Particle methods for branching type signals

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Outline

1. Spatial Branching models
   - A branching-exploration model
   - First moment recursion
   - A Feynman-Kac formulation
   - Multi targets branching signals
2. Normalized distribution flows
3. Some theoretical aspects

Spatial Branching models (time index $n \in \mathbb{N}$, state spaces E_n)

- 2 simple ingredients : Potential $G_n(x) \geq 1$ and $M_n(x_{n-1}, dx_n)$ Markov.
  - Branching rule :

$$x \rightsquigarrow g_n(x) \text{ offsprings, with } \mathbb{E}(g_n(x)) = G_n(x)$$

  - Between branching times : M_n-evolutions
- $\rightsquigarrow$ Random occupation measure (after evolution step)

$$\mathcal{X}_n = \sum_{i=1}^{N_n} \delta_{X_n^i}$$

Spatial Branching models (time index $n \in \mathbb{N}$, state spaces E_n)

- First moment recursion = branching intensity distribution

$$\gamma_{n+1}(f) := \mathbb{E}\left(\mathcal{X}_{n+1}(f)\right) = \gamma_n(G_n\ M_{n+1}(f))$$

$$\text{with} \quad M_{n+1}(f)(x) := \int\ M_{n+1}(x, dx')\ f(x') \quad \text{and} \quad \gamma(\varphi) := \int \varphi(x)\ \gamma(dx)$$

Sketched proof : $$\mathcal{X}_{n+1} = \sum_{i=1}^{N_{n+1}} \delta_{X^i_{n+1}} = \sum_{i=1}^{N_n} \sum_{j=1}^{g^i_n(X^i_n)} \delta_{X^{i,j}_{n+1}}$$

$$\Downarrow$$

$$\mathbb{E}\left(\mathcal{X}_{n+1}(f) \mid \mathcal{X}_n, (g^i_n(X^i_n))_{i,j}\right) = \sum_{i=1}^{N_{n-1}} g^i_n(X^i_n)\ M_{n+1}(f)(X^i_n)$$

$$\Downarrow$$

$$\mathbb{E}\left(\mathcal{X}_{n+1}(f) \mid \mathcal{X}_n\right) = \sum_{i=1}^{N_{n-1}} G_n(X^i_n)\ M_{n+1}(f)(X^i_n) = \mathcal{X}_n(G_n M_{n+1}(f))$$

A Feynman-Kac formulation

- First moment recursion

$$\gamma_{n+1}(f) = \gamma_n(G_n\ M_{n+1}(f)) = \gamma_{n-1}(G_{n-1} M_n\,(G_n\ M_{n+1}(f))) = \ldots$$

- Feynman-Kac formulation $\rightsquigarrow \uparrow$ Application domains

$$\begin{array}{rcl}\gamma_n(f) & = & \gamma_0(1)\ \mathbb{E}_{\eta_0}\left(f(X_n) \displaystyle\prod_{0\leq p<n} G_p(X_p)\right) \\ & \propto & \displaystyle\int \eta_0(dx_0) G_0(x_0) M_1(x_0, dx_1) \ldots G_{n-1}(x_{n-1}) M_n(x_{n-1}, dx_n) f(x_n)\end{array}$$

Spatial Branching models (time index $n \in \mathbb{N}$, state spaces E_n)

- Another formulation : $Q_n(x_{n-1}, dx_n)$ positive integral operator

  Potential branching rule= local mass $\quad G_n(x) = Q_{n+1}(x_n, E_{n+1})$

  and the Markov exploration

$$M_n(x_{n-1}, dx_n) = \frac{Q_{n+1}(x_n, dx_{n+1})}{Q_{n+1}(x_n, E_{n+1})}$$

- Example 1:

$$Q_n(x_{n-1}, dx_n) = \underbrace{B_n(x_{n-1}, dx_n)}_{\text{spawning}} + \underbrace{e_n(x_{n-1})}_{\text{survival probab.}} \underbrace{P_n(x_{n-1}, dx_n)}_{\text{target motion}}$$

- $\neq$ branching model BUT the same first moment recursion

$$\gamma_{n+1}(f) = \gamma_n(Q_{n+1}(f)) \quad \text{with} \quad Q_{n+1}(f)(x) := \int Q_{n+1}(x, dx')\ f(x')$$

Spatial Branching models (time index $n \in \mathbb{N}$, state spaces E_n)

- Standard notation:

$$Q_n(x_{n-1}, dx_n) = \underbrace{Q_{n+1}(x_n, E_{n+1})}_{G_n(x_n)} \times \underbrace{\frac{Q_{n+1}(x_n, dx_{n+1})}{Q_{n+1}(x_n, E_{n+1})}}_{M_{n+1}(x_n, dx_{n+1})}$$

$$\Downarrow$$

$$\gamma_{n+1} = \gamma_n Q_{n+1} \;\Leftrightarrow\; \gamma_{n+1}(dx) = \int \gamma_n(dx') Q_{n+1}(x', dx)$$

- A more general model: Poisson spontaneous births $\sim \mu_{n+1}$ positive measure

$$\Downarrow$$

$$\gamma_{n+1} = \gamma_n Q_{n+1} + \mu_{n+1}$$

Some problems

- Problem 1: Mass process $\gamma_n(1)$ "unstable" $\gamma_n(1) \uparrow \infty$ or $\gamma_n(1) \downarrow 0$ as $n \uparrow \infty$
- Problem 2: $\mathcal{X}_n = \sum_{i=1}^{N_n} \delta_{X_n^i}$ generally NOT POISSON random field.
- Problem 3: $\exists$ non generate numerical sampling method?
- Problem 4: $\exists$ non generate approximation of γ_n?

Some problems

- Problem 5: Filtering problems $\rightsquigarrow \mathrm{Law}\,(\mathcal{X}_n \mid \mathcal{Y}_n)$?

$$\mathcal{X}_n \quad \rightsquigarrow \quad \text{observation measure :} \quad \mathcal{Y}_n := \sum_{i=1}^{N'_n} \delta_{Y_n^i}$$

Using "Poisson Approximations" $\Rightarrow G_n \rightsquigarrow G_{n,\gamma_n}$

$$G_{n,\gamma_n}(x) = (1 - d_n(x)) + d_n(x) \sum_{y \in \mathcal{Y}_n} \frac{g_{n,y}(x)}{\kappa_n(y) + \gamma_n(d_n g_{n,y})}$$

Some references :

Mahler (03), Vo-Singh-Doucet (05), Clark-Vo-Bell (05-06), Johansen-Singh-DouceVo (06) , Singh-Vo-Baddeley-Zuyev (07)

  - Problem 5.1: γ_n-dependent $\rightsquigarrow$ stability problems?
  - Problem 5.2: "Conditional distributions?" γ_n?
  - Problem 5.3: Useful estimates (particle/smc)? "sharp" expo. bounds.
  - ....

Outline

branching distribution flows

- Problem 1: Mass process $\gamma_n(1)$ "unstable" $\gamma_n(1) \uparrow \infty$ or $\gamma_n(1) \downarrow 0$ as $n \uparrow \infty$
- Solution 1 $\rightsquigarrow$ Normalized model :

$$\eta_n(f) := \frac{\gamma_n(f)}{\gamma_n(1)} \rightsquigarrow \eta_n = \text{probability measure}$$

- Normalized constants:

$$\gamma_{n+1}(f) = \gamma_n(G_n\ M_{n+1}(f)) \Rightarrow \gamma_{n+1}(1) = \gamma_n(G_n) = \frac{\gamma_n(G_n)}{\gamma_n(1)}\ \gamma_n(1)$$

$$\Downarrow$$

Product formula :

$$\gamma_{n+1}(1) = \eta_n(G_n)\ \gamma_n(1) \Rightarrow \gamma_{n+1}(1) = \prod_{0 \le p \le n} \eta_p(G_p)$$

Mean field interpretation

- Normalized model recursion:

$$\eta_n = \Phi_n(\eta_{n-1})$$

- **Nonlinear Markov models :** Always $\exists$ Markov process $\overline{X}_n$ s.t.

$$\eta_n = = \mathrm{Law}\left(\overline{X}_n\right) \quad \text{The perfect stochastic algorithm!}$$

with

$$\mathbb{P}(\overline{X}_n \in dx_n \mid \overline{X}_{n-1}) = K_{n,\eta_{n-1}}(\overline{X}_{n-1}, dx_n) \quad \text{and} \quad \eta_{n-1} = \mathrm{Law}\left(\overline{X}_{n-1}\right)$$

$$\Downarrow$$

$$\eta_n(dx) = \int \eta_{n-1}(dx')K_{n,\eta_{n-1}}(x', dx)$$

Mean field particle interpretation

- Markov chain $\xi_n = (\xi_n^1, \ldots, \xi_n^N) \in E_n^N$ s.t.

$$\eta_n^N := \frac{1}{N} \sum_{1 \leq i \leq N} \delta_{\xi_n^i} \simeq_{N \uparrow \infty} \eta_n$$

- Particle approximation transitions ($\forall 1 \leq i \leq N$)

$$\xi_{n-1}^i \rightsquigarrow \xi_n^i \sim K_{n,\eta_{n-1}^N}(\xi_{n-1}^i, dx_n)$$

- Unormalized models :

$$\gamma_{n+1}(1) = \prod_{0 \leq p \leq n} \eta_p(G_p) \simeq_{N \uparrow \infty} \gamma_{n+1}^N(1) := \prod_{0 \leq p \leq n} \eta_p^N(G_p) \quad \text{(Unbias)}$$

and the first "unnormalized" moments

$$\gamma_{n+1}(f) = \gamma_{n+1}(1)\ \eta_{n+1}(f) \simeq_{N \uparrow \infty} \gamma_{n+1}^N(f) = \gamma_{n+1}^N(1)\ \eta_{n+1}^N(f) \quad \text{(Unbias)}$$

Discrete generation mean field particle model

Schematic picture : $\xi_n \in E_n^N \rightsquigarrow \xi_{n+1} \in E_{n+1}^N$

$$\begin{array}{ccc}
\xi_n^1 & \xrightarrow{\quad K_{n+1,\eta_n^N} \quad} & \xi_{n+1}^1 \\
\vdots & & \vdots \\
\xi_n^i & \xrightarrow{\qquad\qquad} & \xi_{n+1}^i \\
\vdots & & \vdots \\
\xi_n^N & \xrightarrow{\qquad\qquad} & \xi_{n+1}^N
\end{array}$$

Rationale :

$$\begin{aligned}
\eta_n^N \simeq_{N\uparrow\infty} \eta_n \quad & \Longrightarrow \quad K_{n+1,\eta_n} \simeq_{N\uparrow\infty} K_{n+1,\eta_n^N} \\
& \Longrightarrow \quad \xi_n^i \text{ almost iid copies of } \overline{X}_n
\end{aligned}$$

Outline

Non asymptotic results

- **Weak estimates** $\leftrightarrow$ **Bias estimates** ($\leftrightarrow$ **Propagations of chaos)**

Law(q particles among N at time n) $\simeq_{N\uparrow\infty}$ **Law(q iid r.v. w.r.t.** η_n)

1. Total variation $= \frac{q^2}{N} c(n)$, Boltzmann entropy $= \frac{q}{N} c(n)$.
2. Stable models: uniform estimates w.r.t. time $\sup_n c(n) < \infty$.
3. Path space and genealogical tree models $c(n) = c \times n$.
4. Explicit weak decompositions at any order $\frac{1}{N^k}$.

$\hookrightarrow$`http-ref : DM-Patras-Rubenthaler, Coalescent tree based functional representations for some Feynman-Kac particle models, Hal-INRIA (2006).`

- $\mathbb{L}_p$**-mean error bounds** [(2),(3) as above]

$$\sup_{N\geq 1} \sqrt{N}\, \mathbb{E}\left(\sup_{f_n\in\mathcal{F}_n} \left|\eta_n^N(f_n) - \eta_n(f_n)\right|^p\right) \leq b(p)\; c(n)$$

- **Exponential estimates** [(2) as above & empirical processes $\sim \mathcal{F}_n$]

$$\mathbb{P}(|\eta_n^N(f_n) - \eta_n(f_n)| > \epsilon) \leq c(n)\ \exp\left\{-\epsilon^2 N/c(n)\right\}$$

Non asymptotic results Unnormalized models

- **$\mathbb{L}_p$-mean error bounds** [as before] and (ok in path spaces)

$$\sup_{N\geq 1} \sqrt{N}\ \mathbb{E}\left(\left[\frac{\gamma_n^N(f_n)}{\gamma_n(1)} - \eta_n(f_n)\right]^2\right) \leq\ c\ n$$

- **Exponential estimates** ["conjecture"]

$$\mathbb{P}(\left|\frac{\gamma_n^N(f_n)}{\gamma_n(1)} - \eta_n(f_n)\right| > \epsilon) \leq c\ \exp\left\{-\epsilon^2 N/c\ n\right\}$$

Fluctuations

- **Central Limit Theorems** [Sharp $\mathbb{L}_p$ estimates]

{http-ref : 1999⇝2004 : DM, Guionnet, Jacod, Ledoux, Tindel}

$$V_n^N(f) := \sqrt{N}\ \left[\eta_n^N(f) - \eta_n(f)\right] \Longrightarrow V_n(f) = \text{Centered Gaussian r.v.}$$

1. **Functional Central Limit Theorems.** $[\forall d,\ \forall (f^i)_{1\leq i\leq d}]$

$$(V_n^N(f^1),\ldots,V_n^N(f^d)) \Longrightarrow \left(V_n(f^1),\ldots,V_n(f^d)\right)$$

2. **Unbounded $\mathbb{L}_2$-functions $\oplus$ algebra sets of functions with some growth conditions.**
   ↪ (Path space models) DM, Guionnet. Annals of Applied Probability, Vol. 9, No. 2, 275-297 (1999).
   ↪ (Donsker+explicit variance) DM, Ledoux, Journal of Theoret. Probability, Vol. 13, No. 1, 225-257 (2000).
   ↪ (marginal approx. models) DM, Jacod, The Fields Institute Communications, Ed. T.J. Lyons, T.S. Salisbury, American Mathematical Society, (2002).
3. **Donsker type theorems, Berry Esseen type theorems, path spaces,...**

Large deviations

- **Large deviations principles** [Sharp asymptotic expo estimates]

$$\lim_{N\to\infty} \frac{1}{N} \log \mathbb{P}\left(\eta_n^N \notin \mathcal{V}(\eta_n)\right)$$

Example : $\mathcal{V}(\eta_n) = \{\mu \ : \ |\eta_n^N(f) - \eta_n(f)| \leq \epsilon\}$ (weak and strong τ-topo).

$\{$`http-ref 1998`$\rightsquigarrow$`2004 :  DM, Dawson, Guionnet, Zajic`$\}$