5.9. *Assume that A is an integral perfect ring of characteristic p. The following holds true:*

i) $p^{n+1}W(A) = I_n(A)$.

ii) The standard topology on $W(A)$ coincides with the p-adic topology.

iii) Each $a = (a_0, a_1, \dots) \in W(A)$ can be written as:

$$(a_0, a_1, a_2, \dots) = \sum_{n=0}^{\infty} [a_n^{p^{-n}}]p^n.$$

Proof. i) Since φ is bijective on A (and therefore on $W(A)$), we can write:

$$p^{n+1}W(A) = V^{n+1}\varphi^{-(n+1)}W(A) = V^{n+1}W(A) = I_n(A).$$

ii) This follows directly from i).

iii) One has:

$$(a_0, a_1, a_2, \dots) = \sum_{n=0}^{\infty} V^n([a_n]) = \sum_{n=0}^{\infty} p^n\varphi^{-n}([a_n]) = \sum_{n=0}^{\infty}[a_n^{p^{-n}}]p^n.$$

□

Theorem 1.5.10. *i) Let A be an integral perfect ring of characteristic p. Then there exists a unique, up to an isomorphism, ring R such that:*

a) R is integral of characteristic 0*;*

b) $R/pR \simeq A$;

c) R is complete for the p-adic topology, namely

$$R \simeq \varprojlim_n R/p^nR;$$

ii) The ring $W(A)$ satisfies properties a-c).

Proof. i) See [142, Chapitre II, Théorème 3].

ii) This follows from Proposition 1.5.9. □

1.5.11. **Examples.** 1) $W(\mathbf{F}_p) \simeq \mathbf{Z}_p$.

2) Let $\overline{\mathbf{F}}_p$ be the algebraic closure of $\mathbf{F}_p$. Then $W(\overline{\mathbf{F}}_p)$ is isomorphic to the ring of integers of the p-adic completion $\widehat{\mathbf{Q}}_p^{\mathrm{ur}}$ of $\mathbf{Q}_p^{\mathrm{ur}}$.

1.6. Non-abelian cohomology.

1.6.1. In this section, we review basic results about non abelian cohomology. We refer the reader to [119, Chapter 2, §2 and Theorem 6.2.1] for further detail.

Let G be a topological group. One says that a (not necessarily abelian) topological group M is a G-group if it is equipped with a continuous action of G, i.e. a continuous map

$$G \times M \to M, \qquad (g,m) \mapsto gm$$

such that

$$g(m_1m_2) = g(m_1)\,g(m_2), \qquad \text{if} \qquad g \in G,\ m_1, m_2 \in M,$$
$$(g_1g_2)(m) = g_1(g_2(m)), \qquad \text{if} \qquad g_1, g_2 \in G,\ m \in M.$$

Let M be a G-group. A 1-cocycle with values in M is a continuous map $f : G \to M$ which satisfies the cocycle condition

$$f(g_1g_2) = f(g_1)\,(g_1 f(g_2)), \qquad g_1, g_2 \in G.$$

Two cocycles f_1 and f_2 are said to be homologous if there exists $m \in M$ such that

$$f_2(g) = m\,f_1(g)\,g(m)^{-1}, \qquad g \in G.$$

This defines an equivalence relation $\sim$ on the set $Z^1(G,M)$ of 1-cocycles. The first cohomology $H^1(G,M)$ of G with coefficients in M is defined to be the quotient set

$Z^1(G,M)/\sim$. It is easy to see that if M is abelian, this construction coincides with the usual definition of the first continuous cohomology. In general, $H^1(G,M)$ is not a group but it has a distinguished element which is the class of the trivial cocycle. This allows to consider $H^1(G,M)$ as a pointed set. The following properties of the non-abelian H^1 are sufficient for our purposes:

1) *Inflation-restriction exact sequence.* Let H be a closed normal subgroup of G. Then there exists an exact sequence of pointed sets:

$$0 \to H^1(G/H, M^H) \xrightarrow{\text{inf}} H^1(G,M) \xrightarrow{\text{res}} H^1(H,M)^{G/H}.$$

2) *Hilbert's Theorem* 90. Let E be a field, and F/E be a finite Galois extension. Then $\mathrm{GL}_n(F)$ is a discrete $\mathrm{Gal}(F/E)$-group, and

$$H^1(\mathrm{Gal}(F/E), \mathrm{GL}_n(F)) = 0, \qquad n \geqslant 1.$$

1.6.2. A direct consequence of the non-abelian Hilbert's Theorem 90 is the following fact. Let V be a finite-dimensional F-vector space equipped with a *semi-linear* action of $\mathrm{Gal}(F/E)$:

$$g(x+y) = g(x) + g(y), \qquad \forall x, y \in V,$$
$$g(ax) = g(a)g(x), \qquad \forall a \in F, \forall x \in V.$$

Let $\{e_1, \dots, e_n\}$ be a basis of V. For any $g \in \mathrm{Gal}(F/E)$, let $A_g \in \mathrm{GL}_n(F)$ denote the unique matrix such that

$$g(e_1, \dots, e_n) = (e_1, \dots, e_n)A_g.$$

Then the map

$$f : \mathrm{Gal}(F/E) \to \mathrm{GL}_n(F), \qquad f(g) = A_g$$

is a 1-cocycle. Hilbert's Theorem 90 shows that there exists a matrix B such that the $(e_1, \dots, e_n)B$ is $\mathrm{Gal}(F/E)$-invariant. To sum up, V always has a $\mathrm{Gal}(F/E)$-invariant basis.

Passing to the direct limit, we obtain the following result.

Proposition 1.6.3. *i)* $H^1(G_E, \mathrm{GL}_n(E^{\mathrm{sep}})) = 0$ *for all* $n \geqslant 1$.

ii) Each finite-dimensional E^{sep}*-vector space* V *equipped with a semi-linear discrete action of* G_E *has a* G_E*-invariant basis.*

1.6.4. Let E be a field of characteristic p, and let $\mathscr{E}$ be a complete unramified field with residue field E. Let $\mathscr{E}^{\mathrm{ur}}$ denote the maximal unramified extension of $\mathscr{E}$. The residue field of $\mathscr{E}^{\mathrm{ur}}$ is isomorphic to E^{sep}, and we have an isomorphism of Galois groups:

$$\mathrm{Gal}(\mathscr{E}^{\mathrm{ur}}/\mathscr{E}) \simeq G_E.$$

Let $\widehat{\mathscr{E}}^{\mathrm{ur}}$ denote the p-adic completion of $\mathscr{E}^{\mathrm{ur}}$ and $\widehat{O}^{\mathrm{ur}}_{\mathscr{E}}$ its ring of integers. The following version of Hilbert's Theorem 90 can be proved from Proposition 1.6.3 by devissage.

Proposition 1.6.5. *i)* $H^1(\mathrm{Gal}(\mathscr{E}^{\mathrm{ur}}/\mathscr{E}), \mathrm{GL}_n(\widehat{O}^{\mathrm{ur}}_{\mathscr{E}})) = 0$ *for all* $n \geqslant 1$.

ii) Each free $\widehat{O}^{\mathrm{ur}}_{\mathscr{E}}$*-module equipped with a semi-linear continuous action of* G_E *has a* G_E*-invariant basis.*

2. Galois groups of local fields

2.1. **Unramified and tamely ramified extensions.**

2.1.1. In this section, we review the structure of Galois groups of local fields. Let K be a local field. Fix a separable closure K^{sep} of K, and set $G_K = \mathrm{Gal}(K^{\mathrm{sep}}/K)$. Set $q = |k_K|$. Since the compositum of two unramified (respectively tamely ramified) extensions of K is unramified (respectively tamely ramified,) we have the well defined notions of the maximal unramified (respectively maximal tamely ramified) extension of K. We denote these extensions by K^{ur} and K^{tr} respectively.

2.1.2. The maximal unramified extension K^{ur} of K is procyclic and its Galois group is generated by the Frobenius automorphism Fr_K:

$$\begin{aligned} \mathrm{Gal}(K^{\mathrm{ur}}/K) &\xrightarrow{\sim} \widehat{\mathbf{Z}}, \\ \mathrm{Fr}_K &\longleftrightarrow 1. \\ \mathrm{Fr}_K(x) &\equiv x^q \pmod{\pi_K}, \qquad \forall x \in O_{K^{\mathrm{ur}}}. \end{aligned}$$

2.1.3. Passing to the direct limit in the diagram (7), we have:

$$\begin{tikzcd} \overline{K} \arrow[d, dash, "P_K"] \arrow[dd, dash, bend right, "I_K"'] \\ K^{\mathrm{tr}} \arrow[d, dash] \\ K^{\mathrm{ur}} \arrow[d, dash, "\widehat{\mathbf{Z}}"'] \\ K \end{tikzcd} \tag{18}$$

Consider the exact sequence:

$$1 \to \mathrm{Gal}(K^{\mathrm{tr}}/K^{\mathrm{ur}}) \to \mathrm{Gal}(K^{\mathrm{tr}}/K) \to \mathrm{Gal}(K^{\mathrm{ur}}/K) \to 1. \tag{19}$$

Here $\mathrm{Gal}(K^{\mathrm{ur}}/K) \simeq \widehat{\mathbf{Z}}$. From the explicit description of tamely ramified extensions, it follows that K^{tr} is generated over K^{ur} by the roots $\pi_K^{1/n}$, $(n,p) = 1$ of any uniformizer π_K of K. This immediately implies that

$$\mathrm{Gal}(K^{\mathrm{tr}}/K^{\mathrm{ur}}) \simeq \prod_{\ell \neq p} \mathbf{Z}_\ell. \tag{20}$$

Let τ_K be a topological generator of $\mathrm{Gal}(K^{\mathrm{tr}}/K^{\mathrm{ur}})$. Fix a lift of the Frobenius automorphism Fr_K to an element $\widehat{\mathrm{Fr}}_K \in \mathrm{Gal}(K^{\mathrm{tr}}/K)$. Analyzing the action of these elements on the elements $\pi_K^{1/n}$, one can easily determine the structure of $\mathrm{Gal}(K^{\mathrm{tr}}/K)$.

Proposition 2.1.4 (Iwasawa)**.** *The group* $\mathrm{Gal}(K^{\mathrm{tr}}/K)$ *is topologically generated by the automorphisms* $\widehat{\mathrm{Fr}}_K$ *and* τ_K *with the only relation:*

$$\widehat{\mathrm{Fr}}_K\, \tau_K\, \widehat{\mathrm{Fr}}_K^{-1} = \tau_K^q. \tag{21}$$

Proof. See [89] or [119, Theorem 7.5.3]. From (19), it follows that $\mathrm{Gal}(K^{\mathrm{tr}}/K)$ is topologically generated by $\widehat{\mathrm{Fr}}_K$ and τ_K. The relation (21) follows from the explicit action of τ_K and $\widehat{\mathrm{Fr}}_K$ on $\pi_K^{1/n}$ for $(n,p)=1$. □

2.2. **Local class field theory.**

2.2.1. Let K^{ab} denote the maximal abelian extension of K. Then $\mathrm{Gal}(K^{\mathrm{ab}}/K)$ is canonically isomorphic to the abelianization $G_K^{\mathrm{ab}} = G_K/[G_K,G_K]$ of G_K. Local class field theory gives an explicit description of $\mathrm{Gal}(K^{\mathrm{ab}}/K)$ in terms of K. Namely, there exists a canonical injective homomorphism (called the reciprocity map) with dense image

$$\theta_K\,:\,K^*\to \mathrm{Gal}(K^{\mathrm{ab}}/K)$$

such that:

i) For any finite abelian extension L/K, the homomorphism θ_K induces an isomorphism

$$\theta_{L/K}\,:\,K^*/N_{L/K}(L^*)\xrightarrow{\sim}\mathrm{Gal}(L/K),$$

where $N_{L/K}$ is the norm map;

ii) If L/K is unramified, then for any uniformizer $\pi\in K^*$ the automorphism $\theta_{L/K}(\pi)$ coincides with the arithmetic Frobenius $\mathrm{Fr}_{L/K}$;

iii) For any $x\in K^*$, the automorphism $\theta_K(x)$ acts on K^{ur} as:

$$\theta_K(x)|_{K^{\mathrm{ur}}} = \mathrm{Fr}_K^{v_K(x)}.$$

The reciprocity map is compatible with the canonical filtrations of K^* and $\mathrm{Gal}(K^{\mathrm{ab}}/K)^{(v)}$. Namely, for any real $v\geqslant 0$ set $U_K^{(v)}=U_K^{(n)}$, where n is the smallest integer $\geqslant v$. Then

$$\theta_K\left(U_K^{(v)}\right)=\mathrm{Gal}(K^{\mathrm{ab}}/K)^{(v)},\qquad \forall v\geqslant 0. \tag{22}$$

For the classical proof of this result, see [142, Chapter XV].

2.2.2. The theory of Lubin–Tate [111] (see also [140]) gives an explicit construction of K^{ab} in terms of torsion points of formal groups with a "big" endomorphism ring, and describes the action of the Galois group $\mathrm{Gal}(K^{\mathrm{ab}}/K)$ on these points. In particular, it gives a simple and natural proof of (22). This theory can be seen as a local analog of the theory of complex multiplication, providing the solution of Hilbert's twelfth problem for local fields. We review it in Section 15 below.

2.2.3. Local class field theory was generalized to the infinite residue field case by Serre, Hazewinkel and Suzuki–Yoshida [53, 138, 149]. In another direction, Parshin and Kato developed the class field theory of higher-dimensional local fields [91, 122, 123]. We refer the reader to [63] for survey articles and further references.

2.3. **The absolute Galois group of a local field.**

2.3.1. First, we review the structure of the Galois group of the maximal p-extension of a local field. A finite Galois extension of K is a p-extension if its degree is a power of $p = \mathrm{char}(k_K)$. It is easy to see that p-extensions form a distinguished class, and we can define the maximal pro-p-extension $K(p)$ of K as the compositum of all finite p-extensions. Set $G_K(p) = \mathrm{Gal}(K(p)/K)$.

First assume that $\mathrm{char}(K) = p$. We have the Artin–Schreier exact sequence

$$0 \to \mathbf{F}_p \to K(p) \xrightarrow{\wp} K(p) \to 0,$$

where $\wp(x) = x^p - x$. Taking the associated long exact cohomology sequence and using the fact that $H^i(G_K(p), K(p)) = 0$ for $i \geqslant 1$, we obtain:

$$H^1(G_K(p), \mathbf{F}_p) = K(p)/\wp(K(p)), \qquad H^2(G_K(p), \mathbf{F}_p) = 0.$$

General results about pro-p-groups (see, for example, [99, Chapter 6] say that
(23)

$$\dim_{\mathbf{F}_p} H^1(G_K(p), \mathbf{F}_p) = \text{cardinality of a minimal system of generators of } G_K(p);$$
$$\dim_{\mathbf{F}_p} H^2(G_K(p), \mathbf{F}_p) = \text{cardinality of a minimal relation system of } G_K(p).$$

This leads to the following theorem:

Theorem 2.3.2. *If* $\mathrm{char}(K) = p$, *then* $G_K(p)$ *is a free pro-p-group of countable infinite rank.*

The situation is more complicated in the inequal characteristic case. Let K be a finite extension of $\mathbf{Q}_p$ of degree N. For any n, let μ_n denote the group of nth roots of unity.

Theorem 2.3.3 (Shafarevich, Demushkin)**.** *Assume that* $\mathrm{char}(K) = 0$.

i) If K doesn't contain the group μ_p, then $G_K(p)$ is a free pro-p-group of rank $N+1$.

ii) If K contains μ_p, then $G_K(p)$ is a pro-p-group of rank $N+2$, and there exists a system of generators $g_1, g_2, \dots, g_{N+2}$ of $G_K(p)$ with the only relation:

$$g_1^{p^s}[g_1, g_2][g_3, g_4] \cdots [g_{N+1}, g_{N+2}] = 1, \tag{24}$$

where p^s denotes the highest p-power such that K contains μ_{p^s}

Comments on the proof. The Poincaré duality in local class field theory gives perfect pairings:

$$H^i(G_K(p), \mathbf{F}_p) \times H^{2-i}(G_K(p), \mu_p) \to H^2(G_K(p), \mu_p) \simeq \mathbf{F}_p, \qquad 0 \leqslant i \leqslant 2.$$

Therefore we have:

$$H^1(G_K(p), \mathbf{F}_p) \simeq (K^*/K^{*p})^\vee, \qquad H^2(G_K(p), \mathbf{F}_p) \simeq \mu_p(K)^\vee,$$

where $^\vee$ denotes the duality of $\mathbf{F}_p$-vector spaces. Assume that K doesn't contain the group μ_p. Then these isomorphisms give:

$$\dim_{\mathbf{F}_p} H^1(G_K(p), \mathbf{F}_p) = N + 1,$$
$$H^2(G_K(p), \mathbf{F}_p) = 0.$$

Now from (23) we obtain that $G_K(p)$ is free of rank $N+1$. Note that this was first proved by Shafarevich [145] by another method.

Assume now that K contains μ_p. In this case, we have:

$$\dim_{\mathbf{F}_p} H^1(G_K(p),\mathbf{F}_p) = N+2,$$
$$H^2(G_K(p),\mathbf{F}_p) = 1.$$

Therefore $G_K(p)$ can be generated by $N+2$ elements $g_1,\dots,g_{N+2}$ with only one relation. In [54], Demushkin proved that $g_1,\dots,g_{N+2}$ can be chosed in such a way that (24) holds. See also [139] and [101].

2.3.4. The structure of the absolute Galois group in the characteristic p case can be determined using the above arguments. One easily sees that the wild inertia subgroup P_K is pro-p-free with a countable number of generators. This allows to describe G_K as an explicit semi-direct product of the tame Galois group $\mathrm{Gal}(K^{\mathrm{tr}}/K)$ and P_K (see [98] or [119, Theorem 7.5.13]). The characteristic 0 case is much more difficult. If K is a finite extension of $\mathbf{Q}_p$, the structure of the G_K in terms of generators and relations was first described by Yakovlev [163] under additional assumption $p \neq 2$. A simpler description was found by Jannsen and Wingberg in [90]. For the case $p=2$, see [164, 165].

2.3.5. The ramification filtration $(G_K^{(v)})$ on G_K has a highly non-trivial structure. We refer the reader to [79, 1, 2, 4, 7] for known results in this direction. Abrashkin [5] and Mochizuki [113] proved that a local field can be completely determined by its absolute Galois group together with the ramification filtration. In another direction, Weinstein [157] interpreted $G_{\mathbf{Q}_p}$ as the fundamental group of some "perfectoid" object.

3. $\mathbf{Z}_p$-EXTENSIONS

3.1. The different in $\mathbf{Z}_p$-extensions.

3.1.1. The results of this section were proved by Tate [151]. We start with illustrating the ramification theory with the example of $\mathbf{Z}_p$-extensions. Let K be a local field of characteristic 0. Set $e = e(K/\mathbf{Q}_p)$. Let $v_K : \overline{K} \to \mathbf{Q} \cup \{+\infty\}$ denote the extension of the discrete valuation on K to $\overline{K}$.

Definition. *A $\mathbf{Z}_p$-extension is a Galois extension whose Galois group is topologically isomorphic to $\mathbf{Z}_p$.*

Let K_∞/K be a $\mathbf{Z}_p$-extension. Set $\Gamma = \mathrm{Gal}(K_\infty/K)$. For any n, $p^n\mathbf{Z}_p$ is the unique open subgroup of $\mathbf{Z}_p$ of index p^n, and we denote by $\Gamma(n)$ the corresponding subgroup of Γ. Set $K_n = K_\infty^{\Gamma(n)}$. Then K_n is the unique subextension of K_∞/K of degree p^n over K, and

$$K_\infty := \bigcup_{n\geqslant 1} K_n, \qquad \mathrm{Gal}(K_n/K) \simeq \mathbf{Z}/p^n\mathbf{Z}.$$

Assume that K_∞/K is totally ramified. Let $(v_n)_{n\geqslant 0}$ denote the increasing sequence of ramification jumps of K_∞/K. Since $\Gamma\simeq \mathbf{Z}_p$, and all quotients $\Gamma^{(v_n)}/\Gamma^{(v_{n+1})}$ are p-elementary, we obtain that

$$\Gamma^{(v_n)}=p^n\mathbf{Z}_p,\qquad \forall n\geqslant 0.$$

Proposition 3.1.2. *Let K_∞/K be a totally ramified $\mathbf{Z}_p$-extension.*
i) There exists n_0 such that

$$v_{n+1}=v_n+e,\qquad \forall n\geqslant n_0.$$

ii) There exists a constant c such that

$$v_K(\mathfrak{D}_{K_n/K})=en+c+p^{-n}a_n,$$

where the sequence $(a_n)_{n\geqslant 0}$ is bounded.

This is [151, Proposition 5]. Below, we reproduce Tate's proof, which uses local class field theory. See also [73, Proposition 1.11].

The following lemma is a classical and well known statement.

Lemma 3.1.3. *i) The series*

$$\log(1+x)=\sum_{m=1}^{\infty}(-1)^{m+1}\frac{x^m}{m}$$

converges for all $x\in\mathfrak{m}_K$.
ii) The series

$$\exp(x)=\sum_{m=0}^{\infty}\frac{x^m}{m!}$$

converges for all x such that $v_K(x)>\frac{e}{p-1}$.
iii) For any integer $n>\frac{e}{p-1}$ we have isomorphisms:

$$\log\,:\,U_K^{(n)}\to\mathfrak{m}_K^n,\qquad \exp\,:\,\mathfrak{m}_K^n\to U_K^{(n)},$$

which are inverse to each other.

Corollary 3.1.4. *For any integer $n>\frac{e}{p-1}$, one has:*

$$\left(U_K^{(n)}\right)^p=U_K^{(n+e)}.$$

Proof. $\left(U_K^{(n)}\right)^p$ and $U_K^{(n+e)}$ have the same image under $\log$. □

3.1.5. *Proof of Proposition 3.1.2.* a) Let $\Gamma=\mathrm{Gal}(K_\infty/K)$. By Galois theory, $\Gamma=G_K^{\mathrm{ab}}/H$, where $H\subset G_K^{\mathrm{ab}}$ is a closed subgroup. Consider the exact sequence

$$\{1\}\to\mathrm{Gal}(K^{\mathrm{ab}}/K^{\mathrm{ur}})\to G_K^{\mathrm{ab}}\xrightarrow{s}\mathrm{Gal}(K^{\mathrm{ur}}/K)\to\{1\}.$$

Since K_∞/K is totally ramified, $(K^{\mathrm{ab}})^H\cap K^{\mathrm{ur}}=K$, and $s(H)=\mathrm{Gal}(K^{\mathrm{ur}}/K)$. Therefore

$$\Gamma\simeq\mathrm{Gal}(K^{\mathrm{ab}}/K^{\mathrm{ur}})/(H\cap\mathrm{Gal}(K^{\mathrm{ab}}/K^{\mathrm{ur}})).$$

By local class field theory, $\mathrm{Gal}(K^{\mathrm{ab}}/K^{\mathrm{ur}}) \simeq U_K$, and there exists a closed subgroup $N \subset U_K$ such that

$$\Gamma \simeq U_K/N.$$

The order of $U_K/U_K^{(1)} \simeq k_K^*$ is coprime with p. Hence the index of $U_K^{(1)}/(N\cap U_K^{(1)})$ in U_K/N is coprime with p. On the other hand, $U_K/N \simeq \Gamma$ is a pro-p-group. Therefore

$$U_K^{(1)}/(N\cap U_K^{(1)}) = U_K/N,$$

and we have an isomorphism:

$$\rho\,:\,\Gamma \simeq U_K^{(1)}/(N\cap U_K^{(1)}).$$

b) To simplify notation, set:

$$\mathscr{U}^{(v)} = U_K^{(v)}/(N\cap U_K^{(v)}), \qquad \forall v \geqslant 1.$$

By (22) and (10), we have:

$$\rho(\Gamma^{(v)}) \simeq \mathscr{U}^{(v)}, \qquad v \geqslant 1.$$

Let γ be a topological generator of Γ. Then $\gamma_n = \gamma^{p^n}$ is a topological generator of $\Gamma(n)$. Let n_0 be an integer such that

$$\rho(\gamma_{n_0}) \in \mathscr{U}^{(m_0)},$$

with some integer $m_0 > \frac{e}{p-1}$. Fix such n_0 and assume that, for this fixed n_0, m_0 is the biggest integer satisfying this condition. Since γ_{n_0} is a generator of $\Gamma(n_0)$, this means that

$$\rho(\Gamma(n_0)) = \mathscr{U}^{(m_0)}, \qquad \text{but} \qquad \rho(\Gamma(n_0)) \neq \mathscr{U}^{(m_0+1)}.$$

Hence m_0 is the n_0-th ramification jump for K_∞/K, i.e.

$$m_0 = v_{n_0}.$$

We can write $\rho(\gamma_{n_0}) = \overline{x}$, where $\overline{x} = x \pmod{(N\cap U_K^{(m_0)})}$ and $x \in U_K^{(m_0)} \setminus U_K^{(m_0+1)}$. By Corollary 3.1.4,

$$x^{p^n} \in U_K^{(m_0+en)} \setminus U_K^{(m_0+en+1)}, \qquad \forall n \geqslant 0.$$

Since $\rho(\gamma_{n_0+n}) = \overline{x}^{p^n}$, and γ_{n_0+n} is a generator of $\Gamma(m_0+n)$, this implies that

$$\rho(\Gamma(n_0+n)) = \mathscr{U}^{(m_0+ne)}, \quad \text{and} \quad \rho(\Gamma(n_0+n)) \neq \mathscr{U}^{(m_0+ne+1)}.$$

This shows that for each integer $n \geqslant 0$, the ramification filtration has a jump at m_0+ne, and

$$\Gamma^{(m_0+ne)} = \Gamma(n_0+n).$$

In other terms, for any *real* $v \geqslant v_{n_0} = m_0$, we have:

$$\Gamma^{(v)} = \Gamma(n_0+n+1) \qquad \text{if} \qquad v_{n_0}+ne < v \leqslant v_{n_0}+(n+1)e.$$

This shows that $v_{n_0+n} = v_{n_0}+en$ for all $n \geqslant 0$, and assertion i) is proved.

c) We prove ii) applying formula (11). For any $n > 0$, set $G(n) = \Gamma/\Gamma(n)$. We have

$$v_K(\mathfrak{D}_{K_n/K}) = \int_{-1}^{\infty}\left(1-\frac{1}{|G(n)^{(v)}|}\right)dv.$$

By Herbrand's theorem, $G(n)^{(v)} = \Gamma^{(v)}/(\Gamma(n) \cap \Gamma^{(v)})$. Since $\Gamma^{(v_n)} = \Gamma(n)$, the ramification jumps of $G(n)$ are $v_0, v_1, \dots, v_{n-1}$, and we have:

$$|G(n)^{(v)}| = \begin{cases} p^{n-i}, & \text{if } v_{i-1} < v \leqslant v_i, \\ 1, & \text{if } v > v_{n-1} \end{cases} \tag{25}$$

(for $i = 0$, we set $v_{i-1} := 0$ to uniformize notation). Assume that $n > n_0$. Then

$$v_K(\mathfrak{D}_{K_n/K}) = A + \int_{v_{i_0}}^{v_{n-1}} \left(1 - \frac{1}{|G(n)^{(v)}|}\right) dv,$$

where $A = \displaystyle\int_{-1}^{v_{n_0}} \left(1 - \frac{1}{|G(n)^{(v)}|}\right) dv$. We evaluate the second integral using i) and (25):

$$\int_{v_{n_0}}^{v_{n-1}} \left(1 - \frac{1}{|G(n)^{(v)}|}\right) dv \;=\; \sum_{i=n_0+1}^{n-1} (v_i - v_{i-1}) \left(1 - \frac{1}{|G(n)^{(v)}|}\right) \;=\; \sum_{i=n_0+1}^{n-1} e\left(1 - \frac{1}{p^{n-i}}\right).$$

Now an easy computation gives:

$$\sum_{i=n_0+1}^{n-1} e\left(1 - \frac{1}{p^{n-i}}\right) = e(n - n_0 - 1) + \frac{e}{p-1}\left(1 - \frac{1}{p^{n-n_0-1}}\right).$$

Setting $c = A - e(n_0 + 1) + \frac{e}{p-1}$, we see that for $n > n_0$,

$$v_K(\mathfrak{D}_{K_n/K}) = c + en - \frac{1}{(p-1)p^{n-n_0-1}}.$$

This implies the proposition.

□

Remark 3.1.6. *Proposition 3.1.2 shows that the ramified $\mathbf{Z}_p$-extensions are arithmetically profinite in the sense of Section 6.1.*

3.2. **The normalized trace.**

3.2.1. In this section, K_∞/K is a totally ramified $\mathbf{Z}_p$-extension. Fix a topological generator γ of Γ. For any $x \in K_n$, set:

$$\mathrm{T}_{K_\infty/K}(x) = \frac{1}{p^n} \mathrm{Tr}_{K_n/K}(x).$$

It is clear that this definition does not depend on the choice of n. Therefore we have a well defined homomorphism

$$\mathrm{T}_{K_\infty/K} \,:\, K_\infty \to K.$$

Note that $\mathrm{T}_{K_\infty/K}(x) = x$ for $x \in K$. Our first goal is to prove that $\mathrm{T}_{K_\infty/K}$ is continuous. It is probably more natural to state the results of this section in terms of absolute values rather that in terms of valuations. Let $|\cdot|_K$ denote the absolute value on $\overline{K}$ associated to v_K.

Proposition 3.2.2. *i) There exists a constant $c > 0$ such that*

$$|\mathrm{T}_{K_\infty/K}(x) - x|_K \leqslant c|\gamma(x) - x|_K, \qquad \forall x \in K_\infty.$$

ii) The map $\mathrm{T}_{K_\infty/K}$ is continuous and extends by continuity to $\widehat{K}_\infty$.

Proof. a) By Proposition 3.1.2, $v_K(\mathfrak{D}_{K_n/K_{n-1}}) = e_K + \alpha_n p^{-n}$, where α_n is bounded. Applying Lemma 1.4.2 to the extension K_n/K_{n-1}, we obtain that

$$|\mathrm{Tr}_{K_n/K_{n-1}}(x)|_K \leqslant |p|_K^{1-b/p^n} |x|_K, \qquad \forall x \in K_n, \tag{26}$$

with some constant $b > 0$ which does not depend on n.

b) Set $\gamma_n = \gamma^{p^n}$. For any $x \in K_n$ we have:

$$\mathrm{Tr}_{K_n/K_{n-1}}(x) = \sum_{i=0}^{p-1} \gamma_{n-1}^i(x),$$

Therefore

$$\mathrm{Tr}_{K_n/K_{n-1}}(x) - px = \sum_{i=0}^{p-1} (\gamma_{n-1}^i(x) - x) = \sum_{i=1}^{p-1} (1 + \gamma_{n-1} + \cdots + \gamma_{n-1}^{i-1})(\gamma_{n-1}(x) - x).$$

and we obtain that

$$\left| \frac{1}{p} \mathrm{Tr}_{K_n/K_{n-1}}(x) - x \right|_K \leqslant |p|_K^{-1} \cdot |\gamma_{n-1}(x) - x|_K, \qquad \forall x \in K_n.$$

Since $\gamma_{n-1}(x) - x = (1 + \gamma + \cdots + \gamma^{p^{n-1}-1})(\gamma(x) - x)$, we also have:

$$\left| \frac{1}{p} \mathrm{Tr}_{K_n/K_{n-1}}(x) - x \right|_K \leqslant |p|^{-1} \cdot |\gamma(x) - x|_K, \qquad \forall x \in K_n. \tag{27}$$

c) By induction on n, we prove that

$$\left| \mathrm{T}_{K_\infty/K}(x) - x \right|_K \leqslant c_n \cdot |\gamma(x) - x|_K, \qquad \forall x \in K_n, \tag{28}$$

where $c_1 = |p|_K$ and $c_n = c_{n-1} \cdot |p|_K^{-b/p^n}$. For $n = 1$, this follows from formula (27). For $n \geqslant 2$ and $x \in K_n$, we write:

$$\mathrm{T}_{K_\infty/K}(x) - x = \left(\frac{1}{p} \mathrm{Tr}_{K_n/K_{n-1}}(x) - x \right) + (\mathrm{T}_{K_\infty/K}(y) - y), \qquad y = \frac{1}{p} \mathrm{Tr}_{K_n/K_{n-1}}(x).$$

The first term can be bounded using formula (27). For the second term, we have:

$$\begin{aligned} |\mathrm{T}_{K_\infty/K}(y) - y|_K \leqslant c_{n-1} |\gamma(y) - y|_K &= c_{n-1} |p|_K^{-1} \cdot |\mathrm{Tr}_{K_n/K_{n-1}}(\gamma(x) - x)|_K \\ &\leqslant c_{n-1} |p|_K^{-b/p^n} |\gamma(x) - x|_K. \end{aligned}$$

(Here the last inequality follows from (26)). This proves (28).

d) Set $c = c_1 \prod_{n=1}^{\infty} |p|_K^{-b/p^n} = c_1 |p|_K^{-b/(p-1)}$. Then $c_n < c$ for all $n \geqslant 1$. From formula (28), we obtain:

$$\left| \mathrm{T}_{K_\infty/K}(x) - x \right|_K \leqslant c \cdot |\gamma(x) - x|_K, \qquad \forall x \in K_\infty.$$

This proves the first assertion of the proposition. The second assertion is its immediate consequence. $\square$

Definition. *The map* $\mathrm{T}_{K_\infty/K} : \widehat{K}_\infty \to K$ *is called the normalized trace.*

3.2.3. Since $\mathrm{T}_{K_\infty/K}$ is an idempotent map, we have:

$$\widehat{K}_\infty = K \oplus \widehat{K}_\infty^\circ,$$

where $K_\infty^\circ = \ker(\mathrm{T}_{K_\infty/K})$.

Theorem 3.2.4 (Tate). *i) The operator $\gamma - 1$ is bijective, with a continuous inverse, on $\widehat{K}_\infty^\circ$.*

ii) For any $\lambda \in U_K^{(1)}$ which is not a root of unity, the map $\gamma - \lambda$ is bijective, with a continuous inverse, on $\widehat{K}_\infty$.

Proof. a) Write $K_n = K \oplus K_n^\circ$, where $K_n^\circ = \ker(\mathrm{T}_{K_\infty/K}) \cap K_n$. Since $\gamma - 1$ is injective on K_n°, and K_n° has finite dimension over K, $\gamma - 1$ is bijective on K_n° and on $K_\infty^\circ = \underset{n\geqslant 0}{\cup} K_n^\circ$. Let $\rho : K_\infty^\circ \to K_\infty^\circ$ denote its inverse. From Proposition 3.2.2, it follows that

$$|x|_K \leqslant c|(\gamma - 1)(x)|_K, \qquad \forall x \in K_\infty^\circ,$$

and therefore

$$|\rho(x)|_K \leqslant c|x|_K, \qquad \forall x \in K_\infty^\circ.$$

Thus ρ is continuous and extends to $\widehat{K}_\infty^\circ$. This proves the theorem for $\lambda = 1$.

b) Assume that $\lambda \in U_K^{(1)}$ is such that

$$|\lambda - 1|_K < c^{-1}.$$

Then $\rho(\gamma - \lambda) = 1 + (1 - \lambda)\rho$, and the series

$$\theta = \sum_{i=0}^{\infty} (\lambda - 1)^i \rho^i$$

converges to an operator θ such that $\rho\theta(\gamma - \lambda) = 1$. Thus $\gamma - \lambda$ is invertible on $\widehat{K}_\infty^\circ$. Since $\lambda \neq 1$, it is also invertible on K.

c) In the general case, we choose n such that $|\lambda^{p^n} - 1|_K < c^{-1}$. By assumptions, $\lambda^{p^n} \neq 1$. Applying part b) to the operator $\gamma^{p^n} - \lambda^{p^n}$, we see that it is invertible on $\widehat{K}_\infty^\circ$. Since

$$\gamma^{p^n} - \lambda^{p^n} = (\gamma - \lambda) \sum_{i=0}^{p^n - 1} \gamma^{p^n - i - 1} \lambda^i,$$

the operator $\gamma - \lambda$ is also invertible, and the theorem is proved. □

3.3. Application to continuous cohomology.

3.3.1. We apply the results of the previous section to the computation of some continuous cohomology of Γ. For any continuous character $\eta : \Gamma \to U_K$, we denote by $\widehat{K}_\infty(\eta)$ the group $\widehat{K}_\infty$ equipped with the natural action of Γ twisted by η:

$$(g, x) \mapsto \eta(g) \cdot g(x), \qquad g \in \Gamma, \quad x \in \widehat{K}_\infty.$$

Let $H^n(\Gamma, -)$ denote the *continuous* cohomology of Γ (see, for example, [119, Chapter II, §7] for definition).

Theorem 3.3.2 (Tate)**.** *i) $H^0(\Gamma, \widehat{K}_\infty) = K$ and $H^0(\Gamma, \widehat{K}_\infty(\eta)) = 0$ for any continuous character $\eta : \Gamma \to U_K$ with infinite image.*

ii) $H^1(\Gamma, \widehat{K}_\infty)$ is a one-dimensional vector space over K, and $H^1(\Gamma, \widehat{K}_\infty(\eta)) = 0$ for any character $\eta : \Gamma \to U_K$ with infinite image.

Proof. i) The first statement follows directly from Theorem 3.2.4.

ii) Since Γ is procyclic, any cocycle $f : \Gamma \to \widehat{K}_\infty(\eta)$ is completely determined by $f(\gamma)$. This gives an isomorphism between $H^1(\Gamma, \widehat{K}_\infty(\eta))$ and the cokernel of $\gamma - \eta(\gamma)$. Applying again Theorem 3.2.4, we obtain ii). □

4. Deeply ramified extensions

4.1. **Deeply ramified extensions.**

4.1.1. In this section, we review the theory of deeply ramified extensions of Coates– Greenberg [37]. This theory goes back to Tate's paper [151], where the case of $\mathbf{Z}_p$-extensions was studied and applied to the proof of the Hodge–Tate decomposition for p-divisible groups.

Let K_∞/K be an infinite algebraic extension of a local field K of characteristic 0. Recall that for each m, the number of algebraic extensions of K of degree m is finite. Hence we can always write K_∞ in the form

$$K_\infty = \bigcup_{n=0}^{\infty} K_n, \qquad K_0 = K, \qquad K_n \subset K_{n+1}, \qquad [K_n : K] < \infty.$$

Following [75], we define the different of K_∞/K as the intersection of the differents of its finite subextensions:

Definition. *The different of K_∞/K is defined as:*

$$\mathfrak{D}_{K_\infty/K} = \bigcap_{n=0}^{\infty} (\mathfrak{D}_{K_n/K} O_{K_\infty}).$$

4.1.2. Let L_∞ be a finite extension of K_∞. Then $L_\infty = K_\infty(\alpha)$, where α is a root of an irreducible polynomial $f(X) \in K_\infty[X]$. The coefficients of $f(X)$ belong to a finite extension K_f of K. Set:

$$n_0 = \min\{n \in \mathbf{N} \mid f(X) \in K_n[X]\}.$$

Let $L_n = K_n(\alpha)$ for all $n \geqslant n_0$. Then

$$L_\infty = \bigcup_{n=n_0}^{\infty} L_n.$$

In what follows, we will assume that $n_0 = 0$ without loss of generality. Note that the degree $[L_n : K_n] = \deg(f)$ does not depend on $n \geqslant 0$.

Proposition 4.1.3. *i) If $m \geqslant n$, then*

$$\mathfrak{D}_{L_n/K_n} O_{L_m} \subset \mathfrak{D}_{L_m/K_m}.$$

ii) One has:

$$\mathfrak{D}_{L_\infty/K_\infty} = \bigcup_{n=0}^{\infty} (\mathfrak{D}_{L_n/K_n} O_{L_\infty}).$$

Proof. i) We consider the trace pairing (3):

$$t_{L_n/K_n} \,:\, L_n \times L_n \to K_n.$$

Let $\{e_k\}_{k=1}^s$ be a basis of O_{L_n} over O_{K_n}, and let $\{e_k^*\}_{k=1}^s$ denote the dual basis. Then

$$\mathfrak{D}_{L_n/K_n} = O_{L_n}e_1^* + \cdots + O_{L_n}e_s^*.$$

Since $\{e_k\}_{k=1}^s$ is also a basis of L_m over K_m, any $x \in \mathfrak{D}_{L_m/K_m}^{-1}$ can be written as

$$x = \sum_{k=1}^{s} a_k e_k^*.$$

Then

$$a_k = t_{L_m/K_m}(x, e_k) \in O_{K_m}, \qquad \forall 1 \leqslant k \leqslant s,$$

and we have:

$$x \in O_{K_m}e_1^* + \cdots + O_{K_m}e_s^* \subset \mathfrak{D}_{L_n/K_n}^{-1} O_{L_m}.$$

Hence $\mathfrak{D}_{L_m/K_m}^{-1} \subset \mathfrak{D}_{L_n/K_n}^{-1} O_{L_m}$, and therefore $\mathfrak{D}_{L_n/K_n} O_{L_m} \subset \mathfrak{D}_{L_m/K_m}$.

ii) By the same argument as in the proof of i), the following holds:

$$\bigcup_{n=0}^{\infty} (\mathfrak{D}_{L_n/K_n} O_{L_\infty}) \subset \mathfrak{D}_{L_\infty/K_\infty}.$$

We need to prove that $\mathfrak{D}_{L_\infty/K_\infty} \subset \bigcup\limits_{n=0}^{\infty} (\mathfrak{D}_{L_n/K_n} O_{L_\infty})$ or, equivalently, that

$$\bigcap_{n=0}^{\infty} (\mathfrak{D}_{L_n/K_n}^{-1} O_{L_\infty}) \subset \mathfrak{D}_{L_\infty/K_\infty}^{-1}.$$

Let $x \in \bigcap\limits_{n=0}^{\infty} (\mathfrak{D}_{L_n/K_n}^{-1} O_{L_\infty})$ and $y \in O_{L_\infty}$. Choosing n such that $x \in \mathfrak{D}_{L_n/K_n}^{-1}$ and $y \in O_{L_n}$, we have:

$$t_{L_\infty/K_\infty}(x, y) = t_{L_n/K_n}(x, y) \in O_{K_n} \subset O_{K_\infty}.$$

The proposition is proved. □

4.1.4. For any algebraic extension M/K of local fields (finite or infinite) we set:

$$v_K(\mathfrak{D}_{M/K}) = \inf\{v_K(x) \mid x \in \mathfrak{D}_{M/K}\}.$$

Definition. *i) We say that K_∞/K has finite conductor if there exists $v \geqslant 0$ such that $K_\infty \subset \overline{K}^{(v)}$. If that is the case, we call the conductor of K_∞/K the number*

$$c(K_\infty) = \inf\{v \mid K_\infty \subset \overline{K}^{(v-1)}\}.$$

ii) We say that K_∞/K is deeply ramified if it does not have finite conductor.

Below, we give some examples of deeply ramified extensions.

4.1.5. **Examples.** 1) The cyclotomic extension $K(\zeta_{p^\infty})/K$ is deeply ramified. This follows from Proposition 3.1.2.

2) Fix a uniformizer π of K and set $\pi_n = \pi^{1/p^n}$. Then the infinite Kummer extension $K(\pi^{1/p^\infty}) = \bigcup\limits_{n=1}^{\infty} K(\pi_n)$ is deeply ramified. This can be proved by a direct computation or alternatively computing the different of this extension and using Theorem 4.1.7 below.

3) Let K_∞/K be a totally ramified infinite Galois extension such that its Galois group $G = \mathrm{Gal}(K_\infty/K)$ is a Lie group. From Theorem 1.3.11, it follows that K_∞/K is deeply ramified. We will come back to this example in Section 6.

4.1.6. Now we state our main theorem about deeply ramified extensions.

Theorem 4.1.7 (Coates–Greenberg)**.** *Let K_∞/K be an algebraic extension of local fields. Then the following assertions are equivalent:*

i) $v_K(\mathfrak{D}_{K_\infty/K}) = +\infty$;

ii) K_∞/K is deeply ramified;

iii) For any finite extension L_∞/K_∞ one has:

$$v_K(\mathfrak{D}_{L_\infty/K_\infty}) = 0;$$

iv) For any finite extension L_∞/K_∞ one has:

$$\mathrm{Tr}_{L_\infty/K_\infty}(\mathfrak{m}_{L_\infty}) = \mathfrak{m}_{K_\infty}.$$

In sections 4.1.8-4.1.12 below, we prove the implications

$$i) \Leftrightarrow ii) \Rightarrow iii) \Rightarrow iv).$$

Lemma 4.1.8. *For any finite extension M/K, one has:*

$$\frac{c(M)}{2} \leqslant v_K(\mathfrak{D}_{M/K}) \leqslant c(M).$$

Proof. We have:

$$[M : M \cap \overline{K}^{(v)}] = 1, \quad \text{for any } v > c(M) - 1;$$

$$[M : M \cap \overline{K}^{(v)}] \geqslant 2, \quad \text{if } -1 \leqslant v < c(M) - 1.$$

Therefore

$$v_K(\mathfrak{D}_{M/K}) = \int_{-1}^{\infty} \left(1 - \frac{1}{[M : M \cap \overline{K}^{(v)}]}\right) dv \leqslant \int_{-1}^{c(M)-1} dv = c(M),$$

and

$$v_K(\mathfrak{D}_{M/K}) = \int_{-1}^{\infty} \left(1 - \frac{1}{[M : M \cap \overline{K}^{(v)}]}\right) dv \geqslant \frac{1}{2} \int_{-1}^{c(M)-1} dv = \frac{c(M)}{2}.$$

The lemma is proved. $\square$

4.1.9. We prove that *i*) $\Leftrightarrow$ *ii*). First assume that $v_K(\mathfrak{D}_{K_\infty/K}) = +\infty$. For any $c > 0$, there exists $K \subset M \subset K_\infty$ such that $v_K(\mathfrak{D}_{M/K}) \geqslant c$. By Lemma 4.1.8, $c(M) \geqslant c$. This shows that K_∞/K doesn't have finite conductor.

Conversely, assume that K_∞/K doesn't have finite conductor. Then for each $c > 0$, there exists a non-zero element $\beta \in K_\infty \cap \overline{K}^{(c)}$. Let $M = K(\beta)$. Then $v_K(\mathfrak{D}_{M/K}) \geqslant \frac{c}{2}$ by Lemma 4.1.8. Therefore $v_K(\mathfrak{D}_{K_\infty/K}) = +\infty$.

Lemma 4.1.10. *Assume that w is such that $L \subset \overline{K}^{(w)}$. Then for any $n \geqslant 0$,*

$$[L_n : L_n \cap \overline{K}^{(w)}] = [K_n : K_n \cap \overline{K}^{(w)}].$$

Proof. Since $\overline{K}^{(w)}/K$ is a Galois extension, K_n and $\overline{K}^{(w)}$ are linearly disjoint over $K_n \cap \overline{K}^{(w)}$. Therefore K_n and $\overline{K}^{(w)} \cap L_n$ are linearly disjoint over $K_n \cap \overline{K}^{(w)}$. We have:

$$[K_n : K_n \cap \overline{K}^{(w)}] = [K_n \cdot (\overline{K}^{(w)} \cap L_n) : (\overline{K}^{(w)} \cap L_n)]. \tag{29}$$

Clearly $K_n \cdot (\overline{K}^{(w)} \cap L_n) \subset L_n$. Conversely, from $L_n = K_n \cdot L$ and $L \subset \overline{K}^{(w)}$, it follows that $L_n \subset K_n \cdot (\overline{K}^{(w)} \cap L_n)$. Thus

$$L_n = K_n \cdot (\overline{K}^{(w)} \cap L_n).$$

Together with (29), this proves the lemma. $\square$

4.1.11. We prove that *ii*) $\Rightarrow$ *iii*). By the multiplicativity of the different, for any $n \geqslant 0$, we have:

$$v_K(\mathfrak{D}_{L_n/K_n}) = v_K(\mathfrak{D}_{L_n/K}) - v_K(\mathfrak{D}_{K_n/K}).$$

Let w be such that $L \subset \overline{K}^{(w)}$. Using formula (11) and Lemma 4.1.10, we obtain:

$$v_K(\mathfrak{D}_{L_n/K_n}) = \int_{-1}^{\infty} \left(\frac{1}{[K_n : (K_n \cap \overline{K}^{(v)})]} - \frac{1}{[L_n : (L_n \cap \overline{K}^{(v)})]} \right) dv =$$
$$\int_{-1}^{w} \left(\frac{1}{[K_n : (K_n \cap \overline{K}^{(v)})]} - \frac{1}{[L_n : (L_n \cap \overline{K}^{(v)})]} \right) dv \leqslant \int_{-1}^{w} \frac{dv}{[K_n : (K_n \cap \overline{K}^{(v)})]}.$$

Since $[K_n : (K_n \cap \overline{K}^{(v)})] \geqslant [K_n : (K_n \cap \overline{K}^{(w)})]$ if $v \leqslant w$, this gives the following estimate for the different:

$$v_K(\mathfrak{D}_{L_n/K_n}) \leqslant \frac{w+1}{[K_n : (K_n \cap \overline{K}^{(w)})]}.$$

Since K_∞/K doesn't have finite conductor, for any $c > 0$ there exists $n \geqslant 0$ such that $[K_n : (K_n \cap \overline{K}^{(w)})] > c$, and therefore $v_K(\mathfrak{D}_{L_n/K_n}) \leqslant (w+1)/c$. This proves that $v_K(\mathfrak{D}_{L_\infty/K_\infty}) = 0$.

4.1.12. We prove that *iii*) $\Rightarrow$ *iv*). We consider two cases.

a) First assume that the set $\{e(K_n/K) \mid n \geqslant 0\}$ is bounded. Then there exists $n_0 \in I$ such that $e(K_n/K_{n_0}) = 1$ for any $n \geqslant n_0$. Therefore $e(L_n/L_{n_0}) = 1$ for any $n \geqslant n_0$, and by the mutiplicativity of the different

$$\mathfrak{D}_{L_n/K_n} = \mathfrak{D}_{L_{n_0}/K_{n_0}} O_{L_n}, \qquad \forall n \geqslant n_0.$$

From Proposition 4.1.3 and assumption iii), it follows that $\mathfrak{D}_{L_n/K_n} = O_{L_n}$ for all $n \geqslant n_0$. Therefore the extensions L_n/K_n are unramified, and Lemma 1.4.2 (or just the well known surjectivity of the trace map for unramified extensions) gives:

$$\mathrm{Tr}_{L_n/K_n}(\mathfrak{m}_{L_n}) = \mathfrak{m}_{K_n}, \qquad \text{for all } n \geqslant n_0.$$

Thus $\mathrm{Tr}_{L_\infty/K_\infty}(\mathfrak{m}_{L_\infty}) = \mathfrak{m}_{K_\infty}$.

b) Now assume that the set $\{e(K_n/K) \mid n \geqslant 0\}$ is unbounded. Let $x \in \mathfrak{m}_{K_\infty}$. Then there exists n such that $x \in \mathfrak{m}_{K_n}$. By Lemma 1.4.2,

$$\mathrm{Tr}_{L_n/K_n}(\mathfrak{m}_{L_n}) = \mathfrak{m}_{K_n}^{r_n}, \qquad r_n = \left[\frac{v_{L_n}(\mathfrak{D}_{L_n/K_n}) + 1}{e(L_n/K_n)}\right].$$

From our assumptions and Proposition 4.1.3, it follows that we can choose n such that in addition

$$v_K(\mathfrak{D}_{L_n/K_n}) + \frac{1}{e(L_n/K)} \leqslant v_K(x).$$

Then

$$r_n \leqslant \frac{v_{L_n}(\mathfrak{D}_{L_n/K_n}) + 1}{e(L_n/K_n)} = \left(v_K(\mathfrak{D}_{L_n/K_n}) + \frac{1}{e(L_n/K)}\right) e(K_n/K) \leqslant v_{K_n}(x).$$

Since $\mathrm{Tr}_{L_n/K_n}(\mathfrak{m}_{L_n})$ is an ideal in O_{K_n}, this implies that $x \in \mathrm{Tr}_{L_n/K_n}(\mathfrak{m}_{L_n})$, and the inclusion $\mathfrak{m}_{K_\infty} \subset \mathrm{Tr}_{L_\infty/K_\infty}(\mathfrak{m}_{L_\infty})$ is proved. Since the converse inclusion is trivial, we have $\mathfrak{m}_{K_\infty} = \mathrm{Tr}_{L_\infty/K_\infty}(\mathfrak{m}_{L_\infty})$.

4.2. **Almost étale extensions.**

4.2.1. In this section, we introduce, in our very particular setting, the notion of an almost étale extension.

Definition. *A finite extension E/F of non archimedean fields is almost étale if and only if*

$$\mathrm{Tr}_{E/F}(\mathfrak{m}_E) = \mathfrak{m}_F.$$

It is clear that an unramified extension of local fields is almost étale. Below, we give two other archetypical examples of almost étale extensions.

4.2.2. **Examples.** 1) Assume that F is a perfect non-archimedean field of characteristic p. Then any finite extension of F is almost étale.

Proof. Let E/F be a finite extension. It is clear that $\mathrm{Tr}_{E/F}(\mathfrak{m}_E) \subset \mathfrak{m}_F$. Moreover, $\mathrm{Tr}_{E/F}(\mathfrak{m}_E)$ is an ideal of O_F, and for any $\alpha \in \mathfrak{m}_E$, one has:

$$\lim_{n\to+\infty} |\mathrm{Tr}_{E/F}\varphi^{-n}(\alpha)|_F = 0.$$

This implies that $\mathfrak{m}_F \subset \mathrm{Tr}_{E/F}(\mathfrak{m}_E)$, and the proposition is proved. $\square$

2) Assume that K_∞ is a deeply ramified extension of a local field K of characteristic 0. By Theorem 4.1.7, any finite extension of K_∞ is almost étale.

4.2.3. Following Tate [151], we apply the theory of almost étale extensions to the proof of the theorem of Ax–Sen–Tate. Let K be a perfect complete non archimedean field, and let $\mathbf{C}_K$ denote the completion of $\overline{K}$. For any topological group G, we denote by $H^n(G,-)$ the continuous cohomology of G.

Theorem 4.2.4. *Assume that F is an algebraic extension of K such that any finite extension of F is almost étale. Then*

$$H^0(G_F,\mathbf{C}_K)=\widehat{F}.$$

We first prove the following lemma. Fix an absolute value $|\cdot|_K$ on $\overline{K}$.

Lemma 4.2.5. *Let E/F be an almost étale Galois extension with Galois group G. Then for any $\alpha\in E$ and any $c>1$, there exists $a\in F$ such that*

$$\left|\alpha-a\right|_K<c\cdot\max_{g\in G}\left|g(\alpha)-\alpha\right|_K.$$

Proof. Let $c>1$. By Theorem 4.1.7 iv), there exists $x\in O_E$ such that $y=\mathrm{Tr}_{E/F}(x)$ satisfies

$$1/c<|y|_K\leqslant 1.$$

Set: $a=\dfrac{1}{y}\sum\limits_{g\in G}g(\alpha x)$. Then

$$|\alpha-a|_K=\left|\frac{\alpha}{y}\sum_{g\in G}g(x)-\frac{1}{y}\sum_{g\in G}g(\alpha x)\right|_K=\left|\frac{1}{y}\sum_{g\in G}g(x)(\alpha-g(\alpha))\right|_K$$
$$\leqslant\frac{1}{|y|_F}\cdot\max_{g\in G}\left|g(\alpha)-\alpha\right|_K.$$

The lemma is proved. □

4.2.6. *Proof of Theorem 4.2.4.* Let $\alpha\in\mathbf{C}_K^{G_F}$. Choose a sequence $(\alpha_n)_{n\in\mathbf{N}}$ of elements $\alpha_n\in\overline{K}$ such that $|\alpha_n-\alpha|_K<p^{-n}$. Then

$$|g(\alpha_n)-\alpha_n|_K=|g(\alpha_n-\alpha)-(\alpha_n-\alpha)|_K<p^{-n},\qquad\forall g\in G_F.$$

By Lemma 4.2.5, for each n, there exists $\beta_n\in F$ such that $|\beta_n-\alpha_n|_K<p^{-n}$. Then

$$\alpha=\lim_{n\to+\infty}\beta_n\in\widehat{F}.$$

The theorem is proved.

□

4.2.7. Now we compute the first cohomology group $H^1(G_F,\mathbf{C}_K)$.

Theorem 4.2.8. *Under the assumptions and notation of Theorem 4.2.4,* $\mathfrak{m}_F H^1(G_F,O_{\mathbf{C}_K})=\{0\}$ *and* $H^1(G_F,\mathbf{C}_K)=\{0\}$.

The proof will be given in Sections 4.2.9–4.2.10 below. For any map $f:X\to O_{\mathbf{C}_K}$, where X is an arbitrary set, we define $|f|:=\sup_{x\in X}|f(x)|_K$.

Lemma 4.2.9. *Let E/F be a finite Galois extension with Galois group G. Then for any map $f:G\to O_{\overline{K}}$ and any $y\in\mathfrak{m}_F$, there exists $\alpha\in O_E$ such that*

$$|yf-h_\alpha|<|\partial(f)|_K,$$

where $h_\alpha:G\to O_{\overline{K}}$ is the 1-coboundary $h_\alpha(g)=g(\alpha)-\alpha$ and $\partial(f):G\times G\to O_{\overline{K}}$ is the 2-coboundary $\partial(f)(g_1,g_2)=g_1f(g_2)-f(g_1g_2)+f(g_1)$.

Proof. Since E/F is almost étale, there exists $x\in O_E$ such that $y=\mathrm{Tr}_{E/F}(x)$. Set:

$$\alpha:=-\sum_{g\in G}g(x)f(g).$$

An easy computation shows that for any $\tau\in G$, one has:

$$\tau(\alpha)-\alpha=yf(\tau)-\sum_{g\in G}\tau g(x)\cdot\partial(f)(\tau,g).$$

This proves the lemma. □

4.2.10. *Proof of Theorem 4.2.8.* Let $f:G_F\to O_{\mathbf{C}_K}$ be a 1-cocycle. Fix $y\in\mathfrak{m}_F$. By continuity of f, for any $n\geqslant 0$ there exists a map $\tilde f:G_F\to O_{\overline{K}}$ such that $|\tilde f-f|<p^{-n}$, and $\tilde f$ factors through a finite quotient of G_F. Note that $|\partial(\tilde f)|<p^{-n}$ because $\partial(f)=0$. By Lemma 4.2.9, there exists $\alpha\in\mathfrak{m}_{\overline{K}}$ such that

$$|yf-h_\alpha|<|\partial(\tilde f)|<p^{-n}.$$

Using this argument together with successive approximation, it is easy to see that $y\cdot\mathrm{cl}(f)=0$. This proves that $\mathfrak{m}_F H^1(G_F,O_{\mathbf{C}_K})=\{0\}$. Now the vanishing of $H^1(G_F,\mathbf{C}_K)$ is obvious.

□

The following corollary should be compared with Theorem 1.1.8.

Corollary 4.2.11. *Let F be a complete perfect non archimedean field of characteristic p. Then the following holds true:*

i) $H^0(G_F,\mathbf{C}_F)=F$;
ii) $\mathfrak{m}_F\cdot H^1(G_F,O_{\mathbf{C}_F})=0$;
iii) $H^1(G_F,\mathbf{C}_F)=0$.

4.3. **Continuous cohomology of G_K.**

4.3.1. Assume that K is a local field of characteristic 0.

Theorem 4.3.2 (Tate)**.** *i) Let K_∞/K be a deeply ramified extension. Then $H^0(G_{K_\infty},\mathbf{C}_K)=\widehat{K}_\infty$ and $H^1(G_{K_\infty},\mathbf{C}_K)=0$.*

ii) $H^0(G_K,\mathbf{C}_K)=K$, and $H^1(G_K,\mathbf{C}_K)$ is the one dimensional K-vector space generated by any totally ramified additive character $\eta: G_K\to\mathbf{Z}_p$.

iii) Let $\eta: G_K\to\mathbf{Z}_p^*$ be a totally ramified character with infinite image. Then $H^0(G_K,\mathbf{C}_K(\eta))=0$, and $H^1(G_K,\mathbf{C}_K(\eta))=0$.

Proof. i) The first assertion follows from Theorems 4.1.7 and 4.2.8.

ii) Let $K_\infty=\overline{K}^{\ker(\eta)}$. Then K_∞/K is a $\mathbf{Z}_p$-extension, and we set $\Gamma=\mathrm{Gal}(K_\infty/K)$. By Proposition 3.1.2, K_∞/K is deeply ramified. Hence $H^0(G_{K_\infty},\mathbf{C}_K)=\widehat{K}_\infty$ by Theorem 4.2.4. Applying Theorem 3.3.2, we obtain that $H^0(G_K,\mathbf{C}_K)=H^0(\Gamma,\widehat{K}_\infty)=K$. To compute the first cohomology, consider the inflation-restriction exact sequence:

$$0\to H^1(\Gamma,\mathbf{C}_K^{G_{K_\infty}})\to H^1(G_K,\mathbf{C}_K)\to H^1(G_{K_\infty},\mathbf{C}_K).$$

By assertion i), $\mathbf{C}_K^{G_{K_\infty}}=\widehat{K}_\infty$, and $H^1(G_{K_\infty},\mathbf{C}_K)=0$. Hence

$$H^1(G_K,\mathbf{C}_K)\simeq H^1(\Gamma,\widehat{K}_\infty).$$

Applying Theorem 3.3.2, we see that $H^1(G_K,\mathbf{C}_K)$ is the one-dimensional K-vector space generated by $\eta: G_K\to\mathbf{Z}_p$.

iii) The last assertion can be proved by the same arguments. □

4.3.3. The group G_K acts on the groups μ_{p^n} of p^n-th roots of unity via the character $\chi_K: G_K\to\mathbf{Z}_p^*$ defined as:

$$g(\zeta)=\zeta^{\chi_K(g)},\qquad \forall g\in G_K,\ \zeta\in\mu_{p^n},\ n\geqslant 1.$$

Definition. *The character $\chi_K: G_K\to\mathbf{Z}_p^*$ is called the cyclotomic character.*

It is clear that $\log\chi_K$ is an additive character of G_K with values in $\mathbf{Z}_p$.

Corollary 4.3.4. *$H^1(G_K,\mathbf{C}_K)$ is the one-dimensional K-vector space generated by $\log\chi_K$.*

4.3.5. Let E/K be a finite extension which contains all conjugates τK of K over $\mathbf{Q}_p$. We say that two multiplicative characters $\psi_1,\psi_2: G_E\to U_K$ are equivalent and write $\psi_1\sim\psi_2$ if $\mathbf{C}_K(\psi_1)\simeq\mathbf{C}_K(\psi_2)$ as G_E-modules. Theorem 4.3.2 implies the following proposition, which will be used in Section 15.

Proposition 4.3.6. *The conditions a) and b) below are equivalent:*

a) $\tau\circ\psi_1\sim\tau\circ\psi_2$ for all $\tau\in\mathrm{Hom}(K,E)$.

b) The characters ψ_1 and ψ_2 coincide on an open subgroup of I_E.

Proof. See [143, Section A2]. □

4.3.7. Using Tate's method, Sen proved the following important result.

Theorem 4.3.8 (Sen). *Assume that K_∞/K is deeply ramified. Then*

$$H^1(G_{K_\infty}, \mathrm{GL}_n(\mathbf{C}_K)) = \{1\}.$$

Proof. For deeply ramified $\mathbf{Z}_p$-extensions, it was proved in [136], and the proof is similar in the general case. □

5. From characteristic 0 to characteristic p and vice versa I: perfectoid fields

5.1. **Perfectoid fields.**

5.1.1. The notion of perfectoid field was introduced in Scholze's fundamental paper [130] as a far-reaching generalization of Fontaine's constructions [66], [70]. Fix a prime number p. Let E be a field equipped with a non-archimedean absolute value $|\cdot|_E : E \to \mathbf{R}_+$ such that $|p|_E < 1$. Note that we don't exclude the case of characteristic p, where the last condition holds automatically. We denote by O_E the ring of integers of E and by $\mathfrak{m}_E$ the maximal ideal of O_E.

Definition. *Let E be a field equipped with an absolute value $|\cdot|_E : E \to \mathbf{R}_+$ such that $|p|_E < 1$. One says that E is perfectoid if the following holds true:*
i) $|\cdot|_E$ is non-discrete;
ii) E is complete for $|\cdot|_E$;
iii) The Frobenius map

$$\varphi : O_E/pO_E \to O_E/pO_E, \qquad \varphi(x) = x^p$$

is surjective.

We give first examples of perfectoid fields, which can be treated directly.

5.1.2. **Examples.** 1) A perfect field of characteristic p, complete for a non-archimedean valuation, is a perfectoid field.

2) Let K be a non archimedean field. The completion $\mathbf{C}_K$ of its algebraic closure is a perfectoid field.

3) Let K be a local field. Fix a uniformizer π of K and set $\pi_n = \pi^{1/p^n}$. Then the completion of the Kummer extension $K(\pi^{1/p^\infty}) = \overset{\infty}{\underset{n=1}{\cup}} K(\pi_n)$ is a perfectoid field. This follows from the congruence

$$\left(\sum_{i=0}^m [a_i]\pi_n^i\right)^p \equiv \sum_{i=0}^m [a_i]^p \pi_{n-1}^i \pmod{p}.$$

5.1.3. The following important result is a particilar case of [78, Proposition 6.6.6].

Theorem 5.1.4 (Gabber–Ramero). *Let K be a local field of characteristic 0. A complete subfield $K \subset E \subset \mathbf{C}_K$ is a perfectoid field if and only if it is the completion of a deeply ramified extension of K.*

5.2. **Tilting.**

5.2.1. In this section, we describe the tilting construction, which functorially associates to any perfectoid field of characteristic 0 a perfect field of characteristic p. This construction first appeared in the pionnering papers of Fontaine [64, 66]. The tilting of arithmetically profinite extensions is closely related to the field of norms functor of Fontaine–Wintenberger [161]. We will come back to this question in Section 6. In the full generality, the tilting was defined in the famous paper of Scholze [130] for perfectoid algebras. This generalization is crucial for geometric application. However, in this introductory paper, we will consider only the arithmetic case.

5.2.2. Let E be a perfectoid field of characteristic 0. Consider the projective limit

$$O_E^\flat := \varprojlim_{\varphi} O_E/pO_E = \varprojlim (O_E/pO_E \xleftarrow{\varphi} O_E/pO_E \xleftarrow{\varphi} \cdots), \tag{30}$$

where $\varphi(x) = x^p$. It is clear that $O_E^\flat$ is equipped with a natural ring structure. An element x of $O_E^\flat$ is an infinite sequence $x = (x_n)_{n\geqslant 0}$ of elements $x_n \in O_E/pO_E$ such that $x_{n+1}^p = x_n$ for all n. Below, we summarize first properties of the ring $O_E^\flat$:

1) For all $m \in \mathbf{N}$, choose a lift $\widehat{x}_m \in O_E$ of x_m. Then for any fixed n, the sequence $(\widehat{x}_{n+m}^{p^m})_{m\geqslant 0}$ converges to an element

$$x^{(n)} = \lim_{m\to\infty} \widehat{x}_{m+n}^{p^m} \in O_E,$$

which does not depend on the choice of the lifts $\widehat{x}_m$. In addition, $\left(x^{(n)}\right)^p = x^{(n-1)}$ fol all $n \geqslant 1$.

Proof. Since $x_{m+n}^p = x_{m+n-1}$, we have $\widehat{x}_{m+n}^p \equiv \widehat{x}_{m+n-1} \pmod p$, and an easy induction shows that $\widehat{x}_{m+n}^{p^m} \equiv \widehat{x}_{m+n-1}^{p^{m-1}} \pmod{p^m}$. Therefore the sequence $(\widehat{x}_{n+m}^{p^m})_{m\geqslant 0}$ converges. Assume that $\widetilde{x}_m \in O_E$ are another lifts of x_m, $m \in \mathbf{N}$. Then $\widetilde{x}_m \equiv \widehat{x}_m \pmod p$ and therefore $\widetilde{x}_{n+m}^{p^m} \equiv \widehat{x}_{n+m}^{p^m} \pmod{p^{m+1}}$. This implies that the limit doesn't depend on the choice of the lifts. □

2) For all $x, y \in O_{E^\flat}$ one has

$$(x+y)^{(n)} = \lim_{m\to+\infty} \left(x^{(n+m)} + y^{(n+m)}\right)^{p^m}, \qquad (xy)^{(n)} = x^{(n)}y^{(n)}. \tag{31}$$

Proof. It is easy to see that $x^{(n)} \in O_E$ is a lift of x_n. Therefore $x^{(n+m)} + y^{(n+m)}$ is a lift of $x_{n+m} + y_{n+m}$, and the first formula follows from the definition of $(x+y)^{(n)}$. The same argument proves the second formula. □

3) The map $x \mapsto (x^{(n)})_{n\geqslant 0}$ defines an isomorphism

$$O_E^\flat \simeq \varprojlim_{x^p \leftarrow x} O_E, \tag{32}$$

where the right hand side is equipped with the addition and the multiplication defined by formula (31).

Proof. This follows from from 2). □

Set:

$$|\cdot|_{E^\flat} : O_E^\flat \to \mathbf{R}\cup\{+\infty\},$$
$$|x|_{E^\flat} = |x^{(0)}|_E.$$

Proposition 5.2.3. *i) $|\cdot|_{E^\flat}$ is a non-archimedean absolute value on $O_{E^\flat}$.*

ii) $O_E^\flat$ is a perfect complete valuation ring of characteristic p, with maximal ideal $\mathfrak{m}_E^\flat = \{x\in O_E^\flat \mid |x|_{E^\flat}<1\}$ and residue field k_E.

iii) Let $E^\flat$ denote the field of fractions of $O_E^\flat$. Then $|E^\flat|_{E^\flat} = |E|_E$.

Proof. i) Let $x,y\in O_E^\flat$. It is clear that

$$|xy|_{E^\flat} = |(xy)^{(0)}|_E = |x^{(0)}y^{(0)}|_E = |x^{(0)}|\cdot|y^{(0)}|_E = |x|_{E^\flat}|y|_{E^\flat}.$$

One has:

$$\begin{aligned}|x+y|_{E^\flat} &= |(x+y)^{(0)}|_E = |\lim_{m\to+\infty}(x^{(m)}+y^{(m)})^{p^m}|_E = \lim_{m\to+\infty}|x^{(m)}+y^{(m)}|_E^{p^m}\\ &\leqslant \lim_{m\to+\infty}\max\{|x^{(m)}|_E,|x^{(m)}|_E\}^{p^m} = \lim_{m\to+\infty}\max\{\left|(x^{(m)})^{p^m}\right|_E,\left|(x^{(m)})^{p^m}\right|_E\}\\ &= \max\{\left|(x^{(0)})\right|_E,\left|(x^{(0)})\right|_E\} = \max\{|x|_{E^\flat},|y|_{E^\flat}\}.\end{aligned}$$

This proves that $|\cdot|_{E^\flat}$ is an non-archimedean absolute value.

ii) We prove the completeness of $O_E^\flat$ (other properties follow easily from i) and properties 1-3) above. First remark that if $y=(y_0,y_1,\dots)\in O_E^\flat$, then

$$y_n = 0 \quad\Leftrightarrow\quad |y|_{E^\flat}\leqslant |p|_E^{p^n}. \tag{33}$$

Let $(x_n)_{n\in\mathbf{N}}$ be a Cauchy sequence in $O_E^\flat$. Then for any $M>0$, there exist N such that for all $n,m\geqslant N$

$$|x_n-x_m|_{E^\flat}\leqslant |p|_E^{p^M}.$$

Write $x_n=(x_{n,0},x_{n,1},\dots)$ and $x_m=(x_{m,0},x_{m,1},\dots)$. Using formula (33), we obtain that for all $n,m\geqslant N$

$$x_{n,i}=x_{m,i} \quad \text{for all} \quad 0\leqslant i\leqslant M.$$

Hence for each $i\geqslant 0$ the sequence $(x_{n,i})_{n\in\mathbf{N}}$ is stationary. Set $a_i=\lim_{n\to+\infty}x_{n,i}$. Then $a=(a_0,a_1,\dots)\in O_E^\flat$, and it is easy to check that $\lim_{n\to+\infty}x_n=a$. □

Definition. *The field $E^\flat$ will be called the tilt of E.*

Proposition 5.2.4. *A perfectoid field E is algebraically closed if and only if $E^\flat$ is.*

Proof. The proposition can be proved by successive approximation. We refer the reader to [60, Proposition 2.1.11] for the proof that $E^\flat$ is algebraically closed if E is and to [130, Proposition 3.8], and [60, Proposition 2.2.19, Corollary 3.1.10] for two different proofs of the converse statement. See also [23]. □

5.3. The ring $\mathbf{A}_{\mathrm{inf}}(E)$.

5.3.1. Let F be a perfect field, complete for a non-archimedean absolute value $|\cdot|_F$. The ring of Witt vectors $W(F)$ is equipped with the p-adic (standard) topology defined in Section 1.5. Now we equip it with a coarser topology, which will be called the *canonical topology*. It is defined as the topology of the infinite direct product

$$W(F) = F^{\mathbf{N}},$$

where each F is equipped with the topology induced by the absolute value $|\cdot|_F$. For any ideal $\mathfrak{a} \subset O_F$ and integer $n \geqslant 0$, the set

$$U_{\mathfrak{a},n} = \{x = (x_0, x_1, \dots) \in W(F) \mid x_i \in \mathfrak{a} \quad \text{for all } 0 \leqslant i \leqslant n\}$$

is an ideal in $W(F)$. In the canonical topology, the family $(U_{\mathfrak{a},n})$ of these ideals form a base of the fundamental system of neighborhoods of 0.

5.3.2. Let E be a perfectoid field of characteristic 0. Set:

$$\mathbf{A}_{\mathrm{inf}}(E) := W(O_E^\flat).$$

Each element of $\mathbf{A}_{\mathrm{inf}}(E)$ is an infinite vector

$$a = (a_0, a_1, a_2, \dots), \qquad a_n \in O_E^\flat,$$

which also can be written in the form

$$a = \sum_{n=0}^{\infty} [a_n^{p^{-n}}] p^n.$$

Proposition 5.3.3 (Fontaine, Fargues–Fontaine)**.** *i) The map*

$$\theta_E : \mathbf{A}_{\mathrm{inf}}(E) \to O_E$$

given by

$$\theta_E\left(\sum_{n=0}^{\infty} [a_n] p^n\right) = \sum_{n=0}^{\infty} a_n^{(0)} p^n$$

is a surjective ring homomorphism.

ii) $\ker(\theta_E)$ *is a principal ideal. An element* $\sum\limits_{n=0}^{\infty} [a_n] p^n \in \ker(\theta_E)$ *is a generator of* $\ker(\theta_E)$ *if and only if* $|a_0|_{E^\flat} = |p|_E$.

Proof. i) For any ring A, set $W_n(A) = W(A)/I_n(A)$. From the definition of Witt vectors, it follows that for any $n \geqslant 0$, the map

$$w_n : W_{n+1}(O_E) \to O_E,$$
$$w_n(a_0, a_1, \dots, a_n) = a_0^{p^n} + p a_1^{p^{n-1}} + \dots + p^n a_n$$

is a ring homomorphism. Consider the map:

$$\eta_n : W_{n+1}(O_E/pO_E) \to O_E/p^{n+1}O_E,$$
$$\eta_n(a_0, a_1, \dots, a_n) = \widehat{a}_0^{p^n} + p \widehat{a}_1^{p^{n-1}} + \dots + p^n \widehat{a}_n,$$

where $\widehat{a}_i$ denotes any lift of a_i in O_E. It's easy to see that the definition of η_n does not depend on the choice of these lifts. Moreover, the diagram

$$\begin{tikzcd} W_{n+1}(O_E) \arrow[r, "w_n"] \arrow[d] & O_E \arrow[d] \\ W_{n+1}(O_E/pO_E) \arrow[r, "\eta_n"] & O_E/p^{n+1}O_E \end{tikzcd}$$

commutes by the functoriality of Witt vectors. This shows that η_n is a ring homomorphism. Let $\theta_{E,n} : W_{n+1}(O_E^\flat) \to O_E/p^{n+1}O_E$ denote the reduction of θ_E modulo p^{n+1}. From the definitions of our maps, it follows that $\theta_{E,n}$ coincides with the composition

$$W_{n+1}(O_E^\flat) \xrightarrow{\varphi^{-n}} W_{n+1}(O_E^\flat) \to W_{n+1}(O_E/pO_E) \xrightarrow{\eta_n} O_E/p^{n+1}O_E.$$

This proves that $\theta_{E,n}$ is a ring homomorphism for all $n \geqslant 0$. Therefore θ_E is a ring homomorphism.

The surjectivity of θ_E follows from the surjectivity of the map

$$\theta_{E,0} : O_E^\flat \to O_E/pO_E.$$

ii) We refer the reader to [66, Proposition 2.4] for the proof of the following statement: an element $\sum_{n=0}^{\infty} [a_n]p^n \in \ker(\theta_E)$ generates $\ker(\theta_E)$ if and only if $|a_0|_{E^\flat} = |p|_E$.

Since $|E^\flat| = |E|$, there exists $a_0 \in O_{E^\flat}$ such that $|a_0|_{E^\flat} = |p|_E$. Then $\theta_E([a_0])/p \in U_E$, and by the surjectivity of θ_E, there exists $b \in \mathbf{A}_{\mathrm{inf}}(E)$ such that $\theta_E(b) = \theta_E([a_0])/p$. Thus $x = [a_0] - pb \in \ker(\theta_E)$. Since $|a_0|_{E^\flat} = |p|_E$, the above criterion shows that x generates $\ker(\theta_E)$. See [60, Proposition 3.1.9] for further detail. □

5.4. **The tilting equivalence.**

5.4.1. We continue to assume that E is a perfectoid field of characteristic 0. Fix an algebraic closure $\overline{E}$ of E and denote by $\mathbf{C}_E$ its completion. By Proposition 5.2.4, $\mathbf{C}_E^\flat$ is algebraically closed and we denote by $\overline{E^\flat}$ the algebraic closure of $E^\flat$ in $\mathbf{C}_E^\flat$. Let $\mathbf{C}_{E^\flat} := \widehat{\overline{E^\flat}}$ denote the p-adic completion of $\overline{E^\flat}$. We have the following picture, where the horizontal arrows denote the tilting:

$$\begin{tikzcd} \mathbf{C}_E \arrow[r, squiggly, "\flat"] \arrow[d, dash] & \mathbf{C}_E^\flat \arrow[d, dash] \\ E \arrow[r, squiggly, "\flat"] & E^\flat. \end{tikzcd}$$

Let $\mathfrak{F}$ be a complete intermediate field $E^\flat \subset \mathfrak{F} \subset \mathbf{C}_E^\flat$. Fix a generator ξ of $\ker(\theta_E)$. Set:

$$O_{\mathfrak{F}}^\sharp := \theta_{\mathbf{C}}(W(O_{\mathfrak{F}})),$$

where we write $\theta_{\mathbf{C}}$ instead $\theta_{\mathbf{C}_E}$ to simplify notation. Consider the diagram:

$$\begin{tikzcd}
0 \arrow[r] & \xi\mathbf{A}_{\mathrm{inf}}(E) \arrow[r] \arrow[d, hook] & \mathbf{A}_{\mathrm{inf}}(E) \arrow[r, "\theta_E"] \arrow[d, hook] & O_E \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & \xi W(O_{\mathfrak{F}}) \arrow[r] \arrow[d, hook] & W(O_{\mathfrak{F}}) \arrow[r] \arrow[d, hook] & O_{\mathfrak{F}}^{\sharp} \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & \xi\mathbf{A}_{\mathrm{inf}}(\mathbf{C}_E) \arrow[r] & \mathbf{A}_{\mathrm{inf}}(\mathbf{C}_E) \arrow[r, "\theta_{\mathbf{C}}"] & O_{\mathbf{C}_E} \arrow[r] & 0
\end{tikzcd}$$

Note that $O_{\mathfrak{F}}^{\sharp} = W(O_{\mathfrak{F}})/\xi W(O_{\mathfrak{F}})$. Set:

$$\mathfrak{F}^{\sharp} = O_{\mathfrak{F}}^{\sharp}[1/p].$$

Proposition 5.4.2. *$\mathfrak{F}^{\sharp}$ is a perfectoid field, and $(\mathfrak{F}^{\sharp})^{\flat} = \mathfrak{F}$.*

Proof. a) First prove that $O_{\mathfrak{F}}^{\sharp}$ is complete. For each $n \geqslant 1$, we have an exact sequence

$$0 \to \xi W_n(O_{\mathfrak{F}}) \to W_n(O_{\mathfrak{F}}) \to O_{\mathfrak{F}}^{\sharp}/p^n O_{\mathfrak{F}}^{\sharp} \to 0,$$

where $W_n = W/p^n W$. Since the projection maps $W_{n+1}(O_{\mathfrak{F}}) \to W_n(O_{\mathfrak{F}})$ are surjective, the passage to inverse limits gives an exact sequence

$$0 \to \xi W(O_{\mathfrak{F}}) \to W(O_{\mathfrak{F}}) \to \varprojlim_n O_{\mathfrak{F}}^{\sharp}/p^n O_{\mathfrak{F}}^{\sharp} \to 0.$$

Hence $O_{\mathfrak{F}}^{\sharp} = \varprojlim_n O_{\mathfrak{F}}^{\sharp}/p^n O_{\mathfrak{F}}^{\sharp}$, and $O_{\mathfrak{F}}^{\sharp}$ is complete.

b) Fix a valuation v_E on E. We prove that for any $x \in W(O_{\mathfrak{F}})$,

$$v_E(\theta_{\mathbf{C}}(x)) \geqslant n \cdot v_E(p) \Rightarrow x \in p^n W(O_{\mathfrak{F}}) + \xi W(O_{\mathfrak{F}}).$$

It's sufficient to prove this assertion for $n = 1$. Let $x = \sum_{k=0}^{\infty} [x_k] p^k$ be such that $v_E(\theta_{\mathbf{C}}(x)) \geqslant v_E(p)$. If $x_0 = 0$, the assertion is clearly true. Assume that $x_0 \neq 0$. Then $v_E(x_0^{(0)}) \geqslant v_E(p)$. By Proposition 5.3.3, $\xi = \sum_{k=0}^{\infty} [a_k] p^k$ with $v_E(a_0^{(0)}) = v_E(p)$. Hence

$$x_0 = a_0 y, \qquad \text{for some } y \in O_{\mathfrak{F}},$$

and

$$x = \xi[y] + pz, \qquad \text{for some } z \in W(O_{\mathfrak{F}}).$$

This shows that $x \in pW(O_{\mathfrak{F}}) + \xi W(O_{\mathfrak{F}})$.

c) Assume that $\alpha \in \mathfrak{F}^{\sharp}$ belongs to the valuation ring of $\mathfrak{F}^{\sharp}$. Write $\alpha = \beta/p^n$ with $\beta = \theta_{\mathbf{C}}(x)$, $x \in W(O_{\mathfrak{F}})$. Then $v_E(\theta_{\mathbf{C}}(x)) \geqslant n \cdot v_E(p)$. By part b), there exists $y \in W(O_{\mathfrak{F}})$ such that $\theta_{\mathbf{C}}(x) = p^n \theta_{\mathbf{C}}(y)$. Therefore $\alpha = \theta_{\mathbf{C}}(y) \in O_{\mathfrak{F}}^{\sharp}$. This proves that $O_{\mathfrak{F}}^{\sharp}$ is the valuation ring of $\mathfrak{F}^{\sharp}$.

d) From a) and c), it follows that $\mathfrak{F}^\sharp$ is a complete field with the valuation ring $O_{\mathfrak{F}}^\sharp$. In addition, the induced valuation on $\mathfrak{F}^\sharp$ is clearly non-discrete. Writing ξ in the form $\xi = \sum_{k=0}^{\infty} [a_k]p^k$, we see that

$$O_{\mathfrak{F}}^\sharp / pO_{\mathfrak{F}}^\sharp \simeq O_{\mathfrak{F}} / a_0 O_{\mathfrak{F}}.$$

This implies that $\mathfrak{F}^\sharp$ is a perfectoid field. Moreover, it is easy to see that the map

$$O_{\mathfrak{F}} \to \varprojlim_{\varphi} O_{\mathfrak{F}}/a_0 O_{\mathfrak{F}}, \qquad z \mapsto (\varphi^{-n}(z) \mod (a_0 O_{\mathfrak{F}}))_{n\geqslant 0}$$

is an isomorphism. Therefore $(\mathfrak{F}^\sharp)^\flat = \mathfrak{F}$, and the proposition is proved.

□

Proposition 5.4.3. *One has* $\mathbf{C}_E^\flat = \mathbf{C}_{E^\flat}$.

Proof. Since $E^\flat \subset \mathbf{C}_E^\flat$, and $\mathbf{C}_E^\flat$ is complete and algebraically closed, we have $\mathbf{C}_{E^\flat} \subset \mathbf{C}_E^\flat$. Set $\mathfrak{F} := \mathbf{C}_{E^\flat}$. By Proposition 5.4.2, $(\mathfrak{F}^\sharp)^\flat = \mathfrak{F}$. Since $\mathfrak{F}$ is complete and algebraically closed, $\mathfrak{F}^\sharp$ is complete and algebraically closed by Proposition 5.2.4. Now from $\mathfrak{F}^\sharp \subset \mathbf{C}_E$, we deduce that $\mathfrak{F}^\sharp = \mathbf{C}_E$. Therefore

$$\mathfrak{F} = (\mathfrak{F}^\sharp)^\flat = \mathbf{C}_E^\flat.$$

The proposition is proved. □

Now we can prove the main result of this section.

Theorem 5.4.4 (Scholze, Fargues–Fontaine)**.** *Let E be a perfectoid field of characteristic* 0. *Then the following holds true:*

i) One has $G_E \simeq G_{E^\flat}$.

ii) Each finite extension of E is a perfectoid field.

iii) The tilt functor $F \mapsto F^\flat$ realizes the Galois correspondence between the categories of finite extensions of E and $E^\flat$ respectively.

iv) The functor

$$\mathfrak{F} \mapsto \mathfrak{F}^\sharp, \qquad \mathfrak{F}^\sharp := (W(O_{\mathfrak{F}})/\xi W(O_{\mathfrak{F}}))[1/p]$$

is a quasi-inverse to the tilt functor.

Proof. The proof below is due to Fargues and Fontaine [60, Theorem 3.2.1].

a) We prove assertion i). The Galois group $G_E = \mathrm{Gal}(\overline{E}/E)$ acts on $\mathbf{C}_E$ and $\mathbf{C}_E^\flat$. To simplify notation, set $\mathbf{F} = \mathbf{C}_{E^\flat}$. By Proposition 5.4.3, $\mathbf{C}_E^\flat = \mathbf{F}$, and we have a map

$$G_E \to \mathrm{Aut}(\mathbf{C}_E^\flat/E^\flat) \xrightarrow{\sim} \mathrm{Aut}(\mathbf{F}/E^\flat) \xrightarrow{\sim} \mathrm{Aut}(\overline{E^\flat}/E^\flat) = G_{E^\flat}. \tag{34}$$

Conversely, again by Proposition 5.4.3, we have an isomorphism

$$W(O_{\mathbf{F}})/\xi W(O_{\mathbf{F}}) \simeq O_{\mathbf{C}_E}, \tag{35}$$

which induces a map

$$G_{E^\flat} \xrightarrow{\sim} \mathrm{Aut}(\mathbf{F}/E^\flat) \to \mathrm{Aut}(\mathbf{C}_E/E) \xrightarrow{\sim} G_E.$$

It is easy to see that the maps (34) and (35) are inverse to each other. Therefore

$$G_E \simeq G_{E^\flat},$$

and by Galois theory we have a one-to-one correspondence

$$\{\text{finite extensions of } E\} \leftrightarrow \{\text{finite extensions of } E^\flat\}. \tag{36}$$

b) Let $\mathfrak{F}/E^\flat$ be a finite extension. By Proposition 5.4.2, $\mathfrak{F}^\sharp$ is a perfectoid field, and

$$(\mathfrak{F}^\sharp)^\flat = \mathfrak{F}.$$

Consider the exact sequence:

$$0 \to W(O_{\mathbf{F}})[1/p] \xrightarrow{\xi} W(O_{\mathbf{F}})[1/p] \to W(O_{\mathbf{F}})/\xi W(O_{\mathbf{F}})[1/p] \to 0.$$

By Corollary 4.2.11 (Ax–Sen–Tate in characteristic p), one has:

$$H^0(G_{\mathfrak{F}}, W(O_{\mathbf{F}})) = W(O_{\mathfrak{F}}).$$

By the same corollary, $\mathfrak{m}_{\mathfrak{F}} \cdot H^1(G_{\mathfrak{F}}, O_{\mathbf{F}}) = 0$. Using successive approximation, one verifies that $[a] \cdot H^1(G_{\mathfrak{F}}, W(O_{\mathbf{F}})) = 0$ for any $a \in \mathfrak{m}_{\mathfrak{F}}$. The generator $\xi \in \ker(\theta_E)$ can be written in the form $\xi = [a] + pu$, where $a \in \mathfrak{m}_{E^\flat}$ and u is invertible in $\mathbf{A}_{\mathrm{inf}}(E)$. If

$$f \in \ker\Big(H^1(G_{\mathfrak{F}}, W(O_{\mathbf{F}}))[1/p] \xrightarrow{\xi} H^1(G_{\mathfrak{F}}, W(O_{\mathbf{F}}))[1/p]\Big),$$

then $[a]f = 0$, $\xi f = 0$, and therefore $f = 0$. Hence

$$\ker(H^1(G_{\mathfrak{F}}, W(O_{\mathbf{F}}))[1/p] \xrightarrow{\xi} H^1(G_{\mathfrak{F}}, W(O_{\mathbf{F}}))[1/p]) = 0.$$

Therefore the long exact sequence of cohomology associated to the above short exact sequence gives an isomorphism:

$$(W(O_{\mathbf{F}})/\xi W(O_{\mathbf{F}})[1/p])^{G_{\mathfrak{F}}} \simeq W(O_{\mathfrak{F}})/\xi W(O_{\mathfrak{F}})[1/p].$$

The isomorphism $G_E \simeq G_{E^\flat}$ identifies $G_{\mathfrak{F}}$ with an open subgroup of G_E. By Theorem 4.3.2 (Ax–Sen–Tate in characteristic 0), $\mathbf{C}_E^{G_{\mathfrak{F}}} \simeq (\overline{E})^{G_{\mathfrak{F}}}$. Since

$$\mathbf{C}_E \simeq (W(O_{\mathbf{F}})/\xi W(O_{\mathbf{F}}))\,[1/p],$$

one has:

$$\overline{E}^{G_{\mathfrak{F}}} \simeq W(O_{\mathfrak{F}})/\xi W(O_{\mathfrak{F}})[1/p] =: \mathfrak{F}^\sharp.$$

We have proved that the Galois correspondence (36) associates to $\mathfrak{F}/E^\flat$ the extension $\mathfrak{F}^\sharp/E$.

c) Conversely, let F be a finite extension of E. Set $\mathfrak{F} = \left(\overline{E^\flat}\right)^{G_F}$. From part b), it follows that $F = \mathfrak{F}^\sharp$. Applying Proposition 5.4.2, we obtain that F is a perfectoid field and that $F^\flat = \left(\mathfrak{F}^\sharp\right)^\flat = \mathfrak{F}$. This concludes the proof of the theorem. □

Remark 5.4.5. *For the theory of almost étale extensions in the geometric setting and Scholze's theory of perfectoid algebras we refer the reader to* [59], [78] *and* [130]. *See also* [95]. *In another direction, further development of these ideas led to the theory of diamonds* [132], *closely related to the theory of Fargues–Fontaine* [60].

6. From characteristic 0 to characteristic p and vice versa II: the field of norms

6.1. **Arithmetically profinite extensions.**

6.1.1. In this section, we review the theory of the arithmetically profinite extensions and the field of norms construction of Fontaine–Wintenberger [161]. Let K be a local field of characteristic 0 with residue field of characteristic p.

Definition. *An algebraic extension L/K is called arithmetically profinite (APF) if and only if*

$$(G_K : G_K^{(v)}G_L) < +\infty, \qquad \forall v \geqslant -1.$$

If L/K is a Galois extension with $G = \mathrm{Gal}(L/K)$, then it is APF if and only if

$$(G : G^{(v)}) < +\infty, \qquad \forall v \geqslant -1.$$

It is clear that any finite extension is APF. Below, we give some archetypical examples of APF extensions.

6.1.2. **Examples.** 1) Any totally ramified $\mathbf{Z}_p$-extension is APF (see Section 3.1).

2) The p-cyclotomic extension $K(\zeta_{p^\infty})/K$ is APF. This easily follows from the fact that $K(\zeta_{p^\infty})/K(\zeta_p)$ is a totally ramified $\mathbf{Z}_p$-extension. See also Proposition 6.1.10 below.

3) Let π be a fixed uniformizer of K, and let K_π be the maximal abelian extension of K such that π is a universal norm in K_π, namely that

$$\pi \in N_{F/K}(F^*), \qquad \text{for all } K \subset F \subset K_\pi.$$

By local class field theory, K_π/K is totally ramified and one has:

$$\mathrm{Gal}(K_\pi/K)^{(v)} \simeq U_K^{(v)}, \qquad \forall v \geqslant 0.$$

Therefore, K_π/K is APF.

4) More generally, from Sen's Theorem 1.3.11 it follows that any totally ramified p-adic Lie extension is APF. The converse is false in general (see [61] for examples).

5) Let π be a fixed uniformizer of K. The associated Kummer extension $K(\sqrt[p^\infty]{\pi})$ is an APF extension, which is not Galois. This can be proved by showing first that the Galois extension $K(\zeta_{p^\infty}, \sqrt[p^\infty]{\pi})$ is APF. The last assertion can be either proved by a direct computation or deduced from Sen's theorem. The extension $K(\sqrt[p^\infty]{\pi})$ plays a key role in Abrashkin's approach to the ramification filtration [4, 5, 7] and in integral p-adic Hodge theory [29],[33], [97].

6.1.3. We analyze the ramification jumps of APF extensions. First we extend the definition of a ramification jump to general (not necessarily Galois) extensions.

Definition. *Let L/K be an algebraic extension. A real number $v \geqslant -1$ is a ramification jump of L/K if and only if*

$$G_K^{(v+\varepsilon)}G_L \neq G_K^{(v)}G_L \qquad \forall \varepsilon > 0.$$

If L/K is a Galois extension, this definition coincides with Definition 1.3.8.

Proposition 6.1.4. *Let L/K be an infinite APF extension, and let B denote the set of ramification jumps of K. Then B is a countably infinite unbounded set.*

Proof. a) Let L/K be an APF extension. First we prove that B is discrete. Let $v_2 > v_1 \geqslant -1$ be two ramification jumps. Then

$$(G_K : G_K^{(v_1)}G_L) \leqslant (G_K : G_K^{(v_2)}G_L) < +\infty,$$

and

$$(G_K^{(v_1)}G_L : G_K^{(v_2)}G_L) < +\infty.$$

Therefore there exists only finitely many subgroups H such that

$$G_K^{(v_2)}G_L \subset H \subset G_K^{(v_1)}G_L.$$

This implies that there are only finitely many ramification jumps in the interval (v_1, v_2).

b) Assume that B is bounded above by a. Then $G_L G_K^{(a)} = \underset{t\geqslant 0}{\cap} G_L G_K^{(a+t)}$. Let $g \in G_L G_K^{(a)}$. Then for any $n \geqslant 0$, we can write $g = x_n y_n$ with $x_n \in G_L$ and $y_n \in G_K^{(a+n)}$. Since G_L is compact, we can assume that $(x_n)_{n\geqslant 0}$ converges. Hence $(y_n)_{n\geqslant 0}$ converges to some $y \in \underset{n\geqslant 0}{\cap} G_K^{(a+n)}$. From $\underset{n\geqslant 0}{\cap} G_K^{(a+n)} = \{1\}$, we obtain that $g \in G_L$. This shows that $G_L G_K^{(a)} = G_L$. Therefore

$$(G_K : G_L G_K^{(a)}) = (G_K : G_L) = +\infty,$$

which is in contradiction with the definition of APF extensions. □

6.1.5. Let L/K be an infinite APF extension. We denote by $B^+ = (b_n)_{n\geqslant 1}$ the set of its *strictly positive* ramification jumps. For all $n \geqslant 1$, set:

$$K_n = \overline{K}^{G_L G_K^{(b_n)}}.$$

Proposition 6.1.6. *The following statements hold true:*

i) $L = \overset{\infty}{\underset{n=1}{\cup}} K_n$.

ii) K_1 is the maximal tamely ramified subextension of L/K.

iii) For all $n \geqslant 1$, K_{n+1}/K_n is a non-trivial finite p-extension.

iv) Assume that L/K is a Galois extension. Then for all $n \geqslant 1$, the group $\mathrm{Gal}(K_{n+1}/K_n)$ *has a unique ramification jump. In particular,* $\mathrm{Gal}(K_{n+1}/K_n)$ *is a p-elementary abelian group.*

Proof. We prove assertion ii). The maximal tamely ramified subextension of L/K is

$$L_{\mathrm{tr}} = \overline{K}^{G_L P_K},$$

where P_K is the wild ramification subgroup. From the definition of the ramification filtration, it follows that P_K is the topological closure of $\underset{v>0}{\cup} G_K^{(v)}$ in G_K. This implies that $G_L P_K = G_L G_K^{(b_1)}$, and ii) is proved.

The assertions i), iii) and iv) are clear. □

Corollary 6.1.7. *An infinite APF extension is deeply ramified.*

Proof. Proposition 6.1.6 shows that such extension does not have finite conductor. □

Remark 6.1.8. *The converse of this corollary is clearly wrong. However Fesenko* [61] *proved that every deeply ramified extension L/K of finite residue degree and with discrete set of ramification jumps is APF.*

6.1.9. We record some general properties of APF extensions.

Proposition 6.1.10. *Let $K \subset F \subset L$ be a tower of extensions.*
i) If F/K is APF and L/F is finite, then L/K is APF.
ii) If F/K is finite and L/F is APF, then L/K is APF.
iii) If L/K is APF, then F/K is APF.

Proof. See [161, Proposition 1.2.3]. □

6.1.11. The definition of Hasse–Herbrand functions can be extended to APF extensions. Namely, for an APF extension L/K, set:

$$\psi_{L/K}(v) = \begin{cases} v, & \text{if } v \in [-1,0], \\ \displaystyle\int_0^v (G_K^{(0)} : G_L^{(0)} G_K^{(t)})dt, & \text{if } v \geqslant 0, \end{cases}$$
$$\varphi_{L/K}(u) = \psi_{L/K}^{-1}(u).$$

It is not difficult to check that if $K \subset F \subset L$ with $[F:K] < +\infty$, then one has:

$$\psi_{L/K} = \psi_{L/F} \circ \psi_{F/K}, \qquad \varphi_{L/K} = \varphi_{F/K} \circ \varphi_{L/F}.$$

6.2. **The field of norms.**

6.2.1. In this section, we review the construction of the field of norms of an APF extension. Let $\mathcal{E}(L/K)$ denote the directed set of finite subextensions of L/K.

Theorem 6.2.2 (Fontaine–Wintenberger)**.** *Let L/F be an infinite APF extension. Set:*

$$\mathscr{X}(L/K) = \varprojlim_{E \in \mathcal{E}(L/K)} E^* \cup \{0\}.$$

Then the following assertions hold true:
i) Let $\alpha = (\alpha_E)_{E \in \mathcal{E}(L/K)}$ and $\beta = (\beta_E)_{E \in \mathcal{E}(L/K)}$. Set:

$$(\alpha\beta)_E := \alpha_E \beta_E,$$
$$(\alpha+\beta)_E := \lim_{E' \in \mathcal{E}(L/E)} N_{E'/E}(\alpha_{E'} + \beta_{E'}).$$

Then $\alpha\beta := ((\alpha\beta)_E)_{E \in \mathcal{E}(L/K)}$ and $\alpha + \beta := ((\alpha+\beta)_E)_{E \in \mathcal{E}(L/K)}$ are well-defined elements of $\mathscr{X}(L/K)$.
ii) The above defined addition and multiplication equip $\mathscr{X}(L/K)$ with a structure of a local field of characteristic p with residue field k_L.
iii) The valuation on $\mathscr{X}(L/K)$ is given by

$$v(\alpha) = v_E(\alpha_E),$$

for any $K_1 \subset E \subset L$. Here K_1 denotes the maximal unramified subextension of L/K.

iv) For any $\xi \in k_L$, let $[\xi]$ denote its Teichmüller lift. For each $K_1 \subset E \subset L$ set:

$$\xi_E := [\xi]^{1/[E:K_1]}.$$

Then the map

$$k_L \to \mathscr{X}(L/K), \qquad \xi \mapsto (\xi_E)_{E\in\mathcal{E}(L/K_1)}$$

is a canonical embedding.

The proof occupies the remainder of this section. See [161, Section 2] for detail.

Definition. *The field $\mathscr{X}(L/K)$ is called the field of norms of the APF extension L/K.*

6.2.3. We start by writing Theorem 6.2.2 in a slightly different form, which also makes more clear its relation to the theory of perfectoid fields.

For any APF extension E/F (finite or infinite), set:

$$i(E/F) = \sup\{v \mid G_E G_F^{(v)} = G_F\}$$

If $E \subset E' \subset E''$ is a tower of finite extensions, then the relation $\psi_{E''/E} = \psi_{E''/E'} \circ \psi_{E'/E}$ implies that

$$i(E''/E) \leqslant \min\{i(E'/E), i(E''/E')\}. \tag{37}$$

Let $B = (b_n)_{n\geqslant 0}$ denote the set of ramification jumps of L/K and let $K_n = \overline{K}^{G_L G_K^{(b_n)}}$. Since

$$\psi_{L/K}(v) = \psi_{K_n/K}(v), \qquad \forall v \in [-1, b_n],$$

from $\psi_{L/K} = \psi_{L/K_n} \circ \psi_{K_n/K}$, it follows that $\psi_{L/K_n}(v) = v$ for $v \in [-1, \psi_{L/K}(b_n)]$, and $\psi_{L/K_n}(v) \neq v$ for $v > \psi_{L/K}(b_n)$. Therefore

$$i(L/K_n) = \psi_{L/K}(b_n), \qquad n \geqslant 1. \tag{38}$$

In particular, $i(L/K_n) \to +\infty$ when $n \to +\infty$.

6.2.4. For any $E \in \mathcal{E}(L/K_1)$, set:

$$r(E) := \text{smallest integer} \geqslant \frac{(p-1)i(L/E)}{p},$$

and

$$\overline{O}_E := O_E/\mathfrak{m}_E^{r(E)}.$$

Theorem 6.2.5. *Let L/K be an infinite APF extension. Then:*

i) For all finite subextensions $E \subset E'$ of L/K, the norm map induces a ring homomorphism

$$N_{E'/E} : \overline{O}_{E'} \to \overline{O}_E.$$

ii) The projective limit

$$A(L/K) := \varprojlim_{E\in\mathcal{E}(L/K_1)} \overline{O}_E$$

is a discrete valuation ring of characteristic p with residue field k_L.

iii) The map

$$k_L \to A(L/K), \qquad \xi \mapsto \left(\xi_E \mod \mathfrak{m}_E^{r(E)}\right)_{E\in\mathcal{E}(L/K_1)}, \qquad \xi_E = [\xi]^{1/[E:K_1]}$$

is a canonical embedding.

6.2.6. The proof of Theorem 6.2.5 relies on the following proposition:

Proposition 6.2.7. *Let E'/E be a finite totally ramified p-extension. Then*
i) For all $\alpha,\beta \in O_{E'}$,

$$v_E(N_{E'/E}(\alpha+\beta)-N_{E'/E}(\alpha)-N_{E'/E}(\beta)) \geqslant \frac{(p-1)i(E'/E)}{p}.$$

ii) For any $a \in O_E$, there exists $\alpha \in O_{E'}$ such that

$$v_E(N_{E'/E}(\alpha)-a) \geqslant \frac{(p-1)i(E'/E)}{p}.$$

Proof. a) Assume first that E'/E is a Galois extension of degree p. From Lemma 1.4.5 it follows that for any $x \in O_{E'}$, one has:

$$v_E(N_{E'/E}(1+x)-1-N_{E'/E}(x)) \geqslant \frac{(p-1)i(E'/E)}{p}.$$

Assume that $v_{E'}(\alpha) \geqslant v_{E'}(\beta)$. Setting $x=\alpha/\beta$, we obtain i).

Let $\pi_{E'}$ be any uniformizer of E'. Set $\pi_E = N_{E'/E}(\pi_{E'})$. Write $a \in O_E$ in the form:

$$a=\sum_{k=0}^{p-1}[\xi_k]\pi_E^k, \qquad \xi_k \in k_E.$$

Then again by Lemma 1.4.5, we have:

$$v_E(N_{E'/E}(\alpha)-a) \geqslant \frac{(p-1)i(E'/E)}{p} \qquad \text{for } \alpha=\sum_{k=0}^{p-1}[\xi_k]^{1/p}\pi_{E'}^k.$$

Therefore the proposition is proved for Galois extensions of degree p.

b) Assume that the proposition holds for finite extensions E''/E' and E'/E. Then for $\alpha,\beta \in O_{E''}$ we have:

$$N_{E''/E'}(\alpha+\beta)=N_{E''/E'}(\alpha)+N_{E''/E'}(\beta)+\gamma,$$

and

$$N_{E''/E}(\alpha+\beta)=N_{E''/E}(\alpha)+N_{E''/E}(\beta)+N_{E'/E}(\gamma)+\delta,$$

where $v_{E'}(\gamma) \geqslant \frac{(p-1)i(E''/E')}{p}$ and $v_E(\delta) \geqslant \frac{(p-1)i(E'/E)}{p}$. Since E'/E is totally ramified, one has $v_E(N_{E'/E}(\gamma)) \geqslant \frac{(p-1)i(E''/E')}{p}$, and from (37) it follows that

$$v_E(N_{E''/E}(\alpha+\beta)-N_{E''/E}(\alpha)-N_{E''/E}(\beta)) \geqslant \frac{(p-1)i(E''/E)}{p}.$$

Therefore the proposition holds for all finite p-extensions.

c) The general case can be reduced to the case b) by passing to the Galois closure of E'. See [161, Section 2.2.2.5] for detail.

□

6.2.8. *Sketch of proof of Theorem 6.2.5.* From Proposition 6.2.7, if follows that $A(L/K)$ is a commutative ring. Let $x=(x_E)_E\in A(L/K)$. If $x\neq 0$, the there exists $E\in\mathcal{E}(L/K_1)$ such that $x_E\neq 0$. For any $E'\in\mathcal{E}(L/E)$, let $\widehat{x}_{E'}\in O_{E'}$ be a lift of $x_{E'}$. Then $v(x):=v_{E'}(\widehat{x}_{E'})$ does not depend on the choice of E' and defines a discrete valuation of $A(L/K)$. It is easy to see that the topology defined by this valuation coincides with the topology of the projective limit of discrete sets on $A(L/K)$. Hence $A(L/K)$ is complete. Lemma 6.2.9 below shows that the element $x=(x_E)_{E\in\mathcal{E}(L/K_1)}$, with $x_E=p \mod \mathfrak{m}_E^{r(E)}$ for all E, is zero in $A(L/E)$. Therefore $A(L/E)$ is a ring of characteristic p. For all $\xi_1,\xi_2\in k_L$, the congruence $[\xi_1+\xi_2]\equiv[\xi_1]+[\xi_2] \pmod p$ together with Lemma 6.2.9 imply that the map

$$k_L\to A(L/K),\qquad \xi\mapsto(\xi_E\mod \pi_E^{r(E)})_{E\in\mathcal{E}(L/K_1)},\qquad \xi_E=[\xi]^{1/[E:K_1]}$$

is an embedding of fields. Finally, from the definition of the valuation on $A(L/K)$, we see that its residue field is isomorphic to k_L. Theorem 6.2.5 is proved.

□

Lemma 6.2.9. *Let L/E be a totally ramified APF pro-p-extension. Then*

$$v_E(p)\geqslant\frac{(p-1)i(L/E)}{p}.$$

Proof. First assume that F/E is a Galois extension of degree p. From elementary properties of the ramification filtration, it follows that $G_i=\{1\}$ for all $i>\frac{e_F}{p-1}$, where e_F is the absolute ramification index of F (see [142, Exercise 3, p. 79]). This implies that $v_E(p)\geqslant\frac{(p-1)i(F/E)}{p}$ for such extensions.

Now we consider the general case. Take the Galois closure M of L over E and denote by M_1/E its maximal tamely ramified subextension. It is clear that M_1/E is linearly disjoint with L/E. From Galois theory, it follows that LM_1/M_1 has a Galois subextension F of degree p over M_1. Then the inequality (37) implies that

$$v_E(p)\geqslant\frac{(p-1)i(F/M_1)}{p}\geqslant\frac{(p-1)i(LM_1/M_1)}{p}.$$

Since the extensions M_1/E and LM_1/L are tamely ramified, from $\psi_{LM_1/M_1}\circ\psi_{M_1/E}=\psi_{LM_1/L}\circ\psi_{L/E}$ it follows that $i(LM_1/M_1)=i(F/E)$. The lemma is proved. □

6.2.10. *Sketch of proof of Theorem 6.2.2.* We will use repeatedly the following inequality: if F/E is a totally ramified p-extension, then for all $x,y\in O_F$ one has:

$$v_E(N_{F/E}(x)-N_{F/E}(y))\geqslant\varphi_{F/E}(t),\qquad \text{if } v_F(x-y)\geqslant t. \tag{39}$$

This estimation can be proved by induction using Corollary 1.4.5. See [142, Chapter V,§6] for the Galois case. The general case can be treated by passing to the Galois closure.

Let $\alpha=(\alpha_E)_{E\in\mathcal{E}(L/K)}$ and $\beta=(\beta_E)_{E\in\mathcal{E}(L/K)}\in\varprojlim_{E\in\mathcal{E}(L/K)}O_E$. From Proposition 6.2.7 and formula (39), it follows that for all intermediate finite subextensions $K\subset E\subset E'\subset E''\subset L$ one has:

$$v_E\left(N_{E''/E}(\alpha_{E''}+\beta_{E''})-N_{E'/E}(\alpha_{E'}+\beta_{E'})\right)\geqslant\varphi_{E'/E}(r(E'))\geqslant\varphi_{L/K}(r(E')).$$

Since $r(E')\to+\infty$ when E' runs over $\mathcal{E}(L/E)$, this proves the existence of the limit

$$(\alpha+\beta)_E := \lim_{E'\in\mathcal{E}(L/E)} N_{E'/E}(\alpha_{E'}+\beta_{E'}).$$

Therefore the addition and the multiplication on $\mathscr{X}(L/E)$ are well defined.

Consider the map

$$\varprojlim_{E\in\mathcal{E}(L/K)} O_E \to A(L/K), \qquad (\alpha_E)_{E\in\mathcal{E}(L/K)} \mapsto (\overline{\alpha}_E)_{E\in\mathcal{E}(L/K_1)}, \tag{40}$$

where $\overline{\alpha}_E=\alpha_E \mod \mathfrak{m}_E^{r(E)}$. Proposition 6.2.7 shows that this map is compatible with the addition and the multiplication on the both sets.

Now let $x=(x_E)_E\in A(L/K)$. For all E, choose a lift $\widehat{x}_E\in O_E$. Applying again the inequality (39), we see that for all E, the sequence $N_{E'/E}(\widehat{x}_{E'})$ converges to some $\alpha_E\in O_E$. From our constructions, it follows that the map

$$A(L/K)\to \varprojlim_{E\in\mathcal{E}(L/K_1)} O_E, \qquad x\mapsto (\alpha_E)_{E\in\mathcal{E}(L/K_1)}$$

is the inverse of the map (40). Now the theorem follows from Theorem 6.2.5.

□

6.3. **Functorial properties.**

6.3.1. In this section, L/K denotes an infinite APF extension. Any finite extension M of L can be written as $M=L(\alpha)$, where α is a root of an irreducible polynomial $f(X)\in L[X]$. The coefficients of $f(X)$ belong to some finite subextension $F\in\mathcal{E}(L/K)$. For any $E\in\mathcal{E}(L/F)$, one has:

$$F(\alpha)\cap E=F,$$

and we set:

$$E'=E(\alpha).$$

The system $(E')_{E\in\mathcal{E}(L/K)}$ is cofinal in $\mathcal{E}(M/K)$. Consider the map

$$j_{M/L}\,:\,\mathscr{X}(L/K)\to\mathscr{X}(M/K)$$

which sends any $\alpha=(\alpha_E)_{E\in\mathcal{E}(L/K)}\in\mathscr{X}(L/K)$ to the element $\beta=(\beta_{E'})_{E'\in\mathcal{E}(M/K)}\in\mathscr{X}(M/K)$ defined by

$$\beta_{E'}=\alpha_E \qquad \text{if } E'=E(\alpha) \text{ with } E\in\mathcal{E}(L/F).$$

The previous remarks show that $j_{M/L}$ is a well-defined embedding.

The following theorem should be compared with Theorem 5.4.4.

Theorem 6.3.2 (Fontaine–Wintenberger)**.** *i) Let M/L be a finite extension. Then $\mathscr{X}(M/K)/\mathscr{X}(L/K)$ is a separable extension of degree* $[M:L]$. *If M/L is a Galois extension, then the natural action of* $\mathrm{Gal}(M/L)$ *on $\mathscr{X}(M/L)$ induces an isomorphism*

$$\mathrm{Gal}(M/L)\simeq\mathrm{Gal}\left(\mathscr{X}(M/K)/\mathscr{X}(L/K)\right).$$

ii) The above construction establishes a one-to-one correspondence

$$\{\textit{finite extensions of } L\}\leftrightarrow\{\textit{finite separable extensions of } \mathscr{X}(L/K)\},$$

which is compatible with the Galois correspondence.

Proof. We only explain how to associate to any finite separable extension $\mathscr{M}$ of $\mathscr{X}(L/K)$ a canonical finite extension M of L of the same degree. Let $\mathscr{M} = \mathscr{X}(L/K)(\alpha)$, where α is a root of an irreducible polynomial $f(X)$ with coefficients in the ring of integers of $\mathscr{X}(L/K)$. We can write $f(X)$ as a sequence $f(X) = (f_E(X))_{E\in\mathcal{E}(L/K)}$, where $f_E(X) \in E[X]$. Then $M = L(\widehat{\alpha})$, where $\widehat{\alpha}$ is a root of $f_E(X)$, and E is of "sufficiently big" degree over K. See [161, Section 3] for a detailed proof. □

6.3.3. From this theorem, it follows that the separable closure $\overline{\mathscr{X}(L/K)}$ of $\mathscr{X}(L/K)$ can de written as:

$$\overline{\mathscr{X}(L/K)} = \bigcup_{[M:L]<\infty} \mathscr{X}(M/K).$$

Corollary 6.3.4. *The field of norms functor induces a canonical isomorphism of absolute Galois groups:*

$$G_{\mathscr{X}(L/K)} \simeq G_L.$$

6.3.5. Let L/K be an infinite totally ramified Galois APF extension. The Galois group $\mathrm{Gal}(L/K)$ acts naturally on $\mathscr{X}(L/K)$. Fixing an uniformizer of $\mathscr{X}(L/K)$, we idenfify $\mathscr{X}(L/K)$ with the local field $k_K((x))$ of Laurent power series. Let τ be an automorphism of $k_K((x))$. If τ acts trivially on k_K, then it is completely determined by the power series $\tau(x) = a_1x + a_2x^2 + \cdots \in k_K[[x]]$ with $a_1 \neq 0$. Consider the group of formal power series

$$\mathrm{Aut}\Big(k_K((x))\Big) = \left\{ f(x) = \sum_{i=1}^{\infty} a_i x^i \mid a_1 \neq 0 \right\}$$

with respect to the substitution group law $f \circ g(x) = f(g(x))$. We have an injective map

$$\mathrm{Gal}(L/K) \hookrightarrow \mathrm{Aut}\Big(k_K((x))\Big). \tag{41}$$

This map encodes important information about the ramification filtration on $\mathrm{Gal}(L/K)$. Recall that for any automorphism g of a local field E we defined:

$$i_E(g) = v_E(g(\pi_E) - \pi_E).$$

Now we define this function on the infinite level, setting:

$$i_x(g) = \mathrm{ord}_x(g(x) - x), \qquad g \in \mathrm{Gal}(L/K).$$

Then there exists $F \in \mathcal{E}(L/K)$ such that for any Galois extension $E \in \mathcal{E}(L/F)$, one has:

$$i_E(g) = i_x(g)$$

(see [161, Proposition 3.3.2]).

6.3.6. The map (41) can be described explicitly for cyclotomic extensions of unramified local fields. Assume that K is unramified, and set $K_\infty = K(\zeta_{p^\infty})$. Let $\Gamma_K = \mathrm{Gal}(K_\infty/K)$. The action of Γ_K on K_∞ is given by the cyclotomic character:

$$\chi_K : \Gamma_K \to \mathbf{Z}_p^*, \qquad \tau(\zeta_{p^n}) = \zeta_{p^n}^{\chi_K(\tau)}, \qquad \tau \in \Gamma_K.$$

Set:

$$\varepsilon = (\zeta_{p^n})_{n\geqslant 0} \in \mathscr{X}(K_\infty/K). \tag{42}$$

Then $x = \varepsilon - 1$ is a uniformizer of $\mathscr{X}(K_\infty/K)$, and $\mathscr{X}(K_\infty/K) = k_K((x))$. The action of Γ_K on $\mathscr{X}(K_\infty/K)$ is given by

$$\tau(x) = (1+x)^{\chi_K(\tau)} - 1 \pmod{p}, \qquad \tau \in \Gamma_K. \tag{43}$$

This explicit formula can be generalized to the case of maximal abelian totally ramified extensions using the Lubin–Tate theory.

We refer the reader to [133], [62], [107], [108], [159], [160] for further results about the connection between Galois groups and automorphisms of local fields of positive characteristic.

6.3.7. We discuss the compatibility of reciprocity maps in characteristics 0 and p with the field of norms functor. Let L/K be an APF extension. For any $E \in \mathcal{E}(L/K)$ we have the reciprocity map

$$\theta_E : E^* \to G_E^{\mathrm{ab}}.$$

Passing to projective limit, and identifying $\varprojlim_{E\in\mathcal{E}(L/K)} E^*$ with $\mathscr{X}(L/K)^*$, we obtain an injective homomorphism:

$$\theta_\infty : \mathscr{X}(L/K)^* \to G_L^{\mathrm{ab}}.$$

By Corollary 6.3.4, the Galois group G_L^{ab} is canonically isomorphic to $G_{\mathscr{X}(L/K)}^{\mathrm{ab}}$. Let

$$\theta_{\mathscr{X}(L/K)} : \mathscr{X}(L/K)^* \to G_{\mathscr{X}(L/K)}^{\mathrm{ab}}$$

denote the reciprocity map for the field of norms $\mathscr{X}(L/K)$.

Theorem 6.3.8 (Laubie)**.** *The diagram*

$$\begin{tikzcd} \mathscr{X}(L/K)^* \arrow[r, "\theta_\infty"] \arrow[dr, "\theta_{\mathscr{X}(L/K)}"'] & G_L^{\mathrm{ab}} \arrow[d, "\simeq"] \\ & G_{\mathscr{X}(L/K)}^{\mathrm{ab}} \end{tikzcd}$$

commutes.

Proof. See [107, Théorème 3.2.2]. □

6.4. **Comparison with the tilting equivalence.**

6.4.1. Recall that an infinite APF extension if deeply ramified, and therefore its completion $\widehat{L}$ is a perfectoid field. We finish this section with comparing the field of norms with the tilting construction. A general result was proved by Fontaine and Wintenberger for APF extensions satisfying some additional condition.

Definition. *A strictly APF extension is an APF extension satisfying the following property:*

$$\liminf_{v\to+\infty} \frac{\psi_{L/K}(v)}{(G_K^{(0)} : G_L^{(0)} G_K^{(v)})} > 0.$$

From Sen's theorem 1.3.11, it follows that if $\mathrm{Gal}(L/K)$ is a p-adic Lie group, then L/K is strictly APF.

6.4.2. Let L/K be an infinite strict APF extension. Recall that we denote by K_1 the maximal tamely ramified subextension of L/K. For $E \in \mathcal{E}(L/K_1)$, set $d(E) = [E : K_1]$. For each $n \geqslant 1$, set:

$$\mathcal{E}_n = \{E \in \mathcal{E}(L/K_1) \mid p^n \text{ divides } d(E)\}.$$

Let $\alpha = (\alpha_E)_{E\in\mathcal{E}(L/K)} \in \mathscr{X}(L/K)$. It can be proved (see [161, Proposition 4.2.1]) that for any $n \geqslant 1$, the family

$$\alpha_E^{d(E)p^{-n}}, \qquad E \in \mathcal{E}_n$$

converges to some $x_n \in \widehat{L}$. Once the convergence is proved, it's clear that $x_n^p = x_{n-1}^p$ for all n, and therefore $x = (x_n)_{n\geqslant 1} \in \widehat{L}^\flat$. This defines an embedding

$$\mathscr{X}(L/K) \hookrightarrow \widehat{L}^\flat.$$

Theorem 6.4.3 (Fontaine–Wintenberger)**.** *Let L/K be an infinite strict APF extension. Then*

$$\widehat{\mathscr{X}(L/K)}^{\mathrm{rad}} = \widehat{L}^\flat.$$

Proof. See [161, Théorème 4.3.2 & Corollaire 4.3.4]. □

Remark 6.4.4. *In* [61]*, Fesenko gave examples of deeply ramified extensions which do not contain infinite APF extensions.*

7. ℓ-ADIC REPRESENTATIONS

7.1. **Preliminaries.**

7.1.1. Let E be a complete normed field, and let V be a finite-dimensional E-vector space. Each choice of a basis of V fixes a topological isomorphism $V \simeq E^n$ and equips V with a product topology. Note that this topology does not depend on the choice of the basis.

Definition. *A representation of a topological group G on V is a continuous homomorphism*

$$\rho : G \to \mathrm{Aut}_E V.$$

Fixing a basis of V, one can view a representation of G as a continuous homomorphism $G \to \mathrm{GL}_n(E)$.

Let K be a field and let $\overline{K}$ be a separable closure of K. We denote by G_K the absolute Galois group $\mathrm{Gal}(\overline{K}/K)$ of K. Recall that G_K is equipped with the inverse limit topology and therefore is a compact and totally disconnected topological group.

Definition. *Let ℓ be a prime number. An ℓ-adic Galois representation is a representation of G_K on a finite dimensional $\mathbf{Q}_\ell$-vector space equipped with the ℓ-adic topology.*

Sometimes it is convenient to consider representations with coefficients with a finite extension E of $\mathbf{Q}_\ell$. Below, we give some archetypical examples of ℓ-adic representations.

7.1.2. *One-dimensional representations.* Let V be a one-dimensional Galois representation. Then the action of G_K on V is given by a continuous character $\eta : G_K \to \mathbf{Z}_p^*$, and we will write $\mathbf{Q}_p(\eta)$ instead V.

7.1.3. *Roots of unity.* The following one-dimensional representations are of particular importance for us. Let $\ell \neq \mathrm{char}(K)$. The group G_K acts on the groups μ_{ℓ^n} of ℓ^n-th roots of unity via the ℓ-adic cyclotomic character $\chi_{K,\ell} : G_K \to \mathbf{Z}_\ell^*$:

$$g(\zeta) = \zeta^{\chi_{K,\ell}(g)}, \qquad \forall g \in G_K,\ \zeta \in \mu_{\ell^n}.$$

Set $\mathbf{Z}_\ell(1) = \varprojlim_n \mu_{\ell^n}$ and $\mathbf{Q}_\ell(1) = \mathbf{Z}_\ell(1) \otimes_{\mathbf{Z}_\ell} \mathbf{Q}_\ell$. Then $\mathbf{Q}_\ell(1)$ is a one dimensional $\mathbf{Q}_\ell$-vector space equipped with a continuous action of G_K. The homomorphism $G_K \to \mathrm{Aut}_{\mathbf{Q}_\ell} \mathbf{Q}_\ell(1) \simeq \mathbf{Q}_\ell^*$ concides with $\chi_{K,\ell}$.

7.1.4. *Abelian varieties.* Let A be an abelian variety over K, and let $\ell \neq \mathrm{char}(K)$. The group $A[\ell^n]$ of ℓ^n-torsion points of $A(\overline{K})$ is a Galois module, which is isomorphic (not canonically) to $(\mathbf{Z}/\ell^n\mathbf{Z})^{2d}$ as an abstract group. The ℓ-adic Tate module of A is defined as the projective limit

$$T_\ell(A) = \varprojlim_n A[\ell^n].$$

$T_\ell(A)$ is a free $\mathbf{Z}_\ell$-module of rank $2d$ equipped with a continuous action of G_K. The associated vector space $V_\ell(A) = T_\ell(A) \otimes_{\mathbf{Z}_\ell} \mathbf{Q}_\ell$ gives rise to an ℓ-adic representation

$$\rho_{A,\ell} : G_K \to \mathrm{Aut}_{\mathbf{Q}_\ell} V_\ell(A).$$

Note that $T_\ell(A)$ is a canonical G_K-lattice of $V_\ell(A)$. The reduction of $T_\ell(A)$ modulo ℓ is isomorphic to $A[\ell]$.

7.1.5. *ℓ-adic cohomology.* Let X be a smooth projective variety over K. Fix $\ell \neq \mathrm{char}(K)$. The Galois group G_K acts on the étale cohomology $H^n_{\text{ét}}(X \times_K \overline{K}, \mathbf{Z}/\ell^n\mathbf{Z})$. Set:

$$H^n_\ell(X) = \varprojlim_n H^n_{\text{ét}}(X \times_K \overline{K}, \mathbf{Z}/\ell^n\mathbf{Z}) \otimes_{\mathbf{Z}_\ell} \mathbf{Q}_\ell.$$

It is known that the $\mathbf{Q}_\ell$-vector spaces $H^n_\ell(X)$ are finite dimensional and therefore can be viewed as ℓ-adic representations of G_K:

$$G_K \to \mathrm{Aut}_{\mathbf{Q}_\ell} H^n_\ell(X). \tag{44}$$

These representations contain fundamental informations about the arithmetic of algebraic varieties. If X is a smooth proper scheme over a finite field $\mathbf{F}_q$ of characteristic p, then the geometric Frobenius Fr_q acts on $H^n_\ell(X)$, and the zeta-function $Z(X/\mathbf{F}_q,t)$ has the following cohomological interpretation envisioned by Weil and proved by Grothendieck:

$$Z(X/\mathbf{F}_q,t)=\prod_{i=0}^{2d}\left(1-\mathrm{Fr}_q t\mid H^n_\ell(X)\right)^{(-1)^{n+1}}.$$

Katz's survey [93] contains an interesting discussion of what is known and not known about ℓ-adic cohomology over finite fields.

Let now X be a smooth projective variety over a number field K. For any finite place $\mathfrak{p}$ of K, we can consider the restriction of the representation (44) on the decomposition group at $\mathfrak{p}$. This gives a representation of the local Galois group $G_{K_\mathfrak{p}}=\mathrm{Gal}(\overline{K}_\mathfrak{p}/K_\mathfrak{p})$:

$$G_{K_\mathfrak{p}}\to \mathrm{Aut}_{\mathbf{Q}_\ell}H^n_\ell(X).$$

If $\mathfrak{p}\nmid\ell$ and X has a good reduction $X_\mathfrak{p}$ at $\mathfrak{p}$, the base change theorem says that $H^n_\ell(X)$ is isomorphic to $H^n_\ell(X_\mathfrak{p})$. In particular, $H^n_\ell(X)$ is unramified at $\mathfrak{p}$, i.e. $G_{K_\mathfrak{p}}$ acts on $H^n_\ell(X)$ through its maximal unramified quotient $\mathrm{Gal}(K_\mathfrak{p}^{\mathrm{ur}}/K_\mathfrak{p})$. The converse holds for abelian varieties: if $V_\ell(A)$ is unramified, then A has good reduction at $\mathfrak{p}\nmid\ell$ (criterion of Néron–Ogg–Shafarevich [144]).

If $\mathfrak{p}\nmid\ell$, and X has bad reduction at $\mathfrak{p}$, an important information about the action of $G_{K_\mathfrak{p}}$ is provided by Grothendieck ℓ-adic monodromy theorem (Theorem 7.2.3 below). The case $\mathfrak{p}\mid\ell$ can be studied by the tools of p-adic Hodge theory. This is the main subject of the remainder of these notes.

7.1.6. We denote by $\mathbf{Rep}_{\mathbf{Q}_\ell}(G_K)$ the category of ℓ-adic representations of the absolute Galois group of a field K. Some of its first properties can be summarized in the following proposition:

Proposition 7.1.7. $\mathbf{Rep}_{\mathbf{Q}_\ell}(G_K)$ *is a neutral Tannakian category.*

We refer the reader to [51] for the tannakian formalism. In particular, $\mathbf{Rep}_{\mathbf{Q}_\ell}(G_K)$ is an abelian tensor category. If V_1 and V_2 are ℓ-adic representations, the Galois group G_K acts on $V_1\otimes_{\mathbf{Q}_\ell}V_2$ by

$$g(v_1\otimes v_2)=gv_1\otimes gv_2,\qquad \forall g\in G_K,\quad v_1\in V_1,\quad v_2\in V_2.$$

$\mathbf{Rep}_{\mathbf{Q}_\ell}(G_K)$ is equipped with the internal Hom:

$$\underline{\mathrm{Hom}}(V_1,V_2):=\mathrm{Hom}_{\mathbf{Q}_\ell}(V_1,V_2).$$

The Galois group acts on $\underline{\mathrm{Hom}}(V_1,V_2)$ by

$$g(f)(v_1)=gf(g^{-1}v_1),\qquad \forall g\in G_K,\quad f\in\underline{\mathrm{Hom}}(V_1,V_2),\quad v_1\in V_1.$$

For any ℓ-adic representation V, we denote by V^* its dual representation

$$V^*:=\underline{\mathrm{Hom}}(V,\mathbf{Q}_\ell):=\mathrm{Hom}_{\mathbf{Q}_\ell}(V,\mathbf{Q}_\ell),$$

where $\mathbf{Q}_\ell$ denotes the trivial representation of dimension one.

For any positive n, we set $\mathbf{Q}_\ell(n)=\mathbf{Q}_\ell(1)^{\otimes n}$ and $\mathbf{Q}_\ell(-n)=\mathbf{Q}_\ell(n)^*$.

7.1.8. We will also consider $\mathbf{Z}_\ell$-representations. Namely, a $\mathbf{Z}_\ell$-representation of G_K is a finitely generated free $\mathbf{Z}_\ell$-module equipped with a continuous linear action of G_K. The category $\mathbf{Rep}_{\mathbf{Z}_\ell}(G_K)$ of $\mathbf{Z}_\ell$-representations is abelian. It contains the tannakian subcategory $\mathbf{Rep}_{\mathbf{F}_\ell}(G_K)$ of representations of G_K over the finite field $\mathbf{F}_\ell = \mathbf{Z}/\ell\mathbf{Z}$. We have the reduction-modulo-ℓ functor

$$\begin{aligned} &\mathbf{Rep}_{\mathbf{Z}_\ell}(G_K) \to \mathbf{Rep}_{\mathbf{F}_\ell}(G_K),\\ &T \mapsto T \otimes_{\mathbf{Z}_\ell} \mathbf{F}_\ell. \end{aligned}$$

The following proposition can be easily deduced from the compactness of G_K :

Proposition 7.1.9. *For any ℓ-adic representation V, there exists a $\mathbf{Z}_\ell$-lattice stable under the action of G_K. In particular, the functor*

$$\begin{aligned} &\mathbf{Rep}_{\mathbf{Z}_\ell}(G_K) \to \mathbf{Rep}_{\mathbf{Q}_\ell}(G_K),\\ &T \mapsto T \otimes_{\mathbf{Z}_\ell} \mathbf{Q}_\ell \end{aligned}$$

is essentially surjective.

7.2. ℓ-adic representations of local fields ($\ell \neq p$).

7.2.1. From now on, we consider ℓ-adic representations of local fields. Let K be a local field with residue field k_K of characteristic p. To distinguish between the cases $\ell \neq p$ and $\ell = p$, we will use in the second case the term p-adic keeping ℓ-adic exclusively for the inequal characteristic case.

7.2.2. We consider the ℓ-adic case. Recall that for the tame quotient of the inertia subgroup we have the isomorphism (20):

$$\mathrm{Gal}(K^{\mathrm{tr}}/K^{\mathrm{ur}}) \simeq \prod_{q \neq p} \mathbf{Z}_q.$$

Let ψ_ℓ denote the projection

$$\psi_\ell : I_K \to \mathrm{Gal}(K^{\mathrm{tr}}/K^{\mathrm{ur}}) \to \mathbf{Z}_\ell.$$

The following general result reflects the Frobenius structure on the tame Galois group.

Theorem 7.2.3 (Grothendieck ℓ-adic monodromy theorem)**.** *Let*

$$\rho : G_K \to \mathrm{GL}(V)$$

be an ℓ-adic representation. Then the following holds true:

i) There exists an open subgroup H of the inertia group I_K such that the automorphism $\rho(g)$ is unipotent for all $g \in H$.

ii) More precisely, there exists a nilpotent operator $N : V \to V$ such that

$$\rho(g) = \exp(N\psi_\ell(g)), \qquad \forall g \in H.$$

iii) Let $\widehat{\mathrm{Fr}}_K \in G_K$ be any lift of the arithmetic Frobenius Fr_K. *Set $F = \rho(\widehat{\mathrm{Fr}}_K)$. Then*

$$FN = qNF,$$

where $q = |k_K|$.

Proof. See [144] for details.

a) By Proposition 7.1.9, ρ can be viewed as an homomorphism

$$\rho : G_K \to \mathrm{GL}_d(\mathbf{Z}_\ell).$$

Let $U = 1 + \ell^2 \mathrm{M}_d(\mathbf{Z}_\ell)$. Then U has finite index in $\mathrm{GL}_d(\mathbf{Z}_\ell)$, and there exists a finite extension K'/K such that $\rho(G_{K'}) \subset U$. Without loss of generality, we may (and will) assume that $K' = K$.

b) The wild ramification subgroup P_K is a pro-p-group. Since U is a pro-ℓ-group with $\ell \neq p$, we have $\rho(P_K) = \{1\}$, and ρ factors through the tame ramification group $\mathrm{Gal}(K^{\mathrm{tr}}/K)$. Since $\mathrm{Gal}(K^{\mathrm{tr}}/K^{\mathrm{ur}}) \simeq \prod_q \mathbf{Z}_q$, the same argument shows that ρ factors through the Galois group of the extension K_ℓ^{tr}/K, where

$$K_\ell^{\mathrm{tr}} = K^{\mathrm{ur}}(\pi^{1/\ell^\infty}), \qquad \pi \text{ is a uniformizer of } K.$$

Let τ_ℓ be the automorphism that maps to 1 under the isomorphism $\mathrm{Gal}(K_\ell^{\mathrm{tr}}/K^{\mathrm{ur}}) \simeq \mathbf{Z}_\ell$. By Proposition 2.1.4, $\mathrm{Gal}(K_\ell^{\mathrm{tr}}/K)$ is the pro-ℓ-group topologically generated by τ_ℓ and by any lift f_ℓ of the Frobenius automorphism, with the single relation:

$$f_\ell \tau_\ell f_\ell^{-1} = \tau_\ell^q. \tag{45}$$

c) Set $X = \rho(\tau_\ell) \in U$. The ℓ-adic logarithm map converges on U, and we set:

$$N := \log(X) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(X-1)^n}{n}.$$

Then for any $g \in I_K$, we have:

$$\rho(g) = \rho(\tau_\ell^{\psi_\ell(g)}) = \exp(N\psi_\ell(g)).$$

Moreover, applying the identity $\log(BAB^{-1}) = B\log(A)B^{-1}$ to (45) and setting $F = \rho(f_\ell)$, we obtain:

$$FNF^{-1} = qN.$$

d) From the last formula, it follows that N and qN have the same eigenvalues. Therefore they are all zero, and N is nilpotent. The theorem is proved. □

8. Classification of p-adic representations

8.1. **The case of characteristic p.**

8.1.1. In this section, we turn to p-adic representations. It turns out, that it is possible to give a full classification of p-adic representations of the Galois group of *any* field K of characteristic p in terms of modules equipped with a semi-linear operator. This can be explained by the existence of the Frobenius structure on K. To simplify the exposition, we will work with the purely inseparable closure $F := K^{\mathrm{rad}}$ of K. However, it is not absolutely necessary (see [69]). On the contrary, it is often preferable to work with non-perfect fields. We will come back to this question in Section 8.2.

8.1.2. Consider the ring of Witt vectors

$$O_{\mathscr{F}} = W(F)$$

Recall that $O_{\mathscr{F}}$ is a complete discrete valuation ring of characteristic 0 with maximal ideal $(p) = pO_{\mathscr{F}}$ and residue field F. Its field of fractions $\mathscr{F} = O_{\mathscr{F}}[1/p]$ is an unramified discrete valuation field. The field $\overline{F} = \overline{K}^{\mathrm{rad}}$ is an algebraic closure of F, and the Galois groups of $\overline{K}/K$ and $\overline{F}/F$ are canonically isomorphic. Set:

$$\widehat{O}^{\mathrm{ur}}_{\mathscr{F}} = W(\overline{F}), \qquad \widehat{\mathscr{F}}^{\mathrm{ur}} = \widehat{O}^{\mathrm{ur}}_{\mathscr{F}}[1/p].$$

Then $\widehat{\mathscr{F}}^{\mathrm{ur}}$ is a complete unramified discrete valuation field with residue field $\overline{F}$ and therefore can be identified with the completion of the maximal unramified extension of $\mathscr{F}$. The field $\overline{F}$ is equipped with the following structures:

- The action of the absolute Galois group G_K;
- The absolute Frobenius automorphism $\varphi : \overline{F} \to \overline{F}$, $\varphi(x) = x^p$.

The actions of G_K and φ commute to each other. On has:

$$\overline{F}^{G_K} = F, \qquad \overline{F}^{\varphi=1} = \mathbf{F}_p.$$

These actions extend naturally from $\overline{F}$ to $\widehat{O}^{\mathrm{ur}}_{\mathscr{F}}$ and $\widehat{\mathscr{F}}^{\mathrm{ur}}$, and one has:

$$(\widehat{O}^{\mathrm{ur}}_{\mathscr{F}})^{G_K} = O_{\mathscr{F}}, \qquad (\widehat{O}^{\mathrm{ur}}_{\mathscr{F}})^{\varphi=1} = \mathbf{Z}_p.$$

Definition. *Let $A = F$, $O_{\mathscr{F}}$ or $\mathscr{F}$. A φ-module over A is a finitely generated A-module D equipped with a semi-linear injective operator $\varphi : D \to D$. Namely, φ satisifies the following properties:*

$$\begin{aligned} &\varphi(x+y) = \varphi(x) + \varphi(y), && \forall x, y \in D,\\ &\varphi(ax) = \varphi(a)\varphi(x), && \forall a \in A, x \in D. \end{aligned}$$

A morphism of φ-modules is an A-linear map $f : D_1 \to D_2$ which commutes with φ :

$$f(\varphi(d)) = \varphi(f(d)), \qquad \forall d \in D_1.$$

8.1.3. Consider A as an A-module via the Frobenius map $\varphi : A \to A$. For a φ-module D, let $D \otimes_{A,\varphi} A$ denote the tensor product of A-modules D and A. We consider $D \otimes_{A,\varphi} A$ as an A-module:

$$\lambda(d \otimes a) = d \otimes \lambda a, \qquad \lambda \in A \quad d \otimes a \in D \otimes_{A,\varphi} A.$$

Then the semi-linear map $\varphi : D \to D$ induces an A-linear map

$$\Phi : D \otimes_{A,\varphi} A \to D, \qquad d \otimes a \mapsto a\varphi(d).$$

Definition. *i) Let $A = F$ or $O_{\mathscr{F}}$. A φ-module D over A is étale if the map $\Phi : D \otimes_{A,\varphi} A \to D$ is an isomorphism.*

ii) A φ-module over $\mathscr{F}$ is étale if it has an étale $O_{\mathscr{F}}$-lattice.

Let $A = F$ or $O_{\mathscr{F}}$, and assume that D if free over A. Then D is étale if and only if the matrix of $\varphi : D \to D$ is invertible over A. Note that this property does not depend on the choice of the A-base of D.

8.1.4. We denote by $\mathbf{M}_A^{\varphi,\text{ét}}$ the category of étale φ-modules over $A = F, O_{\mathscr{F}}, \mathscr{F}$. We refer the reader to [69] for a detailed study of these categories. All these categories are abelian. They are equipped with the tensor product:

$$D_1 \otimes_A D_2, \qquad \varphi(d_1 \otimes d_2) = \varphi(d_1) \otimes \varphi(d_2)$$

and the internal Hom :

$$\underline{\mathrm{Hom}}(D_1, D_2) := \mathrm{Hom}_A(D_1, D_2).$$

The action of φ on $\underline{\mathrm{Hom}}(D_1, D_2)$ is defined as follows. Let $f : D_1 \to D_2$. Then $\varphi(f)$ is the composition of maps:

$$\varphi(f) : D_1 \xrightarrow{\Phi^{-1}} D_1 \otimes_{A,\varphi} A \xrightarrow{f\otimes \mathrm{id}} D_2 \otimes_{A,\varphi} A \xrightarrow{\Phi} D_2.$$

The categories $\mathbf{M}_F^{\varphi,\text{ét}}$ and $\mathbf{M}_{\mathscr{F}}^{\varphi,\text{ét}}$ are neutral tannakian. If $A = F$ or $\mathscr{F}$, then for any $D \in \mathbf{M}_A^{\varphi,\text{ét}}$, we denote by D^* the dual module:

$$D^* = \mathrm{Hom}_A(D, A).$$

8.1.5. The term *étale* can be explained as follows. Let D be a φ-module over F. Fix a basis $\{e_1, \dots, e_n\}$ of D. Write:

$$\varphi(e_i) = \sum_{i=1}^n a_{ij} e_j, \qquad a_{ij} \in F, \qquad 1 \leqslant i \leqslant n.$$

Let $I \subset F[X_1, \dots, X_n]$ denote the ideal generated by

$$X_i^p - \sum_{i=1}^n a_{ij} X_j, \qquad 1 \leqslant i \leqslant n.$$

Then the algebra $A := F[X_1, \dots, X_n]/I$ is étale over F if and only if D is an étale φ-module. Consider the $\mathbf{F}_p$-vector space $\mathrm{Hom}_F(D, \overline{F})^{\varphi=1}$. Let $f \in \mathrm{Hom}_F(D, \overline{F})$. Then $\varphi(f) = f$ if and only if the vector $(f(e_1), \dots f(e_n)) \in \overline{F}^n$ is a solution of the system

$$X_i^p - \sum_{i=1}^n a_{ij} X_j = 0, \qquad 1 \leqslant i \leqslant n.$$

Therefore we have isomorphisms:

$$\mathrm{Hom}_F(D, \overline{F})^{\varphi=1} = \mathrm{Hom}_{F-\mathrm{alg}}(A, \overline{F}) = \mathrm{Spec}(A)(\overline{F}).$$

Note that if D is étale, then the cardinality of $\mathrm{Spec}(A)(\overline{F})$ is p^n, and $\mathrm{Hom}_F(D, \overline{F})^{\varphi=1}$ is a $\mathbf{F}_p$-vector space of dimension n.

8.1.6. For the dual module D^*, we have a canonical isomorphisms:

$$D \otimes_F \overline{F} \simeq \mathrm{Hom}_F(D^*, F) \otimes_F \overline{F} \simeq \mathrm{Hom}_F(D^*, \overline{F}).$$

Then

$$(D \otimes_F \overline{F})^{\varphi=1} \simeq \mathrm{Hom}_F(D^*, \overline{F})^{\varphi=1},$$

and applying the previous remark to D^*, we see that $(D \otimes_F \overline{F})^{\varphi=1}$ is a $\mathbf{F}_p$-vector space of dimension n.

8.1.7. Following Fontaine [69], we construct a canonical equivalence between the category $\mathbf{Rep}_{\mathbf{F}_p}(G_K)$ of modular Galois representations of G_K and $\mathbf{M}_F^{\varphi,\text{ét}}$. For any $V \in \mathbf{Rep}_{\mathbf{F}_p}(G_K)$, set:

$$\mathbf{D}_F(V) = (V \otimes_{\mathbf{F}_p} \overline{F})^{G_K}.$$

Since G_K acts trivially on F, it is clear that $\mathbf{D}_F(V)$ is an F-module equipped with the diagonal action of φ (here φ acts trivially on V). For any $D \in \mathbf{M}_F^{\varphi,\text{ét}}$, set:

$$\mathbf{V}_F(D) = (D \otimes_F \overline{F})^{\varphi=1}.$$

Then $\mathbf{V}_F(D)$ is an $\mathbf{F}_p$-vector space equipped with the diagonal action of G_K (here G_K acts trivially on D).

Theorem 8.1.8. *i) Let $V \in \mathbf{Rep}_{\mathbf{F}_p}(G_K)$ be a modular Galois representation of dimension n. Then $\mathbf{D}_F(V)$ is an étale φ-module of rank n over F.*

ii) Let $D \in \mathbf{M}_F^{\varphi,\text{ét}}$ be an étale φ-module of rank n over F. Then $\mathbf{V}_F(D)$ is a modular Galois representation of G_K of dimension n over $\mathbf{F}_p$.

iii) The functors $\mathbf{D}_F$ and $\mathbf{V}_F$ establish equivalences of tannakian categories

$$\mathbf{D}_F\,:\,\mathbf{Rep}_{\mathbf{F}_p}(G_K) \to \mathbf{M}_F^{\varphi,\text{ét}}, \qquad \mathbf{V}_F\,:\,\mathbf{M}_F^{\varphi,\text{ét}} \to \mathbf{Rep}_{\mathbf{F}_p}(G_K),$$

which are quasi-inverse to each other. Moreover, for all $T \in \mathbf{Rep}_{\mathbf{F}_p}(G_K)$ and $D \in \mathbf{M}_F^{\varphi,\text{ét}}$, we have canonical and functorial isomorphisms compatible with the actions of G_K and φ on the both sides:

$$\mathbf{D}_F(T) \otimes_F \overline{F} \simeq T \otimes_{\mathbf{F}_p} \overline{F},$$
$$\mathbf{V}_F(D) \otimes_{\mathbf{F}_p} \overline{F} \simeq D \otimes_F \overline{F}.$$

Proof. a) Let $V \in \mathbf{Rep}_{\mathbf{F}_p}(G_K)$ be a modular representation of dimension n. The Galois group G_F acts semi-linearly on $V \otimes_{\mathbf{F}_p} \overline{F}$. From Hilbert's Theorem 90 (Theorem 1.6.3), it follows that $\mathbf{D}_F(V) = (V \otimes_{\mathbf{F}_p} \overline{F})^{G_F}$ has dimension n over F, and that the multiplication in $\overline{F}$ induces an isomorphism

$$(V \otimes_{\mathbf{F}_p} \overline{F})^{G_F} \otimes_F \overline{F} \xrightarrow{\sim} V \otimes_{\mathbf{F}_p} \overline{F}.$$

Hence:

$$\mathbf{D}_F(V) \otimes_F \overline{F} \xrightarrow{\sim} V \otimes_{\mathbf{F}_p} \overline{F}.$$

This isomorphism shows that the matrix of φ is invertible in $\mathrm{GL}_n(\overline{F})$ and therefore in $\mathrm{GL}_n(F)$. This proves that $\mathbf{D}_F(V)$ is étale.

Taking the φ-invariants on the both sides, one has:

$$\mathbf{V}_F(\mathbf{D}_F(V)) = (\mathbf{D}_F(V) \otimes_F \overline{F})^{\varphi=1} \xrightarrow{\sim} (V \otimes_{\mathbf{F}_p} \overline{F})^{\varphi=1} = V. \tag{46}$$

b) Conversely, let $D \in \mathbf{M}_F^{\varphi,\text{ét}}$. We already know (see Section 8.1.6) that $\mathbf{V}_F(D)$ is a $\mathbf{F}_p$-vector space of dimension n. Consider the map

$$\alpha\,:\,(D \otimes_F \overline{F})^{\varphi=1} \otimes_{\mathbf{F}_p} \overline{F} \to D \otimes_F \overline{F}, \tag{47}$$

induced by the multiplication in $\overline{F}$. We claim that this map is an isomorpism. Since the both sides have the same dimension over $\overline{F}$, it is sufficient to prove the injectivity. To do that, we use the following argument, known as Artin's trick. Assume that f is not surjective, and take a non-zero element $x \in \ker(\alpha)$ which has a shortest presentation in the form

$$x = \sum_{i=1}^{m} d_i \otimes a_i, \qquad d_i \in \mathbf{V}_F(D), \quad a_i \in \overline{F}.$$

Without loss of generality, we can assume that $a_m = 1$ (dividing by a_m). Note that $\varphi(x) - x \in \ker(\alpha)$. On the other hand, it can be written as:

$$\varphi(x) - x = \sum_{i=1}^{m} d_i \otimes (\varphi(a_i) - a_i) = \sum_{i=1}^{m-1} d_i \otimes (\varphi(a_i) - a_i).$$

By our choice of x, this implies that $\varphi(a_i) = a_i$, and therefore $a_i \in \mathbf{F}_p$ for all i. But in this case $x \in \mathbf{V}_F(D)$, and $x = \alpha(x) = 0$. This proves the injectivity of (47).

c) By part b), we have an isomorphism:

$$\mathbf{V}_F(D) \otimes_{\mathbf{F}_p} \overline{F} \to D \otimes_F \overline{F}.$$

Taking the Galois invariants on the both sides, we obtain:

$$\mathbf{D}_F(\mathbf{V}_F(D)) = (\mathbf{V}_F(D) \otimes_{\mathbf{F}_p} \overline{F})^{G_F} \xrightarrow{\sim} (D \otimes_F \overline{F})^{G_F} = D. \tag{48}$$

From (46) and (48), it follows that the functors $\mathbf{D}_F$ and $\mathbf{V}_E$ are quasi-inverse to each other. In particular, they are equivalences of categories. Other assertions can be checked easily. □

8.1.9. Now we turn to $\mathbf{Z}_p$-representations. For all $T \in \mathbf{Rep}_{\mathbf{Z}_p}(G_K)$ and $D \in \mathbf{M}^{\varphi,\text{ét}}_{O_{\mathscr{F}}}$, set:

$$\mathbf{D}_{O_{\mathscr{F}}}(T) = (T \otimes_{\mathbf{Z}_p} \widehat{O}^{\mathrm{ur}}_{\mathscr{F}})^{G_K},$$
$$\mathbf{V}_{O_{\mathscr{F}}}(D) = (D \otimes_{O_{\mathscr{F}}} \widehat{O}^{\mathrm{ur}}_{\mathscr{F}})^{\varphi=1}.$$

The following theorem can be deduced from Theorem 8.1.8 by devissage.

Theorem 8.1.10 (Fontaine). *i) Let $T \in \mathbf{Rep}_{\mathbf{Z}_p}(G_K)$ be a $\mathbf{Z}_p$-representation. Then $\mathbf{D}_{O_{\mathscr{F}}}(T)$ is an étale φ-module over $O_{\mathscr{F}}$.*

ii) Let $D \in \mathbf{M}^{\varphi,\text{ét}}_{O_{\mathscr{F}}}$ be an étale φ-module over $O_{\mathscr{F}}$. Then $\mathbf{V}_{O_{\mathscr{F}}}(D)$ is a $\mathbf{Z}_p$-representation of G_K.

iii) The functors $\mathbf{D}_{O_{\mathscr{F}}}$ and $\mathbf{V}_{O_{\mathscr{F}}}$ establish equivalences of categories

$$\mathbf{D}_{O_{\mathscr{F}}} : \mathbf{Rep}_{\mathbf{Z}_p}(G_K) \to \mathbf{M}^{\varphi,\text{ét}}_{O_{\mathscr{F}}}, \qquad \mathbf{V}_{O_{\mathscr{F}}} : \mathbf{M}^{\varphi,\text{ét}}_{O_{\mathscr{F}}} \to \mathbf{Rep}_{\mathbf{Z}_p}(G_K),$$

which are quasi-inverse to each other. Moreover, for all $T \in \mathbf{Rep}_{\mathbf{Z}_p}(G_K)$ and $D \in \mathbf{M}^{\varphi,\text{ét}}_{O_{\mathscr{F}}}$, we have canonical and functorial isomorphisms compatible with the actions of G_K and φ on the both sides:

$$\mathbf{D}_{O_{\mathscr{F}}}(T) \otimes_{O_{\mathscr{F}}} \widehat{O}^{\mathrm{ur}}_{\mathscr{F}} \simeq T \otimes_{\mathbf{Z}_p} \widehat{O}^{\mathrm{ur}}_{\mathscr{F}},$$
$$\mathbf{V}_{O_{\mathscr{F}}}(D) \otimes_{\mathbf{Z}_p} \widehat{O}^{\mathrm{ur}}_{\mathscr{F}} \simeq D \otimes_{O_{\mathscr{F}}} \widehat{O}^{\mathrm{ur}}_{\mathscr{F}}.$$

For p-adic representations, we have the following:

Theorem 8.1.11. *i) Let V be a p-adic representation of G_K of dimension n. Then $\mathbf{D}_{\mathscr{F}}(V) = (V \otimes_{\mathbf{Q}_p} \widehat{\mathscr{F}}^{\mathrm{ur}})^{G_K}$ is an étale φ-module of dimension n over $\mathscr{F}$.*

ii) Let $D \in \mathbf{M}_{\mathscr{F}}^{\varphi,\text{ét}}$ be an étale φ-module of dimension n over $\mathscr{F}$. Then $\mathbf{V}_{\mathscr{F}}(D) = (D \otimes_{\mathbf{Q}_p} \widehat{\mathscr{F}}^{\mathrm{ur}})^{\varphi=1}$ is a p-adic Galois representation of G_K of dimension n over $\mathbf{Q}_p$.

iii) The functors

$$\mathbf{D}_{\mathscr{F}} : \mathbf{Rep}_{\mathbf{Q}_p}(G_K) \to \mathbf{M}_{\mathscr{F}}^{\varphi,\text{ét}},$$
$$\mathbf{V}_{\mathscr{F}} : \mathbf{M}_{\mathscr{F}}^{\varphi,\text{ét}} \to \mathbf{Rep}_{\mathbf{Q}_p}(G_K),$$

are equivalences of tannakian categories, which are quasi-inverse to each other. Moreover, for all $V \in \mathbf{Rep}_{\mathbf{Q}_p}(G_K)$ and $D \in \mathbf{M}_{\mathscr{F}}^{\varphi,\text{ét}}$, we have canonical and functorial isomorphisms compatible with the actions of G_K and φ on the both sides:

$$\mathbf{D}_{\mathscr{F}}(V) \otimes_{\mathscr{F}} \widehat{\mathscr{F}}^{\mathrm{ur}} \simeq V \otimes_{\mathbf{Q}_p} \widehat{\mathscr{F}}^{\mathrm{ur}},$$
$$\mathbf{V}_{\mathscr{F}}(D) \otimes_{\mathbf{Q}_p} \widehat{\mathscr{F}}^{\mathrm{ur}} \simeq D \otimes_{\mathscr{F}} \widehat{\mathscr{F}}^{\mathrm{ur}}.$$

8.2. **The case of characteristic** 0.

8.2.1. In this section, K is a local field of characteristic 0 with residual characteristic p. Let $K_\infty = K(\zeta_{p^\infty})$ denote the p-cyclotomic extension of K. Set $H_K = \mathrm{Gal}(\overline{K}/K_\infty)$ and $\Gamma_K = \mathrm{Gal}(K_\infty/K)$. Then K_∞/K is a deeply ramified (even an APF) extension, and we can consider the tilt of its completion:

$$F := \widehat{K}_\infty^\flat.$$

The field F is perfect, of characteristic p, and we apply to F the contructions of Section 8.1. Namely, set $O_{\mathscr{F}} = W(F)$ and $\mathscr{F} = O_{\mathscr{F}}[1/p]$. These rings are equipped with the weak topology, defined in Section 5.3. By Proposition 5.4.3, the separable closure $\overline{F}$ of F is dense in $\mathbf{C}_K^\flat$ and we have a natural inclusion $\widehat{O}_{\mathscr{F}}^{\mathrm{ur}} \subset W(\mathbf{C}_K^\flat)$. The Galois group G_K acts naturally on the maximal unramified extension $\mathscr{F}^{\mathrm{ur}}$ of $\mathscr{F}$ in $W(\mathbf{C}_K^\flat)[1/p]$ and on its p-adic completion $\widehat{\mathscr{F}}^{\mathrm{ur}}$. By Theorem 5.4.4, this action induces a canonical isomorphism:

$$H_K \simeq \mathrm{Gal}(\mathscr{F}^{\mathrm{ur}}/\mathscr{F}). \tag{49}$$

In particular, $(\widehat{\mathscr{F}}^{\mathrm{ur}})^{H_K} = \mathscr{F}$. The cyclotomic Galois group Γ_K acts on F and therefore on $O_{\mathscr{F}}$ and $\mathscr{F}$.

Definition. *Let $A = F, O_{\mathscr{F}}$, or $\mathscr{F}$. A (φ,Γ_K)-module over A is a φ-module over A equipped with a continuous semi-linear action of Γ_K commuting with φ. A (φ,Γ_K)-module is étale if it is étale as a φ-module.*

We denote by $\mathbf{M}_A^{\varphi,\Gamma,\text{ét}}$ the category of (φ,Γ_K)-modules over A. It can be easily seen that $\mathbf{M}_A^{\varphi,\Gamma,\text{ét}}$ is an abelian tensor category. Moreover, if $A = F$ or $\mathscr{F}$, it is neutral tannakian.

8.2.2. Now we are in position to introduce the main constructions of Fontaine's theory of (φ,Γ_K)-modules. Let T be a $\mathbf{Z}_p$-representation of G_K. Set:

$$\mathbf{D}_{\mathscr{O}_{\mathscr{F}}}(T) = (T\otimes_{\mathbf{Z}_p}\widehat{\mathscr{O}}^{\mathrm{ur}}_{\mathscr{F}})^{H_K}.$$

Thanks to the isomorphism (49) and the results of Section 8.1, $\mathbf{D}_{\mathscr{O}_{\mathscr{F}}}(T)$ is an étale φ-module. In addition, it is equipped with a natural action of Γ_K, and therefore we have a functor

$$\mathbf{D}_{\mathscr{O}_{\mathscr{F}}}\,:\, \mathbf{Rep}_{\mathbf{Z}_p}(G_K)\to \mathbf{M}^{\varphi,\Gamma,\text{ét}}_{\mathscr{O}_{\mathscr{F}}}.$$

Conversely, let D be an étale (φ,Γ_K)-module over $\mathscr{O}_{\mathscr{F}}$. Set:

$$\mathbf{V}_{\mathscr{O}_{\mathscr{F}}}(D) = (D\otimes_{\mathbf{Z}_p}\widehat{\mathscr{O}}^{\mathrm{ur}}_{\mathscr{F}})^{\varphi=1}.$$

By the results of Section 8.1, $\mathbf{V}_{\mathscr{O}_{\mathscr{F}}}(D)$, is a free $\mathbf{Z}_p$-module of the same rank. Moreover, it is equipped with a natural action of G_K, and we have a functor

$$\mathbf{V}_{\mathscr{O}_{\mathscr{F}}}\,:\, \mathbf{M}^{\varphi,\Gamma,\text{ét}}_{\mathscr{O}_{\mathscr{F}}}\to \mathbf{Rep}_{\mathbf{Z}_p}(G_K).$$

Theorem 8.2.3 (Fontaine)**.** *i) The functors $\mathbf{D}_{\mathscr{O}_{\mathscr{F}}}$ and $\mathbf{V}_{\mathscr{O}_{\mathscr{F}}}$ are equivalences of categories, which are quasi-inverse to each other.*

ii) For all $T\in \mathbf{Rep}_{\mathbf{Z}_p}(G_K)$ and $D\in \mathbf{M}^{\varphi,\text{ét}}_{\mathscr{O}_{\mathscr{F}}}$, we have canonical and functorial isomorphisms compatible with the actions of G_K and φ on the both sides:

$$\begin{aligned}\mathbf{D}_{\mathscr{O}_{\mathscr{F}}}(T)\otimes_{\mathscr{O}_{\mathscr{F}}}\widehat{\mathscr{O}}^{\mathrm{ur}}_{\mathscr{F}} &\simeq T\otimes_{\mathbf{Z}_p}\widehat{\mathscr{O}}^{\mathrm{ur}}_{\mathscr{F}},\\ \mathbf{V}_{\mathscr{O}_{\mathscr{F}}}(D)\otimes_{\mathbf{Z}_p}\widehat{\mathscr{O}}^{\mathrm{ur}}_{\mathscr{F}} &\simeq D\otimes_{\mathscr{O}_{\mathscr{F}}}\widehat{\mathscr{O}}^{\mathrm{ur}}_{\mathscr{F}}.\end{aligned}\tag{50}$$

Here G_K acts on (φ,Γ_K)-modules through Γ_K.

Proof. Theorem 8.1.10 provide us with the canonial and functorial isomorphisms (50), which are compatible with the action of φ and H_K. From construction, it follows that they are compatible with the action of the whole Galois group G_K on the both sides. This implies that the functors $\mathbf{D}_{\mathscr{O}_{\mathscr{F}}}$ and $\mathbf{V}_{\mathscr{O}_{\mathscr{F}}}$ are quasi-inverse to each other, and the theorem is proved. □

Remark 8.2.4. *We invite the reader to formulate and prove the analogous statements for the categories $\mathbf{Rep}_{\mathbf{F}_p}(G_K)$ and $\mathbf{Rep}_{\mathbf{Q}_p}(G_K)$.*

8.2.5. One can refine this theory working with the field of norms rather that with the perfectoid field $\widehat{K}^{\flat}_{\infty}$. To simplify notation, let $\mathbf{E}_K$ denote the field of norms of K_∞/K. We recall that by Theorem 6.4.3, $\mathbf{E}^{\mathrm{rad}}_K$ is dense in $\widehat{K}^{\flat}_{\infty}$. We want to lift $\mathbf{E}_K$ to characteristic 0. First, we consider the maximal unramified subextension K_0 of K. Let $K_{0,\infty}/K_0$ denote its p-cyclotomic extension. Set $\Gamma_{K_0}=\mathrm{Gal}(K_{0,\infty}/K_0)$ and $H_{K_0}=\mathrm{Gal}(\overline{K}/K_{0,\infty})$. Let $\mathbf{E}_{K_0}$ denote the field of norms of $K_{0,\infty}/K_0$. Then $\mathbf{E}_{K_0}=k_K((x))$, where $x=\varepsilon-1$ and $\varepsilon=(\zeta_{p^n})_{n\geqslant 0}$ (see (43)). Take the Teichmüller lift $[\varepsilon]\in \mathscr{O}_{\mathscr{F}}$ of ε and set $X=[\varepsilon]-1$. The Galois group and the Frobenius automorphism act on $[\varepsilon]$ and X through Γ_{K_0} as follows:

$$\begin{aligned} &g([\varepsilon])=[\varepsilon]^{\chi_0(g)}, && g\in G_{K_0}, && \varphi([\varepsilon])=[\varepsilon]^p,\\ &g(X)=(1+X)^{\chi_0(g)}-1, && g\in G_{K_0}, && \varphi(X)=(1+X)^p-1,\end{aligned}$$

where $\chi_0 : G_{K_0} \to \mathbf{Z}_p^*$ denote the p-adic cyclotomic character for K_0. The ring of integers $O_{K_0} = W(k_K)$ is a subring of $O_{\mathscr{F}}$. We define the following subrings of $O_{\mathscr{F}}$:

$$\mathbf{A}_{K_0}^+ = O_{K_0}[[X]],$$

$$\mathbf{A}_{K_0} = \widehat{\mathbf{A}_{K_0}^+[1/X]} = p\text{-adic completion of } \mathbf{A}_{K_0}^+[1/X].$$

Note that $\mathbf{A}_{K_0}$ is an unramified discrete valuation ring with residue field $\mathbf{E}_{K_0}$. It can be described explicitely as the ring of power series of the form

$$\sum_{n\in\mathbf{Z}} a_n X^n, \qquad a_n \in O_{K_0} \text{ and } \lim_{n\to-\infty} a_n = 0.$$

It is crucial that $\mathbf{A}_{K_0}$ is stable under the actions of Γ_{K_0} and φ. Set $\mathbf{B}_{K_0} = \mathbf{A}_{K_0}[1/p]$. Then $\mathbf{B}_{K_0}$ is an unramified discrete valuation field with the ring of integers $\mathbf{A}_{K_0}$.

8.2.6. By Hensel's lemma, for each finite separable extension $E/\mathbf{E}_0$, there exists a unique complete subring $A \subset \widehat{O}_{\mathscr{F}}^{\mathrm{ur}}$ containing $\mathbf{A}_{K_0}$ and such that its residue field A/pA is isomorphic to E. We denote by $\mathbf{A}_{K_0}^{\mathrm{ur}}$ the compositum of all such extensions in $\widehat{O}_{\mathscr{F}}^{\mathrm{ur}}$ and set $\mathbf{B}_{K_0}^{\mathrm{ur}} = \mathbf{A}_{K_0}^{\mathrm{ur}}[1/p]$. Then $\mathbf{B}_{K_0}^{\mathrm{ur}}$ is the maximal unramified extension of $\mathbf{B}_{K_0}$ and $\mathbf{A}_{K_0}^{\mathrm{ur}}$ is its ring of integers. Let $\mathbf{B}$ and $\mathbf{A}$ denote the p-adic completions of $\mathbf{B}_{K_0}^{\mathrm{ur}}$ and $\mathbf{A}_{K_0}^{\mathrm{ur}}$ respectively. All these rings are stable under the natural action of G_{K_0}. By the theory of fields of norms, this action induces canonical isomorphisms:

$$H_{K_0} \simeq \mathrm{Gal}(\overline{\mathbf{E}}_{K_0}/\mathbf{E}_{K_0}) \simeq \mathrm{Gal}(\mathbf{B}_{K_0}^{\mathrm{ur}}/\mathbf{B}_{K_0}).$$

8.2.7. Recall that K is a totally ramified extension of K_0. Set:

$$\mathbf{A}_K = \mathbf{A}^{H_K}, \qquad \mathbf{B}_K = \mathbf{A}_K[1/p].$$

Then $\mathbf{B}_K$ is an unramified extension of $\mathbf{B}_{K_0}$ with residue field $\mathbf{E}_K$. One has:

$$[\mathbf{B}_K : \mathbf{B}_{K_0}] = [\mathbf{E}_K : \mathbf{E}_{K_0}] = [K_\infty : K_{0,\infty}].$$

These constructions can be summarized in the following diagram, where the horizontal maps are reductions modulo p :

$$\begin{tikzcd}
\mathbf{A} \arrow[r] \arrow[d, dash] & \mathbf{E} \arrow[d, dash] \\
\mathbf{A}_K \arrow[r] \arrow[d, dash] & \mathbf{E}_K \arrow[d, dash] \\
\mathbf{A}_{K_0} \arrow[r] & \mathbf{E}_{K_0}
\end{tikzcd}$$

8.2.8. The notion of an (étale) (φ,Γ_K)-module extends verbatim to the case of modules over $\mathbf{A}_K$ (respectively $\mathbf{B}_K$). We denote by $\mathbf{M}_{\mathbf{A}_K}^{\varphi,\text{ét}}$ and $\mathbf{M}_{\mathbf{B}_K}^{\varphi,\text{ét}}$ the resulting categories. For any $\mathbf{Z}_p$-representation T of G_K, set:

$$\mathbf{D}(T) = (T \otimes_{\mathbf{Z}_p} \mathbf{A})^{H_K}.$$

Conversely, for any étale (φ,Γ_K)-module D over $\mathbf{A}_K$, set:

$$\mathbf{V}(D) = (D \otimes_{\mathbf{Z}_p} \mathbf{A})^{\varphi=1}.$$

Theorem 8.2.9 (Fontaine)**.** *The functors* $\mathbf{D}$ *and* $\mathbf{V}$ *define equivalences of categories*

$$\mathbf{D} : \mathbf{Rep}_{\mathbf{Z}_p}(G_K) \to \mathbf{M}^{\varphi,\text{ét}}_{\mathbf{A}_K}, \qquad \mathbf{V} : \mathbf{M}^{\varphi,\text{ét}}_{\mathbf{A}_K} \to \mathbf{Rep}_{\mathbf{Z}_p}(G_K),$$

which are quasi-inverse to each other.

ii) For all $T \in \mathbf{Rep}_{\mathbf{Z}_p}(G_K)$ *and* $D \in \mathbf{M}^{\varphi,\text{ét}}_{\mathbf{A}_K}$*, we have canonical and functorial isomorphisms compatible with the actions of* G_K *and* φ *on the both sides:*

$$\mathbf{D}(T) \otimes_{\mathbf{A}_K} \mathbf{A} \simeq T \otimes_{\mathbf{Z}_p} \mathbf{A},$$
$$\mathbf{V}(D) \otimes_{\mathbf{Z}_p} \mathbf{A} \simeq D \otimes_{\mathbf{A}_K} \mathbf{A}.$$

Proof. The theorem can be proved by the same arguments as used in the proofs of Theorems 8.1.8 and 8.2.3 above. For details, see [69, Théorème 3.4.3]. □

Remark 8.2.10. *We invite the reader to formulate and prove the analogous statements for the categories* $\mathbf{Rep}_{\mathbf{F}_p}(G_K)$ *and* $\mathbf{Rep}_{\mathbf{Q}_p}(G_K)$*.*

8.2.11. We remark that for all $T \in \mathbf{Rep}_{\mathbf{Z}_p}(G_K)$, one has:

$$\mathbf{D}_{O_{\mathscr{F}}}(T) \simeq \mathbf{D}(T) \otimes_{\mathbf{A}_K} O_{\mathscr{F}}.$$

Analogously, for all $D \in \mathbf{M}^{\varphi,\text{ét}}_{\mathbf{A}_K}$, one has:

$$\mathbf{V}(D) \simeq \mathbf{V}_{O_{\mathscr{F}}}(D \otimes_{\mathbf{A}_K} O_{\mathscr{F}}).$$

Contrary to $\mathbf{D}_{O_{\mathscr{F}}}(T)$, the module $\mathbf{D}(T)$ is defined over a ring of formal power series. This allows to use the tools of p-adic analysis and relate (φ,Γ_K)-modules to the theory p-adic differential equations (Fontaine's program). See also Section 13 for further comments.

9. B-admissible representations

9.1. **General approach.**

9.1.1. The classification of all p-adic representations of local fields of characteristic 0 in terms of (φ,Γ_K)-modules is a powerful result. However, the representations arising in algebraic geometry have very special properties and form some natural subcategories of $\mathbf{Rep}_{\mathbf{Q}_p}(G_K)$. Moreover, as was first observed by Grothendieck in the good reduction case, it should be possible to classify them in terms of some objects of semi-linear algebra, such as filtered Dieudonné modules (Grothendieck's mysterious functor). In this section, we consider Fontaine's general approach to this problem. See [71] for a detailed exposition.

9.1.2. In this section, K is a local field. As usual, we denote by $\overline{K}$ its separable closure and set $G_K = \mathrm{Gal}(\overline{K}/K)$. To simplify notation, in the remainder of this paper we will write $\mathbf{C}$ instead of $\mathbf{C}_K$ for the p-adic completion of $\overline{K}$. Since the field of complex numbers will appear only occasionally, this convention should not lead to confusion.

Let B be a commutative $\mathbf{Q}_p$-algebra without zero divisors, equipped with a $\mathbf{Q}_p$-linear action of G_K. Let C denote the field of fractions of B. Set $E = B^{G_K}$. We adopt the definition of a regular algebra provided by Brinon and Conrad in [32], which differs from the original definition in [71].

Definition. *The algebra B is G_K-regular if it satisfies the following conditions:*
i) $B^{G_K} = C^{G_K}$*;*
ii) Each non-zero $b \in B$ such that the line $\mathbf{Q}_p b$, is stable under the action of G_K, is invertible in B.

If B is a field, these conditions are satisfied automatically.

9.1.3. In the remainder of this section, we assume that B is G_K-regular. From the condition ii), it follows that E is a field. For any p-adic representation V of G_K we consider the E-module

$$\mathbf{D}_B(V) = (V \otimes_{\mathbf{Q}_p} B)^{G_K}.$$

The multiplication in B induces a natural map

$$\alpha_B : \mathbf{D}_B(V) \otimes_E B \to V \otimes_{\mathbf{Q}_p} B.$$

Proposition 9.1.4. *i) The map α_B is injective for all $V \in \mathbf{Rep}_{\mathbf{Q}_p}(G_K)$.*
ii) $\dim_E \mathbf{D}(V) \leqslant \dim_{\mathbf{Q}_p} V$.

Proof. See [32, Theorem 5.2.1]. Set $\mathbf{D}_C(V) = (V \otimes_{\mathbf{Q}_p} C)^{G_K}$. Since $B^{G_K} = C^{G_K}$, $\mathbf{D}_C(V)$ is an E-vector space, and we have the following diagram with injective vertical maps:

$$\begin{tikzcd}
\mathbf{D}_B(V) \arrow[r, "\alpha_B"] \arrow[d] & V \otimes_{\mathbf{Q}_p} B \arrow[d] \\
\mathbf{D}_C(V) \arrow[r, "\alpha_C"] & V \otimes_{\mathbf{Q}_p} B.
\end{tikzcd}$$

Therefore it is sufficient to prove that α_C is injective. We prove it applying Artin's trick. Assume that $\ker(\alpha_C) \neq 0$ and choose a non-zero element

$$x = \sum_{i=1}^{m} d_i \otimes c_i \in \ker(\alpha_C)$$

of the shortest length m. It is clear that in this formula, $d_i \in \mathbf{D}_C(V)$ are linearly independent. Moreover, since C is a field, one can assume that $c_m = 1$. Then for all $g \in G_K$

$$g(x) - x = \sum_{i=1}^{m-1} d_i \otimes (g(c_i) - c_i) \in \ker(\alpha_C).$$

This shows that $g(x) = x$ for all $g \in G_K$, and therefore that $c_i \in C^{G_K} = E$ for all $1 \leqslant i \leqslant m$. Thus $x \in \mathbf{D}_C(V)$. From the definition of α_C, it follows that $\alpha_C(x) = x$, hence $x = 0$. □

Definition. *A p-adic representation V is called B-admissible if*

$$\dim_E \mathbf{D}_B(V) = \dim_{\mathbf{Q}_p} V.$$

Proposition 9.1.5. *If V is admissible, then the map α_B is an isomorphism.*

Proof. See [71, Proposition 1.4.2]. Let $v = \{v_i\}_{i=1}^n$ and $d = \{d_i\}_{i=1}^n$ be arbitrary bases of V and $\mathbf{D}_B(V)$ respectively. Then $v = Ad$ for some matrix A with coefficients in B. The bases $x = \bigwedge_{i=1}^n d_i \in \bigwedge^n \mathbf{D}_B(V)$ and $y = \bigwedge_{i=1}^n v_i \in \bigwedge^n V$ are related by $x = \det(A)y$. Since G_K acts on $y \in \bigwedge^n V$ as multiplication by a character, the $\mathbf{Q}_p$-vector space generated by $\det(A)$ is stable under the action of G_K. This shows that A is invertible, and α_B is an isomorphism. □

9.1.6. We denote by $\mathbf{Rep}_B(G_K)$ the category of B-admissible representations. The following proposition summarizes some properties of this category.

Proposition 9.1.7. *The category $\mathbf{Rep}_B(G_K)$ is a tannakian subcategory of all p-adic representations $\mathbf{Rep}_{\mathbf{Q}_p}(G_K)$. In particular, the following holds true:*

i) If in an exact sequence

$$0 \to V' \to V \to V'' \to 0$$

V is B-admissible, then V' and V'' are B-admissible.

ii) If V' and V'' are B admissible, then $V' \otimes_{\mathbf{Q}_p} V''$ is B admissible.

iii) V is B-admissible if and only if the dual representation V^* is B-admissible, and in that case $\mathbf{D}_B(V^*) = \mathbf{D}_B(V)^*$.

iv) The functor

$$\mathbf{D}_B : \mathbf{Rep}_B(G_K) \to \mathbf{Vect}_E$$

to the category of finite dimensional E-vector spaces, is exact and faithful.

Proof. The proof is formal. See [71, Proposition 1.5.2]. □

9.1.8. We can also work with the contravariant version of the functor $\mathbf{D}_B$:

$$\mathbf{D}_B^*(V) = \mathrm{Hom}_{G_K}(V, B).$$

From definitions, it is clear that

$$\mathbf{D}_B^*(V) = \mathbf{D}_B(V^*).$$

In particular, if V (and therefore V^*) is admissible, then

$$\mathbf{D}_B^*(V) = \mathbf{D}_B(V)^* := \mathrm{Hom}_E(\mathbf{D}_B(V), E).$$

The last isomorphism shows that the covariant and contravariant theories are equivalent. For an admissible V, we have the canonical non-degenerate pairing

$$\langle\, , \rangle : V \times \mathbf{D}^*(V) \to B, \qquad \langle v, f \rangle = f(v),$$

which can be seen as an abstract p-adic version of the canonical duality between singular homology and de Rham cohomology of a complex variety.

9.2. **First examples.**

9.2.1. *$B = \overline{K}$, where K is of characteristic* 0. The $\overline{K}$-admissible representations are p-adic representations having finite image. Indeed, since the action of G_K is discrete, each $\overline{K}$-admissible representation has finite image. Conversely, if V has finite image, it is $\overline{K}$-admissible by Hilbert's theorem 90.

9.2.2. $B = W(\overline{k}_K)[1/p]$. The B-admissible representations are unramified p-adic representations. This follows from Proposition 1.6.5.

9.2.3. $B = \widehat{\mathscr{F}}^{\mathrm{ur}}$. Let K be a local field of characteristic p, and let $\widehat{\mathscr{F}}^{\mathrm{ur}} = W(K^{\mathrm{rad}})[1/p]$. By Theorem 8.1.11, each p-adic representation of G_K is $\widehat{\mathscr{F}}^{\mathrm{ur}}$-admissible.

9.2.4. $B = \mathbf{C}$*, where* K *is of characteristic* 0*.* Sen proved (see Corollary 10.2.12 below) that V is $\mathbf{C}$-admissible if and only of I_K acts on V through a finite quotient. The sufficiency of this condition can be proved as follows. Set $n = \dim_{\mathbf{Q}_p} V$. Assume that $\rho(I_K)$ is finite. Let $U \subset I_K$ be a subgroup of finite index such that $\rho(U) = \{1\}$. By the theorem of Ax–Sen–Tate, $(V \otimes_{\mathbf{Q}_p} \mathbf{C})^U = V \otimes_{\mathbf{Q}_p} \widehat{L}$, where $L = \overline{K}^U$. Applying Hilbert's Theorem 90 to the extension $\widehat{L}/\widehat{K}^{\mathrm{ur}}$, we obtain that $(V \otimes_{\mathbf{Q}_p} \mathbf{C})^{I_K}$ is a n-dimensional vector space over $\widehat{K}^{\mathrm{ur}}$ equipped with a semi-linear action of $\mathrm{Gal}(K^{\mathrm{ur}}/K)$. Now from Proposition 1.6.5 it follows that $(V \otimes_{\mathbf{Q}_p} \mathbf{C})^{I_K}$ has a $\mathrm{Gal}(K^{\mathrm{ur}}/K)$-invariant basis, and therefore $\dim_K \mathbf{D}_{\mathbf{C}}(V) = n$.

The necessity is the difficult part of Sen's theorem, and we prove it only for one-dimensional representations.

Proposition 9.2.5. *If the one-dimensional representation* $\mathbf{Q}_p(\eta)$ *is* $\mathbf{C}$*-admissible, then* $\eta(I_K)$ *is finite.*

Proof. a) If $\eta(I_K)$ is infinite, then from Theorem 4.3.2, it follows that $\mathbf{C}(\eta)^{G_K} = 0$. Hence $\mathbf{Q}_p(\eta)$ is not $\mathbf{C}$-admissible. □

9.2.6. Consider the multiplicative group $\mathbb{G}_m$ over the field of complex numbers $\mathbb{C}$. Then $\mathbb{G}_m(\mathbb{C}) = \mathbb{C}^*$ is the punctured complex plane, and the Betti homology $H_1(\mathbb{G}_m)$ is the one-dimensional $\mathbf{Q}$-vector space generated by the counter-clockwise circle centered at 0. The de Rham cohomology $H^1_{\mathrm{dR}}(\mathbb{G}_m)$ is generated over K by the class of the differential form $\frac{dX}{X}$. The integration yields a non-degenerate bilinear map:

$$\begin{aligned} &\langle\, ,\, \rangle_{\mathbf{C}} \,:\, H_1(\mathbb{G}_m) \times H^1_{\mathrm{dR}}(\mathbb{G}_m) \to \mathbb{C}, \\ &\langle \gamma, \omega \rangle_{\mathbf{C}} = \int_\gamma \omega. \end{aligned} \tag{51}$$

The p-adic realization of $\mathbb{G}_m$ is its Tate module:

$$T_p(\mathbb{G}_m) := \varprojlim_n \mu_{p^n} \simeq \mathbf{Z}_p(1).$$

The p-adic analog of the pairing (51) should be a non-degenerate bilinear map

$$\langle\, ,\, \rangle \,:\, T_p(\mathbb{G}_m) \times H^1_{\mathrm{dR}}(\mathbb{G}_m) \to B,$$

with values in some ring B of "p-adic periods", compatible with the Galois action on $T_p(\mathbb{G}_m)$ and B. Proposition 9.2.5 shows that in the field $\mathbf{C}$, there doesn't exist a non-zero element t such that

$$g(t) = \chi_K(g)t, \qquad g \in G_K.$$

Therefore the ring of p-adic periods should be in some sense "bigger" that $\mathbf{C}$.

10. Tate–Sen theory

10.1. **Hodge–Tate representations.**

10.1.1. We maintain notation and conventions of Section 9.1. The notion of a Hodge–Tate representation was introduced in Tate's paper [151]. We use the formalism of admissible representations. Let K be a local field of characteristic 0. Let

$$\mathbf{B}_{\mathrm{HT}} = \mathbf{C}[t, t^{-1}]$$

denote the ring of polynomials in the variable t with integer exponents and coefficients in $\mathbf{C}$. We equip $\mathbf{B}_{\mathrm{HT}}$ with the action of G_K given by

$$g\left(\sum a_i t^i\right) = \sum g(a_i)\chi_K^i(g)\,t^i, \qquad g \in G_K,$$

where χ_K denotes the cyclotomic character. Therefore G_K acts naturally on $\mathbf{C}$, and t can be viewed as the "p-adic $2\pi i$" – the p-adic period of the multiplicative group $\mathbb{G}_m$. For any p-adic representation V of G_K, we set:

$$\mathbf{D}_{\mathrm{HT}}(V) = (V \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{HT}})^{G_K}.$$

Proposition 10.1.2. *The ring* $\mathbf{B}_{\mathrm{HT}}$ *is* G_K*-regular and* $\mathbf{B}_{\mathrm{HT}}^{G_K} = K$.

Proof. a) The field of fractions $\mathrm{Fr}(\mathbf{B}_{\mathrm{HT}})$ of $\mathbf{B}_{\mathrm{HT}}$ is isomorphic to the field of rational functions $\mathbf{C}(t)$. Embedding it in $\mathbf{C}((t))$, we have:

$$\mathbf{B}_{\mathrm{HT}}^{G_K} \subset \mathrm{Fr}(\mathbf{B}_{\mathrm{HT}})^{G_K} \subset \mathbf{C}((t))^{G_K}.$$

From Theorem 4.3.2, it follows that $(\mathbf{C}t^i)^{G_K} = K$ if $i = 0$, and $(\mathbf{C}t^i)^{G_K} = 0$ otherwise. Hence $\mathbf{B}_{\mathrm{HT}}^{G_K} = \mathbf{C}((t))^{G_K} = K$. Therefore

$$\mathrm{Fr}(\mathbf{B}_{\mathrm{HT}})^{G_K} = \mathbf{B}_{\mathrm{HT}}^{G_K} = K.$$

b) Let $b \in \mathbf{B}_{\mathrm{HT}} \setminus \{0\}$. Assume that $\mathbf{Q}_p b$ is stable under the action of G_K. This means that

$$g(b) = \eta(g)b, \qquad \forall g \in G_K \tag{52}$$

for some character $\eta : G_K \to \mathbf{Z}_p^*$. Write b in the form

$$b = \sum_i a_i t^i.$$

We will prove by contradiction that all, except one monomials in this sum are zero. From formula (52), if follows that for all i one has:

$$g(a_i)\chi_K^i(g) = a_i\eta(g), \qquad g \in G_K.$$

Assume that a_i and a_j are both non-zero for some $i \neq j$. Then

$$\frac{g(a_i)\chi_K^i(g)}{a_i} = \frac{g(a_j)\chi_K^j(g)}{a_j}, \qquad \forall g \in G_K.$$

Set $c = a_i/a_j$ and $m = i - j \neq 0$. Then c is a non-zero element of $\mathbf{C}$ such that

$$g(c)\chi_K^m(g) = c, \qquad \forall g \in G_K.$$

This is in contradiction with the fact that $\mathbf{C}(m)^{G_K} = 0$ if $m \neq 0$.

Therefore $b = a_i t^i$ for some $i \in \mathbf{Z}$ and $a_i \neq 0$. This implies that b is invertible in $\mathbf{B}_{\mathrm{HT}}$. The proposition is proved. □

10.1.3. Let $\mathbf{Grad}_K$ denote the category of finite-dimensional graded K-vector spaces. The morphisms in this category are linear maps preserving the grading. We remark that $\mathbf{D}_{\mathrm{HT}}(V)$ has a natural structure of a graded K-vector space:

$$\mathbf{D}_{\mathrm{HT}}(V) = \underset{i\in\mathbf{Z}}{\oplus} \mathrm{gr}^i \mathbf{D}_{\mathrm{HT}}(V), \qquad \mathrm{gr}^i \mathbf{D}_{\mathrm{HT}}(V) = \left(V \otimes_{\mathbf{Q}_p} \mathbf{C}t^i\right)^{G_K}.$$

Therefore we have a functor

$$\mathbf{D}_{\mathrm{HT}} : \mathbf{Rep}_{\mathbf{Q}_p}(G_K) \to \mathbf{Grad}_K.$$

Note that this functor is clearly left exact but not right exact (see Example 10.2.13 below).

Definition. *A p-adic representation V is a Hodge–Tate representation if it is $\mathbf{B}_{\mathrm{HT}}$-admissible.*

We denote by $\mathbf{Rep}_{\mathrm{HT}}(G_K)$ the category of Hodge–Tate representations. From the general formalism of B-admissible representations, it follows that the restriction of $\mathbf{D}_{\mathrm{HT}}$ on $\mathbf{Rep}_{\mathrm{HT}}(G_K)$ is exact and faithful.

10.1.4. Set:

$$V^{(i)} = \{x \in V \otimes_{\mathbf{Q}_p} \mathbf{C} \mid g(x) = \chi_K(g)^i x, \quad \forall g \in G_K\}, \qquad i \in \mathbf{Z},$$
$$V\{i\} = V^{(i)} \otimes_K \mathbf{C}.$$

It is clear that $V^{(i)} \simeq \mathrm{gr}^{-i}\mathbf{D}_{\mathrm{HT}}(V)$. Moreover, the multiplication in $\mathbf{C}$ induces linear maps of $\mathbf{C}$-vector spaces $V\{i\} \to V \otimes_{\mathbf{Q}_p} \mathbf{C}$. Therefore one has a $\mathbf{C}$-linear map:

$$\underset{i\in\mathbf{Z}}{\oplus} V\{i\} \to V \otimes_{\mathbf{Q}_p} \mathbf{C}. \tag{53}$$

The following proposition shows that our definition of a Hodge–Tate representation coincides with Tate's original definition:

Proposition 10.1.5. *i) For any representation V, the map (53) is injective.*
ii) V is a Hodge–Tate if and only if (53) is an isomorphism.

Proof. i) By Proposition 9.1.4, for *any* p-adic representation V, the map

$$\alpha_{\mathrm{HT}} : \mathbf{D}_{\mathrm{HT}}(V) \otimes_K \mathbf{B}_{\mathrm{HT}} \to V \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{HT}}$$

is injective. The restriction of α_{HT} on the homogeneous subspaces of degree 0 coincides with the map (53). Therefore (53) is injective.

ii) By Proposition 9.1.5, V is a Hodge–Tate if and only if α_{HT} is an isomorphism. We remark that α_{HT} is an isomorphism if and only if the map (53) is. This proves the proposition. □

Definition. *Let V be a Hodge–Tate representation. The isomorphism*

$$V \otimes_{\mathbf{Q}_p} \mathbf{C} \simeq \underset{i\in\mathbf{Z}}{\oplus} V\{i\}$$

is called the Hodge–Tate decomposition of V. If $V\{i\} \neq 0$, one says that the integer i is a Hodge–Tate weight of V, and that $d_i = \dim_{\mathbf{C}} V\{i\}$ is the multiplicity of i.

We will use the standard notation $\mathbf{C}(i) = \mathbf{C}(\chi_K^i)$ for the cyclotomic twists of $\mathbf{C}$. Then $V\{i\} = \mathbf{C}(i)^{d_i}$ as a Galois module. The Hodge–Tate decomposition of V can be written in the following form:

$$V \otimes_{\mathbf{Q}_p} \mathbf{C} = \underset{i\in\mathbf{Z}}{\oplus} \mathbf{C}(i)^{d_i}.$$

10.1.6. **Examples.** 1) Let $\psi : G_K \to \mathbf{Z}_p^*$ be a continuous character. Then $\mathbf{Q}_p(\psi)$ is a Hodge–Tate of weight i if and only if

$$\psi|_{I'_K} = \chi_K^i|_{I'_K}$$

for some open subgroup I'_K of the inertia group I_K. This follows from Proposition 9.2.5.

2) Assume that E is a subextension K such that $\tau E \subset K$ for each conjugate of E over $\mathbf{Q}_p$. Let $\psi : G_K \to O_E^*$ be a continuous character. Then $E(\psi)$ can be seen as a p-adic representation of dimension $[E : \mathbf{Q}_p]$ with coefficients in $\mathbf{Q}_p$ and

$$E(\psi) \otimes_{\mathbf{Q}_p} \mathbf{C} = \bigoplus_{\tau\in \mathrm{Hom}_{\mathbf{Q}_p}(E,K)} \mathbf{C}(\tau \circ \psi).$$

Therefore $E(\psi)$ is of Hodge–Tate if and only if for each τ

$$\mathbf{C}(\tau \circ \psi) = \mathbf{C}(\chi_K^{n_\tau}), \qquad \text{for some } n_\tau \in \mathbf{Z}.$$

We come back to this example in Section 15.

10.2. **Sen's theory.**

10.2.1. Let V be a Galois representation of G_K. Then $V \otimes_{\mathbf{Q}_p} \mathbf{C}$ can be viewed as an object of the category $\mathbf{Rep}_{\mathbf{C}}(G_K)$ of finite-dimensional $\mathbf{C}$-vector spaces equipped with a *semi-linear* action of G_K. This category was first studied by Sen [136]. Let $K_\infty = K(\zeta_{p^\infty})$ denote the cyclotomic extension of K. Set $\Gamma_K = \mathrm{Gal}(K_\infty/K)$ and $H_K = \mathrm{Gal}(\overline{K}/K_\infty)$. Let $W \in \mathbf{Rep}_{\mathbf{C}}(G_K)$. Sen's method decomposes into 3 steps:

10.2.2. *Descent to* $\widehat{K}_\infty$. Set $\widehat{W}_\infty = W^{H_K}$. By Theorem 4.3.8 and the inflation-restriction exact sequence, one has:

$$H^1(\Gamma_K, \mathrm{GL}_n(\widehat{K}_\infty)) \simeq H^1(G_K, \mathrm{GL}_n(\mathbf{C})).$$

Therefore the natural map

$$\widehat{W}_\infty \otimes_{\widehat{K}_\infty} \mathbf{C} \to W$$

is an isomorphism. Let $\mathbf{Rep}_{\widehat{K}_\infty}(\Gamma_K)$ be the category of finite-dimensional $\widehat{K}_\infty$-vector spaces equipped with a semi-linear action of Γ_K. Then the functor

$$\mathbf{Rep}_{\mathbf{C}}(G_K) \to \mathbf{Rep}_{\widehat{K}_\infty}(\Gamma_K), \qquad W \mapsto \widehat{W}_\infty$$

is an equivalence of categories. Its quasi-inverse is given by extension of scalars $X \mapsto X \otimes_{\widehat{K}_\infty} \mathbf{C}$.

10.2.3. *Undoing the completion.* For any $\widehat{K}_\infty$-representation X, let X_f denote the union of all finite-dimensional K-vector subspaces of X. Sen proves that the map

$$X_f \otimes_{K_\infty} \widehat{K}_\infty \to X$$

is an isomorphism. The key tool here is the canonical isomorphism

$$H^1(\Gamma_K, \mathrm{GL}_n(K_\infty)) \simeq H^1(\Gamma_K, \mathrm{GL}_n(\widehat{K}_\infty))$$

(see [136, Proposition 6]). This implies that the functors $X \mapsto X_f$ and $U \to U \otimes_{K_\infty} \widehat{K}_\infty$ are mutually quasi-inverse equivalences between $\mathbf{Rep}_{\widehat{K}_\infty}(\Gamma_K)$ and $\mathbf{Rep}_{K_\infty}(\Gamma_K)$.

10.2.4. *Infinitesimal action of* Γ_K. Let U be a K_∞-representation of Γ_K. If $\gamma \in \Gamma_K$ is close to 1, the formal power series

$$\frac{\log(\gamma)}{\log(\chi_K(\gamma))} = \frac{1}{\log(\chi_K(\gamma))} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(\gamma-1)^n}{n}$$

defines a K_∞*-linear* operator Θ on U, which does not depend on the choice of γ. There exists an open subgroup $\Gamma' \subset \Gamma_K$ such that

$$\gamma(x) = \exp(\log(\chi_K(\gamma))\,\Theta)\,(x) \qquad \forall \gamma \in \Gamma', x \in U.$$

Let $\mathbf{S}_{K_\infty}$ denote the category of finite dimensional K_∞-vector spaces equipped with a linear operator. The morphisms of $\mathbf{S}_{K_\infty}$ are defined as K_∞-linear maps which commute with the action of underlying operators. Using Hilbert's Theorem 90, it can be checked that the functor

$$\mathbf{Rep}_{K_\infty}(\Gamma_K) \to \mathbf{S}_{K_\infty}, \qquad U \mapsto (U, \Theta)$$

is exact and fully faithful.

10.2.5. Combining previous results, one can associate to any $\mathbf{C}$-representation W the K_∞-vector space $W_\infty = (\widehat{W}_\infty)_f$ equipped with the operator Θ. The main result of Sen's theory states as follows:

Theorem 10.2.6 (Sen)**.** *The functor*

$$\mathbf{\Delta}_{\mathrm{Sen}} : \mathbf{Rep}_{\mathbf{C}}(G_K) \to \mathbf{S}_{K_\infty}, \qquad W \mapsto (W_\infty, \Theta)$$

is exact and fully faithful.

Proof. See [136]. □

Remark 10.2.7. *Let* $\Theta_{\mathbf{C}} : W \to W$ *denote the linear operator obtained from* Θ *by extension of scalars. The map*

$$W^{G_K} \otimes_K \mathbf{C} \to W$$

is injective and identifies $W^{G_K} \otimes_K \mathbf{C}$ *with* $\ker(\Theta_{\mathbf{C}})$. *In particular,* W^{G_K} *is a finite-dimensional* K*-vector space.*

10.2.8. We discuss some applications of Sen's theory to p-adic representations. To any p-adic representation $\rho : G_K \to \mathrm{Aut}_{\mathbf{Q}_p} V$, we associate the $\mathbf{C}$-representation $W = V \otimes_{\mathbf{Q}_p} \mathbf{C}$ and set:

$$\mathbf{D}_{\mathrm{Sen}}(V) = \mathbf{\Delta}_{\mathrm{Sen}}(W).$$

Hodge-Tate representations have the following characterization in terms of the operator Θ :

Proposition 10.2.9. *V is a Hodge–Tate representation if and only if the operator $\Theta : \mathbf{D}_{\mathrm{Sen}}(V) \to \mathbf{D}_{\mathrm{Sen}}(V)$ is semi-simple and its eigenvalues belong to $\mathbf{Z}$.*

Proof. See [136, Section 2.3]. □

10.2.10. We come back to general p-adic representations. The operator Θ allows to recover the Lie algebra of the image $\rho(I_K)$ of the inertia group:

Theorem 10.2.11 (Sen)**.** *The Lie algebra $\mathfrak{g}$ of $\rho(I_K)$ is the smallest of the $\mathbf{Q}_p$-subspaces S of $\mathrm{End}_{\mathbf{Q}_p}(V)$ such that $\Theta \in S \otimes_{\mathbf{Q}_p} \mathbf{C}$.*

Proof. See [136, Theorem 11]. □

The following corollary of this theorem generalizes Proposition 9.2.5.

Corollary 10.2.12. *$\rho(I_K)$ is finite if and only if $\Theta = 0$.*

10.2.13. **Example.** Let V be a two dimensional $\mathbf{Q}_p$-vector space with a fixed basis $\{e_1, e_2\}$. Let $\rho : G_K \to \mathrm{GL}(V)$ be the representation given by

$$\rho(g) = \begin{pmatrix} 1 & \log(\chi_K(g)) \\ 0 & 1 \end{pmatrix} \qquad \text{in the basis } \{e_1, e_2\}.$$

Prove that V is not Hodge–Tate. Let $\overline{e}_2 = e_2 \pmod{\mathbf{Q}_p e_1}$. Since V sits in the exact sequence

$$0 \to \mathbf{Q}_p e_1 \to V \to \mathbf{Q}_p \overline{e}_2 \to 0,$$

we have an exact sequence:

$$0 \to \mathbf{D}_{\mathrm{HT}}(\mathbf{Q}_p e_1) \to \mathbf{D}_{\mathrm{HT}}(V) \to \mathbf{D}_{\mathrm{HT}}(\mathbf{Q}_p \overline{e}_2).$$

Here $\mathbf{Q}_p e_1$ and $\mathbf{Q}_p \overline{e}_2$ are trivial p-adic representations, and

$$\mathbf{D}_{\mathrm{HT}}(\mathbf{Q}_p e_1) = K e_1, \qquad \mathbf{D}_{\mathrm{HT}}(\mathbf{Q}_p \overline{e}_2) = K \overline{e}_2.$$

Therefore $\mathbf{D}_{\mathrm{HT}}(V)$ has dimension 2 if and only if $\overline{e}_2$ lifts to an element

$$x = e_2 + \lambda \otimes e_1 \in \mathbf{D}_{\mathrm{HT}}(V), \qquad \lambda \in \mathbf{B}_{\mathrm{HT}}.$$

The condition $x \in \mathbf{D}_{\mathrm{HT}}(V)$ reads:

$$g(\lambda) - \lambda = \log \chi_K(g), \qquad \forall g \in G_K.$$

Therefore $\log \chi_K$ is a coboundary in $\mathbf{C}$, but this contradicts to Theorem 4.3.2. Hence V is not Hodge–Tate. This example also shows that $\mathbf{Rep}_{\mathrm{HT}}(G_K)$ is not stable under extensions.

Finally remark that in the same basis, the operator Θ reads:

$$\Theta = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

In particular, it is not semi-simple, and the above arguments agree with Proposition 10.2.9.

11. Rings of p-adic periods

11.1. The field $\mathbf{B}_{\mathrm{dR}}$.

11.1.1. In this section, we define Fontaine's rings of p-adic periods $\mathbf{B}_{\mathrm{dR}}$, $\mathbf{B}_{\mathrm{st}}$ and $\mathbf{B}_{\mathrm{cris}}$. For proofs and more detail, we refer the reader to [66], [68] and [70].

Let K be a local field of characteristic 0. Recall that the ring of integers of the tilt $\mathbf{C}^\flat$ of $\mathbf{C}$ was defined as the projective limit

$$O_{\mathbf{C}}^\flat = \varprojlim_{\varphi} O_{\mathbf{C}}/p\,O_{\mathbf{C}}, \qquad \varphi(x) = x^p$$

(see Section 5.2). By Propositions 5.2.3 and 5.2.4, $O_{\mathbf{C}}^\flat$ is a complete perfect valuation ring of characteristic p with residue field $\overline{k}_K$. The field $\mathbf{C}^\flat$ is a complete algebraically closed field of characteristic p.

11.1.2. We will denote by $\mathbf{A}_{\mathrm{inf}}$ the ring of Witt vectors

$$\mathbf{A}_{\mathrm{inf}}(\mathbf{C}) = W(O_{\mathbf{C}}^\flat).$$

Recall that $\mathbf{A}_{\mathrm{inf}}$ is equipped with the surjective ring homomorphism $\theta : \mathbf{A}_{\mathrm{inf}} \to O_{\mathbf{C}}$ (see Proposition 5.3.3, where it is denoted by θ_E). The kernel of θ is the principal ideal generated by any element $\xi = \sum\limits_{n=0}^{\infty} [a_n]p^n \in \ker(\theta)$ such that a_1 is a unit in $O_{\mathbf{C}^\flat}$. Useful canonical choices are:

- $\xi = [\tilde p] - p$, where $\tilde p = (p^{1/p^n})_{n\geqslant 0}$;
- $\omega = \sum\limits_{i=0}^{p-1} [\varepsilon]^{i/p}$, where $\varepsilon = (\zeta_{p^n})_{n\geqslant 0}$.

Let K_0 denote the maximal unramified subextension of K. Then $O_{K_0} = W(k_K) \subset \mathbf{A}_{\mathrm{inf}}$, and we set $\mathbf{A}_{\mathrm{inf},K} = \mathbf{A}_{\mathrm{inf}} \otimes_{O_{K_0}} K$. Then θ extends by linearity to a sujective homomorphism

$$\theta \otimes \mathrm{id}_K \ :\ \mathbf{A}_{\mathrm{inf}}(\mathbf{C}) \otimes_{O_{K_0}} K \to \mathbf{C}.$$

Again, the kernel $J_K := \ker(\theta\otimes \mathrm{id}_K)$ is a principal ideal. It is generated, for example, by $[\tilde\pi] - \pi$, where π is any uniformizer of K and $\tilde\pi = (\pi^{1/p^n})_{n\geqslant 0}$. The action of G_K extends naturally to $\mathbf{A}_{\mathrm{inf},K}$, and it's easy to see that J_K is stable under this action. Let $\mathbf{B}^+_{\mathrm{dR},K}$ denote the completion of $\mathbf{A}_{\mathrm{inf},K}$ for the J_K-adic topology:

$$\mathbf{B}^+_{\mathrm{dR},K} = \varprojlim_{n} \mathbf{A}_{\mathrm{inf},K}/J_K^n.$$

The action of G_K extends to $\mathbf{B}^+_{\mathrm{dR},K}$. The main properties of $\mathbf{B}^+_{\mathrm{dR},K}$ are summarized in the following proposition:

Proposition 11.1.3. *i)* $\mathbf{B}^+_{\mathrm{dR},K}$ *is a discrete valuation ring with maximal ideal*

$$\mathfrak{m}_{\mathrm{dR},K} = J_K \mathbf{B}^+_{\mathrm{dR},K}.$$

The residue field $\mathbf{B}^+_{\mathrm{dR},K}/\mathfrak{m}_{\mathrm{dR},K}$ *is isomorphic to* $\mathbf{C}$ *as a Galois module.*

ii) The series

$$t=\log([\varepsilon])=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{([\varepsilon]-1)^n}{n}$$

converges in the J_K-adic topology to a uniformizer of $\mathbf{B}^+_{\mathrm{dR},K}$, and the Galois group acts on t as follows:

$$g(t)=\chi_K(g)t, \qquad g\in G_K.$$

iii) If L/K is a finite extension, then the natural map $\mathbf{B}^+_{\mathrm{dR},K}\to\mathbf{B}^+_{\mathrm{dR},L}$ is an isomorphism. In particular, $\mathbf{B}^+_{\mathrm{dR},K}$ depends only on the algebraic closure $\overline{K}$ of K.

iv) There exists a natural G_K-equivariant embedding of $\overline{K}$ in $\mathbf{B}^+_{\mathrm{dR},K}$, and

$$\left(\mathbf{B}^+_{\mathrm{dR},K}\right)^{G_K}=K.$$

11.1.4. We refer the reader to [66] and [70] for detailed proofs of these properties. Note that if L is a finite extension of K, then one checks first that $\mathbf{B}^+_{\mathrm{dR},K}\subset\mathbf{B}^+_{\mathrm{dR},L}$. From assertions i) and ii), it follows that this is an unramified extension of discrete valuation rings with the same residue field. This implies that $\mathbf{B}^+_{\mathrm{dR},K}=\mathbf{B}^+_{\mathrm{dR},L}$. Since $L\subset\mathbf{B}^+_{\mathrm{dR},L}$ for all L/K, this proves that $\overline{K}\subset\mathbf{B}^+_{\mathrm{dR},K}$.

11.1.5. The above proposition shows that $\mathbf{B}^+_{\mathrm{dR},K}$ depends only on the residual characteristic of the local field K. By this reason, we will omit K from notation and write $\mathbf{B}^+_{\mathrm{dR}}:=\mathbf{B}^+_{\mathrm{dR},K}$.

Definition. *The field of p-adic periods $\mathbf{B}_{\mathrm{dR}}$ is defined to be the field of fractions of $\mathbf{B}^+_{\mathrm{dR}}$.*

11.1.6. The field $\mathbf{B}_{\mathrm{dR}}$ is equipped with the canonical filtration induced by the discrete valuation, namely

$$\mathrm{Fil}^i\mathbf{B}_{\mathrm{dR}}=t^i\mathbf{B}^+_{\mathrm{dR}}, \qquad i\in\mathbf{Z}.$$

In particular, $\mathrm{Fil}^0\mathbf{B}_{\mathrm{dR}}=\mathbf{B}^+_{\mathrm{dR}}$ and $\mathrm{Fil}^1\mathbf{B}_{\mathrm{dR}}=\mathfrak{m}_{\mathrm{dR}}$. From Proposition 11.1.3, it follows that

$$\mathrm{Fil}^i\mathbf{B}_{\mathrm{dR}}/\mathrm{Fil}^{i+1}\mathbf{B}_{\mathrm{dR}}\simeq\mathbf{C}(i).$$

Therefore for the associated graded module we have

$$\mathrm{gr}^\bullet(\mathbf{B}_{\mathrm{dR}})\simeq\mathbf{B}_{\mathrm{HT}}.$$

Note that from this isomorphism it follows that $\mathbf{B}_{\mathrm{dR}}^{G_K}=K$ as claimed in Proposition 11.1.3, iii).

11.1.7. Recall that $\mathbf{A}_{\mathrm{inf}}$ is equipped with the canonical Frobenius operator φ. Set $X=[\varepsilon]-1$. Then

$$\varphi(\omega)=\frac{\varphi(X)}{X}=\frac{(1+X)^p-1}{X}=p+\binom{p}{2}X+\cdots+X^{p-1}.$$

From this formula it follows that $\ker(\theta)$ is *not* stable under the action of φ, and therefore φ can not be naturally extended to $\mathbf{B}_{\mathrm{dR}}$.

11.1.8. The field $\mathbf{B}_{\mathrm{dR}}$ is equipped with the topology induced by the discrete valuation. Now we equip it with a coarser topology, which is better adapted to the study of $\mathbf{B}_{\mathrm{dR}}$. Recall that the valuation topology on $\mathbf{C}^\flat$ induces a topology on $\mathbf{A}_{\mathrm{inf}}$, which we call the canonical topology (see Section 5.3). This topology induces a topology on $\mathbf{A}_{\mathrm{inf},K}$. The *canonical topology* on $\mathbf{B}_{\mathrm{dR}}^+ = \varprojlim_n \mathbf{A}_{\mathrm{inf},K}/J_K^n$ is defined as the topology of the inverse limit, where $\mathbf{A}_{\mathrm{inf},K}/J_K^n$ are equipped with the quotient topology. We refer the reader to [32, Exercise 4.5.3] for further detail.

11.2. The rings $\mathbf{B}_{\mathrm{cris}}$ and $\mathbf{B}_{\max}$.

11.2.1. We define the ring $\mathbf{B}_{\mathrm{cris}}$ of crystalline p-adic periods, which is a subring of $\mathbf{B}_{\mathrm{dR}}$ equipped with a natural Frobenius structure. The map $\theta : \mathbf{A}_{\mathrm{inf}} \to O_{\mathbf{C}}$ is the universal formal thickening of $O_{\mathbf{C}}$ in the sense of [70], and we denote by $\mathbf{A}_{\mathrm{inf}}^{\mathrm{PD}}$ the PD-envelop of $\ker(\theta)$ in $\mathbf{A}_{\mathrm{inf}}$ (see, for example, [22] for definition and basic properties of divided powers). Recall that

$$\xi = [\widetilde{p}] - p \in \mathbf{A}_{\mathrm{inf}}.$$

is a generator of the $\ker(\theta)$. Then $\mathbf{A}_{\mathrm{inf}}^{\mathrm{PD}}$ can be seen as the submodule of $\mathbf{B}_{\mathrm{dR}}^+$ defined as:

$$\mathbf{A}_{\mathrm{inf}}^{\mathrm{PD}} = \mathbf{A}_{\mathrm{inf}}\left[\frac{\xi^2}{2!}, \frac{\xi^3}{3!}, \dots, \frac{\xi^n}{n!}, \dots\right].$$

From the formula

$$\frac{\xi^n}{n!}\frac{\xi^m}{m!} = \binom{n+m}{n}\frac{\xi^{n+m}}{(n+m)!}$$

it follows that $\mathbf{A}_{\mathrm{inf}}^{\mathrm{PD}}$ is a subring of $\mathbf{B}_{\mathrm{dR}}$. Let

$$\mathbf{A}_{\mathrm{cris}}^+ := \widehat{\mathbf{A}}_{\mathrm{inf}}^{\mathrm{PD}} = \varprojlim_n \mathbf{A}_{\mathrm{inf}}^{\mathrm{PD}}/p^n\mathbf{A}_{\mathrm{inf}}^{\mathrm{PD}}$$

denote its p-adic completion.

Proposition 11.2.2. *$\mathbf{A}_{\mathrm{inf}}^{\mathrm{PD}}$ is stable under the action of φ. Moreover, the action of φ extends to a continuous injective map $\varphi : \mathbf{A}_{\mathrm{cris}}^+ \to \mathbf{A}_{\mathrm{cris}}^+$.*

Proof. We have

$$\varphi(\xi) = [\widetilde{p}]^p - p = (\xi + p)^p - p = \xi^p + pz$$

for some $z \in \mathbf{A}_{\mathrm{inf}}$. Hence

$$\frac{\varphi(\xi^n)}{n!} = \frac{p^n}{n!}\left(1 + (p-1)!\frac{\xi^p}{p!}\right)^n.$$

Since $\mathbf{A}_{\mathrm{inf}}^{\mathrm{PD}}$ is a ring, and $\frac{p^n}{n!} \in \mathbf{Z}_p$, this implies the proposition. □

11.2.3. It can be shown that the inclusion $\mathbf{A}_{\mathrm{inf}}^{\mathrm{PD}} \subset \mathbf{B}_{\mathrm{dR}}^{+}$ extends to a continuous embedding

$$\mathbf{A}_{\mathrm{cris}}^{+} \subset \mathbf{B}_{\mathrm{dR}}^{+},$$

where $\mathbf{A}_{\mathrm{cris}}^{+}$ and $\mathbf{B}_{\mathrm{dR}}^{+}$ are equipped with the p-adic and canonical topology respectively. In more explicit terms, $\mathbf{A}_{\mathrm{cris}}^{+}$ can be viewed as the subring

$$\mathbf{A}_{\mathrm{cris}}^{+} = \left\{ \sum_{n=0}^{\infty} a_n \frac{\xi^n}{n!} \mid a_n \in \mathbf{A}_{\mathrm{inf}}, \quad \lim_{n\to+\infty} a_n = 0 \right\} \subset \mathbf{B}_{\mathrm{dR}}^{+}.$$

The element $t = \log[\varepsilon]$ belongs to $\mathbf{A}_{\mathrm{cris}}^{+}$, and one has:

$$\varphi(t) = pt.$$

Definition. *Set* $\mathbf{B}_{\mathrm{cris}}^{+} = \mathbf{A}_{\mathrm{cris}}^{+}[1/p]$ *and* $\mathbf{B}_{\mathrm{cris}} = \mathbf{B}_{\mathrm{cris}}^{+}[1/t]$. *The ring* $\mathbf{B}_{\mathrm{cris}}$ *is called the ring of crystalline periods.*

It is easy to see that the rings $\mathbf{B}_{\mathrm{cris}}^{+}$ and $\mathbf{B}_{\mathrm{cris}}$ are stable under the action of G_K. The actions of G_K and φ on $\mathbf{B}_{\mathrm{cris}}$ commute to each other. The inclusion $\mathbf{B}_{\mathrm{cris}} \subset \mathbf{B}_{\mathrm{dR}}$ induces a filtration on $\mathbf{B}_{\mathrm{cris}}$ which we denote by $\mathrm{Fil}^i\mathbf{B}_{\mathrm{cris}}$. Note that $\mathbf{B}_{\mathrm{cris}}^{+} \subset \mathrm{Fil}^0\mathbf{B}_{\mathrm{cris}}$ but the latter space is much bigger. Also the action of φ on $\mathbf{B}_{\mathrm{cris}}$ is not compatible with filtration i.e. $\varphi(\mathrm{Fil}^i\mathbf{B}_{\mathrm{cris}}) \not\subset \mathrm{Fil}^i\mathbf{B}_{\mathrm{cris}}$. We summarize some properties of $\mathbf{B}_{\mathrm{cris}}$ in the following proposition.

Proposition 11.2.4. *The following holds true:*

i) The map

$$K \otimes_{K_0} \mathbf{B}_{\mathrm{cris}} \to \mathbf{B}_{\mathrm{dR}}, \qquad a \otimes x \to ax$$

is injective.

ii) $\mathbf{B}_{\mathrm{cris}}^{G_K} = K_0$.

iii) $\mathrm{Fil}^0\mathbf{B}_{\mathrm{cris}}^{\varphi=1} = \mathbf{Q}_p$.

iv) $\mathbf{B}_{\mathrm{cris}}$ *is* G_K*-regular.*

Proof. See [70], especially Theorems 4.2.4 and 5.3.7. □

11.2.5. The main information about the relationship between the filtration on $\mathbf{B}_{\mathrm{cris}}$ and the Frobenius map is contained in the *fundamental exact sequence*:

$$0 \to \mathbf{Q}_p \to \mathbf{B}_{\mathrm{cris}}^{\varphi=1} \to \mathbf{B}_{\mathrm{dR}}/\mathrm{Fil}^0\mathbf{B}_{\mathrm{dR}} \to 0. \tag{54}$$

The exactness in the middle term is equivalent to Proposition 11.2.4, iii) above. In addition, (54) says that and the projection $\mathbf{B}_{\mathrm{cris}}^{\varphi=1} \to \mathbf{B}_{\mathrm{dR}}/\mathbf{B}_{\mathrm{dR}}^{+}$ is surjective. We refer to [70] and [28] for proofs and related results.

11.2.6. The importance of the ring $\mathbf{B}_{\mathrm{cris}}$ relies on its connection to the crystalline cohomology [74]. On the other hand, the natural topology on $\mathbf{B}_{\mathrm{cris}}$ is quite ugly (see [40]). Sometimes, it is more convenient to work with the rings

$$\begin{aligned}
\mathbf{A}_{\max}^{+} &= \left\{ \sum_{n=0}^{\infty} a_n \frac{\xi^n}{p^n} \mid a_n \in \mathbf{A}_{\mathrm{inf}}, \quad \lim_{n\to+\infty} a_n = 0 \right\}, \\
\mathbf{B}_{\max}^{+} &= \mathbf{A}_{\max}^{+} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p, \\
\mathbf{B}_{\max} &= \mathbf{B}_{\max}^{+}[1/t],
\end{aligned}$$

which are equipped with a natural action of φ and have better topological properties. One has:

$$\varphi(\mathbf{B}_{\max}) \subset \mathbf{B}_{\mathrm{cris}} \subset \mathbf{B}_{\max}.$$

In particular, $\mathbf{B}_{\max}^{\varphi=1} = \mathbf{B}_{\mathrm{cris}}^{\varphi=1}$, and in the fundamental exact sequence $\mathbf{B}_{\mathrm{cris}}$ can be replaced by $\mathbf{B}_{\max}$. Note that the periods of crystalline representations (see Section 13) live in the ring

$$\widetilde{\mathbf{B}}_{\mathrm{rig}} = \bigcap_{i=0}^{\infty} \varphi^n(\mathbf{B}_{\mathrm{cris}}) = \bigcap_{i=0}^{\infty} \varphi^n(\mathbf{B}_{\max}).$$

We refer the reader to [40] for proofs and further results about these rings.

11.3. The ring $\mathbf{B}_{\mathrm{st}}$.

11.3.1. Morally $\mathbf{B}_{\mathrm{st}}$ is the ring of p-adic periods of varieties having semi-stable reduction modulo p. The simplest example of such a variety is provided by Tate elliptic curves E_q/K. Tate's original paper dated 1959 appeared only in [152], but an exposition of his theory can be found in [127]. See also [147] and [142]. For each $q \in K^*$ with $|q|_p < 1$, Tate constructs an elliptic curve E_q with modular invariant given by the usual formula

$$j(q) = \frac{1}{q} + 744 + 196884q + \dots$$

and having multiplicative split reduction modulo p. If E is an elliptic curve with modular invariant $j(E)$ such that $|j(E)|_p > 1$, then $j(E) = j(q)$ for some q, and E is isomorphic to E_q over a quadratic extension of K. The group of points $E_q(\overline{K})$ of E_q is isomorphic to $\overline{K}^*/q^{\mathbf{Z}}$, and the associated p-adic representation $V_p(E)$ is reducible and sits in an exact sequence

$$0 \to \mathbf{Q}_p(1) \to V_p(E) \to \mathbf{Q}_p \to 0.$$

There exists a basis $\{e_1, e_2\}$ of $V_p(E)$ such that the action of G_K is given by

$$g(e_1) = \chi_K(g)\, e_1, \quad g(e_2) = e_2 + \psi_q(g) e_1, \qquad g \in G_K,$$

where $\psi_q : G_K \to \mathbf{Z}_p$ is the cocycle defined by

$$g(\sqrt[p^m]{q}) = \zeta_{p^n}^{\psi_q(g)} \sqrt[p^m]{q}.$$

11.3.2. The ring $\mathbf{B}_{\mathrm{st}}$ is defined as the ring $\mathbf{B}_{\mathrm{cris}}[u]$ of polynomials with coefficients in $\mathbf{B}_{\mathrm{cris}}$. The Frobenius map extends to $\mathbf{B}_{\mathrm{st}}$ by $\varphi(u) = pu$. One equips $\mathbf{B}_{\mathrm{st}}$ by a *monodromy operator* N defined by $N = -\dfrac{d}{du}$. The operators φ and N are related by the formula:

$$N\varphi = p\varphi N.$$

This formula should be compared with the formulation of the ℓ-adic monodromy theorem (Theorem 7.2.3). One extends the Galois action on $\mathbf{B}_{\mathrm{st}}$ setting:

$$g(u) = u + \psi_p(g)t, \qquad g \in G_K,$$

where $\psi_p : G_K \to \mathbf{Z}_p$ is the cocycle defined by

$$g([\widetilde{p}]) = [\varepsilon]^{\psi_p(g)}[\widetilde{p}], \qquad g \in G_K.$$

There exists a G_K-equivariant embedding of $\mathbf{B}_{\mathrm{st}}$ in $\mathbf{B}_{\mathrm{dR}}$ which sends u onto the element

$$\log[\tilde{p}] = \log p + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \left(\frac{[\tilde{p}]}{p} - 1 \right)^n.$$

We remark that this embedding is not canonical and depends on the choice of $\log p$. In particular, there is no canonical filtration on $\mathbf{B}_{\mathrm{st}}$. Note that it is customary to choose $\log p = 0$.

Finally we remark that sometimes it is more natural to work with the ring $\mathbf{B}_{\mathrm{mst}} = \mathbf{B}_{\mathrm{max}}[u]$ instead $\mathbf{B}_{\mathrm{st}}$, which is equipped with the same structures but has better topological properties.

12. Filtered (φ, N)-modules

12.1. Filtered vector spaces.

12.1.1. In this section, we review the theory of filtered Dieudonné modules. The main reference is [71]. We also refer the reader to [8] for the general formalism of slope filtrations. Let K be an arbitrary field.

Definition. *A filtered vector space over K is a finite dimensional K-vector space Δ equipped with an exhaustive separated decreasing filtration by K-subspaces $(\mathrm{Fil}^i\Delta)_{i\in\mathbf{Z}}$:*

$$\dots \supset \mathrm{Fil}^{i-1}\Delta \supset F^i\Delta \supset F^{i+1}\Delta \supset \dots, \qquad \bigcap_{i\in\mathbf{Z}} \mathrm{Fil}^i\Delta = \{0\}, \quad \bigcup_{i\in\mathbf{Z}} \mathrm{Fil}^i\Delta = \Delta.$$

A morphism of filtered spaces is a linear map $f : \Delta' \to \Delta''$ which is compatible with filtrations:

$$f(\mathrm{Fil}^i\Delta') \subset \mathrm{Fil}^i\Delta'', \qquad \forall i \in \mathbf{Z}.$$

If Δ' and Δ'' are two filtered spaces, one defines the filtered space $\Delta' \otimes_K \Delta''$ as the tensor product of Δ' and Δ'' equipped with the filtration

$$\mathrm{Fil}^i(\Delta' \otimes_K \Delta'') = \sum_{i'+i''=i} \mathrm{Fil}^{i'}\Delta' \otimes_K \mathrm{Fil}^{i''}\Delta''.$$

The one-dimensional vector space $\mathbf{1}_K = K$ with the filtration

$$\mathrm{Fil}^i\mathbf{1}_K = \begin{cases} K & \text{if } i \leqslant 0 \\ 0 & \text{if } i > 0 \end{cases}$$

is a unit object with respect to the tensor product defined above, namely

$$\Delta \otimes_K \mathbf{1}_K \simeq \Delta \qquad \text{for any filtered module } \Delta.$$

One defines the internal Hom in the category of filtered vector spaces as the vector space $\underline{\mathrm{Hom}}_K(\Delta', \Delta'')$ of K-linear maps $f : \Delta' \to \Delta''$ equipped with the filtration

$$\mathrm{Fil}^i\left(\underline{\mathrm{Hom}}_K(\Delta', \Delta'')\right) = \{f \in \underline{\mathrm{Hom}}_K(\Delta', \Delta'') \mid f(\mathrm{Fil}^j\Delta') \subset \mathrm{Fil}^{j+i}(\Delta'') \quad \forall j \in \mathbf{Z}\}.$$

In particular, we consider the dual space $\Delta^* = \underline{\mathrm{Hom}}_K(\Delta, \mathbf{1}_K)$ as a filtered vector space.

We denote by $\mathbf{MF}_K$ the category of filtered K-vector spaces. It is easy to check that the category $\mathbf{MF}_K$ is an additive tensor category with kernels and cokernels, but it is not abelian.

12.1.2. **Example.** Let W be a non-zero K-vector space. Let Δ' and Δ'' denote W equipped with the following filtrations:

$$\mathrm{Fil}^i\Delta' = \begin{cases} W, & \text{if } i \leqslant 0, \\ 0, & \text{if } i \geqslant 1, \end{cases} \qquad \mathrm{Fil}^i\Delta'' = \begin{cases} W, & \text{if } i \leqslant 1, \\ 0, & \text{if } i \geqslant 2. \end{cases}$$

The identity map $\mathrm{id}_W : W \to W$ defines a morphism $f : \Delta' \to \Delta''$ in $\mathbf{MF}_K$. It is easy to check that $\ker(f) = 0$ and $\mathrm{coker}(f) = 0$. Therefore f is both a monomorphism and an epimorphism, but $\Delta' \not\simeq \Delta''$.

12.1.3. We adopt the following general definition:

Definition. *Let $\mathscr{C}$ be an additive category with kernels and cokernels. A sequence*

$$0 \to X' \xrightarrow{f} X \xrightarrow{g} X'' \to 0$$

of objects in $\mathscr{C}$ is exact if $X' = \ker(g)$ and $X'' = \mathrm{coker}(f)$.

The following proposition describes short exact sequences in $\mathbf{MF}_K$:

Proposition 12.1.4. *i) Let $f : \Delta' \to \Delta''$ be a morphism of filtered vector spaces. The canonical isomorphism*

$$\mathrm{coim}(f) = \Delta'/\ker(f) \to \mathrm{Im}(f)$$

is an isomorphism if and only if

$$f(\mathrm{Fil}^i\Delta') = f(\Delta') \cap \mathrm{Fil}^i\Delta'', \qquad \forall i \in \mathbf{Z}. \tag{55}$$

ii) A short sequence of filtered spaces

$$0 \to \Delta' \to \Delta \to \Delta'' \to 0 \tag{56}$$

is exact if and only if for each $i \in \mathbb{Z}$ the sequence

$$0 \to \mathrm{Fil}^i\Delta' \to \mathrm{Fil}^i\Delta \to \mathrm{Fil}^i\Delta'' \to 0$$

is exact.

Proof. The proof is left as an exercise. See also [50, Section 1]. □

12.1.5. For each filtered space, set:

$$t_{\mathrm{H}}(\Delta) = \sum_{i\in\mathbf{Z}} i \dim_K\left(\mathrm{gr}^i\Delta\right),$$

where $\mathrm{gr}^i\Delta = \mathrm{Fil}^i\Delta/\mathrm{Fil}^{i+1}\Delta$.

Proposition 12.1.6. *i) The function t_{H} is additive, i.e. for any exact sequence of filtetred spaces (56) one has:*

$$t_{\mathrm{H}}(\Delta) = t_{\mathrm{H}}(\Delta') + t_{\mathrm{H}}(\Delta'').$$

ii) $t_{\mathrm{H}}(\Delta) = t_{\mathrm{H}}(\wedge^d\Delta)$, where $d = \dim_K \Delta$.

Proof. i) From the definition of an exact sequence it follows that the sequence

$$0 \to \mathrm{gr}^i\Delta' \to \mathrm{gr}^i\Delta \to \mathrm{gr}^i\Delta'' \to 0$$

is exact for all i. Therefore

$$\dim_K(\mathrm{gr}^i\Delta) = \dim_K(\mathrm{gr}^i\Delta') + \dim_K(\mathrm{gr}^i\Delta'').$$

This implies i).

ii) For each i, choose a base $\{\overline{e}_{ij}\}_{j=1}^{d_i}$ of $\mathrm{gr}^i\Delta$ and denote by $\{e_{ij}\}_{j=1}^{d_i}$ its arbitrary lift in $\mathrm{Fil}^i\Delta$. Then $e = \underset{i,j}{\wedge} e_{ij}$ is a basis of $\wedge^d\Delta$. This description shows that $t_{\mathrm{H}}(\Delta)$ is the unique filtration break of $\wedge^d\Delta$. □

12.2. **φ-modules.**

12.2.1. In this section, we study in more detail the category of φ-modules over the field of fractions of Witt vectors, which was defined in Section 8.1. Here we change notation slightly and denote by k a perfect field of characteristic p and by K_0 the field $W(k)[1/p]$. This notation is consistent with the applications to the classification of p-adic representations of local fields of characteristic 0 which will be discussed in Section 13. As before, φ denotes the automorphism of Frobenius acting on K_0. Recall that a φ-module (or an φ-isocrystal) over K_0 is a finite dimensional K_0-vector space D equipped with a φ-semi-linear bijective map $\varphi : D \to D$. The category of φ-modules $\mathbf{M}_{K_0}^{\varphi}$ is a neutral tannakian category. In particular, it is abelian.

12.2.2. The structure of φ-modules is described by the theory of Dieudonné–Manin. Let v_p denote the valuation on K_0. First assume that D is a φ-module of dimension 1 over K_0. If d is a basis of D, then $\varphi(d) = \lambda d$ for some non-zero $\lambda \in K_0$, and we set $t_{\mathrm{N}}(D) = v_p(\lambda)$. Note that $v_p(\lambda)$ does not depend on the choice of d. Now, if D is a φ-module of arbitrary dimension n, its top exterior power $\wedge^n D$ is a one-dimensional vector space and we set

$$t_{\mathrm{N}}(D) = t_{\mathrm{N}}(\wedge^n D).$$

More explicitly, $t_{\mathrm{N}}(D) = v_p(A)$, where A is the matrix of φ with respect to any basis of M. The function t_{N} is additive on short exact sequences: if

$$0 \to D' \to D \to D'' \to 0$$

is exact, then $t_{\mathrm{N}}(D) = t_{\mathrm{N}}(D') + t_{\mathrm{N}}(D'')$.

Definition. *i) The slope of a non-zero φ-module D is the rational*

$$s(D) = \frac{t_{\mathrm{N}}(D)}{\dim_{K_0} D}.$$

ii) A φ-module D is pure (or isoclinic) of slope λ if $s(D') = \lambda$ for any non-zero submodule $D' \subset D$.

If D is isoclinic, we will write its slope λ in the form:

$$\lambda = \frac{a}{b}, \qquad (a,b) = 1, \quad b > 0.$$

Theorem 12.2.3 (Dieudonné–Manin)**.** *i) D is isoclinic of slope $\lambda = a/b$ if and only if there exists an O_{K_0}-lattice $L \subset D$ such that $\varphi^b(L) = p^a L$.*

ii) For all $a, b \in \mathbf{Z}$ such that $b > 0$ and $(a, b) = 1$, the φ-module

$$D_\lambda = K_0[\varphi]/(\varphi^b - p^a)$$

is isoclinic of slope $\lambda = a/b$. Moreover, if k is algebraically closed, then each isoclinic φ-module is isomorphic to a direct sum of copies of D_λ.

iii) Each φ-module D over K_0 has a unique decomposition into a direct sum

$$D = \underset{\lambda \in \mathbf{Q}^*}{\oplus} D(\lambda),$$

where $D(\lambda)$ is isoclinic of slope λ.

Proof. See [112, Section 2]. See also [56]. □

Corollary 12.2.4. *If k is algebraically closed, the category of φ-modules over K_0 is semi-simple. Its simple objects are Dieudonné modules which are isomorphic to D_λ.*

Remark 12.2.5. *1) A φ-module is étale in the sense of Section 8.1 if and only if it is isoclinic of slope* 0*.*

2) The theorem of Dieudonné–Manin allows to write $t_{\mathrm{N}}(D)$ in the form

$$t_{\mathrm{N}}(D) = \sum_{\lambda} \lambda \dim_{K_0} D(\lambda).$$

3) Kedlaya [94] *extended the theory of slopes to the category of φ-modules over the Robba ring.*

12.3. **Slope filtration.**

12.3.1. Slope functions appear in several theories. Important examples are provided by the theory of vector bundles (Harder–Narasimhan theory [85]), differential modules [155],[110] and euclidian lattices [80],[148]. A unified axiomatic treatement of the theory of slopes was proposed by Y. André [8]. In this section, we discuss this formalism in relation with the examples seen in the previous sections. We work with additive categories and refer to [8] for the general treatement.

Definition. *Let $\mathscr{C}$ be an additive category with kernels and cokernels.*

i) A monomorphism $f : X \to Y$ is strict if there exists $g : Y \to Z$ such that $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ is exact.

ii) An epimorphism $g : Y \to Z$ is strict if there exists $f : X \to Y$ such that $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ is exact.

iii) $\mathscr{C}$ is quasi-abelian if every pull-back of a strict epimorphism is a strict epimorphism and every push-out of a strict monomorphism is a strict monomorphism.

Note that in the category $\mathbf{MF}_K$, a monomorphism (respectively epimorphism) $f : X \to Y$ is strict if and only if it satisfies the condition (55).

12.3.2. Let $\mathscr{C}$ be a quasi-abelian category. Assume that $\mathscr{C}$ is essentially small, i.e. that it is equivalent to a small category. A rank function on $\mathscr{C}$ is a function $\mathrm{rk} : \mathscr{C} \to \mathbf{N}$ such that:

1) $\mathrm{rk}(X) = 0$ if and only if $X = 0$;
2) rk is additive, i.e. for any exact sequence

$$0 \to X' \xrightarrow{f} X \xrightarrow{g} X'' \to 0$$

one has:

$$\mathrm{rk}(X) = \mathrm{rk}(X') + \mathrm{rk}(X'').$$

We can now define the notion of a slope function.

Definition. *A slope function on $\mathscr{C}$ is a function $\mu : \mathscr{C} \setminus \{0\} \to \mathbf{R}$ such that:*
1) The associated degree function

$$\deg = \mathrm{rk} \cdot \mu : \mathscr{C} \to \mathbf{N}$$

(taking value 0 at the zero object) is additive on short exact sequences;
2) For any morphism $f : X \to Y$ which is both a monomorphism and an epimorphism, one has:

$$\mu(X) \leqslant \mu(Y).$$

An object $Y \in \mathscr{C}$ is called semi-stable if for any subobject X of Y, $\mu(X) \leqslant \mu(Y)$. We can now state the main theorem of this section.

Theorem 12.3.3 (Harder–Narasimhan, André)**.** *For any $X \in \mathscr{C}$, there exists a unique filtration*

$$X = X_0 \supset X_1 \supset \ldots \supset X_k = \{0\}$$

such that:
1) X_{i+1} is a strict subobject of X_i for all i;
2) The quotients X_i/X_{i+1} are semi-stable, and the sequence $\mu(X_i/X_{i+1})$ is strictly increasing.

Proof. The theorem was first proved for the category of vector bundles on a smooth projective curve over $\mathbf{C}$ [85]. André [8] extended the proof to the case of general quasi-abelian (and even proto-abelian) categories. □

We call the canonical filtration provided by Theorem 12.3.3 the Harder–Narasimhan filtration.

12.3.4. **Examples.** 1) Let $\mathscr{C} = \mathbf{MF}_K$. Set $\mathrm{rk}(\Delta) = \dim_K \Delta$ and $\deg(\Delta) = t_{\mathrm{H}}(\Delta)$. Then

$$\mu_{\mathrm{H}}(\Delta) = \frac{t_{\mathrm{H}}(\Delta)}{\dim_K \Delta}$$

is a slope function. Semi-stable objects are filtered vector spaces with a unique filtration break. The Harder–Narasimhan filtration coincides (up to enumeration) with the canonical filtration on Δ.

2) Let $\mathscr{C} = \mathbf{M}^{\varphi}_{K_0}$. Set $\mathrm{rk}(D) = \dim_{K_0} D$ and $\deg(D) = -t_{\mathrm{N}}(D)$. Then

$$\mu_{\mathrm{N}}(D) = s(D) = \frac{t_{\mathrm{N}}(D)}{\dim_{K_0} D}$$

is a slope function. Semi-simple objects are isoclinic φ-modules. On the other hand, it's easy to see that $-s(D)$ is also a slope function, which provides the opposite filtration on M and therefore its splitting in the direct sum of isoclinic components. This gives an interpretation of the decomposition of Dieudonné–Manin in terms of the slope filtration.

3) Let $\mathscr{C} = \mathbf{Bun}(X)$ be the category of vector bundles on a smooth projective curve $X/\mathbf{C}$. To each object E of this category one associates its rank rk(E) and degree $\deg(E) := \deg(\wedge^{\mathrm{rk}(E)}E)$. Then

$$\mu_{\mathrm{HN}}(E) = \frac{\deg(E)}{\mathrm{rk}(E)}$$

is a slope function. This is the classical setting of the Harder–Narasimhan theory [85]. The semi-stable objects of $\mathscr{C}$ are described in [118]. The analog of this filtration in the setting of the curve of Fargues–Fontaine plays an important role in [60].

12.4. **Filtered (φ, N)-modules.**

12.4.1. Let K be a complete discrete valuation field of characteristic 0 with perfect residue field k of characteristic p, and let K_0 denote the maximal unramified subfield of K.

Definition. *i) A filtered φ-module over K is a φ-module D over K_0 together with a structure of filtered K-vector space on $D_K = D\otimes_{K_0} K$.*

ii) A filtered (φ, N)-module over K is a filtered φ-module D over K equipped with a K_0-linear operator $N : D \to D$ such that

$$N\varphi = p\varphi N.$$

Note that the relation $N\varphi = p\varphi N$ implies that $N : D \to D$ is *nilpotent*.

12.4.2. A morphism of filtered φ-modules (respectively (φ, N)-modules) is a K_0-linear map $f : D' \to D''$ which is compatible with all additional structures. Filtered φ-modules (respectively (φ, N)-modules) form additive tensor categories which we denote by $\mathbf{MF}_K^{\varphi}$ and $\mathbf{MF}_K^{\varphi,N}$ respectively. Note that these categories are not abelian.

12.4.3. We define some subcategories of $\mathbf{MF}_K^{\varphi}$ and $\mathbf{MF}_K^{\varphi,N}$, which play an important role in the classification of p-adic representations. Equip $\mathbf{MF}_K^{\varphi}$ and $\mathbf{MF}_K^{\varphi,N}$ with the functions

$$\mathrm{rk}(D) := \dim_{K_0} K, \qquad \deg(D) := t_{\mathrm{H}}(D) - t_{\mathrm{N}}(D).$$

Proposition 12.4.4. $\mu(D) = \deg(D)/\mathrm{rk}(D)$ *is a slope function.*

Proof. We only need to prove that if $f : D' \to D''$ is both a monomorphism and an epimorphism, then $\mu(D') \leqslant \mu(D'')$. We remark that such f is an isomorphism of φ-modules; hence $\mu_{\mathrm{N}}(D') = \mu_{\mathrm{N}}(D'')$. Set $d := \dim_{K_0} D' = \dim_{K_0} D''$. Then we have a monomorphism of one-dimensional filtered spaces $\wedge^d D' \to \wedge^d D''$, and therefore

$$t_{\mathrm{H}}(D') = t_{\mathrm{H}}\left(\wedge^d D'\right) \leqslant t_{\mathrm{H}}\left(\wedge^d D''\right) = t_{\mathrm{H}}(D'').$$

Hence $\mu(D') \leqslant \mu(D'')$, and the proposition is proved. $\square$

Definition. *A filtered φ-module (respectively (φ,N)-module) is weakly admissible if it is semi-stable of slope* 0.

More explicitly, D is weakly admissible if it satisfies the following conditions:

1) $t_{\mathrm{H}}(D_K) = t_{\mathrm{N}}(D)$;
2) $t_{\mathrm{H}}(D'_K) \leqslant t_{\mathrm{N}}(D')$ for any submodule D' of D.

This is the classical definition of the weak admissibility [65], [71]. We denote by $\mathbf{MF}_K^{\varphi,f}$ and $\mathbf{MF}_K^{\varphi,N,f}$ the resulting subcategories of $\mathbf{MF}_K^{\varphi}$ and $\mathbf{MF}_K^{\varphi,N}$.

Proposition 12.4.5. *i) The categories $\mathbf{MF}_K^{\varphi,f}$ and $\mathbf{MF}_K^{\varphi,N,f}$ are abelian.*
ii) If D is weakly admissible, then its dual D^* is weakly admissible.
iii) If in a short exact sequence

$$0 \to D' \to D \to D'' \to 0$$

two of the three modules are weakly admissible, then so is the third.

Proof. This is [65, Proposition 4.2.1]. See also [32, Proposition 8.2.10 &Theorem 8.2.11] for a detailed proof. □

Remark 12.4.6. *The tensor product of two weakly admissible modules is weakly admissible. See* [153] *for a direct proof of this result. It also follows from the theorem "weakly admissible ⇒ admissible" of Colmez–Fontaine* [48]. *Therefore the categories $\mathbf{MF}_K^{\varphi,f}$ and $\mathbf{MF}_K^{\varphi,N,f}$ are neutral tannakian.*

13. The hierarchy of p-adic representations

13.1. de Rham representations.

13.1.1. In this section, we come back to classification of p-adic representations. Let K be a local field. We apply the general formalism of Section 9.1 to the rings of p-adic periods constructed in Section 11.

13.1.2. Recall that $\mathbf{B}_{\mathrm{dR}}$ is a field with $\mathbf{B}_{\mathrm{dR}}^{G_K} = K$. In particuler, it is G_K-regular. To any p-adic representation V of G_K we associate the finite-dimensional K-vector space

$$\mathbf{D}_{\mathrm{dR}}(V) = (V \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{dR}})^{G_K}.$$

We equip it with the filtration induced from $\mathbf{B}_{\mathrm{dR}}$:

$$\mathrm{Fil}^i\mathbf{D}_{\mathrm{dR}}(V) = (V \otimes_{\mathbf{Q}_p} \mathrm{Fil}^i\mathbf{B}_{\mathrm{dR}})^{G_K}.$$

The mapping which assigns $\mathbf{D}_{\mathrm{dR}}(V)$ to each V defines a functor of tensor categories

$$\mathbf{D}_{\mathrm{dR}} : \mathbf{Rep}_{\mathbf{Q}_p}(G_K) \to \mathbf{MF}_K.$$

Definition. *A p-adic representation V is called de Rham if it is $\mathbf{B}_{\mathrm{dR}}$-admissible, i.e. if*

$$\dim_K \mathbf{D}_{\mathrm{dR}}(V) = \dim_{\mathbf{Q}_p}(V).$$

We denote by $\mathbf{Rep}_{\mathrm{dR}}(G_K)$ the category of de Rham representations. By Proposition 9.1.7, it is tannakian and the the restriction of $\mathbf{D}_{\mathrm{dR}}$ on $\mathbf{Rep}_{\mathrm{dR}}(G_K)$ is exact and faithful.

Proposition 13.1.3. *Each de Rham representation is Hodge–Tate.*

Proof. Recall that we have exact sequences

$$0 \to \mathrm{Fil}^{i+1}\mathbf{B}_{\mathrm{dR}} \to \mathrm{Fil}^i\mathbf{B}_{\mathrm{dR}} \to \mathbf{C}t^i \to 0.$$

Tensoring with V and taking Galois invariants we have

$$\dim_K\left(\mathrm{gr}^i\mathbf{D}_{\mathrm{dR}}(V)\right) \leqslant \dim_K(V\otimes_{\mathbf{Q}_p}\mathbf{C}t^i).$$

From $\mathbf{B}_{\mathrm{HT}} = \underset{i\in\mathbf{Z}}{\oplus}\mathbf{C}t^i$ it follows that

$$\dim_K \mathbf{D}_{\mathrm{dR}}(V) = \sum_{\imath\in\mathbb{Z}}\dim_K\left(\mathrm{gr}^i\mathbf{D}_{\mathrm{dR}}(V)\right) \leqslant \dim_K\mathbf{D}_{\mathrm{HT}}(V) \leqslant \dim_{\mathbf{Q}_p}(V).$$

The proposition is proved. □

Remark 13.1.4. *The functor* $\mathbf{D}_{\mathrm{dR}}$ *is not fully faithful. A p-adic representation cannot be recovered from its filtered module.*

13.1.5. Using the fundamental exact sequence, one can construct Hodge–Tate representations which are not de Rham. Fix an integer $r \geqslant 1$ and consider an extension V of $\mathbf{Q}_p$ by $\mathbf{Q}_p(-r)$:

$$0 \to \mathbf{Q}_p(-r) \to V \to \mathbf{Q}_p \to 0.$$

Such extensions are classified by the first Galois cohomology group $H^1(G_K, \mathbf{Q}_p(-r))$, which is a one-dimensional K-vector space. Assume that V is a non-trivial extension. Since the Hodge–Tate weights of $\mathbf{Q}_p$ and $\mathbf{Q}_p(-r)$ are distinct, V is Hodge–Tate. However it is not de Rham (see [28, Section 4] for the proof).

13.2. Crystalline and semi-stable representations.

13.2.1. Recall that $\mathbf{B}_{\mathrm{cris}}$ is G_K-regular with $\mathbf{B}_{\mathrm{cris}}^{G_K} = K_0$. Therefore for each p-adic representation V, the K_0-vector space

$$\mathbf{D}_{\mathrm{cris}}(V) = (V\otimes_{\mathbf{Q}_p}\mathbf{B}_{\mathrm{cris}})^{G_K}$$

is finite-dimensional with $\dim_{K_0}\mathbf{D}_{\mathrm{cris}}(V) \leqslant \dim_{\mathbf{Q}_p}(V)$. The action on φ on $\mathbf{B}_{\mathrm{cris}}$ induces a semi-linear operator on $\mathbf{D}_{\mathrm{cris}}(V)$, which we denote again by φ. Since φ is injective on $\mathbf{B}_{\mathrm{cris}}$, it is bijective on the finite-dimensional vector space $\mathbf{D}_{\mathrm{cris}}(V)$. The embedding $K\otimes_{K_0}\mathbf{B}_{\mathrm{cris}} \hookrightarrow \mathbf{B}_{\mathrm{dR}}$ induces an inclusion

$$K\otimes_{K_0}\mathbf{D}_{\mathrm{cris}}(V) \hookrightarrow \mathbf{D}_{\mathrm{dR}}(V).$$

This equips $\mathbf{D}_{\mathrm{cris}}(V)_K = K\otimes_{K_0}\mathbf{D}_{\mathrm{cris}}(V)$ with the induced filtration:

$$\mathrm{Fil}^i\mathbf{D}_{\mathrm{cris}}(V)_K = \mathbf{D}_{\mathrm{cris}}(V)_K \cap \mathrm{Fil}^i\mathbf{D}_{\mathrm{dR}}(V).$$

Thereore $\mathbf{D}_{\mathrm{cris}}$ can be viewed as a functor

$$\mathbf{D}_{\mathrm{cris}} : \mathbf{Rep}_{\mathbf{Q}_p}(G_K) \to \mathbf{MF}_K^{\varphi}.$$

Definition. *A p-adic representation V is crystalline if it is* $\mathbf{B}_{\mathrm{cris}}$*-admissible, i.e. if*

$$\dim_{K_0}\mathbf{D}_{\mathrm{cris}}(V) = \dim_{\mathbf{Q}_p} V.$$

By Proposition 9.1.5, V is crystalline if and only if the map

$$\alpha_{\mathrm{cris}} : \mathbf{D}_{\mathrm{cris}}(V)\otimes_{K_0}\mathbf{B}_{\mathrm{cris}} \to V\otimes_{\mathbf{Q}_p}\mathbf{B}_{\mathrm{cris}} \tag{57}$$

is an isomorphism. We denote by $\mathbf{Rep}_{\mathrm{cris}}(G_K)$ the category of crystalline representations. From the general formalism of B-admissible representations it follows that $\mathbf{Rep}_{\mathrm{cris}}(G_K)$ is tannakian.

13.2.2. Similar arguments show that for each p-adic representation V the K_0-vector space

$$\mathbf{D}_{\mathrm{st}}(V) = (V\otimes_{\mathbf{Q}_p}\mathbf{B}_{\mathrm{st}})^{G_K}$$

is finite-dimensional and equipped with a natural structure of filtered (φ,N)-module. Since $\mathbf{B}_{\mathrm{st}}^{N=0} = \mathbf{B}_{\mathrm{cris}}$, we have:

$$\mathbf{D}_{\mathrm{cris}}(V) = \mathbf{D}_{\mathrm{st}}(V)^{N=0}.$$

Definition 13.2.3. *A p-adic representation is called semi-stable if it is* $\mathbf{B}_{\mathrm{st}}$*-admissible, i.e. if* $\dim_{K_0}\mathbf{D}_{\mathrm{st}}(V) = \dim_{\mathbf{Q}_p} V$.

By Proposition 9.1.5, V is semi-stable if and only if

$$\alpha_{\mathrm{st}} : \mathbf{D}_{\mathrm{st}}(V)\otimes_{K_0}\mathbf{B}_{\mathrm{st}} \to V\otimes_{\mathbf{Q}_p}\mathbf{B}_{\mathrm{st}} \tag{58}$$

is an isomorphism. We denote by $\mathbf{Rep}_{\mathrm{st}}(G_K)$ the tannakian category of semi-stable representations. The inclusions

$$K\otimes_{K_0}\mathbf{B}_{\mathrm{cris}} \hookrightarrow K\otimes_{K_0}\mathbf{B}_{\mathrm{st}} \hookrightarrow \mathbf{B}_{\mathrm{dR}}$$

show that

$$K\otimes_{K_0}\mathbf{D}_{\mathrm{cris}}(V) \hookrightarrow K\otimes_{K_0}\mathbf{D}_{\mathrm{st}}(V) \hookrightarrow \mathbf{D}_{\mathrm{dR}}(V).$$

Therefore each crystalline representation is semi-stable, and each semi-stable representation is de Rham.

13.2.4. **Example.** The representation $V_p(E)$ constructed in Section 11.3 gives an example of semi-stable representation which is not crystalline.

Definition. *A filtered φ-module (respectively (φ,N)-module) D is called admissible if it belongs to the essential image of* $\mathbf{D}_{\mathrm{cris}}$ *(respectively* $\mathbf{D}_{\mathrm{st}}$*). In other words, D is admissible if $D\simeq \mathbf{D}_{\mathrm{cris}}(V)$ (respectively $D\simeq\mathbf{D}_{\mathrm{st}}(V)$) for some crystalline (respectively semi-stable) representation V.*

We denote by $\mathbf{MF}_K^{\varphi,a}$ and $\mathbf{MF}_K^{\varphi,N,a}$ the resulting subcategories. The following proposition shows that semi-stable representations can be recovered from their (φ,N)-modules.

Proposition 13.2.5. *The functors*

$$\mathbf{D}_{\mathrm{cris}} : \mathbf{Rep}_{\mathrm{cris}}(G_K) \to \mathbf{MF}_K^{\varphi,a}, \qquad \mathbf{D}_{\mathrm{st}} : \mathbf{Rep}_{\mathrm{st}}(G_K) \to \mathbf{MF}_K^{\varphi,N,a}$$

are equivalences of categories. The mappings

$$\mathbf{V}_{\mathrm{cris}} : D \to \mathrm{Fil}^0(D\otimes_{K_0}\mathbf{B}_{\mathrm{st}})^{\varphi=1}, \qquad \mathbf{V}_{\mathrm{st}} : D\to \mathrm{Fil}^0(D\otimes_{K_0}\mathbf{B}_{\mathrm{st}})^{N=0,\varphi=1}$$

define quasi-inverse functors of $\mathbf{D}_{\mathrm{cris}}$ *and* $\mathbf{D}_{\mathrm{st}}$.

Proof. This follows from the equalities

$$\mathrm{Fil}^0(\mathbf{B}_{\mathrm{st}})^{N=0,\varphi=1} = \mathrm{Fil}^0(\mathbf{B}_{\mathrm{cris}})^{\varphi=1} = \mathbf{Q}_p.$$

Namely, assume that V is crystalline. Then using (58), we have

$$\mathbf{V}_{\mathrm{cris}}(\mathbf{D}_{\mathrm{cris}}(V)) = \mathrm{Fil}^0(\mathbf{D}_{\mathrm{cris}}(V)\otimes_{K_0}\mathbf{B}_{\mathrm{cris}})^{\varphi=1} = \mathrm{Fil}^0(V\otimes_{\mathbf{Q}_p}\mathbf{B}_{\mathrm{cris}})^{\varphi=1} = V.$$

The same argument applies in the semi-stable case. □

13.2.6. As in Section 9.1.8, one can also consider the contravariant functors

$$\mathbf{D}^*_{\mathrm{cris}} : \mathbf{Rep}_{\mathbf{Q}_p}(G_K) \to \mathbf{MF}^{\varphi}_K, \quad \mathbf{D}^*_{\mathrm{cris}}(V) = \mathrm{Hom}_{G_K}(V, \mathbf{B}_{\mathrm{cris}}),$$
$$\mathbf{D}^*_{\mathrm{st}} : \mathbf{Rep}_{\mathbf{Q}_p}(G_K) \to \mathbf{MF}^{\varphi,N}_K, \quad \mathbf{D}^*_{\mathrm{cris}}(V) = \mathrm{Hom}_{G_K}(V, \mathbf{B}_{\mathrm{st}}).$$

If V is crystalline (respectively semi-stable), there is a canonical isomorphism

$$\mathbf{D}^*_{\mathrm{cris}}(V) \simeq \mathbf{D}_{\mathrm{cris}}(V)^*$$

(respectively $\mathbf{D}^*_{\mathrm{st}}(V) \simeq \mathbf{D}_{\mathrm{st}}(V)^*$). The tautological map

$$V\otimes_{\mathbf{Q}_p}\mathbf{D}^*_{\star}(V) \to \mathbf{B}_{\star}, \qquad \star \in \{\mathrm{cris}, \mathrm{st}\}$$

can be viewed as an abstract p-adic integration pairing.

Proposition 13.2.7. *Each admissible* (φ, N)*-module is weakly admissible.*

Proof. This is [65, Proposition 4.4.5]. We refer the reader to [32, Theorem 9.3.4] for a detailed proof. □

13.2.8. The converse statement is a fundamental theorem of the p-adic Hodge theory, which was first formulated as a conjecture in [65].

Theorem 13.2.9 (Colmez–Fontaine)**.** *Each filtered weakly admissible module is admissible, i.e. we have equivalences of categories:*

$$\mathbf{MF}^{\varphi,a}_K \simeq \mathbf{MF}^{\varphi,f}_K, \qquad \mathbf{MF}^{\varphi,N,a}_K \simeq \mathbf{MF}^{\varphi,N,f}_K.$$

This theorem was first proved in [48]. Further development of ideas of this proof leads to the theory of p-adic Banach spaces [41] and almost $\mathbf{C}_p$-representations [72], [17]. Another proof, based on the theory of (φ, Γ)-modules was found by Berger [18]. A completely new insight on this theorem is provided by the theory of Fargues–Fontaine [60]. See [55] and [114] for an introduction to the work of Fargues and Fontaine.

Remark 13.2.10. *The theorem of Colmez–Fontaine implies that the tensor product of two weakly admissible modules is weakly admissible. Recall that there exists a direct proof of this result* [153].

13.3. **The hierarchy of p-adic representations.**

13.3.1. Let L be a finite extension of K. If $\rho : G_K \to \mathrm{Aut}_{\mathbf{Q}_p} V$ is a p-adic representation, one can consider its restriction on G_L and ask for the behavior of the functors $\mathbf{D}_{\mathrm{dR}}$, $\mathbf{D}_{\mathrm{st}}$ and $\mathbf{D}_{\mathrm{cris}}$ under restriction. Set:

$$\mathbf{D}_{\star/L}(V) = (V \otimes_{\mathbf{Q}_p} B_\star)^{G_L}, \qquad \star \in \{\mathrm{dR}, \mathrm{st}, \mathrm{cris}\}.$$

Applying Hilbert's theorem 90 (Theorem 1.6.3), we obtain that

$$\mathbf{D}_{\mathrm{dR}/L}(V) = \mathbf{D}_{\mathrm{dR}}(V) \otimes_K L.$$

In particular, V is a de Rham representation if and only if its restriction on G_L is a de Rham.

13.3.2. One says that a p-adic representation ρ is *potentially semi-stable* (respectively *potentially crystalline*) if there exists a finite extension L/K such that the restriction of ρ on G_L is semi-stable (respectively crystalline). Applying Hilbert's theorem 90 (Theorem 1.6.3), we obtain that in the case L/K is unramified, ρ is crystalline (respectively semi-stable) if and only if it's restriction on G_L is. The following proposition shows that ramified representations with finite image provide examples of potentially semi-stable representations that are not semi-stable.

Proposition 13.3.3. *A p-adic representation $\rho : G_K \to \mathrm{Aut}_{\mathbf{Q}_p} V$ with finite image is semi-stable if and only if it is unramified.*

Proof. Let ρ be a representation with a finite image. Let L/K be a finite extension such that $V^{G_L} = V$. Then

$$\mathbf{D}_{\mathrm{st}/L}(V) = V \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{st}}^{G_L} = V \otimes_{\mathbf{Q}_p} L_0,$$

where L_0 is the maximal unramified subfield of L. One has:

$$\mathbf{D}_{\mathrm{st}}(V) = (\mathbf{D}_{\mathrm{st}/L}(V))^{G_K} = (V \otimes_{\mathbf{Q}_p} L_0)^{\mathrm{Gal}(L/K)}.$$

Therefore V is semi-stable if and only if it is L_0-admissible if and only if it is unramified (see Example 9.2.2). □

13.3.4. Set:

$$\mathbf{D}_{\mathrm{pst}}(V) = \varinjlim_{L/K} \mathbf{D}_{\mathrm{st}/L}(V),$$

where L runs all finite extensions of K. Then $\mathbf{D}_{\mathrm{pst}}(V)$ is a finite dimensional K_0^{ur}-vector space endowed with a natural structure of filtered (φ, N)-module. In addition, it is equipped with a discrete action of the Galois group G_K such that $\mathbf{D}_{\mathrm{st}}(V) = \mathbf{D}_{\mathrm{pst}}(V)^{G_K}$. This Galois action allows to define on $\mathbf{D}_{\mathrm{pst}}(V)$ the stucture of a Weil–Deligne representation. One can see $\mathbf{D}_{\mathrm{pst}}$ as a functor to the category of *filtered* (φ, N, G_K)*-modules*. One says that V is potentially semi-stable if and only if $\dim_{K_0^{\mathrm{ur}}} \mathbf{D}_{\mathrm{st}}(V) = \dim_{\mathbf{Q}_p}(V)$. The functor $\mathbf{D}_{\mathrm{pcris}}$ can be defined by the same way. See [71] for more detail.

The hierarchy of p-adic representations can be represented by the following diagram of full subcategories of $\mathbf{Rep}_{\mathbf{Q}_p}(G_K)$:

$$
\begin{tikzcd}[arrows=dash]
& \mathbf{Rep}_{\mathbf{Q}_p}(G_K) \arrow[d] & \\
& \mathbf{Rep}_{\mathrm{HT}}(G_K) \arrow[d] & \\
& \mathbf{Rep}_{\mathrm{dR}}(G_K) \arrow[d] & \\
& \mathbf{Rep}_{\mathrm{pst}}(G_K) \arrow[dl] \arrow[dr] & \\
\mathbf{Rep}_{\mathrm{pcris}}(G_K) \arrow[dr] & & \mathbf{Rep}_{\mathrm{st}}(G_K) \arrow[dl] \\
& \mathbf{Rep}_{\mathrm{cris}}(G_K) &
\end{tikzcd}
$$

Finally, the categories $\mathbf{Rep}_{\mathrm{pst}}(G_K)$ and $\mathbf{Rep}_{\mathrm{dR}}(G_K)$ coincide as the following fundamental theorem shows:

Theorem 13.3.5 (p-adic monodromy conjecture)**.** *Each de Rham representation is potentially semi-stable.*

This theorem was formulated as a conjecture by Fontaine. It can be seen as a highly non-trivial analog of Grothendieck's ℓ-adic monodromy theorem in the case $\ell = p$. The first proof, found by Berger [15], uses the theory of (φ,Γ_K)-modules (see below). Colmez [43] gave a completely different proof, based on the theory of p-adic Banach Spaces. See [60, Chapter 10] for the insight provided by the theory of Fargues–Fontaine.

13.3.6. Recall that Theorem 8.2.9 classifies *all* p-adic representations in terms of (φ,Γ_K)-modules. It is natural to ask how to recover $\mathbf{D}_{\mathrm{cris}}(V)$, $\mathbf{D}_{\mathrm{st}}(V)$ and $\mathbf{D}_{\mathrm{dR}}(V)$ from the étale (φ,Γ_K)-module $\mathbf{D}(V)$. This question is known as Fontaine's program. As a first step, Cherbonnier and Colmez [35] proved that each p-adic representation is overconvergent. As a second step, Berger [15] showed how to construct $\mathbf{D}_{\mathrm{cris}}(V)$, $\mathbf{D}_{\mathrm{st}}(V)$ and $\mathbf{D}_{\mathrm{dR}}(V)$ in terms of the overconvergent lattice $\mathbf{D}^\dagger(V)$ of $\mathbf{D}(V)$ using the Robba ring $\mathscr{R}_K$. Moreover, the infinitesimal action of Γ_K on $\mathbf{D}^\dagger(V)\otimes_{\mathbf{Q}_p}\mathscr{R}_K$ gives rise to a structure of a differential φ-module and associates to V a p-adic differential equation. This reduces the p-adic monodromy conjecture to a conjecture of Crew on p-adic differential equations. This last conjecture was proved by Kedlaya [94]. We refer the reader to [42] for a survey of these results. In another direction, the theory of (φ,Γ_K)-modules is closely related to the p-adic Langlands program for $\mathrm{GL}_2(\mathbf{Q}_p)$ [45, 46, 47].

13.4. **Comparison theorems.**

13.4.1. In [151] Tate considered the p-adic analog of the following situation. Let X be a smooth proper scheme over the field of complex numbers $\mathbb{C}$. To the analytic space $X(\mathbb{C})$ on can associate on the one hand, the singular cohomology $H^n(X(\mathbb{C}),\mathbf{Q})$ and on the other hand, the de Rham cohomology $H^n_{\mathrm{dR}}(X/\mathbb{C})$ defined as the hypercohomology of the complex $\Omega^\bullet_X$ of differential forms on X. The integration of differental forms against simplexes gives a non-degenerate pairing

$$H_n(X(\mathbb{C}),\mathbf{Q})\times H^n_{\mathrm{dR}}(X/\mathbb{C})\to\mathbb{C}, \tag{59}$$

which induces an isomorphism (comparison isomorphism):

$$H^n(X(\mathbb{C}),\mathbf{Q})\otimes_{\mathbf{Q}}\mathbb{C}\simeq H^n_{\mathrm{dR}}(X/\mathbb{C})$$

The spectral sequence

$$E_1^{ij}=H^j(X,\Omega^i_{X/\mathbb{C}})\Rightarrow H^{i+j}_{\mathrm{dR}}(X/\mathbb{C})$$

defines a decreasing exhaustive filtration $F^iH^n_{\mathrm{dR}}(X/\mathbb{C})$ on $H^n_{\mathrm{dR}}(X/\mathbb{C})$ such that

$$\mathrm{gr}^iH^n_{\mathrm{dR}}(X/\mathbb{C})=H^{n-i}(X,\Omega^i_X).$$

By Hodge theory, this filtration splits canonically and gives the decomposition of $H^n_{\mathrm{dR}}(X/\mathbb{C})$ into direct sum (Hodge decomposition):

$$H^n_{\mathrm{dR}}(X/\mathbb{C})=\underset{i+j=n}{\oplus}H^j(X,\Omega^i_X).$$

Therefore one has the decomposition:

$$H^n(X(\mathbb{C}),\mathbf{Q})\otimes_{\mathbf{Q}}\mathbb{C}\simeq\underset{i+j=n}{\oplus}H^j(X,\Omega^i_X).$$

13.4.2. Now assume that X is a smooth proper scheme over a local field K of characteristic 0. The de Rham cohomologies $H^n_{\mathrm{dR}}(X/K)$ are still defined as the hypercohomology of $\Omega^\bullet_{X/K}$. Contrary to the complex case, the filtration $F^iH^n_{\mathrm{dR}}(X/K)$ has no canonical splitting [2]. One has:

$$\mathrm{gr}^\bullet H^n_{\mathrm{dR}}(X/K)=\underset{i+j=n}{\oplus}H^j(X,\Omega^i_{X/K}).$$

In the p-adic situation the singular cohomology is not defined, but it can be replaced by the p-adic étale cohomology $H^n_p(X)$, which has the additional structure of a p-adic representation. The following result formulated by Tate as a conjecture was proved in full generality by Faltings [57].

Theorem 13.4.3 (Faltings)**.** *There exists a functorial isomorphism*

$$H^n_p(X)\otimes_{\mathbf{Q}_p}\mathbf{C}\simeq\underset{i+j=n}{\oplus}\Big(H^j(X,\Omega^i_{X/K})\otimes_K\mathbf{C}(-i)\Big).$$

In particular, $H^n_p(X)$ is of Hodge–Tate, and

$$\mathbf{D}_{\mathrm{HT}}\Big(H^n_p(X)\Big)\simeq\mathrm{gr}^\bullet H^n_{\mathrm{dR}}(X/K).$$

[2]However, see [162].

Tate proved this conjecture for abelian varieties having good reduction using his results about the continuous cohomology of G_K (see Section 4.3). Faltings' proof relies on the higher-dimensional generalization of Tate's method of almost étale extensions. The theory of almost étale extensions was systematically developped in [78]. See [130] for further generalization of Faltings' almost purity theorems.

13.4.4. Inspired by Grothendieck's problem of mysterious functor [83], [84], Fontaine [66], [71] formulated more precise conjectures, relating étale cohomology to other cohomology theories via the rings $\mathbf{B}_{\mathrm{cris}}$, $\mathbf{B}_{\mathrm{st}}$ and $\mathbf{B}_{\mathrm{dR}}$. These conjectures are actually theorems, which can be formulated as follows:

13.4.5. *Étale cohomology vs. de Rham cohomology.* Recall that the ring $\mathbf{B}_{\mathrm{dR}}$ is equipped with a canonical filtration and a continuous action of the Galois group G_K.

Theorem 13.4.6 (C_{dR}-conjecture)**.** *Let X/K be a smooth proper scheme. There exists a functorial isomorphism*

$$H^i_p(X)\otimes_{\mathbf{Q}_p}\mathbf{B}_{\mathrm{dR}} \simeq H^i_{\mathrm{dR}}(X/K)\otimes_K \mathbf{B}_{\mathrm{dR}}, \tag{60}$$

which is compatible with the filtration and the Galois action. In particular, $H^i_p(X)$ is de Rham, and

$$\mathbf{D}_{\mathrm{dR}}\left(H^n_p(X)\right)\simeq H^n_{\mathrm{dR}}(X/K).$$

Using the isomorphism $\mathrm{gr}^\bullet\mathbf{B}_{\mathrm{dR}}\simeq \underset{i\in\mathbf{Z}}{\oplus}\mathbf{C}(i)$ it is easy to see that this theorem implies Theorem 13.4.3.

13.4.7. *Étale cohomology vs. crystalline cohomology.* Let X/O_K be a smooth proper scheme having good reduction. The theory of crystalline cohomology [20] associates to the special fiber of X finite-dimensional K_0-vector spaces $H^i_{\mathrm{cris}}(X)$ equipped with a semi-linear Frobenius φ. By a theorem of Berhtelot–Ogus [22], there exists a canonical isomorphism

$$H^i_{\mathrm{dR}}(X/K)\simeq H^i_{\mathrm{cris}}(X)\otimes_{K_0}K,$$

which equips $H^i_{\mathrm{cris}}(X)\otimes_{K_0}K$ with a canonical filtration.

Theorem 13.4.8 (C_{cris}-conjecture)**.** *Let X/O_K be a smooth proper scheme having good reduction.*

i) There exists a functorial isomorphism

$$H^i_p(X)\otimes_{\mathbf{Q}_p}\mathbf{B}_{\mathrm{cris}}\simeq H^i_{\mathrm{cris}}(X)\otimes_{K_0}\mathbf{B}_{\mathrm{cris}}, \tag{61}$$

which is compatible with the Galois action and the action of φ. In particular, $H^i_p(X)$ is crystalline, and

$$\mathbf{D}_{\mathrm{cris}}\left(H^n_p(X)\right)\simeq H^n_{\mathrm{cris}}(X).$$

ii) The isomorphism (60) can be obtained from (62) by the extension of scalars $\mathbf{B}_{\mathrm{cris}}\otimes_{K_0}K\subset\mathbf{B}_{\mathrm{dR}}$.

13.4.9. *Étale cohomology vs. log-crystalline cohomology.* Let X/O_K be a proper scheme having semi-stable reduction. The theory of log-crystalline cohomology [92] associates to X a finite-dimensional K_0-vector spaces $H^i_{\log-\mathrm{cris}}(X)$ equipped with a semi-linear Frobenius φ and a monodromy operator N such that $N\varphi = p\varphi N$. A theorem of Hyodo–Kato [87] shows the existence of an isomorphism

$$H^i_{\mathrm{dR}}(X/K) \simeq H^i_{\log-\mathrm{cris}}(X) \otimes_{K_0} K,$$

which equips $H^i_{\log-\mathrm{cris}}(X) \otimes_{K_0} K$ with the induced filtration. Note that if X has good reduction, then $N = 0$, and the log-crystalline cohomology coincides with the classical crystalline cohomology of X.

Theorem 13.4.10 (C_{st}-conjecture of Fontaine–Jannsen)**.** *Let X/O_K be a proper scheme having semi-stable reduction.*

i) There exists a functorial isomorphism

$$H^i_p(X) \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{st}} \simeq H^i_{\log-\mathrm{cris}}(X) \otimes_{K_0} \mathbf{B}_{\mathrm{st}}, \tag{62}$$

which is compatible with the Galois action and the actions of φ and N. In particular, $H^i_p(X)$ is semistable, and

$$\mathbf{D}_{\mathrm{st}}\left(H^n_p(X)\right) \simeq H^n_{\log-\mathrm{cris}}(X).$$

13.4.11. These conjectures were first proved by two completely different methods:

- The method of almost étale extensions (Faltings [58, 59]);
- The method of syntomic cohomology of Fontaine–Messing (Fontaine–Messing, Hyodo–Kato, Tsuji [74], [154]).

Alternative proofs were found by Nizioł[120, 121] and Beilinson [26, 27]. The theory of perfectoids gave a new impetus to this subject [24, 25, 34, 49, 131]. The generalization of comparison theorems to cohomology with coefficients is intimately related to the theory of p-adic representations of affinoid algebras [31, 9, 95, 96, 115].

13.4.12. Over the field of complex numbers, the comparison isomorphism can be alternatively seen as the non-degenerate pairing of complex periods (59). In the p-adic case, such an interpretation exists for abelian varieties. Namely, if A is an abelian variety over K, then the p-adic analog of $H_1(A(\mathbb{C}), \mathbf{Q})$ is the p-adic representation $V_p(A) := T_p(A) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$. For the first p-adic cohomology of A, one has:

$$H^1_p(A) \simeq V_p(A)^*.$$

The theory of p-adic integration [38, 39, 67] provides us with a non-degenerate pairing

$$H^1_{\mathrm{dR}}(A) \times T_p(A) \to \mathbf{B}_{\mathrm{dR}},$$

which gives an explicit approach to the comparison theorems for abelian varieties. The simplest case of p-divisible formal groups will be studied in the next section.

14. p-DIVISIBLE GROUPS

14.1. **Formal groups.**

14.1.1. In this section, we make first steps in studing p-adic representations arising from p-divisible groups. Such representations are crystalline and the associated filtered modules have an explicit description in geometric terms. We will focus our attention on formal groups because in this case many results can be proved by elementary methods, without using the theory of finite group schemes. We start with a short review of the theory of formal groups.

Definition. *Let A be an integral domain. A one-dimensional commutative formal group over A is a formal power series $F(X,Y) \in A[[X,Y]]$ satisfying the following conditions:*
i) $F(F(X,Y),Z) = F(X,F(Y,Z))$;
ii) $F(X,Y) = F(Y,X)$;
iii) $F(X,0) = X$ and $F(0,Y) = Y$;
iv) There exists $i(X) \in XA[[X]]$ such that $F(X,i(X)) = 0$.

It can be proved that ii) and iv) follow from i) and iii) (see [109]). We will often write $X +_F Y$ instead $F(X,Y)$.

14.1.2. **Examples.** 1) The additive formal group $\widehat{\mathbb{G}}_a(X,Y) = X+Y$. Here $i(X) = -X$.
2) The multiplicative formal group $\widehat{\mathbb{G}}_m(X,Y) = X+Y+XY$. Note that $\widehat{\mathbb{G}}_m(X,Y) = (1+X)(1+Y)-1$. Here $i(X) = -\dfrac{X}{1+X}$.
3) More generally, for each $a \in A$, the power series

$$F(X,Y) = X+Y+aXY$$

is a formal group over A. Here $i(X) = -\dfrac{X}{1+aX}$.

14.1.3. We introduce basic notions of the theory of formal groups. An homomorphism of formal groups $F \to G$ over A is a power series $f \in XA[[X]]$ such that $f \circ F(X,Y) = G(f(X), f(Y))$. The set $\mathrm{Hom}_A(F,G)$ of homomorphisms $F \to G$ is an abelian group with respect to the addition defined by the formula

$$f \oplus g = G(f(X), g(X)).$$

We set $\mathrm{End}_A(F) = \mathrm{Hom}_A(F,F)$. Then $\mathrm{End}_A(F)$ is a ring with respect to the addition defined above and the multiplication defined as the composition of power series:

$$f \circ g(X) = f(g(X)).$$

14.1.4. The module $\widehat{\Omega}^1_{A[[X]]}$ of formal Kähler differentials of $A[[X]]$ over A is the free $A[[X]]$-module generated by dX.

Definition. *We say that $\omega(X) = f(X)dX \in \widehat{\Omega}^1_{A[[X]]}$ is an invariant differential form on the formal group F if*

$$\omega(X +_F Y) = \omega(X).$$

14.1.5. The next proposition describes invariant differential forms on one-dimensional formal groups. We will write $F_1'(X,Y)$ (respectively $F_2'(X,Y)$) for the formal derivative of $F(X,Y)$ with respect to the first (respectively second) variable.

Proposition 14.1.6. *The space of invariant differential forms on a one-dimensional formal group $F(X,Y)$ is the free A-module of rank one generated by*

$$\omega_F(X) = \frac{dX}{F_1'(0,X)}.$$

Proof. See, for example, [88, Section 1.1].

a) Since $F(Y,X) = Y + X + (\text{terms of degree} \geqslant 2)$, the series $F_1'(0,X)$ is invertible in $A[[X]]$, and one has:

$$\omega(X) := \frac{dX}{F_1'(0,X)} \in A[[X]].$$

Differentiating the identity

$$F(Z,F(X,Y)) = F(F(Z,X),Y)$$

with respect ot Z, one has:

$$F_1'(Z,F(X,Y)) = F_1'(F(Z,X),Y)\cdot F_1'(Z,X).$$

Setting $Z=0$, we obtain that

$$\frac{F_1'(X,Y)}{F_1'(0,F(X,Y))} = \frac{1}{F_1'(0,X)},$$

or equivalently, that

$$\frac{dF(X,Y)}{F_1'(0,F(X,Y))} = \frac{dX}{F_1'(0,X)}.$$

This shows that $\omega(X)$ is invariant.

b) Conversely, assume that $\omega(X) = f(X)dX$ is invariant. Then

$$f(F(X,Y))F_1'(X,Y) = f(X).$$

Setting $X=0$, we obtain that $f(Y) = F_1'(0,Y)f(0)$. Therefore

$$\omega(X) = f(0)\omega_F(X),$$

and the proposition is proved. □

Remark 14.1.7. *We can write ω_F in the form:*

$$\omega_F(X) = \left(\sum_{n=0}^{\infty} a_n X^n\right) dX, \qquad \text{where } a_n \in A \text{ and } a_0 = 1.$$

14.1.8. Let K denote the field of fractions of A. We say that a power series $\lambda(X) \in K[[X]]$ is a logarithm of F, if

$$\lambda(X +_F Y) = \lambda(X) + \lambda(Y).$$

Proposition 14.1.9. *Assume that* $\mathrm{char}(K) = 0$. *Then the map*

$$\omega \mapsto \lambda_\omega(X) := \int_0^X \omega$$

establishes an isomorphism between the one-dimensional K-vector space generated by ω_F and the K-vector space of logarithms of F.

Proof. a) Let $\omega(X) = g(X)dX$ be a non-zero invariant differential form on F. Set $g(X) = \sum\limits_{n=0}^{\infty} b_n X^n$. Since $\mathrm{char}(K) = 0$, the series $f(X)$ has the formal primitive

$$\lambda_\omega(X) := \int_0^X \omega = \sum_{n=1}^{\infty} \frac{b_{n-1}}{n} X^n \in K[[X]].$$

The invariance of ω reads

$$g(F(X,Y))F_1'(X,Y) = g(X),$$

and taking the primitives, we obtain:

$$\lambda_\omega(X +_F Y) = \lambda_\omega(X) + h(Y)$$

for some $h(Y) \in K[[Y]]$. Putting $X = 0$ in the last formula, we have $h(Y) = \lambda_\omega(Y)$, and $\lambda_\omega(X +_F Y) = \lambda_\omega(X) + \lambda_\omega(Y)$. Therefore λ_ω is a logarithm of F.

b) Conversely, let $\lambda(X)$ be a logarithm of F. Differentiating the identity $\lambda(Y +_F X) = \lambda(Y) + \lambda(X)$ with respect to Y and setting $Y = 0$, one has:

$$\lambda'(X) = \frac{\lambda'(0)}{F_1(0,X)}.$$

Set $\omega = \lambda'(X)dX$. Then $\omega = \lambda'(0)\omega_F$, and the proposition is proved. □

Definition 14.1.10. *Set*

$$\lambda_F(X) = \int_0^X \omega_F.$$

Note that $\lambda_F(X)$ is the unique logarithm of F such that

$$\lambda_F(X) \equiv X \pmod{\deg 2}.$$

From Proposition 14.1.9 if follows that over a field of characteristic 0 all formal goups are isomorphic to the additive formal group. Indeed, λ_F is an isomorphism $F \simeq \widehat{\mathbb{G}}_a$.

14.1.11. **Example.** For the multiplicative group we have

$$\omega_{\mathbb{G}_m}(X) = \frac{dX}{1+X}, \qquad \lambda_{\mathbb{G}_m}(X) = \log(1+X) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{X^n}{n}.$$

14.1.12. We consider formal groups over the ring of integers of a local field K of characteristic 0 and residue caracteristic p.

For each $n \in \mathbf{Z}$, we denote by $[n]$ the formal multiplication by n:

$$[n] = \begin{cases} \underbrace{X +_F + X_F +_F \cdots + X}_{n}, & \text{if } n \geqslant 0, \\ \\ i([-n]), & \text{if } n < 0. \end{cases}$$

This defines an injection

$$[\ ] : \mathbf{Z} \to \mathrm{End}_{O_K}(F), \qquad n \to [n](X) = nX + \cdots .$$

It can be easily checked that this map can be extended by continuity to an injective map

$$[\ ] : \mathbf{Z}_p \to \mathrm{End}_{O_K}(F), \qquad a \to [a](X) = aX + \cdots .$$

Proposition 14.1.13. *Let F be a formal group over O_K. Then either*

$$[p](X) \equiv 0 \pmod{\mathfrak{m}_K}$$

or there exists an integer $h \geqslant 1$ and a power series $g(X) = c_1 X + \cdots$ such that $c_1 \not\equiv 0$ $\pmod{\mathfrak{m}_K}$ *and*

$$[p](X) \equiv g(X^{p^h}) \pmod{\mathfrak{m}_K}. \tag{63}$$

Proof. The proof is not difficult. See, for example, [76, Chapter I, § 3, Theorem 2]. □

Definition 14.1.14. *If* $[p](X) \equiv 0 \pmod{\mathfrak{m}_K}$, *we say that F has infinite height. Otherwise, we say that F is p-divisible and call the height of F the unique $h \geqslant 1$ satisfying condition (63).*

14.1.15. Now we can explain the connection between formal groups and p-adic representations. Recall that we write $\mathbf{C}$ for the completion of $\overline{K}$. We denote by $O_{\mathbf{C}}$ the ring of integers of $\mathbf{C}$ and by $\mathfrak{m}_{\mathbf{C}}$ the maximal ideal of $O_{\mathbf{C}}$. Any formal group law $F(X,Y)$ over O_K defines a structure of $\mathbf{Z}_p$-module on $\mathfrak{m}_{\mathbf{C}}$ of $\overline{K}$:

$$\alpha +_F \beta := F(\alpha,\beta), \quad \alpha,\beta \in \mathfrak{m}_{\mathbf{C}},$$
$$\mathbf{Z}_p \times \mathfrak{m}_{\mathbf{C}} \to \mathfrak{m}_{\mathbf{C}}, \qquad (a,\alpha) \mapsto [a](\alpha).$$

We will denote by $F(\mathfrak{m}_{\mathbf{C}})$ the ideal $\mathfrak{m}_{\mathbf{C}}$ equipped with this $\mathbf{Z}_p$-module structure. The analogous notation will be used for O_K-submodules of $\mathfrak{m}_{\mathbf{C}}$.

Proposition 14.1.16. *Assume that F is a formal group of finite height h. Then:*

i) The map $[p] : F(\mathfrak{m}_{\mathbf{C}}) \to F(\mathfrak{m}_{\mathbf{C}})$ is surjecive.

ii) The kernel $\ker([p])$ *is a free $\mathbf{F}_p$-module of rank h.*

Proof. i) Consider the equation

$$[p](X) = \alpha, \qquad \alpha \in F(\mathfrak{m}_{\mathbf{C}}).$$

A version of the Weierstrass preparation theorem (see, for example, the proof of [105, Theorem 4.2]) shows that this equation can be written in the form $f(X) =$

$g(\alpha)$, where $f(X)\in O_K[X]$ is a polynomial of degree p^h such that $f(X)\equiv X^{p^h} \pmod{\mathfrak{m}_K}$, and $g\in O_K[[X]]$. Therefore the roots of this equation are in $\mathfrak{m}_{\mathbf{C}}$.

ii) To prove that $\ker([p])$ is a free $\mathbf{Z}/p\mathbf{Z}$-module of rank h, we only need to show that the roots of the equation $[p](X)=0$ are all of multiplicity one. Differentiating the identity

$$[p](F(X,Y))=F([p](X),[p](Y))$$

with respect to Y and setting $Y=0$, we get:

$$[p]'(X)\cdot F_2'(X,0)=F_2'([p](X),0).$$

Let $[p](\xi)=0$. Since $F_2'(X,0)$ is invertible in $O_K[[X]]$ and $\xi\in\mathfrak{m}_{\mathbf{C}}$, we have $F_2'(\xi,0)\neq 0$ and $[p]'(\xi)\neq 0$. Therefore ξ is a simple root. □

14.1.17. For $n\geqslant 1$, let $T_{F,n}$ denote the p^n-torsion subgroup of $F(fm_{\mathbf{C}})$. From Proposition 14.1.16 it follows that as abelian group, it is not canonically isomorphic to $(\mathbf{Z}/p^n\mathbf{Z})^h$ and sits in the exact sequence

$$0\to T_{F,n}\to F(\mathfrak{m}_{\mathbf{C}})\xrightarrow{[p^n]}F(\mathfrak{m}_{\mathbf{C}})\to 0.$$

As in the case of abelian varieties, the Tate module of F is defined as the projective limit

$$T(F)=\varprojlim_n T_{F,n}$$

with respect to the multiplication-by-p maps. Since the series $[p^n](X)$ have coefficients in O_K, the Galois group G_K acts on $E_{F,n}$, and this action gives rise to a $\mathbf{Z}_p$-adic representation:

$$\rho_F\,:\,G_K\to \operatorname{Aut}_{\mathbf{Z}_p}(T(F))\simeq \mathrm{GL}_h(\mathbf{Z}_p).$$

We will denote by $V(F)=T(F)\otimes_{\mathbf{Z}_p}\mathbf{Q}_p$ the associated p-adic representation.

14.1.18. **Examples.** 1) $F=\widehat{\mathbb{G}}_m$. One has $[p^n]=(1+X)^{p^n}-1$. Therefore

$$T_{\widehat{\mathbb{G}}_m,n}=\left\{\zeta-1\mid \zeta^{p^n}=1\right\},$$

and the map

$$\mu_{p^n}\to T_{\widehat{\mathbb{G}}_m,n},\qquad \zeta\mapsto\zeta-1$$

is an isomorphism of G_K-modules. In particular, $T(\widehat{\mathbb{G}}_m)\simeq\mathbf{Z}_p(1)$.

2) Let E/O_K be an elliptic curve having good reduction modulo $\mathfrak{m}_K$. Writing the group law on E in terms of a local parameter at 0, one obtains a formal power series $F(X,Y)$, which is a formal group law over O_K. One can prove that F is of height 1 if E has ordinary reduction, and of height 2 if E has supersingular reduction. We have a canonical injection of $T(F)$ in the Tate module $T_p(E)$ of E, which is an isomorphism in the supersingular case. See [146, Chapter 4] for further detail and applications.

14.1.19. The notion of a formal group can be generalized to higher dimensions. Let $X = (X_1, \dots, X_d)$ and $Y = (Y_1, \dots, Y_d)$ be d-vectors of variables. A d-dimensional formal group over O_K is a d-tuple $F(X,Y) = (F_1(X,Y), \dots, F_d(X,Y))$ with

$$F_i(X,Y) \in O_K[[X,Y]], \qquad 1 \leqslant i \leqslant d,$$

which satisfies the direct analogs of conditions i), iii) and iv) in the definition of a one-dimensional formal group. We remark that contrary to the one-dimensional case, there are non-commutative formal groups of dimension $\geqslant 2$. Non-commutative formal groups appear in Lie theory. Below, without special mentioning, we consider only commutative formal groups.

14.1.20. Propositions 14.1.6 and 14.1.9 generalize directly to the higher-dimensional case. Namely, let $I = (X_1, \dots, X_d) \subset O_K[[X]]$. We set:

$$t_F^*(O_K) = I/I^2$$

and call it the cotangent space of F over O_K. The module of invariant differential forms on F is canonically isomorphic to $t_F^*(O_K)$. Namely:

1) For each $a_1X_1 + \cdots + a_dX_d \mod I^2 \in t_F^*(O_K)$, there exists a unique invariant differential form ω such that

$$\omega(0) = a_1dX_1 + \cdots + a_ddX_d.$$

This correspondence gives an isomorphism:

$$t_F^*(O_K) \simeq \{\text{invariant differential forms on } F\}.$$

2) Each invariant differential form ω is *closed*, i.e. there exists a unique $\lambda_\omega(X) \in K[X]$ such that $\lambda_\omega(0, \dots, 0) = 0$ and

$$d\lambda_\omega(X) = \omega.$$

3) The map $\omega \mapsto \lambda_\omega$ establishes an isomorphism between the K-vector space Ω_F^1 generated by invariant differential forms on F and the K-vector space of logarithms of F.

The notion of the height of a formal group generalizes as follows:

Definition 14.1.21. *A formal group F is p-divisible if the morphism*

$$[p]^* : O_K[[X]] \to O_K[[X]], \qquad f(X) \mapsto f \circ [p](X)$$

makes $O_K[[X]]$ into a free module of finite rank over itself.

If F is p-divisible, then the degree of the map $[p]^*$ is of the form p^h for some $h \geqslant 1$. This follows from the fact that any finite connected group over k_K is of order p^h for some h (see, for example [64, Chapitre I, § 9]). We call *h the height* of F. A formal group of dimension d defines a structure of $\mathbf{Z}_p$-module on $\mathfrak{m}_{\mathbf{C}}^d$, which we will denote by $F(\mathfrak{m}_{\mathbf{C}})$. The definition of the Tate module $T(F)$ and the p-adic representation $V(F)$ generalizes directly to p-divisible formal groups.

14.2. **p-divisible groups.**

14.2.1. The category of formal groups is to small to develop a satisfactory theory. In particular, it is not closed under taking duals. To remedy this problem, it is more convenient to work in the category of p-divisible groups, introduced by Tate [151].

Definition. *A p-divisible group of height h over O_K is a system $\mathscr{G} = (\mathscr{G}_n)_{n\in\mathbf{N}}$ of finite group schemes $\mathscr{G}_n$ of order p^{hn} equipped with injective maps $i_n : \mathscr{G}_n \to \mathscr{G}_{n+1}$ such that the sequences*

$$0 \to \mathscr{G}_n \xrightarrow{i_n} \mathscr{G}_{n+1} \xrightarrow{p^n} \mathscr{G}_{n+1}, \qquad n \geqslant 1$$

are exact.

From the theory of finite group schemes, it is known that each $\mathscr{G}_n$ sits in an exact sequence

$$0 \to \mathscr{G}_n^0 \to \mathscr{G}_n \to \mathscr{G}_n^{\text{ét}} \to 0, \tag{64}$$

where $\mathscr{G}_n^0$ is a connected and $\mathscr{G}_n^{\text{ét}}$ is an étale group scheme. We will say that $\mathscr{G} = (\mathscr{G}_n)_{n\in\mathbf{N}}$ is connected (respectively étale) if each $\mathscr{G}_n$ is. The exact sequences (64) give rise to an exact sequence of p-divisible groups

$$0 \to \mathscr{G}^0 \to \mathscr{G} \to \mathscr{G}^{\text{ét}} \to 0, \tag{65}$$

where $\mathscr{G}^0$ and $\mathscr{G}^{\text{ét}}$ are connected and étale respectively.

14.2.2. To each p-divisible group $\mathscr{G}$, one can naturally associate its Tate module, setting:

$$T(\mathscr{G}) = \varprojlim_n \mathscr{G}_n(O_{\mathbf{C}}).$$

Then $T(\mathscr{G})$ is a free $\mathbf{Z}_p$-module of rank h equipped with a natural action of G_K. We denote by $V(\mathscr{G}) := \mathbf{Q}_p \otimes_{\mathbf{Z}_p} T(\mathscr{G})$ the associated p-adic representation. From the exact sequence (65), one has an exact sequence of p-adic representations:

$$0 \to V(\mathscr{G}^0) \to V(\mathscr{G}) \to V(\mathscr{G}^{\text{ét}}) \to 0.$$

14.2.3. If $F(X,Y)$ is a p-divisible formal group, then the kernels $F[p^n]$ of the isogenies $[p^n] : F \to F$ form a system $F(p) = (F[p^n])_{n\in\mathbf{N}}$ of finite group schemes satisfying the above definition, and we have a functor $F \mapsto F(p)$ from the category of formal groups to the category of p-divisible groups.

Proposition 14.2.4 (Tate)**.** *The functor $F \mapsto F(p)$ induces an equivalence between the category of p-divisible formal groups and the category of connected p-divisible groups.*

Proof. See [151, Proposition 1] and the references in *op. cit.* □

14.2.5. If $\mathscr{G}$ is a p-divisible group, we call the dimension of $\mathscr{G}$ the dimension of the formal group F corresponding to its connected component. We also define the tangent space $t_{\mathscr{G}}(O_K)$ of $\mathscr{G}$ as the tangent space of F.

14.2.6. The Cartier duality for finite group schemes allows to associate to $\mathscr{G}$ a dual p-divisible group $\mathscr{G}^\vee$. We have fundamental relations between the heights and dimensions of $\mathscr{G}$ and $\mathscr{G}^\vee$:

$$\mathrm{ht}(\mathscr{G}) = \mathrm{ht}(\mathscr{G}^\vee), \qquad \dim(\mathscr{G}) + \dim(\mathscr{G}^\vee) = \mathrm{ht}(\mathscr{G})$$

([151, Proposition 3]). Moreover, the duality induces a non-degenerate pairing on Tate modules:

$$T(\mathscr{G}) \times T(\mathscr{G}^\vee) \to \mathbf{Z}_p(1).$$

14.2.7. **Example.** Let E/O_K be an elliptic curve having a good reduction modulo $\mathfrak{m}_K$. The kernel $E[p^n]$ of the multiplication-by-p^n map is a finite group scheme of order p^{2n}. The system $(E[p^n])_{n\in\mathbf{N}}$ is a p-divisible group of height 2. The connected component of this p-divisible group corresponds to the formal group associated by E in Example 14.1.18, 2).

14.3. **Classification of p-divisible groups.**

14.3.1. In [64], Fontaine classified p-divisible groups over O_K up to isogeny in terms of filtered φ-modules. The idea of such classification goes back to Grothendieck [83], [84] and relies on the following principles:

1) One associates to any p-divisible group $\mathscr{G}$ of dimension d and height h a φ-module $M(\mathscr{G})$ together with a d-dimensional subspace $L(\mathscr{G}) \subset M(\mathscr{G})_K$.
2) The φ-module $M(\mathscr{G})$ is the Dieudonné module associated to the reduction $\overline{\mathscr{G}}$ of $\mathscr{G}$ modulo $\mathfrak{m}_K$ by the theory of formal group schemes in characteristic p (see, for example, [112]).
3) The subspace $L(\mathscr{G}) \subset M(\mathscr{G})_K$ depends on the lift of $\overline{\mathscr{G}}$ in characteristic 0. The filtration on $M(\mathscr{G})_K$ is defined as follows:

$$\mathrm{Fil}^0 M(\mathscr{G})_K = M(\mathscr{G})_K, \qquad \mathrm{Fil}^1 M(F)_K = L(\mathscr{G}), \qquad \mathrm{Fil}^2 M(\mathscr{G})_K = \{0\}.$$

14.3.2. We give an interpretation of the module $(M(\mathscr{G}), L(\mathscr{G}))$ for formal p-divisible groups in terms of differential forms. This description is equivalent to Fontaine's general construction (see [64, Chapter V] for the proofs of the results stated below). Let F be a formal p-divisible group of dimension d and height h. Recall that a differential form

$$\omega = \sum_{i=1}^d a_i(X_1,\dots,X_d)dX_i, \qquad a_i(X_1,\dots,X_d) \in K[[X_1,\dots,X_d]]$$

is closed if there exists a power series $\lambda_\omega \in K[[X_1,\dots,X_d]]$ such that $\lambda_\omega(0,\dots,0)=0$ and $d\lambda_\omega = \omega$. Note that if ω is an invariant form, then λ_ω is the associated logarithm of F. As before, we set $X=(X_1,\dots,X_d)$ and $Y=(Y_1,\dots,Y_d)$ to simplify notation.

Definition. *A closed differential form ω is*

i) of the second kind on F, if there exists $r \geqslant 0$ such that

$$\lambda_\omega(X +_F Y) - \lambda_\omega(X) - \lambda_\omega(Y) \in p^{-r} O_K[[X,Y]];$$

ii) exact, if there exists $r \geqslant 0$ such that $\lambda_\omega \in p^{-r} O_K[[X]]$.

It is easy to see that each exact form is of the second kind. Consider the quotient:

$$H^1_{\mathrm{dR}}(F) = \frac{\{\text{differential forms of the second kind}\}}{\{\text{ exact forms}\}}.$$

Then $H^1_{\mathrm{dR}}(F)$ is a K-vector space of dimension h, which can be viewed as the first de Rham cohomology group of F. Let K_0 denote the maximal unramified subfield of K, and let $M(F)$ be the K_0-subspace of $H^1_{\mathrm{dR}}(F)$ generated by the forms with coefficients in K_0. Then $M(F)$ depends only on the reduction of F modulo $\mathfrak{m}_K$ and one has:

$$H^1_{\mathrm{dR}}(F) = M(F)_K.$$

Moreover, $M(F)$ is equipped with the Frobenius operator φ which acts as the absolute Frobenius on the coefficients of power series and such that $\varphi(X_i) = X_i^p$:

$$\varphi\left(\sum_{i=1}^d a_i(X_1,\dots,X_d)dX_i\right) = \sum_{i=1}^d a_i^{\varphi}(X_1^p,\dots,X_d^p)dX_i^p.$$

Consider the K-vector space Ω^1_F generated by invariant forms on F. Recall that $\dim_K \Omega^1_F = d$. Each invariant form is clearly of the second kind, and Ω^1_F injects into $H^1_{\mathrm{dR}}(F)$. Set:

$$L(F) := \text{image of } \Omega^1_F \text{ in } H^1_{\mathrm{dR}}(F).$$

These data define a structure of filtered module on $M(F)$.

14.3.3. Assume that the local field K is absolutely unramified. In that case, formal groups over O_K were classified up isomorphism by Honda [88], purely in terms of their logarithms. In this section, we review Honda's classification. To simplify the exposition, we restrict our discussion to the one-dimensional case.

The ring of power series $K[[X]]$ is equipped with the Frobenius operator φ :

$$\varphi\left(\sum_{i=0}^\infty a_iX^i\right) = \sum_{i=0}^\infty \varphi(a_i)X^{ip}.$$

Assume that $\alpha_1,\dots,\alpha_{h-1},\alpha_h \in O_K$ satisfy the following conditions:

$$\begin{aligned}&\alpha_1,\dots,\alpha_{h-1} \equiv 0 \pmod{p},\\ &\alpha_h \in U_K.\end{aligned} \tag{66}$$

Set:

$$\mathscr{A}(\varphi) := \sum_{i=0}^h \alpha_i\varphi^i,$$

and consider the power series

$$\lambda(X) := \left(1 - \frac{\mathscr{A}(\varphi)}{p}\right)^{-1}(X) \in K[[X]].$$

For formal p-divisible groups of dimension one, the result of Honda states as follows:

Theorem 14.3.4 (Honda)**.** *i) Assume that $\alpha_1,\dots,\alpha_h$ satisfy conditions (66). Then $\lambda(X)=\lambda_G(X)$ for some one-dimensional formal group G of height h.*

ii) Let F be a one-dimensional formal group over O_K of height h. Then there exists a unique system $\alpha_1,\dots,\alpha_h$ satisfying (66) such that

$$\left(1-\frac{\mathscr{A}(\varphi)}{p}\right)\lambda_F(X)\in O_K[[X]].$$

Let G be the formal group associated to $\alpha_1,\dots,\alpha_h$ by part i). Then $F\simeq G$.

The relation between this theorem and Fontaine's classification is given by the following:

Proposition 14.3.5. *Assume that K is absolutely unramified. Let F be a one-dimensional formal group over O_K of height h. Denote by b_F the image of ω_F in $M(F)$. Then the following holds true:*

i) The elements $b_F,\varphi(b_F),\dots,\varphi^{h-1}(b_F)$ form a basis of $M(F)$ over K.

ii) Let $\alpha_1,\dots,\alpha_h$ be the parameters associated to F by Honda's theorem. Then

$$\alpha_1\varphi(b_F)+\alpha_2\varphi(b_F)+\cdots+\alpha_h\varphi^h(b_F)=pb_F.$$

iii) One has an isomorphism of filtered φ-modules

$$M(F)\simeq K[\varphi]/(\mathscr{A}(\varphi)-p),$$

which sends $L(F)=K\cdot b_F$ to the one-dimensional K-vector space generated by 1.

Proof. See [64, Chapitre V]. □

Remark 14.3.6. *In fact, Fontaine's theory* [64] *gives more precise results that those that we have stated. Namely, if the absolute ramification index of K is* $\leqslant p-1$, *it allows to classify p-divisible groups up to isomorphism and not only up to isogeny. Using new ideas, Breuil* [30] *classified p-divisible groups up to isomorphism without any restriction on ramification. See* [97] *and* [33] *for further developments.*

14.4. p-adic integration on formal groups.

14.4.1. We maintain assumptions and conventions of the previous section. Let F be a formal p-divisible group of dimension d and height h. We denote by $T(F)$ the Tate module of F. Let $\xi=(\xi_n)_{n\geqslant 0}\in T(F)$, where $\xi_n\in T_{F,n}$ for each $n\geqslant 0$. Recall that we have the canonical map $\theta:\mathbf{A}_{\mathrm{inf}}\to O_{\mathbf{C}}$. For each n, choose $\widehat{\xi_n}\in\mathbf{A}_{\mathrm{inf}}^d$ such that $\theta(\widehat{\xi_n})=\xi_n$.

Theorem 14.4.2 (Colmez, Fontaine)**.** *i) Let ω be a differential form of second kind. Then the sequence $(p^n\lambda_\omega(\widehat{\xi_n}))_{n\geqslant 0}$ converges in $\mathbf{B}^+_{\mathrm{cris},K}=K\otimes_{K_0}\mathbf{B}^+_{\mathrm{cris}}$. Its limit does not depend on the choice of $\widehat{\xi_n}$ and therefore defines the "p-adic integral":*

$$\int_\xi\omega:=-\lim_{n\to+\infty}p^n\lambda_\omega(\widehat{\xi_n}). \tag{67}$$

If ω has coefficients in K_0, then $\int_\xi\omega\in\mathbf{B}^+_{\mathrm{cris}}$.

ii) If ω is exact, then $\int_\xi\omega=0$.

iii) The p-adic integration (67) is compatible with the actions of the Galois group and the Frobenius φ. Namely, one has:

$$\int_\xi \varphi(\omega) = \varphi\left(\int_\xi \omega\right),$$
$$\int_{g(\xi)} \omega = g\left(\int_\xi \omega\right), \qquad g \in G_K.$$

iv) The p-adic integration induces a non-degenerated pairing

$$M(F) \times T(F) \to \mathbf{B}_{\mathrm{cris}},$$

which is compatible with the Frobenius operator and the Galois action, and a non-degenerated pairing

$$H^1_{\mathrm{dR}}(F) \times T(F) \to \mathbf{B}^+_{\mathrm{dR}},$$

which is compatible with the Galois action and filtration.

Proof. See [64, Chapitre V, §1], [66, Théorème 6.2] and [38, Proposition 3.1]. We remark that the delicate part here is the non-degeneracy of the constructed pairings. The proof of other points is straightforward. □

14.4.3. **Example.** Consider the case of the multiplicative formal group $\widehat{\mathbb{G}}_m$. Recall that $T(\widehat{\mathbb{G}}_m) \simeq \mathbf{Z}_p(1)$ is generated by any compatible system $(\xi_n)_{n\geqslant 0}$ such that $\xi_n = \zeta_{p^n} - 1$ and $\zeta_p \neq 1$. The space $H^1_{\mathrm{dR}}(\widehat{\mathbb{G}}_m)$ is generated over K by $\omega = \frac{dX}{1+X}$, and the formal primitive of ω is $\log(1+X)$. Take $\hat{\xi}_n = [\varepsilon]^{1/p^n} - 1$. One has:

$$\int_\xi \omega = -\lim_{n\to+\infty} p^n \log[\varepsilon]^{1/p^n} = -t.$$

This formula can be seen as the p-adic analog of the following computation. Let C denote the unit circle on the complex plane parametrized by $e^{2\pi i x}$, $x \in [0,1]$. Then

$$\int_C \frac{dz}{z} = n \log(z)\Big|_0^{e^{\frac{2\pi i}{n}}} = 2\pi i.$$

Corollary 14.4.4. *The representation $V(F)$ is crystalline, and there exist canonical isomorphisms:*

$$\mathbf{D}^*_{\mathrm{cris}}(V(F)) \simeq M(F), \qquad \mathbf{D}^*_{\mathrm{dR}}(V(F)) \simeq H^1_{\mathrm{dR}}(F).$$

Corollary 14.4.5 (Tate)**.** *The representation $V(F)$ is Hodge–Tate and there exists a canonical isomorphism*

$$V(F) \otimes_{\mathbf{Q}_p} \mathbf{C} \simeq (t^*_{F^\vee}(K) \otimes_K \mathbf{C}) \oplus (t_F(K) \otimes_K \mathbf{C}(1)). \tag{68}$$

Proof. This follows from the previous corollary and the isomorphisms

$$t^*_F(K) \simeq \Omega^1_F, \qquad H^1_{\mathrm{dR}}(F)/\Omega^1_F \simeq t_{F^\vee}(K)$$

(the second isomorphism is provided by duality). □

Remark 14.4.6. *1) Corollary 14.4.4 holds for all p-divisible groups (see* [66, Théorème 6.2]*). Conversely, Breuil* [30] *proved that each crystalline representation with Hodge–Tate weights* 0 *and* 1 *arises from a p-divisible group.*

2) The Hodge–Tate decomposition (68) was first proved by Tate [151] *for all p-divisible groups. Some constructions of this paper will be revewed in Section 16. The case of abelian variety with bad reduction follows from the semi-stable reduction theorem (Raynaud). A completely different proof was found by Fontaine* [67].

3) The construction of p-adic integration in Theorem 14.4.2 generalizes to the case of abelian varieties [38], [39].

15. Formal complex multiplication

15.1. **Lubin–Tate theory.**

15.1.1. In this section, we discuss the theory of complex multiplication in formal groups. We start with a brief overview of Lubin–Tate theory [111]. Let K is a local field of arbitrary characteristic. Set $q = |k_K| = p^f$. Fix an uniformizer π of K.

Theorem 15.1.2. *i) Let $f(X) \in O_K[[X]]$ be a power series satisfying the following conditions:*

$$
\begin{aligned}
&f(X) \equiv \pi X \pmod{\deg 2},\\
&f(X) \equiv X^q \pmod{\mathfrak{m}_K}.
\end{aligned}
\tag{69}
$$

Then the following holds true:

i) There exists a unique formal group $F_f(X,Y)$ over O_K such that $f(X) \in \mathrm{End}_{O_K}(F)$. Moreover, for each $a \in O_K$, there exists a unique endomorphism $[a](X) \in \mathrm{End}_{O_K}(F)$ such that $[a](X) \equiv aX \pmod{\deg 2}$.

ii) Let $g(X)$ be another power series satisfying conditions (69) with the same uniformizer π. Then F_g and F_f are isomorphic over O_K. In the isomorphism class of F_f, there exists a formal group F_{LT} with the logarithm

$$\lambda_{\mathrm{LT}}(X) = X + \frac{X^q}{\pi} + \frac{X^{q^2}}{\pi^2} + \cdots .$$

iii) Let π' be another uniformizer of O_K, and let $g(X)$ be a power series satisfying conditions (69) with π' in the place of π. Then F_f and F_g are isomorphic over the ring $\widehat{O}_K^{\mathrm{ur}}$.

Proof. All these statements can be proved by successive approximation in the rings of formal power series. We refer the reader to [111] or to [140] for detailed proofs. □

Definition. *F_f is called the Lubin–Tate formal group associated to f.*

15.1.3. Let F_f be the Lubin–Tate formal group associated to $f(X) = \pi X + X^q$. The group of points $F_f(\mathfrak{m}_{\mathbf{C}})$ is an O_K-module with the action of O_K given by

$$(a, \alpha) \mapsto [a](\alpha), \qquad a \in O_K, \quad \alpha \in F_f(\mathfrak{m}_{\mathbf{C}}).$$

In particular, $[\pi](X)=f(X)$, and for any $n\geqslant 1$, one has:

$$[\pi^n](X)=\underbrace{f\circ f\circ\cdots\circ f(X)}_{n}.$$

The polynomial

$$[\pi^n]/[\pi^{n-1}]=\pi+[\pi^{n-1}](X)^{q-1}, \tag{70}$$

is Eisenstein of degree $q^{n-1}(q-1)$. Let $T_{f,n}$ denote the group of π^n-torsion points of F_f. An easy induction together with the previous remark show that $T_{f,n}$ is an abelian group of order q^n. The endomorphism ring $\mathrm{End}_{O_K}(F_f)\simeq O_K$ acts on $T_{f,n}$ through the quotient O_K/π^nO_K, and $T_{f,n}$ is free of rank one over O_K/π^nO_K. The generators of $T_{f,n}$ are the roots of the polynomial (70). Let $K_{f,n}$ be the field generated over K by $T_{f,n}$. Then

$$K_{f,n}=K(\pi_n),$$

where π_n is any generator of $T_{f,n}$. In particular, $[K_{f,n}:K]=(q-1)q^{n-1}$, and π_n is a uniformizer of $K_{f,n}$.

15.1.4. Let g be another power series satisfying (69) with the same π. Then $F_g\simeq F_f$, $T_{g,n}\simeq T_{f,n}$, and $K_{f,n}=K_{g,n}$. Since the field generated by π^n-torsion points of a Lubin–Tate formal group depends only on the choice of the uniformizer π, we will write $K_{\pi,n}$ in the place of $K_{f,n}$. Set:

$$K_\pi=\bigcup_{n=1}^{\infty}K_{\pi,n}.$$

From the explicit form of Eisenstein polynomials (70), it follows that π is a universal norm in K_π/K.

The following theorem gives an explicit approach to local class field theory:

Theorem 15.1.5 (Lubin–Tate)**.** *i) One has:*

$$K^{\mathrm{ab}}=K^{\mathrm{ur}}\cdot K_\pi.$$

ii) Let $\theta_K\,:\,K^*\to\mathrm{Gal}(K^{\mathrm{ab}}/K)$ denote the reciprocity map. For any $u\in U_K$, the automorphism $\theta_K(u)$ acts on the torsion points of F_f by the formula:

$$\theta_K(u)(\xi)=[u^{-1}](\xi),\qquad \forall\xi,\quad [\pi^n](\xi)=0,\quad n\in\mathbf{N}.$$

Proof. See [111] or [140]. □

Remark 15.1.6. *1) The torsion points of a one dimensional formal group are the roots of its logarithm (see Proposition 16.1.2 below). Therefore K^{ab} is generated over K^{ur} by the roots of the power series $\lambda_{\mathrm{LT}}(X)$. This can be seen as a solution of Hilbert 12th problem for local fields. Theorem 15.1.5 is the local analog of the theory of complex multiplication.*

2) Let $K=\mathbf{Q}_p$. The multiplicative formal group $\widehat{\mathbb{G}}_m$ is the Lubin–Tate group associated to the series $f(X)=(X+1)^p-1$. In that case, Theorem 15.1.5 says that $\mathbf{Q}_p^{\mathrm{ab}}=\bigcup_{n}^{\infty}\mathbf{Q}_p(\zeta_n)$ and that

$$\theta_{\mathbf{Q}_p}(u)(\zeta_{p^n})=\zeta_{p^n}^{u^{-1}},\qquad \forall u\in U_{\mathbf{Q}_p}.$$

This can be proved without using the theory of formal groups.

3) Let π_n be a generator of the group of π^n-torsion points of F_f. Since π_n is a uniformizer of $K_{\pi,n}$, and Theorem 15.1.5 describes the action of $\mathrm{Gal}(K^{\mathrm{ab}}/K)$ *on π_n, this allows to compute the ramification filtration on* $\mathrm{Gal}(K^{\mathrm{ab}}/K)$. *One has:*

$$\theta_K\left(U_K^{(v)}\right) = \mathrm{Gal}(K^{\mathrm{ab}}/K)^{(v)}, \qquad \forall v \geqslant 0.$$

See [140] *for a detailed proof.*

15.2. Hodge–Tate decomposition for Lubin–Tate formal groups.

15.2.1. In this section, we assume that K has characteristic 0. We fix a uniformizer π and write F for an unspecified Lubin–Tate formal group associated to π. Since $p = \pi^e u$ with $e = e(K/\mathbf{Q}_p)$, and $u \in U_K$, we see that F is a p-divisible group of height $h = ef = [K : \mathbf{Q}_p]$. Its Tate module $T(F)$ can be written as the projective limit of π^n-torsion subgroups with respect to the multiplication-by-π map. Since $T(F)$ is an O_K-module of rank one, the action of G_K on $T(F)$ is given by a character

$$\chi_\pi : G_K \to U_K.$$

The theory of Lubin–Tate (Theorem 15.1.5) says that $\chi_\pi^{-1} \circ \theta_K$ coincides with the projection of K^* onto U_K under the decomposition $K^* \simeq U_K \times \langle \pi \rangle$.

15.2.2. Let E be a finite extension of K containing all conjugates τK of K over $\mathbf{Q}_p$. By local class field theory, one has a commutative diagram

$$\begin{tikzcd} E^* \arrow[r, "\theta_E"] \arrow[d, "N_{E/K}"'] & \mathrm{Gal}(E^{\mathrm{ab}}/E) \arrow[d] \\ K^* \arrow[r, "\theta_K"] & \mathrm{Gal}(K^{\mathrm{ab}}/K). \end{tikzcd}$$

Therefore G_E acts on $T(F)$ via the character $\rho_E = \chi_\pi \circ N_{E/K}$. Consider the vector space $V(F) = T(F) \otimes_{O_K} K$ as a G_E-module. By the previous remark, $V(F) \simeq K(\rho_E)$, and one has:

$$V(F) \otimes_{\mathbf{Q}_p} \mathbf{C} \simeq \bigoplus_{\tau \in \mathrm{Hom}(K,E)} \mathbf{C}(\tau \circ \rho_E).$$

Compare this decomposition with the Hodge–Tate decomposition:

$$V(F) \otimes_{\mathbf{Q}_p} \mathbf{C} \simeq t^*_{F^\vee}(\mathbf{C}) \oplus t_F(\mathbf{C})(1).$$

These decompositions are compatible with the K-module structures on the both sides . Since K acts on $t_F(E)$ via the embedding $K \hookrightarrow E$, one has:

$$\mathbf{C}(\tau \circ \rho_E) \simeq \begin{cases} \mathbf{C}(1), & \text{if } \tau = \mathrm{id}, \\ \mathbf{C}, & \text{if } \tau \neq \mathrm{id}. \end{cases} \tag{71}$$

Proposition 15.2.3. *For any continuous character $\psi : G_E \to U_K$, the following conditions are equivalent:*

a) ψ concides with $\prod_{\tau \in \mathrm{Hom}(K,E)} \tau^{-1} \circ \rho_{\tau E}^{n_\tau}$ on some open subgroup of I_E;

b) $\mathbf{C}(\tau \circ \psi) = \mathbf{C}(\chi_E^{n_\tau})$ for all $\tau \in \mathrm{Hom}(K, E)$.

Proof. See [143, Section A5]. Recall that for two continuous characters ψ_1 and ψ_2 we write $\psi_1 \sim \psi_2$ if $\mathbf{C}(\psi_1)$ and $\mathbf{C}(\psi_2)$ are isomorphic as continuous Galois modules. From (71), one has:

$$\begin{aligned} &\tau \circ \sigma^{-1} \circ \rho_{\sigma K} \sim \chi_E, \quad \text{if } \tau = \sigma, \\ &\tau \circ \sigma^{-1} \circ \rho_{\sigma K} \sim \mathrm{id}, \quad \text{if } \tau \neq \sigma. \end{aligned}$$

Set:

$$\psi_1 = \prod_{\tau \in \mathrm{Hom}(K,E)} \tau^{-1} \circ \rho_{\tau K}^{n_\tau}.$$

Then the previous formula gives:

$$\tau \circ \psi_1 \sim \chi_K^{n_\tau}, \qquad \forall \tau \in \mathrm{Hom}(E,K).$$

Now the proposition follows from Proposition 4.3.6. □

15.3. **Formal complex multiplication for p-divisible groups.**

15.3.1. Using Proposition 15.2.3, we can prove a general result about formal complex multiplication for p-divisible groups.

Definition. *Let $\mathscr{G}$ be a p-divisible group over O_E of dimension d and height h. We say that $\mathscr{G}$ has a formal complex multiplication by a p-adic field $K \subset E$ if $[K : \mathbf{Q}_p] = h$ and there exists an injective ring map*

$$K \to \mathrm{End}_{O_E}(\mathscr{G}) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p.$$

If $\mathscr{G}$ has a complex multiplication by K, the p-adic representation $V(\mathscr{G})$ is a K-vector space of dimension 1, and G_E acts on $V(\mathscr{G})$ via a character $\psi_{\mathscr{G}} : G_E \to U_K$. On the other hand, the tangent space $t_{\mathscr{G}}(E)$ is a (E,K)-module, and the multiplication by E in $t_{\mathscr{G}}(E)$ gives rise to a map

$$\det_{\mathscr{G}} : E^* \to \mathrm{Aut}_K(t_{\mathscr{G}}(E)) \xrightarrow{\det} K^*.$$

Recall that $\theta_E : E^* \to \mathrm{Gal}(E^{\mathrm{ab}}/E)$ denotes the reciprocity map.

Theorem 15.3.2. *Let $\mathscr{G}$ be a p-divisible group having a formal complex multiplication by K. Assume that E contains all conjugates of K. Then one has:*

$$\psi_{\mathscr{G}}(\theta_E(u)) = \det_{\mathscr{G}}(u)^{-1}, \qquad u \in U,$$

for some open subgroup U of U_E.

Proof. Comparing the decomposition

$$V(\mathscr{G}) \otimes_{\mathbf{Q}_p} \mathbf{C} \simeq \bigoplus_{\tau \in \mathrm{Hom}(K,E)} \mathbf{C}(\tau \circ \psi_{\mathscr{G}})$$

and the Hodge–Tate decomposition of $V(\mathscr{G})$, we see that there exists a subset $S \subset \mathrm{Hom}(K,E)$ such that $t_{\mathscr{G}}(E) \simeq \bigoplus_{\tau \in S} \tau(K)$ as a K-module and that

$$\begin{aligned} &\tau \circ \psi_{\mathscr{G}} \sim \chi_E \quad \text{if } \tau \in S, \\ &\tau \circ \psi_{\mathscr{G}} \sim 1 \quad\ \ \text{if } \tau \notin S. \end{aligned}$$

Proposition 15.2.3 implies that $\psi_{\mathscr{G}}$ concides on an open subgroup of I_E with the character

$$\prod_{\tau\in\mathrm{Hom}(K,E)}\tau^{-1}\circ\rho_{\tau E}.$$

Now the theorem follows from the theory of Lubin–Tate together with the formula

$$\det{}_{\mathscr{G}}(u)=\prod_{\tau\in S}\tau^{-1}\circ N_{E/\tau(K)}(u).$$

□

15.3.3. **Remark.** Theorem 15.3.2 is mentioned in [144]. We remark that it implies the main theorem of complex multiplication of abelian varieties in the *global* setting.

16. The exponential map

16.1. The group of points of a formal group.

16.1.1. In this section, we study the group of points of a formal group in more detail. Let F be a formal p-divisible group. We denote by $T_{F,\infty}$ the group of torsion points of F. Note that $T_{F,\infty}=\bigcup\limits_{n=0}^{\infty}T_{F,n}$, and that there is a canonical isomorphism

$$T_{F,\infty}\simeq V(F)/T(F).$$

Proposition 16.1.2. *i) For any invariant differential form ω on F, the logarithm $\lambda_\omega(X)$ converges on $\mathfrak{m}_{\mathbf{C}}$.*

ii) The map

$$\log_F\,:\,F(\mathfrak{m}_{\mathbf{C}})\to t_F(\mathbf{C}),$$
$$\log_F(\alpha)(\omega)=\lambda_\omega(\alpha),\qquad \forall\omega\in\Omega^1_F$$

is an homomorphism.

iii) One has an exact sequence

$$0\to T_{F,\infty}\to F(\mathfrak{m}_{\mathbf{C}})\xrightarrow{\log_F}t_F(\mathbf{C})\to 0. \tag{72}$$

Moreover, $\log_F$ *is a local isomorphism.*

Proof. i) The space of invariant differential forms on F is generated by the forms $\omega_1,\dots,\omega_d$ such that $\omega_i(0)=dX_i$. Let $\lambda_1,\dots,\lambda_d$ denote the logarithms of these forms. Since ω_i have coefficients in O_K, the series λ_i can be written as

$$\lambda_i(X)=X_i+\sum_{n\geqslant 2}\left(\sum_{n_1+\cdots n_d=n}a_{n_1,\dots,n_d}X_1^{n_1}\cdot\ldots\cdot X_d^{n_d}\right),$$

where

$$n\cdot a_{n_1,\dots,n_d}\in O_K,\qquad n=n_1+\cdots+n_d. \tag{73}$$

This implies that the series λ_i converge on $\mathfrak{m}_{\mathbf{C}}^d$. Moreover, any logarithm can be written as a linear combination of λ_i. Therefore for any ω, the series λ_ω converges on $\mathfrak{m}_{\mathbf{C}}^d$. This proves that the map $\log_F$ is well defined.

ii) Since $\lambda_\omega(X+_F Y)=\lambda_\omega(X)+\lambda_\omega(Y)$, we have

$$\log_F(\alpha+_F\beta)=\log_F(\alpha)+\log_F(\beta).$$

iii) Fix $c\in O_K$ such that

$$v_K(c)>\frac{v_K(p)}{p-1}.$$

Then from (73) it follows that

$$c^{-1}\lambda_i(cX_1,\dots,cX_d)=X_i+\sum_{n\geqslant 2}\left(\sum_{n_1+\dots+n_d=n} b_{n_1,\dots,n_d}X_1^{n_1}\cdot\ldots\cdot X_d^{n_d}\right),$$

where $b_{n_1,\dots,n_d}\in O_K$. Applying the p-adic version of the inverse function theorem to the function $\boldsymbol{\lambda}(X)=(\lambda_1,\dots,\lambda_n)$ (see, for example, [129, Chapter 1, Proposition 5.9]), we see that it establishes an analytic homeomorphism between $F(c\mathfrak{m}_{\mathbf{C}})$ and $(c\mathfrak{m}_{\mathbf{C}})^d$. This shows that $\log_F$ is a local analytic homeomorphism.

We show the exactness of the short exact sequence. Assume that $\alpha\in T_{F,\infty}$. Then there exists n such that $[p^n](\alpha)=0$, and therefore for each invariant differential form ω one has $p^n\lambda_\omega(\alpha)=\lambda_\omega([p^n](\alpha))=0$. This shows that $\lambda_\omega(\alpha)=0$ for all ω; hence $\alpha\in\ker(\log_F)$. Conversely, assume that $\alpha\in\ker(\log_F)$. Take n such that $[p^n](\alpha)\in F(c\mathfrak{m}_{\mathbf{C}})$. Then $\log_F([p^n](\alpha))=p^n\log_F(\alpha)=0$. Since $\log_F$ is an isomorphism on $F(c\mathfrak{m}_{\mathbf{C}})$, this shows that $\alpha\in T_{F,n}$. Thus $\ker(\log_F)=T_{F,\infty}$. Finally, since $\log_F$ is a local isomorphism and $F(\mathfrak{m}_{\mathbf{C}})$ is p-divisible, $\log_F$ is surjective. $\square$

Corollary 16.1.3. *For each c such that $v_K(c)>\dfrac{v_K(p)}{p-1}$, the local inverse of* $\log_F$ *induces an isomorphism*

$$\exp_F\,:\,t_F(c\mathfrak{m}_{\mathbf{C}})\simeq F(c\mathfrak{m}_{\mathbf{C}}).$$

Tensoring this local isomorphism with $\mathbf{Q}_p$, we obtain an isomorphism (which we denote again by $\exp_F$*):*

$$\exp_F\,:\,t_F(\mathbf{C})\simeq F(\mathfrak{m}_{\mathbf{C}})\otimes_{\mathbf{Z}_p}\mathbf{Q}_p. \tag{74}$$

Definition. *We call* $\log_F$ *and* $\exp_F$ *the logarithmic map and the exponential map respectively.*

16.1.4. **Example.** For the multiplicative formal group, the exact sequence (72) reads:

$$0\to\mu_{p^\infty}\to U_{\mathbf{C}}^{(1)}\xrightarrow{\log}\mathbf{C}\to 0, \tag{75}$$

where $U_{\mathbf{C}}^{(1)}=(1+\mathfrak{m}_{\mathbf{C}})^*$ is the multiplicative group of principal units of $\mathbf{C}$.

16.1.5. Following Tate [151], we give a description of the group of points $F(\mathfrak{m}_{\mathbf{C}})$ in terms of the Tate module of the dual p-divisible group $F^\vee$. Let $F(p)=(F[p^n])_{n\geqslant 1}$ be the p-divisible group associated to F. Then $F[p^n](O_{\mathbf{C}})=T_{F,n}$. Recall the injective maps $i_n\,:\,F[p^n]\to F[p^{n+1}]$. It's easy to see that for any s, one has:

$$F(\mathfrak{m}_{\mathbf{C}}/p^s)=\varinjlim_{i_n}F[p^n](O_{\mathbf{C}}/p^s).$$

Therefore $F(\mathfrak{m}_{\mathbf{C}})$ can be defined in terms of the p-divisible group $F(p)$:

$$F(\mathfrak{m}_{\mathbf{C}})=\varprojlim_s F(\mathfrak{m}_{\mathbf{C}}/p^s)=\varprojlim_s \varinjlim_{i_n} F[p^n](O_{\mathbf{C}}/p^s).$$

16.1.6. By Cartier duality, for any O_K-algebra R, we have a canonical isomorphism

$$F[p^n](R)\simeq \mathrm{Hom}_R(F^\vee[p^n],\mathbb{G}_m).$$

Taking $R=O_{\mathbf{C}}/p^s$ and passing to the limits on the both sides, we obtain a morphism

$$F(\mathfrak{m}_{\mathbf{C}})\to \mathrm{Hom}\Big(T(F^\vee),U^{(1)}_{\mathbf{C}}\Big). \tag{76}$$

Theorem 16.1.7 (Tate). *i) We have a commutative diagram with exact rows*

$$\begin{tikzcd}
0 \arrow[r] & V(F)/T(F) \arrow[r] \arrow[d, "="] & F(\mathfrak{m}_{\mathbf{C}}) \arrow[r, "\log_F"] \arrow[d, "f"] & t_F(\mathbf{C}) \arrow[r] \arrow[d, "g"] & 0 \\
0 \arrow[r] & V(F)/T(F) \arrow[r] & \mathrm{Hom}\Big(T(F^\vee),U^{(1)}_{\mathbf{C}}\Big) \arrow[r] & \mathrm{Hom}(T(F^\vee),\mathbf{C}) \arrow[r] & 0,
\end{tikzcd}$$

where the morphisms are defined as follows:
- the upper row is the short exact sequence (72);
- the bottom row is induced by the short exact sequence (75) and the isomorphism $V(F)/T(F)\simeq \mathrm{Hom}(T(F^\vee),\mathbf{Q}_p/\mathbf{Z}_p(1))$*;*
- the middle vertical map is (76).

ii) The maps f and g are injective.

iii) The map g agrees with the Hodge–Tate decomposition of $V(F)$. Namely, the diagram

$$\begin{tikzcd}
t_F(\mathbf{C}(1)) \arrow[r, "g"] \arrow[dr, "\text{Hodge–Tate}"'] & \mathrm{Hom}(T(F^\vee),\mathbf{C}(1)) \arrow[d, "\simeq\ \text{duality}"] \\
& T(F)\otimes_{\mathbf{Q}_p}\mathbf{C}
\end{tikzcd}$$

commutes.

iv) The middle vertical row of the diagram induces an isomorphism

$$F(\mathfrak{m}_K)\simeq \mathrm{Hom}_{G_K}\Big(T(F^\vee),U^{(1)}_{\mathbf{C}}\Big).$$

Proof. i) The commutativity of the diagram and the exactness of rows is clear from construction.

We omit the proof of ii-iv), which are the key assertions of the proposition. We remark that assertions ii) and iv) are proved in [151, Proposition 11 and Theorem 3] without any referring to p-adic integration on formal groups. They imply immediately the Hodge–Tate decomposition for $V(F)$. Assertion iii) says, roughly speaking, that the Hodge–Tate decomposition arising from p-adic integration agrees with Tate's one. See [64, Chapter V, §1].

□

Corollary 16.1.8. *The map f can be identified with the canonical injection*

$$F(\mathfrak{m}_{\mathbf{C}}) \hookrightarrow T(F)\otimes_{\mathbf{Z}_p} U^{(1)}_{\mathbf{C}}(-1)$$

which gives rise to an isomorphism

$$F(\mathfrak{m}_K) \simeq \left(T(F)\otimes_{\mathbf{Z}_p} U^{(1)}_{\mathbf{C}}(-1)\right)^{G_K}.$$

Proof. This follows from Theorem 16.1.7 and the Cartier duality. □

16.2. **The universal covering.**

16.2.1. In this section, we introduce the notion of the universal covering of a formal group, and relate it to the p-adic representation $V(F)$.

Definition. *We call the universal covering of $F(\mathfrak{m}_{\mathbf{C}})$ the projective limit*

$$CF(\mathfrak{m}_{\mathbf{C}}) = \varprojlim_{[p]} F(\mathfrak{m}_{\mathbf{C}})$$

taken with respect to the multiplication-by-p map $[p]\,:\,F(\mathfrak{m}_{\mathbf{C}}) \to F(\mathfrak{m}_{\mathbf{C}})$.

We have an exact sequence

$$0 \to T(F) \to CF(\mathfrak{m}_{\mathbf{C}}) \xrightarrow{\mathrm{pr}_0} F(\mathfrak{m}_{\mathbf{C}}) \to 0, \tag{77}$$

where pr_0 denotes the projection map

$$\mathrm{pr}_0(\xi) = \xi_0, \qquad \forall \xi = (\xi_0, \xi_1, \ldots), \qquad [p](\xi_n) = \xi_{n-1}.$$

Combining this exact sequence with (72), we obtain an exact sequence

$$0 \to V(F) \to CF(\mathfrak{m}_{\mathbf{C}}) \xrightarrow{\log_F \circ \mathrm{pr}_0} t_F(\mathbf{C}) \to 0. \tag{78}$$

16.2.2. Let F_k denote the reduction of F modulo $\mathfrak{m}_K$, and let $S = \mathfrak{m}_{\mathbf{C}}/\mathfrak{m}_K$. Set:

$$CF_k(S) = \varprojlim_{[p]} F_k(S).$$

Proposition 16.2.3. *The canonical map $F(\mathfrak{m}_{\mathbf{C}}) \to F_k(S)$ induces an isomorphism*

$$CF(\mathfrak{m}_{\mathbf{C}}) \simeq CF_k(S).$$

In particular, $CF(\mathfrak{m}_{\mathbf{C}})$ depends only on the reduction of F.

Proof. a) The map $F(\mathfrak{m}_{\mathbf{C}}) \to F_k(S)$ is clearly an epimorphism. Let $y = (y_n)_{n\geqslant 0} \in CF_k(S)$. Let $\widehat{y}_n \in F(\mathfrak{m}_{\mathbf{C}})$ be any lift of y_n. It is easy to see that for each n, the sequence $[p^m](\widehat{y}_{n+m})$ converges to some $x_n \in F(\mathfrak{m}_{\mathbf{C}})$ and that $[p](x_{n+1}) = x_n$. This proves the surjectivity.

b) The injectivity follows from the fact that for any non-zero $x = (x_n)_{n\geqslant 0} \in CF(\mathfrak{m}_{\mathbf{C}})$, there exists N such that $v_K(x_n) < 1$ for $n \geqslant N$. □

16.2.4. From Corollary 16.1.8 if follows that there exists a canonical isomorphism

$$CF(\mathfrak{m}_{\mathbf{C}}) \simeq T(F)\otimes_{\mathbf{Z}_p} CU^{(1)}_{\mathbf{C}}(-1). \tag{79}$$

16.2.5. **Example.** Consider the universal covering of $\widehat{\mathbb{G}}_m$. One has:

$$\widehat{\mathbb{G}}_m(\mathfrak{m}_{\mathbf{C}}) \simeq U^{(1)}_{\mathbf{C}}, \qquad U^{(1)}_{\mathbf{C}} := (1+\mathfrak{m}_{\mathbf{C}})^*,$$

and

$$C\widehat{\mathbb{G}}_m(\mathfrak{m}_{\mathbf{C}}) \simeq CU^{(1)}_{\mathbf{C}}, \qquad CU^{(1)}_{\mathbf{C}} := \varprojlim_{x^p \leftarrow x} U^{(1)}_{\mathbf{C}}.$$

The universal covering of the reduction of $\widehat{\mathbb{G}}_m$ is

$$C\widehat{\mathbb{G}}_{m,k}(S) = \varprojlim_{x^p \leftarrow x} (1+S)^* \simeq (1+\mathfrak{m}_{\mathbf{C}^\flat})^*,$$

and the isomorphism $C\mathbb{G}_m(\mathfrak{m}_{\mathbf{C}}) \simeq C\mathbb{G}_{m,k}(S)$ is induced by the isomorphism (32) for $E = \mathbf{C}$:

$$\varprojlim_{x^p \leftarrow x} O_{\mathbf{C}} \simeq O^\flat_{\mathbf{C}}.$$

The short exact sequence (77) reads:

$$0 \to \mathbf{Z}_p(1) \to CU^{(1)}_{\mathbf{C}} \to U^{(1)}_{\mathbf{C}} \to 0. \tag{80}$$

16.3. Application to Galois cohomology.

16.3.1. In this section, we consider the sequence

$$0 \to \mathbf{Q}_p(1) \to (\mathbf{B}^+_{\mathrm{cris}})^{\varphi=p} \xrightarrow{\theta} \mathbf{C} \to 0, \tag{81}$$

where the first map is the canonical identification of $\mathbf{Q}_p(1)$ with the submodule $\mathbf{Q}_p t$ of $(\mathbf{B}^+_{\mathrm{cris}})^{\varphi=p}$. The fundamental exact sequence 54 shows that the sequence (81) is also exact. Consider the diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathbf{Z}_p(1) \arrow[r] \arrow[d, "="] & CU^{(1)}_{\mathbf{C}} \arrow[r] \arrow[d, "\log[\cdot]"] & U^{(1)}_{\mathbf{C}} \arrow[r] \arrow[d, "\log"] & 0 \\
0 \arrow[r] & \mathbf{Q}_p(1) \arrow[r] & (\mathbf{B}^+_{\mathrm{cris}})^{\varphi=p} \arrow[r, "\theta"] & \mathbf{C} \arrow[r] & 0,
\end{tikzcd} \tag{82}$$

Here we use the isomorphism $CU^{(1)}_{\mathbf{C}} \simeq 1+\mathfrak{m}_{\mathbf{C}^\flat}$ to define the middle vertical arrow as follows:

$$x \mapsto \log([x]) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{([x]-1)^n}{n}.$$

We omit the proof of convergence of this series in $\mathbf{B}^+_{\mathrm{cris}}$.

Proposition 16.3.2. *The diagram (82) commutes, and the middle vertical map is an isomorphism.*

Proof. a) The proof of commutativity is straightforward.

b) The map $\log[\cdot]$ is surjective because the right vertical map log is surjective, and $CU^{(1)}_{\mathbf{C}}$ is a $\mathbf{Q}_p$-vector space. Since $\log[x] = 0$ implies that $[x]$ is a root of unity, and $CU^{(1)}_{\mathbf{C}}$ is torsion free, $\log[\cdot]$ is injective. □

16.3.3. The exact sequence (81) induces a long exact sequence of continuous Galois cohomology:

$$0 \to H^0(G_K, \mathbf{Q}_p(1)) \to H^0(G_K, (\mathbf{B}^+_{\mathrm{cris}})^{\varphi=p}) \to H^0(G_K, \mathbf{C}) \xrightarrow{\partial_0} H^1(G_K, \mathbf{Q}_p(1))$$
$$\to H^1(G_K, (\mathbf{B}^+_{\mathrm{cris}})^{\varphi=p}) \to H^1(G_K, \mathbf{C}) \xrightarrow{\partial_1} H^2(G_K, \mathbf{Q}_p(1)).$$

We use Proposition 16.3.2 to compute the connecting homomorphisms ∂_0 and ∂_1.

16.3.4. Recall that μ_{p^n} denotes the group of p^nth roots of unity. For each n, the Kummer exact sequence

$$0 \to \mu_{p^n} \to \overline{K}^* \xrightarrow{p^n} \overline{K}^* \to 0$$

gives rise to the connecting map

$$\delta_n : K^* = H^0(G_K, \overline{K}^*) \to H^1(G_K, \mu_{p^n}).$$

Passing to the projective limit on n, we obtain a map

$$\delta : K^* \to H^1(G_K, \mathbf{Z}_p(1)).$$

The following proposition gives an interpretation of the Kummer map in terms of the fundamental exact sequence:

Proposition 16.3.5. *i) The diagram*

$$\begin{tikzcd}
U_K^{(1)} \arrow[r, "\delta"] \arrow[d, "\log"] & H^1(G_K, \mathbf{Z}_p(1)) \arrow[d] \\
K \arrow[r, "\partial_1"] & H^1(G_K, \mathbf{Q}_p(1))
\end{tikzcd}$$

is commutative.

ii) The diagram

$$\begin{tikzcd}
H^1(G_K, \mathbf{C}) \arrow[r, "\partial_2"] & H^2(G_K, \mathbf{Q}_p(1)) \arrow[d, "\mathrm{inv}_K"] \\
K \arrow[u, "\simeq"] \arrow[r, "-\mathrm{Tr}_K"] & \mathbf{Q}_p
\end{tikzcd}$$

is commutative. Here the left vertical isomorphism is $a \mapsto a \log \chi_K$ (see Theorem 4.3.2), and the right vertical map is the canonical isomorphism of local class field theory [140, Theorem 3].

Proof. i) The commutative diagram (82) gives a commutative square:

$$\begin{tikzcd}
H^0(G_K, U_{\mathbf{C}}^{(1)}) \arrow[r] \arrow[d] & H^1(G_K, \mathbf{Z}_p(1)) \arrow[d] \\
H^0(G_K, \mathbf{C}) \arrow[r, "\partial_i"] & H^1(G_K, \mathbf{Q}_p(1)).
\end{tikzcd}$$

Here $H^0(G_K, U^{(1)}_{\mathbf{C}}) = U^{(1)}_K$, and $H^0(G_K, \mathbf{C}) = K$ by Ax–Sen–Tate theorem. The explicit description of the connecting map shows that in this diagram, the upper row coincides with δ. This proves the first assertion.

ii) Assertion ii) is proved in [12, Proposition 1.7.2]. □

16.4. **The Bloch–Kato exponential map.**

16.4.1. We maintain previous notation and conventions. Our first goal is to extend the definition of the Kummer map to the case of general p-divisible formal groups. Let $\mathfrak{m}_{\overline{K}}$ denote the maximal ideal of the ring of integers of $\overline{K}$.

For all $n \geqslant 1$, we have an exact sequence

$$0 \to T_{F,n} \to F(\mathfrak{m}_{\overline{K}}) \xrightarrow{[p^n]} F(\mathfrak{m}_{\overline{K}}) \to 0,$$

which can be seen as the analog of the Kummer exact sequence for formal groups. It induces a long exact sequence of Galois cohomology:

$$0 \to H^0(G_K, T_{F,n}) \to H^0(G_K, F(\mathfrak{m}_{\overline{K}})) \to H^0(G_K, F(\mathfrak{m}_{\overline{K}})) \xrightarrow{\delta_{F,n}} H^1(G_K, T_{F,n}) \to \dots.$$

Since $H^0(K, F(\mathfrak{m}_{\overline{K}})) = F(\mathfrak{m}_K)$, this exact sequence gives an injection

$$\delta_{F,n} \,:\, F(\mathfrak{m}_K)/p^n F(\mathfrak{m}_K) \to H^1(G_K, T_{F,n}).$$

Passing to the projective limit, we obtain a map

$$\delta_F \;:\; F(\mathfrak{m}_K) \to H^1(K, T(F)),$$

which is referred to as the Kummer map for F. This map plays an important role in the Iwasawa theory of elliptic curves (see, for example, [81] for an introduction to this topic).

16.4.2. Bloch and Kato [28] found a remarkable description of δ_F in terms of p-adic periods, which also allows to construct an analog of the Kummer map for a wide class of p-adic representations.

Definition. *Let V be a de Rham representation of G_K. The quotient*

$$t_V(K) = \mathbf{D}_{\mathrm{dR}}(V)/\mathrm{Fil}^0\mathbf{D}_{\mathrm{dR}}(V)$$

is called the tangent space of V.

Using the isomorphisms $\mathrm{gr}_i(\mathbf{B}_{\mathrm{dR}}) \simeq \mathbf{C}(i)$, one can prove by devissage that the tautological exact sequence

$$0 \to \mathrm{Fil}^0\mathbf{B}_{\mathrm{dR}} \to \mathbf{B}_{\mathrm{dR}} \to \mathbf{B}_{\mathrm{dR}}/\mathrm{Fil}^0\mathbf{B}_{\mathrm{dR}} \to 0$$

induces an isomorphism

$$t_V(K) \simeq H^0(G_K, V \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{dR}}/\mathrm{Fil}^0\mathbf{B}_{\mathrm{dR}}).$$

Consider the fundamental exact sequence (54):

$$0 \to \mathbf{Q}_p \to \mathbf{B}_{\mathrm{cris}}^{\varphi=1} \to \mathbf{B}_{\mathrm{dR}}/\mathrm{Fil}^0\mathbf{B}_{\mathrm{dR}} \to 0.$$

Tensoring this sequence with V and taking Galois cohomology, we obtain a long exact sequence

$$0 \to H^0(G_K,V) \to \mathbf{D}_{\mathrm{cris}}(V)^{\varphi=1} \to t_V(K) \xrightarrow{\exp_V} H^1(G_K,V).$$

Definition. *The connecting homomorphism*

$$\exp_V\,:\, t_V(K) \to H^1(G_K,V)$$

is called the exponential map of Bloch and Kato.

16.4.3. We come back to representations arising from p-divisible formal groups. Since the Hodge–Tate weights of $V(F)$ are 0 and 1, we have

$$t_{V(F)}(K) \simeq H^0(G_K, V\otimes_{\mathbf{Q}_p} \mathbf{C}(-1)).$$

The Hodge–Tate decomposition of $V(F)$ provides us with a canonical isomorphism

$$t_F(K) \simeq t_{V(F)}(K). \tag{83}$$

In Proposition 16.1.2, we constructed the logarithmic map $\log_F\,:\, F(\mathfrak{m}_K) \to t_F(K)$. Taking the composition, we obtain a map $F(\mathfrak{m}_K) \to t_{V(F)}(K)$.

Theorem 16.4.4 (Bloch–Kato)**.** *The diagram*

$$\begin{tikzcd} F(\mathfrak{m}_K) \arrow[r, "\delta_F"] \arrow[d] & H^1(G_K,T(F)) \arrow[d] \\ t_{V(F)}(K) \arrow[r, "\exp_{V(F)}"] & H^1(G_K,V(F)), \end{tikzcd}$$

where the left vertical map is the composition of the exponential map $\exp_F$ *with the isomorphism (83), is commutative.*

Proof. This is [28, Example 3.10.1]. We first prove the following lemma, which gives an interpretation of the Kummer map in terms of universal coverings.

Lemma 16.4.5. *i) One has a commutative diagram with exact rows and injective vertical maps:*

$$\begin{tikzcd} 0 \arrow[r] & T(F) \arrow[r] \arrow[d, "\simeq"] & CF(\mathfrak{m}_{\mathbf{C}}) \arrow[r] \arrow[d, hook] & F(\mathfrak{m}_{\mathbf{C}}) \arrow[r] \arrow[d, hook] & 0 \\ 0 \arrow[r] & T(F) \arrow[r] & T(F)\otimes_{\mathbf{Z}_p} CU_{\mathbf{C}}^{(1)}(-1) \arrow[r] & T(F)\otimes_{\mathbf{Z}_p} U_{\mathbf{C}}^{(1)}(-1) \arrow[r] & 0. \end{tikzcd}$$

ii) This diagram gives rise to a commutative diagram

$$\begin{tikzcd} F(\mathfrak{m}_K) \arrow[r, "\delta_F"] \arrow[d, "\simeq"] & H^1(G_K,T(F)) \arrow[d, "="] \\ H^0\left(G_K,T(F)\otimes_{\mathbf{Z}_p} U_{\mathbf{C}}^{(1)}(-1)\right) \arrow[r] & H^1(G_K,T(F)). \end{tikzcd}$$

Proof. i) The first statement follows from the exactness of the sequence (80) and Corollary 16.1.8.

ii) Directly from construction, it follows that the upper connecting map is δ_F. Taking into account the isomorphism from Corollary 16.1.8, we obtain the lemma. $\square$

16.4.6. *Proof of the theorem.* Consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbf{Z}_p \arrow[r] \arrow[d, hook] & CU^{(1)}_{\mathbf{C}}(-1) \arrow[r] \arrow[d, "\log[\cdot]"] & U^{(1)}_{\mathbf{C}}(-1) \arrow[r] \arrow[d, "\log"] & 0 \\
0 \arrow[r] & \mathbf{Q}_p \arrow[r] \arrow[d, "="] & (\mathbf{B}^+_{\mathrm{cris}})^{\varphi=p}(-1) \arrow[r] \arrow[d, hook, "\varepsilon\otimes t^{-1}"] & \mathbf{C}(-1) \arrow[r] \arrow[d, hook, "\varepsilon\otimes t^{-1}"] & 0, \\
0 \arrow[r] & \mathbf{Q}_p \arrow[r] & \mathbf{B}^{\varphi=1}_{\mathrm{cris}} \arrow[r] & \mathbf{B}_{\mathrm{dR}}/\mathrm{Fil}^0\mathbf{B}_{\mathrm{dR}} \arrow[r] & 0.
\end{tikzcd}$$

The upper part of the diagram is diagram (82) twisted by χ_K^{-1}. Therefore the two upper squares commute. It is easy to check that the two lower squares commute too. Tensoring the diagram with $T(F)$ and taking Galois cohomology, we obtain a commutative diagram

$$\begin{tikzcd}
H^0\left(G_K, T(F)\otimes_{\mathbf{Z}_p} U^{(1)}_{\mathbf{C}}(-1)\right) \arrow[r] \arrow[d] & H^1(G_K, T(F)) \arrow[d] \\
t_{V(F)}(K) \arrow[r, "\exp_{V(F)}"] & H^1(G_K, V(F)).
\end{tikzcd}$$

Combining this diagram with Lemma 16.4.5, we obtain the theorem. $\square$

16.5. **Hilbert symbols for formal groups.**

16.5.1. To illustrate the theory developed in previous sections, we sketch its application to an explicit description of Hilbert symbols on formal groups. Fix $n \geqslant 1$. Let L/K be a finite extension containing the coordinates of all points of $T_{F,n}$. Recall that $\theta_L : L^* \to G_L^{\mathrm{ab}}$ denotes the reciprocity map.

Definition. *The Hilbert symbol on F is the pairing*

$$(\ ,\ )_{F,n} \,:\, L^* \times F(\mathfrak{m}_L) \to T_{F,n} \tag{84}$$

defined by the formula

$$(\alpha,\beta)_{F,n} = x^{\theta_L(\alpha)} -_F x,$$

where $x \in F(\mathfrak{m}_{\overline{K}})$ is any solution of the equation $[p^n](x) = \beta$.

It is easy to see that this pairing is well defined, i.e. that $(\alpha,\beta)_{F,n}$ does not depend on the choice of x. If $F = \widehat{\mathbb{G}}_m$, and L contains the group μ_{p^n} of p^nth roots of unity, it reduces to the classical Hilbert symbol:

$$(\ ,\ )_{L,n} \,:\, L^* \times L^* \to \mu_{p^n},$$

$$(\alpha,\beta)_{L,n} = \left(\sqrt[p^n]{\beta}\right)^{\theta_L(\alpha)} / \sqrt[p^n]{\beta}.$$

16.5.2. By local class field theory, there exists a canonical isomorphism

$$H^2(G_L,\mu_{p^n}) \simeq \mathbf{Z}/p^n\mathbf{Z}$$

(see, for example, [142], Chapter VI). Since $T_{F,n}$ is a trivial G_L-module, one has:

$$H^2(G_L,\mu_{p^n}\otimes T_{F,n}) \simeq T_{F,n}.$$

Consider the cup product

$$H^1(G_L,\mu_{p^n})\times H^1(G_L,T_{F,n}) \xrightarrow{\cup} H^2(G_L,\mu_{p^n}\otimes T_{F,n}) \simeq T_{F,n}.$$

Composing this pairing with the Kummer maps $\delta_{F,n} : F(\mathfrak{m}_L) \to H^1(G_L,T_{F,n})$ and $\delta_n : L^* \to H^1(G_L,\mu_{p^n})$, we obtain a pairing

$$L^*\times F(\mathfrak{m}_L) \to T_{F,n}.$$

From the cohomological description of the reciprocity map (see for example, [142], Chapter VI), it follows that this pairing coincides with the Hilbert symbol (84).

16.5.3. Fix an uniformizer π_L of L. Let $f(X) \in O_K[X]$ denote the minimal polynomial of π_L over K. Writing O_L as $O_K[X]/(f(X))$ and taking into account that $\mathfrak{D}_{L/K} = (f'(\pi_L))$, we obtain an explicit description of the module of differentials $\Omega^1_{O_L/\mathbf{Z}_p}$ (see [142, Chapter III,§7]):

$$\Omega^1_{O_L/\mathbf{Z}_p} \simeq \left(O_L/\mathfrak{D}_{L/\mathbf{Q}_p}\right) d\pi_L$$

(recall that $\mathfrak{D}_{L/\mathbf{Q}_p}$ denotes the different of $L/\mathbf{Q}_p$). For any $\alpha \in O_L$ we write $\dfrac{d\alpha}{d\pi_L}$ for an element $a \in O_L$ such that $d\alpha = a\cdot d\pi_L$. Note that a is well defined modulo $\mathfrak{D}_{L/\mathbf{Q}_p}$. Set $d\log(\alpha) = \alpha^{-1}\dfrac{d\alpha}{d\pi_L}$.

16.5.4. Fix a base $(\xi_i)_{1\leqslant i\leqslant h}$ of $T_{F,n}$ over $\mathbf{Z}/p^n\mathbf{Z}$ and a basis $(\omega_j)_{1\leqslant j\leqslant h}$ of $H^1_{\mathrm{dR}}(F)$ in such a way that $(\omega_j)_{1\leqslant j\leqslant d}$ is a basis of Ω^1_F. Set:

$$\Theta_{L,n} = p^n \begin{pmatrix} \lambda'_{\omega_1}(\xi_1)\dfrac{d\xi_1}{d\pi_L} & \lambda'_{\omega_1}(\xi_2)\dfrac{d\xi_2}{d\pi_L} & \cdots & \lambda'_{\omega_1}(\xi_h)\dfrac{d\xi_h}{d\pi_L} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda'_{\omega_d}(\xi_1)\dfrac{d\xi_1}{d\pi_L} & \lambda'_{\omega_d}(\xi_2)\dfrac{d\xi_2}{d\pi_L} & \cdots & \lambda'_{\omega_d}(\xi_h)\dfrac{d\xi_h}{d\pi_L} \\ \lambda_{\omega_{d+1}}(\xi_1) & \lambda_{\omega_{d+1}}(\xi_2) & \cdots & \lambda_{\omega_{d+1}}(\xi_h) \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{\omega_h}(\xi_1) & \lambda_{\omega_h}(\xi_2) & \cdots & \lambda_{\omega_h}(\xi_h) \end{pmatrix},$$

where we adopt the notation:

$$\lambda'_{\omega_j}(\xi_i)\frac{d\xi_i}{d\pi_L} := \sum_{k=1}^{d} \frac{d\lambda_{\omega_j}(\xi_i)}{dX_k}\frac{d\xi_i^{(k)}}{d\pi_L}, \qquad \text{if } \xi_i = \left(\xi_i^{(1)},\dots,\xi_i^{(d)}\right).$$

Let $X=(X_{ij})_{1\leqslant i,j\leqslant h}$ denote the inverse matrix of $\Theta_{L,n}$. The theory of p-adic integration together with Bloch–Kato's interpretation of the Kummer map allow to give the following explicit formula for this pairing:

Theorem 16.5.5. *For all $\alpha\in L^*$ and $\beta\in F(\mathfrak{m}_L)$ such that $v_p(\beta)>\dfrac{2}{p-1}$, one has:*

$$(\alpha,\beta)_{F,n}=\sum_{i=1}^{h}\sum_{j=1}^{d}\Big[\mathrm{Tr}_{L/\mathbf{Q}_p}\Big(X_{ij}d\log(\alpha)\lambda_{\omega_j}(\beta)\Big)\Big](\xi_j).$$

Corollary 16.5.6. *Applying this formula to the multiplicative formal group we obtain the explicit formula of Sen* [137] *for the classical Hilbert symbol:*

$$(\alpha,\beta)_{L,n}=\zeta_{p^n}^{[\alpha,\beta]_n},\qquad \textit{where } [\alpha,\beta]_n:=\frac{1}{p^n}\mathrm{Tr}_{L/\mathbf{Q}_p}\left(\frac{d\log(\alpha)}{d\log(\zeta_{p^n})}\log(\beta)\right).$$

For Lubin–Tate formal groups, this formula impoves the explicit reciprocity law of Wiles [158].

Comments on the proof. a) This formula was proved in [12] assuming that $v_p(\beta)>c$ for some constant c independent of n. In [77], it was noticed that one can take $c=\frac{2}{p-1}$.

b) Let $\widehat{\pi}_L\in\mathbf{A}_{\mathrm{inf}}$ be any lift of π_L under the map $\theta:\mathbf{A}_{\mathrm{inf}}\to O_{\mathbf{C}}$. Note that $\pi_L-\widehat{\pi}_L\in\mathrm{Fil}^1\mathbf{B}_{\mathrm{dR}}$. Take $u=(u_i)_{i\geqslant 0}\in T(F)$, where $[p](u_{i+1})=u_i$. Let ω be a differential form of the second kind. From the definition of the p-adic integration in Theorem 14.4.2, it follows that the p-adic period $\int_u\omega$ can be approximated as follows:

$$\int_u\omega\approx\begin{cases}p^n\lambda'_\omega(u_n)\dfrac{du_n}{d\pi_L}(\pi_L-\widehat{\pi}_L)\pmod{\mathrm{Fil}^2\mathbf{B}_{\mathrm{dR}}}, & \text{if } \omega\in\Omega^1_F,\\ -p^n\lambda_\omega(u_n)\pmod{\mathrm{Fil}^1\mathbf{B}_{\mathrm{dR}}}, & \text{otherwise}\end{cases}$$

(see [12, Section 2.4] for precise statements). Therefore the matrix $\Theta_{L,n}$ can be seen as "the matrix of p-adic periods of F modulo p^n".

c) The Hodge–Tate decomposition gives an isomorphism

$$t_F(L)\simeq H^0(G_L,T(F)\otimes_{\mathbf{Z}_p}\mathbf{C}(-1)),$$

which can be described in terms of the matrix of p-adic periods. We consider an integral mod p^n version of this isomorphism. Namely, set:

$$\mathfrak{m}'_{\mathbf{C}}=\left\{x\in\mathbf{C}\mid v_K(x)>\frac{v_K(p)}{p-1}\right\},$$

and $\mathfrak{m}'_L=\mathfrak{m}'_{\mathbf{C}}\cap\mathfrak{m}_L$. Since $T_{F,n}$ is a trivial G_L-module, we have a map

$$\eta_n:t_F(\mathfrak{m}'_L)\to H^0(G_L,T_{F,n}\otimes_{\mathbf{Z}_p}\mathfrak{m}'_{\mathbf{C}}(-1))\simeq H^0(G_L,\mathfrak{m}'_{\mathbf{C}}(-1))\otimes_{\mathbf{Z}_p}T_{F,n},$$

which has an explicit description in terms of the matrix $\Theta_{L,n}$.

c) The plan of the proof is the following. Using a mod p^n version $\exp_{F,n}$ of the Bloch–Kato exponential map, we construct a commutative diagram

$$\begin{tikzcd}
L^*\times t_F(\mathfrak{m}'_L) \arrow[r, "\delta_{p^n}\cup\eta_n"] \arrow[d, "(\delta_{p^n}{,}\exp_{F,n})"] & H^1(G_L,\mathfrak{m}'_{\mathbf{C}}/p^n)\otimes_{\mathbf{Z}_p} T_{F,n} \arrow[d, "(\tau_n{,}\mathrm{id})"] & \\
H^1(G_L,\mu_{p^n})\times H^1(G_L,T_{F,n}) \arrow[r, "\cup"] & H^2(G_L,\mu_{p^n})\otimes T_{F,n} \arrow[r, "\sim"] & T_{F,n}.
\end{tikzcd}$$

From the cohomological interpretation of the Hilbert symbol and Theorem 16.4.4, it follows that the Hilbert symbol $(\alpha,\beta)_{F,n}$ can be computed as the image of $(\alpha,\log_F)$ under the map $\delta_{p^n}\cup\exp_{F,n}$. We compute it using the above diagram, as the image of $(\alpha,\log_F)$ under the composition $(\tau_n,\mathrm{id})\circ(\delta_{p^n}\cup\eta_n)$. From construction, τ_n is the integral mod p^n version of the connecting map $\partial_2\,:\,H^1(G_L,\mathbf{C})\to H^2(G_L,\mathbf{Q}_p(1))$ associated to the exact sequence

$$0\to\mathbf{Q}_p(1)\to(\mathbf{B}^+_{\mathrm{cris}})^{\varphi=p}\to\mathbf{C}\to 0.$$

Therefore it can be computed in terms of the trace map using Proposition 16.3.5. The computation of the cup product $\delta_{p^n}\cup\eta_n$ is more subtle, and we refer the reader to [12] for further details. □

Remark 16.5.7. *1) Explicit formulas of other types are proved in* [3] *and* [150]. *They generalize the explicit reciprocity law of Vostokov* [156] *and also use information about the matrix of p-adic periods.*

2) The exponential map of Bloch–Kato is closely related to special values of L-functions and Iwasawa theory [28, 125]. *For further reading, see* [13, 14, 16, 40, 116, 117, 124].

17. The weak admissibility: the case of dimension one

17.1. Formal groups of dimension one.

17.1.1. In this section, we assume that K is a finite totally ramified extension of $K_0=\widehat{\mathbf{Q}}_p^{\mathrm{ur}}$. Assume that M is an irreducible filtered φ-module over K of rank h satisfying the following conditions:

1) $M=M_{1/h}$.
2) $\mathrm{Fil}^0M_K=M_K$, $\mathrm{Fil}^2M_K=\{0\}$, and $\dim_K\mathrm{Fil}^1M_K=1$.

The first condition means that $M\simeq K_0[\varphi]/(\varphi^h-p)$, and by the theory of Dieudonné–Manin, M is the unique irreducible φ-module with $\mu_{\mathrm{N}}(M)=1/h$. Since $t_{\mathrm{H}}(M)=1/h$, we see that M is weakly admissible.

17.1.2. Let F_{LT} denote the Lubin–Tate formal group with the logarithm

$$\lambda_{\mathrm{LT}}(X)=\left(1-\frac{\varphi^h}{p}\right)^{-1}(X)=X+\frac{X^{p^h}}{p}+\frac{X^{p^{2h}}}{p^2}+\cdots.$$

Extending scalars, we consider F_{LT} as a formal group over K. The filtered φ-module $M(F_{\mathrm{LT}})$ has the following description. The class b_{LT} of the canonical differential $\omega_{\mathrm{LT}} = d\lambda_{\mathrm{LT}}$ in $M(F_{\mathrm{LT}})$ satisfies the relation

$$\varphi^h(b_{\mathrm{LT}}) = pb_{\mathrm{LT}},$$

and the vectors $b_{\mathrm{LT}}, \varphi(b_{\mathrm{LT}}), \dots, \varphi^{h-1}(b_{\mathrm{LT}})$ form a basis of $M(F_{\mathrm{LT}})$ over K_0. The filtration on $M(F_{\mathrm{LT}})_K$ is given by

$$\mathrm{Fil}^1 M(F_{\mathrm{LT}})_K = K \cdot b_{\mathrm{LT}}.$$

In particular, $M(F_{\mathrm{LT}})$ and M are isomorphic as φ-modules. Let v_p denote the valuation normalized as $v_p(p) = 1$.

Theorem 17.1.3 (Laffaille)**.** *Assume that M is a filtered φ-module satisfying the conditions 1-2) above. The following holds true:*

i) There exists $b \in M$ such that:

a) *b is a generator of M as a φ-module, and $\varphi^h(b) = pb$;*

b) *There exist $c_0 = 1, c_1, \dots, c_{h-1} \in K$ such that*

$$v_p(c_i) \geqslant -i/h \qquad \textit{for all} \quad 1 \leqslant i \leqslant h-1, \tag{85}$$

and

$$\ell := \sum_{i=0}^{h-1} c_i \varphi^i(b) \in \mathrm{Fil}^1 M_K.$$

ii) For all $c_0 = 1, c_1, \dots, c_{h-1} \in K$ satisfying condition (85), the series

$$\lambda(X) = \sum_{i=0}^{h-1} c_i \lambda_{\mathrm{LT}}(X^{p^{ih}})$$

is the logarithm of some formal p-divisible group over O_K of height h.

iii) M is admissible. More precisely, there exists a formal group F of dimension one over O_K such that $M(F) \simeq M$ as filtered φ-modules.

Proof. This theorem is proved in [102].

i) By the discussion preceding the theorem, there exists a generator b' of M such that $\varphi^h(b') = pb'$ and $b', \varphi(b'), \dots, \varphi^{h-1}(b')$ is a K_0-basis of M. Then for any non-zero $\ell \in \mathrm{Fil}^1 M_K$ one has:

$$\ell' = \sum_{i=0}^{h-1} c_i' \varphi^i(b'), \qquad \text{for some } c_i' \in K. \tag{86}$$

Note that $c_i' \neq 0$ for some i. Replacing, if necessary, b' by $\varphi^i(b')$ and dividing ℓ by c_i', we can assume that in (86), $c_0' = 1$. Let j be such that

$$v_p(c_j') + j/h \leqslant v_p(c_i') + i/h, \qquad \forall i = 0, \dots, h-1.$$

If $v_p(c_j') + j/h \geqslant 0$, then $v_p(c_i') \geqslant -i/h$ for all i, and we can take

$$c_i = c_i', \qquad \ell = \ell'.$$

Otherwise $c_j' \neq 0$. In that case, set:

$$b = \varphi^j(b'), \qquad \ell = \ell'/c_j'.$$

Then

$$\ell = \sum_{i=0}^{h-1} c_i \varphi^i(b),$$

where the coefficients c_i are given by

$$c_i = \begin{cases} c'_{i+j}/c'_j, & \text{if } 0 \leqslant i \leqslant h-j-1 \\ c'_{i+j-h}/pc'_j, & \text{if } h-j \leqslant i \leqslant h-1. \end{cases}$$

For $0 \leqslant i \leqslant h-j-1$, one has:

$$v_p(c_i) + i/h = v_p(c'_{i+j}) - v_p(c'_j) + i/h = (v_p(c'_{i+j}) + (i+j)/h) - (v_p(c'_j) + j/h) \geqslant 0.$$

For $h-j \leqslant i \leqslant h-1$, one has:

$$\begin{aligned} v_p(c_i) + i/h &= v_p(c'_{i+j-h}) - v_p(c'_j) - 1 + i/h \\ &= (v_p(c'_{i+j-h}) + (i+j-h)/h) - (v_p(c'_j) + j/h) \geqslant 0. \end{aligned}$$

This shows that $c_0, c_1, \dots, c_{h-1}$ satisfy (85).

ii) By [86, §15.2], a power series of the form $\sum\limits_{n=0}^{\infty} a_n X^{p^n}$ with $a_0 = 1$ is the logarithm of a formal group if and only if the sums

$$\begin{aligned} &A_1 := pa_1, \\ &A_2 := pa_2 - a_1 A_1^p, \\ &\dots\dots\dots\dots\dots\dots \\ &A_n := pa_n - \sum_{i=0}^{n-1} a_i A^{p^i}, \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \end{aligned}$$

are in O_K. The verification of these conditions for the series $\lambda(X)$ is quite technical and is omitted here. See [102, proof of Proposition 2.4].

iii) Let M be a filtered φ-module satisfying conditions 1-2). By part i), there exists a generator b of M such that conditions a-b) hold for some $c_1, \dots, c_{h-1}$. By part ii), the formal power series $\lambda(X) = \sum\limits_{i=0}^{h-1} c_i \lambda_{\mathrm{LT}}(X^{p^{ih}})$ is the logarithm of some formal group F of height h. Then $M(F) \simeq M$ as filtered φ-modules. By Theorem 14.4.2, one has $M(F) \simeq \mathbf{D}^*_{\mathrm{cris}}(V(F))$. Hence M is admissible. $\square$

Remark 17.1.4. *This theorem implies the surjectivity of the Gross–Hopkins period map* [82]. *See also* [103] *for the case of Drinfeld spaces.*

17.2. Geometric interpretation of $(\mathbf{B}^+_{\mathrm{cris}})^{\varphi^h = p}$.

17.2.1. We maintain previous notation and consider the Lubin–Tate formal group F_{LT} of height h with the logarithm $\lambda_{\mathrm{LT}}(X)$. Note that F_{LT} is defined over $\mathbf{Z}_p$. Let $F_{\mathrm{LT},k}$ denote the reduction of F_{LT} modulo p. We have the following interpretation of the universal covering of F_{LT}, which generalizes Example 16.2.5:

Proposition 17.2.2. *There is a canonical isomorphism*

$$CF_{\mathrm{LT}}(\mathfrak{m}_{\mathbf{C}}) \simeq F_{\mathrm{LT},k}(\mathfrak{m}_{\mathbf{C}^\flat}).$$

Proof. Since $[p](X) \equiv X^q \pmod{p}$, the multiplication by p in $F_{\mathrm{LT},k}$ is given by φ^h. Set $S = \mathfrak{m}_{\overline{K}}/(p)$. Then

$$CF_{\mathrm{LT},k}(S) \simeq \varprojlim_{\varphi^h} F_{\mathrm{LT},k}(S) \simeq F_{\mathrm{LT},k}(\varprojlim_{\varphi^h} S) \simeq F_{\mathrm{LT},k}(\mathfrak{m}_{\mathbf{C}^\flat}).$$

Now the proposition follows from Proposition 16.2.3. □

17.2.3. Since $\mathbf{A}_{\mathrm{inf}}/(p) \simeq O_{\mathbf{C}}^\flat$, we have a well defined composition

$$\kappa : F_{\mathrm{LT},k}(\mathfrak{m}_{\mathbf{C}^\flat}) \xrightarrow{\sim} F_{\mathrm{LT},k}(\mathbf{A}_{\mathrm{inf}}/(p)) \xrightarrow{\sim} CF_{\mathrm{LT}}(\mathbf{A}_{\mathrm{inf}}) \xrightarrow{\mathrm{pr}_0} F_{\mathrm{LT}}(\mathbf{A}_{\mathrm{inf}}).$$

Here $CF_{\mathrm{LT}}(\mathbf{A}_{\mathrm{inf}}) := \varprojlim_{[p]} F_{\mathrm{LT}}(\mathbf{A}_{\mathrm{inf}})$, and pr_0 denotes the projection on the ground level.

Theorem 17.2.4 (Fargues–Fontaine)**.** *The map*

$$\mathrm{Log}(x) := \lambda_{\mathrm{LT}}(\kappa(x))$$

establishes an isomorphism $F_{\mathrm{LT},k}(\mathfrak{m}_{\mathbf{C}^\flat}) \simeq (\mathbf{B}^+_{\mathrm{cris}})^{\varphi^h=p}$.

Sketch of the proof. The proof of the convergence of the series $\lambda_{\mathrm{LT}}(y)$ in $\mathbf{B}^+_{\mathrm{cris}}$ for $y \in F_{\mathrm{LT}}(\mathbf{A}_{\mathrm{inf}})$ is routine, and we omit it. Since , $F_{\mathrm{LT}}(\mathbf{A}_{\mathrm{inf}})$ does not contain torsion points of F_{LT}, the map Log is injective.

The series $F_{\mathrm{LT}}(X, Y)$ has coefficients in $\mathbf{Z}_p$. Hence the formal group law commutes with φ, and one has:

$$\varphi^h \lambda_{\mathrm{LT}}(\kappa(x)) = \lambda_{\mathrm{LT}}(\varphi^h(\kappa(x))) = \lambda_{\mathrm{LT}}(\kappa(\varphi^h(x))).$$

On the other hand, $\varphi^h(x) = [p](x)$ in $F_{\mathrm{LT},k}(\mathfrak{m}_{\mathbf{C}^\flat})$, and therefore

$$\lambda_{\mathrm{LT}}(\kappa(\varphi^h(x))) = \lambda_{\mathrm{LT}}([p](\kappa(x))) = p\lambda_{\mathrm{LT}}(\kappa(x)).$$

This proves that $\mathrm{Log}(x) \in (\mathbf{B}^+_{\mathrm{cris}})^{\varphi^h=p}$.

The proof of the surjectivity is more subtle and we refer the reader to [60, Chapter 4], where this map is studied in all detail and in a more general setting. □

17.2.5. **Example.** If $h = 1$, then F_{LT} is isomorphic to $\widehat{\mathbb{G}}_m$. Therefore $F_{\mathrm{LT},k}(\mathfrak{m}_{\mathbf{C}^\flat}) \simeq (1 + \mathfrak{m}_{\mathbf{C}^\flat})^*$, and the map κ can be identified with the map $\log[\,\cdot\,]$ introduced in Proposition 16.3.2.

17.2.6. The next theorem furnishes further information about the structure of $(\mathbf{B}^+_{\mathrm{cris}})^{\varphi^h=p}$.

Theorem 17.2.7 (Fargues–Fontaine)**.** *For any family of elements*

$$\alpha_0, \alpha_1, \alpha_2, \dots, \alpha_{h-1} \in \mathbf{C},$$

not all zero, consider the map:

$$f : (\mathbf{B}^+_{\mathrm{cris}})^{\varphi^h=p} \to \mathbf{C}, \qquad f(x) = \sum_{i=1}^{h-1} \alpha_i \theta(\varphi^i(x)).$$

Then f is surjective, and $\ker(f)$ *is a* $\mathbf{Q}_p$*-vector space of dimension* h.

Proof. See [60, Théorème 8.1.2]. Without loss of generality, we can assume that $v_p(\alpha_i) \geqslant 0$ and $\alpha_0 = 1$. The arguments used in the proof of Theorem 17.1.3 apply and show that there exists a formal group F over $O_{\mathbf{C}}$ such that

$$\lambda_F = \sum_{i=0}^{h-1} \alpha_i \varphi^i(\lambda_{\mathrm{LT}}).$$

Consider the diagram

$$\begin{tikzcd} 0 \arrow[r] & V(F) \arrow[r] & CF(\mathfrak{m}_{\mathbf{C}}) \arrow[r, "\lambda_F \circ \mathrm{pr}_0"] & \mathbf{C} \arrow[r] & 0, \\ & & (\mathbf{B}^+_{\mathrm{cris}})^{\varphi^h = p} \arrow[u, "\simeq"] \arrow[ur, "f"'] & & \end{tikzcd}$$

where the first line is the exact sequence (78) for F, and the vertical isomorphism is provided by Theorem 17.2.4. Since $\dim_{\mathbf{Q}_p} V(F) = h$, the theorem is proved. $\square$

We refer the reader to [41] for the interpretation of this result in terms of the theory of Banach Spaces, and to [60] and [55] for applications to the theory of Fargues–Fontaine.

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