

# Particle methods for branching type signals

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- 1 Spatial Branching models
  - A branching-exploration model
  - First moment recursion
  - A Feynman-Kac formulation
  - Multi targets branching signals
- 2 Normalized distribution flows
- 3 Some theoretical aspects

## Spatial Branching models (time index $n \in \mathbb{N}$ , state spaces $E_n$ )

- **2 simple ingredients** : Potential  $G_n(x) \geq 1$  and  $M_n(x_{n-1}, dx_n)$  Markov.

- Branching rule :

$$x \rightsquigarrow g_n(x) \text{ offsprings, with } \mathbb{E}(g_n(x)) = G_n(x)$$

- Between branching times :  $M_n$ -evolutions
- $\rightsquigarrow$  Random occupation measure (after evolution step)

$$\mathcal{X}_n = \sum_{i=1}^{N_n} \delta_{X_n^i}$$

- First moment recursion = **branching intensity distribution**

$$\gamma_{n+1}(f) := \mathbb{E}(\mathcal{X}_{n+1}(f)) = \gamma_n(G_n M_{n+1}(f))$$

with  $M_{n+1}(f)(x) := \int M_{n+1}(x, dx') f(x')$  and  $\gamma(\varphi) := \int \varphi(x) \gamma(dx)$

**Sketched proof :** 
$$\mathcal{X}_{n+1} = \sum_{i=1}^{N_{n+1}} \delta_{X_{n+1}^i} = \sum_{i=1}^{N_n} \sum_{j=1}^{g_n^i(X_n^i)} \delta_{X_{n+1}^{i,j}}$$

$\Downarrow$

$$\mathbb{E}(\mathcal{X}_{n+1}(f) \mid \mathcal{X}_n, (g_n^i(X_n^i))_{i,j}) = \sum_{i=1}^{N_{n-1}} g_n^i(X_n^i) M_{n+1}(f)(X_n^i)$$

$\Downarrow$

$$\mathbb{E}(\mathcal{X}_{n+1}(f) \mid \mathcal{X}_n) = \sum_{i=1}^{N_{n-1}} G_n(X_n^i) M_{n+1}(f)(X_n^i) = \mathcal{X}_n(G_n M_{n+1}(f))$$

## A Feynman-Kac formulation

- First moment recursion

$$\gamma_{n+1}(f) = \gamma_n(G_n M_{n+1}(f)) = \gamma_{n-1}(G_{n-1} M_n(G_n M_{n+1}(f))) = \dots$$

- Feynman-Kac formulation  $\rightsquigarrow$   $\uparrow$  Application domains

$$\begin{aligned} \gamma_n(f) &= \gamma_0(1) \mathbb{E}_{\eta_0} \left( f(X_n) \prod_{0 \leq p < n} G_p(X_p) \right) \\ &\propto \int \eta_0(dx_0) G_0(x_0) M_1(x_0, dx_1) \dots G_{n-1}(x_{n-1}) M_n(x_{n-1}, dx_n) f(x_n) \end{aligned}$$

## Spatial Branching models (time index $n \in \mathbb{N}$ , state spaces $E_n$ )

- **Another formulation** :  $Q_n(x_{n-1}, dx_n)$  positive integral operator

Potential branching rule = local mass  $G_n(x) = Q_{n+1}(x_n, E_{n+1})$

and the Markov exploration

$$M_n(x_{n-1}, dx_n) = \frac{Q_{n+1}(x_n, dx_{n+1})}{Q_{n+1}(x_n, E_{n+1})}$$

- **Example 1:**

$$Q_n(x_{n-1}, dx_n) = \underbrace{B_n(x_{n-1}, dx_n)}_{\text{spawning}} + \underbrace{e_n(x_{n-1})}_{\text{survival probab.}} \underbrace{P_n(x_{n-1}, dx_n)}_{\text{target motion}}$$

- $\neq$  **branching model** BUT the same first moment recursion

$$\gamma_{n+1}(f) = \gamma_n(Q_{n+1}(f)) \quad \text{with} \quad Q_{n+1}(f)(x) := \int Q_{n+1}(x, dx') f(x')$$

## Spatial Branching models (time index $n \in \mathbb{N}$ , state spaces $E_n$ )

- Standard notation:

$$Q_n(x_{n-1}, dx_n) = \underbrace{Q_{n+1}(x_n, E_{n+1})}_{G_n(x_n)} \times \underbrace{\frac{Q_{n+1}(x_n, dx_{n+1})}{Q_{n+1}(x_n, E_{n+1})}}_{M_{n+1}(x_n, dx_{n+1})}$$

↓

$$\gamma_{n+1} = \gamma_n Q_{n+1} \Leftrightarrow \gamma_{n+1}(dx) = \int \gamma_n(dx') Q_{n+1}(x', dx)$$

- **A more general model:** Poisson spontaneous births  $\sim \mu_{n+1}$  positive measure

↓

$$\gamma_{n+1} = \gamma_n Q_{n+1} + \mu_{n+1}$$

## Some problems

- **Problem 1:** Mass process  $\gamma_n(1)$  "unstable"  $\gamma_n(1) \uparrow \infty$  or  $\gamma_n(1) \downarrow 0$  as  $n \uparrow \infty$
- **Problem 2:**  $\mathcal{X}_n = \sum_{i=1}^{N_n} \delta_{X_n^i}$  generally **NOT POISSON** random field.
- **Problem 3:**  $\exists$  non generate numerical sampling method?
- **Problem 4:**  $\exists$  non generate approximation of  $\gamma_n$ ?

## Some problems

- **Problem 5:** Filtering problems  $\rightsquigarrow$  Law  $(\mathcal{X}_n \mid \mathcal{Y}_n)$  ?

$$\mathcal{X}_n \rightsquigarrow \text{observation measure : } \mathcal{Y}_n := \sum_{i=1}^{N'_n} \delta_{Y_n^i}$$

Using "Poisson Approximations"  $\Rightarrow G_n \rightsquigarrow G_{n,\gamma_n}$

$$G_{n,\gamma_n}(x) = (1 - d_n(x)) + d_n(x) \sum_{y \in \mathcal{Y}_n} \frac{g_{n,y}(x)}{\kappa_n(y) + \gamma_n(d_n g_{n,y})}$$

*Some references :*

*Mahler (03), Vo-Singh-Doucet (05), Clark-Vo-Bell (05-06),  
Johansen-Singh-DouceVo (06) , Singh-Vo-Baddeley-Zuyev (07)*

- **Problem 5.1:**  $\gamma_n$ -dependent  $\rightsquigarrow$  stability problems?
- **Problem 5.2:** "Conditional distributions?"  $\gamma_n$ ?
- **Problem 5.3:** Useful estimates (particle/smc)? "sharp" expo. bounds.
- ....

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## branching distribution flows

- **Problem 1:** Mass process  $\gamma_n(1)$  "unstable"  $\gamma_n(1) \uparrow \infty$  or  $\gamma_n(1) \downarrow 0$  as  $n \uparrow \infty$
- **Solution 1**  $\rightsquigarrow$  **Normalized model** :

$$\eta_n(f) := \frac{\gamma_n(f)}{\gamma_n(1)} \rightsquigarrow \eta_n = \text{probability measure}$$

- **Normalized constants:**

$$\gamma_{n+1}(f) = \gamma_n(G_n M_{n+1}(f)) \Rightarrow \gamma_{n+1}(1) = \gamma_n(G_n) = \frac{\gamma_n(G_n)}{\gamma_n(1)} \gamma_n(1)$$

$\Downarrow$

**Product formula :**

$$\gamma_{n+1}(1) = \eta_n(G_n) \gamma_n(1) \Rightarrow \gamma_{n+1}(1) = \prod_{0 \leq p \leq n} \eta_p(G_p)$$

## Mean field interpretation

- Normalized model recursion:

$$\eta_n = \Phi_n(\eta_{n-1})$$

- Nonlinear Markov models** : Always  $\exists$  Markov process  $\bar{X}_n$  s.t.

$$\eta_n = \text{Law}(\bar{X}_n) \quad \text{The perfect stochastic algorithm!}$$

with

$$\mathbb{P}(\bar{X}_n \in dx_n \mid \bar{X}_{n-1}) = K_{n, \eta_{n-1}}(\bar{X}_{n-1}, dx_n) \quad \text{and} \quad \eta_{n-1} = \text{Law}(\bar{X}_{n-1})$$

$\Downarrow$

$$\eta_n(dx) = \int \eta_{n-1}(dx') K_{n, \eta_{n-1}}(x', dx)$$

## Mean field particle interpretation

- Markov chain  $\xi_n = (\xi_n^1, \dots, \xi_n^N) \in E_n^N$  s.t.

$$\eta_n^N := \frac{1}{N} \sum_{1 \leq i \leq N} \delta_{\xi_n^i} \simeq_{N \uparrow \infty} \eta_n$$

- Particle approximation transitions ( $\forall 1 \leq i \leq N$ )

$$\xi_{n-1}^i \rightsquigarrow \xi_n^i \sim K_{n, \eta_{n-1}^N}(\xi_{n-1}^i, dx_n)$$

- Unnormalized models :

$$\gamma_{n+1}(1) = \prod_{0 \leq p \leq n} \eta_p(G_p) \simeq_{N \uparrow \infty} \gamma_{n+1}^N(1) := \prod_{0 \leq p \leq n} \eta_p^N(G_p) \quad (\text{Unbias})$$

and the first "unnormalized" moments

$$\gamma_{n+1}(f) = \gamma_{n+1}(1) \eta_{n+1}(f) \simeq_{N \uparrow \infty} \gamma_{n+1}^N(f) = \gamma_{n+1}^N(1) \eta_{n+1}^N(f) \quad (\text{Unbias})$$

Schematic picture :  $\xi_n \in E_n^N \rightsquigarrow \xi_{n+1} \in E_{n+1}^N$

$$\begin{array}{ccc}
 \xi_n^1 & \xrightarrow{K_{n+1, \eta_n^N}} & \xi_{n+1}^1 \\
 \vdots & & \vdots \\
 \xi_n^i & \longrightarrow & \xi_{n+1}^i \\
 \vdots & & \vdots \\
 \xi_n^N & \longrightarrow & \xi_{n+1}^N
 \end{array}$$

Rationale :

$$\begin{aligned}
 \eta_n^N \simeq_{N \uparrow \infty} \eta_n &\implies K_{n+1, \eta_n} \simeq_{N \uparrow \infty} K_{n+1, \eta_n^N} \\
 &\implies \xi_n^i \text{ almost iid copies of } \bar{X}_n
 \end{aligned}$$

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  - Non asymptotic results
  - Asymptotic results

- **Weak estimates**  $\leftrightarrow$  **Bias estimates** ( $\leftrightarrow$  **Propagations of chaos**)

Law(q particles among N at time n)  $\simeq_{N \uparrow \infty}$  Law(q iid r.v. w.r.t.  $\eta_n$ )

- 1 Total variation =  $\frac{q^2}{N} c(n)$ , Boltzmann entropy =  $\frac{q}{N} c(n)$ .
- 2 **Stable models: uniform estimates w.r.t. time**  $\sup_n c(n) < \infty$ .
- 3 Path space and genealogical tree models  $c(n) = c \times n$ .
- 4 Explicit weak decompositions at any order  $\frac{1}{N^k}$ .

$\hookrightarrow$ http-ref : DM-Patras-Rubenthaler, Coalescent tree based functional representations for some Feynman-Kac particle models, Hal-INRIA (2006).

- **$\mathbb{L}_p$ -mean error bounds** [(2),(3) as above]

$$\sup_{N \geq 1} \sqrt{N} \mathbb{E} \left( \sup_{f_n \in \mathcal{F}_n} |\eta_n^N(f_n) - \eta_n(f_n)|^p \right) \leq b(p) c(n)$$

- **Exponential estimates** [(2) as above & empirical processes  $\sim \mathcal{F}_n$ ]

$$\mathbb{P}(|\eta_n^N(f_n) - \eta_n(f_n)| > \epsilon) \leq c(n) \exp \{-\epsilon^2 N / c(n)\}$$

## Non asymptotic results Unnormalized models

- $\mathbb{L}_p$ -mean error bounds [as before] and (ok in path spaces)

$$\sup_{N \geq 1} \sqrt{N} \mathbb{E} \left( \left[ \frac{\gamma_n^N(f_n)}{\gamma_n(1)} - \eta_n(f_n) \right]^2 \right) \leq c n$$

- Exponential estimates ["conjecture"]

$$\mathbb{P} \left( \left| \frac{\gamma_n^N(f_n)}{\gamma_n(1)} - \eta_n(f_n) \right| > \epsilon \right) \leq c \exp \{ -\epsilon^2 N / c n \}$$

- **Central Limit Theorems** [Sharp  $\mathbb{L}_p$  estimates]

{http-ref : 1999~2004 : DM, Guionnet, Jacod, Ledoux, Tindel}

$$V_n^N(f) := \sqrt{N} [\eta_n^N(f) - \eta_n(f)] \implies V_n(f) = \text{Centered Gaussian r.v.}$$

- 1 **Functional Central Limit Theorems.**  $[\forall d, \forall (f^i)_{1 \leq i \leq d}]$

$$(V_n^N(f^1), \dots, V_n^N(f^d)) \implies (V_n(f^1), \dots, V_n(f^d))$$

- 2 **Unbounded  $\mathbb{L}_2$ -functions  $\oplus$  algebra sets of functions with some growth conditions.**

$\hookrightarrow$  (Path space models) DM, Guionnet. Annals of Applied Probability, Vol. 9, No. 2, 275-297 (1999).

$\hookrightarrow$  (Donsker+explicit variance) DM, Ledoux, Journal of Theoret. Probability, Vol. 13, No. 1, 225-257 (2000).

$\hookrightarrow$  (marginal approx. models) DM, Jacod, The Fields Institute Communications, Ed. T.J. Lyons, T.S. Salisbury, American Mathematical Society, (2002).

- 3 **Donsker type theorems, Berry Esseen type theorems, path spaces,...**

## Large deviations

- **Large deviations principles** [Sharp asymptotic expo estimates]

$$\lim_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{P} (\eta_n^N \notin \mathcal{V}(\eta_n))$$

*Example* :  $\mathcal{V}(\eta_n) = \{\mu : |\eta_n^N(f) - \eta_n(f)| \leq \epsilon\}$  (weak and strong  $\tau$ -topo).

{[http-ref 1998~2004](#) : DM, Dawson, Guionnet, Zajic}