Constructing general dual-feasible functions

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Abstract

Dual-feasible functions have proved to be very effective for generating fast lower bounds and valid inequalities for integer linear programs with knapsack constraints. However, a significant limitation is that they are defined only for positive arguments. Extending the concept of dual-feasible function to the general domain and range $\mathbb{R}$ is not straightforward. In this paper, we propose the first construction principles to obtain general functions with domain and range $\mathbb{R}$, and we show that they lead to non-dominated maximal functions.

Keywords: Integer linear programming; dual-feasible functions; generalization.

1 Introduction

The concept of dual-feasible function was introduced first by Johnson in [7]. Recent studies have shown the utility of these functions for computing efficiently lower bounds and valid inequalities for integer linear optimization problems with knapsack constraints [4, 9, 11, 10]. For many problems involving this type of constraints, dual-feasible functions lead indeed to current state-of-the-art results. Despite these contributions, the functions that are currently known still apply only to very specific domains restricted to positive arguments, thus limiting the applicability of dual-feasible functions.

The first attempt to extend dual-feasible functions to less restricted domains was done in [1]. New functions were proposed for multidimensional domains, leading to bounds and inequalities that consider simultaneously different knapsack constraints that may apply to a given integer linear optimization problem. Because these functions apply directly to the vector packing problem, they were called *vector packing dual-feasible functions*. However, the domain of these multidimensional functions remains confined to positive values. In [13], the authors showed that extending the concept of dual-feasible function to domain and range $\mathbb{R}$ is far from straightforward. Many properties of standard dual-feasible functions are lost in this process, thus complicating the task of deriving strong non-dominated functions.

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In this paper, we introduce the first general construction principles that allow to generate dual-feasible functions with domain and range $\mathbb{R}$. These principles can be seen as general methods defining families of different dual-feasible functions. We show in particular that these principles lead to families of non-dominated dual-feasible functions from which strong lower bounds and inequalities can be obtained for integer linear optimization problems with knapsack constraints and general coefficients. To complete the analysis of these general construction principles, we describe and analyse specific instances of functions obtained from each principle.

In [6], Gomory and Johnson showed how valid inequalities can be obtained from knapsack constraints by analysing corner polyhedra, and how to use them to generate other valid inequalities by interpolation. This idea was further explored by Dash *et al.* in [5]. However, the approaches described in [5] require rational problem data, while the contributions described in this paper do not impose such prerequisites. Furthermore, while the related Theorem 1.5 of [6] does not need rational data, it is a characterisation of subadditive functions on a subgroup of the unit interval with addition modulo 1. This theorem does not explain how to construct these functions, like it is done in our contribution.

In Section 2, we recall some of the definitions related to standard and general dual-feasible functions. The construction principles are introduced and analysed in Section 3. To further illustrate these general construction principles, we introduce in Section 4 specific examples of dual-feasible functions with domain and range $\mathbb{R}$ obtained by applying each procedure. The paper ends with concluding remarks at Section 5.

2 General dual-feasible functions

The vast majority of the dual-feasible functions described in the literature is defined for positive arguments only. Most of the time, these functions are declared on the domain $[0, 1]$, although their extension to the domain $\mathbb{R}^+$ is usually straightforward. The formal definition of these standard dual-feasible functions stands as follows.

Definition 1. *A function* $f : [0, 1] \to [0, 1]$ *is a* dual-feasible function (DFF)*, if for any finite set* $\{x_i \in \mathbb{R}_+ : i \in I\}$ *of nonnegative numbers, it holds that*

$$\sum_{i \in I} x_i \leq 1 \Longrightarrow \sum_{i \in I} f(x_i) \leq 1. \quad (1)$$

We will use the term *general dual-feasible function* to refer to a dual-feasible function whose domain is not restricted to positive arguments, but that considers instead the domain $\mathbb{R}$ of any real value. In [13], we showed that this generalization is far from straightforward. Indeed, several properties that apply to standard functions are lost when one goes into the domain of real arguments. The counterpart is that general dual-feasible functions can be used on problems whose knapsack constraints have general coefficients and hence their applicability increases significantly. General dual-feasible functions are defined as follows.

Definition 2. *A function* $f : \mathbb{R} \to \mathbb{R}$ *is a* general dual-feasible function*, if for any finite set* $\{x_i \in \mathbb{R} : i \in I\}$ *of real numbers, it holds that*

$$\sum_{i \in I} x_i \leq 1 \Longrightarrow \sum_{i \in I} f(x_i) \leq 1. \quad (2)$$

In this context, maximality is an important property that distinguishes dominated from non-dominated functions. In practice, a (general) dual-feasible function f is *maximal*, if there is no other (general) dual-feasible function g with $f(x) \leq g(x)$ for all possible values of x. For the sake of completeness, we recall in the sequel the properties of standard maximal dual-feasible functions from $[0,1] \to [0,1]$. For a function $f : [0,1] \to [0,1]$ to be a maximal dual-feasible function (MDFF), it is necessary and sufficient [4, 9] that f is symmetric:

$$f(x) + f(1-x) = 1 \quad \text{for all} \quad x \in [0, 1/2], \tag{3}$$

$f(0) = 0$, and f satisfies the superadditivity condition:

$$f(x_1 + x_2) \geq f(x_1) + f(x_2) \text{ for all } x_1, x_2 \text{ with } 0 < x_1 \leq x_2 < 1/2 \text{ and } x_1 + x_2 \leq 2/3.$$

For general dual-feasible functions, the symmetry rule (3) stands as follows

$$f(x) + f(1-x) = 1, \quad \text{for all} \quad x \leq 1/2. \tag{4}$$

An example of how standard dual-feasible functions may be different from general dual-feasible functions is the fact that symmetry (4), which is an important property of standard maximal dual-feasible functions, has not to be necessarily fulfilled by a general maximal dual-feasible function. In [12], we stated the conditions for a general dual-feasible function to be maximal. We recall these conditions next.

Theorem 1. *[12] Let $f : \mathbb{R} \to \mathbb{R}$ be a given function.*

(a) If f satisfies the following conditions, then f is a general MDFF:

1. *$f(0) = 0$;*
2. *f is superadditive, i.e., for all $x, y \in \mathbb{R}$, it holds that*

$$f(x+y) \geq f(x) + f(y); \tag{5}$$

3. *there is an $\varepsilon > 0$, such that $f(x) \geq 0$ for all $x \in (0, \varepsilon)$;*
4. *f obeys the symmetry rule (4).*

(b) If f is a general MDFF, then the above properties (1.)–(3.) hold for f, but not necessarily (4.).

(c) If f satisfies the above conditions (1.)–(3.), then f is monotone increasing.

(d) If the symmetry rule (4) holds and f obeys the inequality (5) for all $x, y \in \mathbb{R}$ with $x \leq y \leq \frac{1-x}{2}$, then f is superadditive.

Proof. The proof provided in [12] is repeated here for the sake of clarity. The proof is made in the following order: first, we prove (c), then (b), (a), and finally (d) is proved.

(c) If f satisfies the first three conditions, then for any $x > 0$ it follows that $n := \lfloor x/\varepsilon \rfloor + 1 \in \mathbb{N} \setminus \{0\}$ and $0 < x/n < \varepsilon$. Hence, we have $f(x/n) \geq 0$ and $f(x) \geq n \times f(x/n) \geq 0$. Therefore, the monotonicity follows immediately from $f(x_2) \geq f(x_1) + f(x_2 - x_1)$ for any $x_1, x_2 \in \mathbb{R}$ with $x_1 \leq x_2$.

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be a general MDFF. We prove the properties (1.)–(3.). One has $f(0) \leq 0$ due to the condition for general dual-feasible functions. On the other hand, $f(x) < 0$ for a certain $x \geq 0$ is impossible, because f is maximal and setting $f(x)$ to zero cannot violate the condition for general dual-feasible functions. Assume that $f(x_1 + x_2) < f(x_1) + f(x_2)$ for certain $x_1, x_2 \in \mathbb{R}$. Define a function $g : \mathbb{R} \to \mathbb{R}$ as

$$g(x) := \begin{cases} f(x) & \text{if } x \neq x_1 + x_2 \\ f(x_1) + f(x_2) & \text{otherwise} \end{cases}.$$

Since f is a general MDFF, g must violate the defining condition for a general dual-feasible function. Replacing $g(x_1 + x_2)$ by $g(x_1) + g(x_2)$ and $x_1 + x_2$ by two ones x_1 and x_2 leads to a violation if $x_1, x_2 \neq 0$, because of the definition of g. That is a contradiction.

(a) The converse direction is to prove that if f satistfies the conditions (1.)–(4.), then f is a general MDFF. For any $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in \mathbb{R}$ with $\sum_{i=1}^{n} x_i \leq 1$, the superadditivity condition (2.) yields $\sum_{i=1}^{n} f(x_i) \leq f(\sum_{i=1}^{n} x_i)$. Let $x_0 := 1 - \sum_{i=1}^{n} x_i \geq 0$. Therefore, we have $f(x_0) \geq 0$. Because of $f(1 - x_0) + f(x_0) = 1$, it follows that f is a general dual-feasible function. Let $g : \mathbb{R} \to \mathbb{R}$ be a general dual-feasible function with $g(x) > f(x)$ for a certain $x \in \mathbb{R}$. Since g is a general dual-feasible function, one has $g(1-x) + g(x) \leq 1$. It follows that $g(1-x) \leq 1 - g(x) < 1 - f(x) = f(1-x)$ due to (4), hence g does not dominate f. Therefore, f is a general MDFF.

(d) If $x > 1/2$, then $z := 1 - x < 1/2$, and hence $f(z) + f(1-z) = 1$ due to (4). That implies $f(x) + f(1-x) = 1$. This symmetry will be assumed for the entire remaining proof. The condition $x \leq y \leq \frac{1-x}{2}$ implies $x + y \leq 2/3$ and $x \leq 1/3$, because $x \leq \frac{1-x}{2}$ leads to $3x \leq 1$ and therefore $x + y \leq \frac{1+x}{2} \leq \frac{1+1/3}{2} = \frac{2}{3}$. Obviously, the inequality (5) is valid if and only if it is true after exchanging x against y. Therefore, $x \leq y$ can be enforced without loss of generality. Now we prove that the inequality (5) holds for all $x, y \in \mathbb{R}$, if it is true for all $x, y \in \mathbb{R}$ with $x + y \leq 2/3$. If $x + y > 2/3$, then $y > 1/3$ due to $x \leq y$. Hence, $1 - y < 2/3$ and $f(x) + f(1-y-x) \leq f(1-y)$ according to the inequality (5). The symmetry (4) yields $f(x) + 1 - f(x+y) \leq 1 - f(y)$, and hence $f(x) + f(y) \leq f(x+y)$, as needed. Therefore, $x + y \leq 2/3$ can be assumed in the rest of the proof, and hence $x \leq \frac{1}{3} \leq \frac{1-x}{2}$. If $y > \frac{1-x}{2}$, then let $z := 1 - x - y < \frac{1-x}{2}$. Due to the previous parts of the proof of point (d) and the prerequisites, the superadditivity rule (5) can be used, implying $f(x) + f(z) \leq f(x+z)$. The symmetry rule (4) yields $f(x) + 1 - f(1-z) \leq 1 - f(1-x-z)$, and hence $f(x) + f(1-x-z) = f(x) + f(y) \leq f(1-z) = f(x+y)$. □

It is well known that standard dual-feasible functions generate solutions that are feasible for the dual of instances of the 1-dimensional cutting stock problem. The same happens with general dual-feasible functions for the case where the sizes of the items and the variables of the dual problem are not restricted in sign. Negative sizes may happen when it is possible to use a certain fixed quantity of extra space in containers in limited number (which can be seen as items of negative size).

Negative sizes also occur when balance constraints are considered. For example, let us consider a process assignment problem, where p processes have to be assigned to a minimum number of identical machines. Each process i has a given demand c_i in CPU, and a given demand d_i in RAM. We will assume that all machines have the same CPU and RAM availability C and D. One would like to assign the processes to processors such that the latter are well-balanced, *i.e.* there is not a large quantity of CPU available while the RAM is full. That would be a waste of resources, and would not allow to assign any new process to the machine. Configurations may be represented by a column vector $\mathbf{a}^j \in \mathbb{Z}^p$. The balance constraint can be written as $\mathbf{c}^\top \mathbf{a} - \mathbf{d}^\top \mathbf{a} \leq T$, where $T > 0$ is a threshold above which the machine is not considered well balanced. Using the shortcut $\ell = (1/T)(\mathbf{c} - \mathbf{d})$, the constraint can be rewritten as $\mathbf{l}^\top \mathbf{a} \leq 1$, where the elements of $\mathbf{c} - \mathbf{d}$ can be positive or negative.

Similarly to the cutting-stock problem, this variant can be modelled as follows. Let $\mathbf{b} \in \mathbb{R}^p_+$ and $\mathbf{l} \in \mathbb{R}^p$ be some fixed vectors denoting respectively the demands and the sizes of a set of p items. Each configuration may be represented by a column vector $\mathbf{a}^j \in \mathbb{Z}^p$, and the set of all the patterns by the matrix $\mathbf{A}$. A pattern is valid if it satisfies the capacity constraint $\mathbf{l}^\top \mathbf{a}^j \leq 1$. The continuous relaxation of this variant of the cutting stock problem stands as follows

$$\min \sum_{j=1}^{n} x_j \text{ s.t. } \mathbf{A}\mathbf{x} = \mathbf{b},\ \mathbf{x} \in \mathbb{Z}^n_+. \tag{6}$$

Obviously, overproduction cannot be allowed, otherwise if one item has a negative size, a trivial solution with one bin would always be possible. Therefore, the demand constraints must be satisfied as equalities, and the dual of (6) has variables unrestricted in sign

$$\max\ \mathbf{b}^\top \mathbf{u} \text{ s.t. } \mathbf{u} \in \mathbb{R}^p, \mathbf{u}^\top \mathbf{a}^j \leq 1,\ j = 1, \ldots, n,$$

whose solutions can be obtained through general dual-feasible functions $f : \mathbb{R} \to \mathbb{R}$ as follows

$$u_i := f(\ell_i),\ i = 1, \ldots, p. \tag{7}$$

3 Construction principles

In this section, we describe the first general procedures to build non-dominated dual-feasible functions for general domains and ranges, and we provide the proofs that these procedures lead indeed to maximal functions. To simplify the presentation, we will use the abbreviation $\mathsf{frac}(\cdot)$ to denote the non-integer part of a real expression, *i.e.*, $\mathsf{frac}(x) \equiv x - \lfloor x \rfloor$.

3.1 Principle I

Proposition 1 describes how to extend a maximal dual-feasible function $g : [0, 1] \to [0, 1]$ to domain and range $\mathbb{R}$ as a sum of a periodic and a monotone staircase function.

Proposition 1. *Let $g : [0, 1] \to [0, 1]$ be a MDFF, and $b \in \mathbb{R}$ such that*

$$b \geq b_0 := \sup\{g(x) + g(y) - g(x + y - 1) | x, y \in [0, 1] \wedge x + y \geq 1\}. \tag{8}$$

Then $1 \leq b_0 \leq 2$, and g can be extended to a general MDFF $f : \mathbb{R} \to \mathbb{R}$ as follows

$$f(x) := \begin{cases} g(\mathsf{frac}(x)) + b * \lfloor x \rfloor, & \textit{if } x < 1, \\ 1 - f(1 - x), & \textit{otherwise.} \end{cases}$$

Proof. For $x \in [0, 1)$, the definition of f implies immediately $f(x) = g(x)$. Since g is a MDFF with domain $[0, 1]$, it follows that $g(0) = 0$ and $g(1) = 1 = f(1)$, such that f is really an extension of g and fulfills the conditions (1.) and (3.) in part (a) of Theorem 1. Furthermore, g is also symmetric. One has for all $x, y \in [0, 1]$ with $x + y \geq 1$ that $g(x) \leq 1$, $g(y) \leq 1$ and $g(x + y - 1) \geq 0$, and hence, $b_0 \leq 2$. Since $g(1/2) = 1/2$ and $g(0) = 0$, setting $x := y := 1/2$ in (8) yields $b_0 \geq 1/2 + 1/2 - 0 = 1$. By construction, f is symmetric. Only the superadditivity remains to be proved, *i.e.*, it must be verified according to part (d) of Theorem 1 that $h(x, y) := f(x+y) - f(x) - f(y)$ is not negative for any $x, y \in \mathbb{R}$ with $x \leq y \leq \frac{1-x}{2}$. The latter inequality chain implies $3x \leq 1$ and $x + y \leq \frac{1+x}{2} \leq \frac{2}{3}$. Some cases and subcases must be distinguished.

1. $y < 1$ yields $h(x, y) = g(\mathsf{frac}(x + y)) - g(\mathsf{frac}(x)) - g(\mathsf{frac}(y)) + b * (\lfloor x + y \rfloor - \lfloor x \rfloor - \lfloor y \rfloor)$.
   (a) If $\mathsf{frac}(x) + \mathsf{frac}(y) < 1$, then $\mathsf{frac}(x + y) = \mathsf{frac}(x) + \mathsf{frac}(y)$ and $\lfloor x + y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor$, and hence $h(x, y) = g(\mathsf{frac}(x) + \mathsf{frac}(y)) - g(\mathsf{frac}(x)) - g(\mathsf{frac}(y)) \geq 0$ due to the superadditivity of g.
   (b) If $\mathsf{frac}(x) + \mathsf{frac}(y) \geq 1$, then $h(x, y) = g(\mathsf{frac}(x) + \mathsf{frac}(y) - 1) - g(\mathsf{frac}(x)) - g(\mathsf{frac}(y)) + b * 1 \geq 0$.
2. $y \geq 1$ yields $h(x, y) = g(\mathsf{frac}(x + y)) - g(\mathsf{frac}(x)) - 1 + g(\mathsf{frac}(1 - y)) + b * (\lfloor x + y \rfloor - \lfloor x \rfloor + \lfloor 1 - y \rfloor)$.
   (a) If $y \in \mathbb{N}$, then $\mathsf{frac}(x + y) = \mathsf{frac}(x)$ and $\lfloor x + y \rfloor = \lfloor x \rfloor + y$, and finally $h(x, y) = -1 + b \geq 0$.
   (b) If $y \notin \mathbb{N}$ and $\mathsf{frac}(x) + \mathsf{frac}(y) < 1$, then $\lfloor x + y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor$, $\lfloor 1 - y \rfloor = -\lfloor y \rfloor$, $\mathsf{frac}(x + y) = \mathsf{frac}(x) + \mathsf{frac}(y)$ and $g(\mathsf{frac}(1 - y)) - 1 = g(1 - \mathsf{frac}(y)) - 1 = -g(\mathsf{frac}(y))$. Hence, $h(x, y) = g(\mathsf{frac}(x) + \mathsf{frac}(y)) - g(\mathsf{frac}(x)) - g(\mathsf{frac}(y)) + b * 0 \geq 0$ due to the superadditivity of g.

(c) If $\mathsf{frac}(x) + \mathsf{frac}(y) \geq 1$, then $\lfloor x+y \rfloor - \lfloor x \rfloor = \lfloor y \rfloor + 1$, $\lfloor 1-y \rfloor = -\lfloor y \rfloor$ and $g(\mathsf{frac}(1-y)) - 1 = -g(\mathsf{frac}(y))$, such that $h(x,y) = g(\mathsf{frac}(x) + \mathsf{frac}(y) - 1) - g(\mathsf{frac}(x)) - g(\mathsf{frac}(y)) + b * 1 \geq 0$.

□

Of course, if $b > b_0$, then f will not be extreme, *i.e.*, f will be a non-trivial convex combination of MDFFs. To see this, we may use the proposition with b_0 and $b_1 := 2b - b_0 > b_0$ to get the non-identical functions f_0, f_1. One obtains for any $x < 1$ that

$$\begin{aligned} 2f(x) - f_0(x) - f_1(x) &= 2 * (g(\mathsf{frac}(x)) + b * \lfloor x \rfloor) - g(\mathsf{frac}(x)) - b_0 * \lfloor x \rfloor - g(\mathsf{frac}(x)) - b_1 * \lfloor x \rfloor \\ &= \lfloor x \rfloor * (2b - b_0 - b_1) \\ &= 0. \end{aligned}$$

If $x \geq 1$, then the symmetry leads to the same result.

3.2 Principle II

The next proposition gives the possibility to extend a Lipschitz-continuous MDFF $g : [0,1] \to [0,1]$ to domain and range $\mathbb{R}$. A function $f : X \subseteq \mathbb{R} \to \mathbb{R}$ is Lipschitz-continuous, if there is a constant $L \geq 0$ with $|f(x) - f(y)| \leq L * |x - y|$ for all $x, y \in X$. Note that Lipschitz-continuity implies the usual continuity, while the converse does not hold.

Proposition 2. *Let $p, t \in \mathbb{R}$ and $g : [0,1] \to [0,1]$ be a MDFF with $|g(x) - g(y)| \leq t * |x - y|$ for all $x, y \in [0,1]$, i.e., the function g is Lipschitz-continuous with $L := t$. Then, we have $t \geq 1$, and, for $1 \leq p \leq t$, the following function $f : \mathbb{R} \to \mathbb{R}$ is a general MDFF:*

$$f(x) := \begin{cases} tx + 1 - p, & \text{if } x < 0, \\ tx + p - t, & \text{if } x > 1, \\ g(x), & \text{otherwise.} \end{cases}$$

Proof. Since g is a MDFF with domain and range $[0,1]$, it is necessarily superadditive, symmetric and non-decreasing. In particular, one has $g(0) = 0$, and hence, $g(x) \leq tx$ for all $x \in [0,1]$ due to the Lipschitz-condition. The symmetry of g implies $g(1) = 1$. Therefore, we have $t \geq 1$, and $t * (1 - x) \geq g(1 - x) = 1 - g(x)$, or equivalently

$$g(x) \geq tx + 1 - t, \text{ for all } x \in [0,1]. \tag{9}$$

The proof is based on Theorem 1. Its conditions (1.), (3.) and (4.) are obviously fulfilled, because for $x < 0$ one gets $f(x) + f(1-x) = tx + 1 - p + t * (1 - x) + p - t = 1$. To show the superadditivity, let $x, y \in \mathbb{R}$ with $x \leq y \leq \frac{1-x}{2}$ be given. Hence, we have $x \leq 1/3$ and $x + y \leq \frac{1+x}{2} \leq 2/3$. It must be verified that $d := f(x + y) - f(x) - f(y)$ is not negative. The following six cases have to be checked.

1. $x \geq 0$: then $x, y, x + y \in [0,1]$ and $d = g(x + y) - g(x) - g(y) \geq 0$ due to the superadditivity of g.

2. $y < 0$: then $x < 0$ and $x + y < 0$, and hence, $d = t*(x+y)+1-p-tx+p-1-ty+p-1 = p-1 \geq 0$.

3. $x + y < 0 \leq y \leq 1$: then $x < 0$ and $d = t * (x + y) + 1 - p - tx - 1 + p - g(y) = ty - g(y) \geq 0$.

4. $0 \leq x + y < y \leq 1$: then $x < 0$. The Lipschitz-condition yields

$$|g(x + y) - g(y)| = g(y) - g(x + y) \leq t * |x| = -tx,$$

and hence, $d = g(x + y) - g(y) - tx - 1 + p \geq 0$.

5. $x + y < 0$ and $y > 1$: then $x < 0$ and $d = t * (x + y) + 1 - p - tx - 1 + p - ty - p + t = t - p \geq 0$.
6. $0 \leq x+y < 1 < y$: then $x < 0$ and $d = g(x+y)-tx-1+p-ty-p+t = g(x+y)-t*(x+y)-1+t \geq 0$.

□

The Lipschitz-continuity cannot be weakened to the more general Hölder-continuity, otherwise the superadditivity of f can be lost in the previous case (4.).

3.3 Principle III

The construction principle based on Proposition 2 relies on Lipschitz-continuous MDFFs with domain and range $[0, 1]$. It is possible to overcome the constraint related to Lipschitz-continuity at the cost of losing some generality in the construction (now, only one parameter applies), as shown in the following proposition.

Proposition 3. *Let $t \in \mathbb{R}$ and $g : [0, 1] \to [0, 1]$ be a MDFF with $g(x) \leq tx$ for all $x \in [0, 1]$. Then, we have $t \geq 1$, and the following function $f : \mathbb{R} \to \mathbb{R}$ is a general MDFF:*

$$f(x) := \begin{cases} tx + 1 - t, & \text{if } x < 0, \\ tx, & \text{if } x > 1, \\ g(x), & \text{otherwise.} \end{cases}$$

Proof. The proof is very similar to the one of Proposition 2 with $p := t$, except for the case (4.). Therefore, only the necessary change in case (4.) of the superadditivity proof is given here. We have $0 \leq x + y < y \leq 1$, and hence $x < 0$. Then (9) yields

$$d = g(x + y) - g(y) - tx - 1 + t \geq t * (x + y) + 1 - t - g(y) - tx - 1 + t = ty - g(y) \geq 0.$$

□

Note that every MDFF $g : [0, 1] \to [0, 1]$ fulfills the prerequisites of this proposition for $t \geq 2$. This is obvious for $x \in [\frac{1}{2}, 1] \cup \{0\}$, because $g(0) = 0$ and $g(x) \leq 1$. If $0 < x < \frac{1}{2}$, then we refer to point 1 of Lemma 3 in [10], which states that for MDFFs $f_0, g, h_0 : [0, 1] \to [0, 1]$ with $2f_0(x) = g(x) + h_0(x)$ for all $x \in [0, 1]$, it holds that $f_0(x) \leq 1/\lfloor 1/x \rfloor$, for all $x \in (0, 1]$. Our function g obeys the prerequisites of that lemma. Therefore, we have

$$g(x) \leq 1/\lfloor \frac{1}{x} \rfloor < 1/(\frac{1}{x} - 1) = \frac{x}{1 - x} < 2x.$$

4 Examples

In this section, we illustrate the results described above by providing specific maximal general dual-feasible functions built using these construction principles.

4.1 From principle I

In the following proposition, we describe how to compute the value of b_0 referred to in Proposition 1 for different functions g. For each function g, we obtain a specific family of general dual-feasible functions built according to Principle I.

Proposition 4. *Let $g : [0, 1] \to [0, 1]$ be a MDFF and $h(x, y) := g(x + y) - g(x) - g(y)$ with $x, y \in [0, 1]$ and $x + y \leq 1$. Then, the following holds:*

1. $b_0 = 1 + \sup\{h(x, y) : 0 < x \le y < 1/2 \wedge x + y \le 2/3\}$.

2. If $g \equiv f_{MT,0}(\cdot;\lambda) : [0, 1] \to [0, 1]$*, proposed implicitly in [8], with* $\lambda \in (0, 1/2)$*, i.e.,*

$$g(x) = \begin{cases} 0, & \text{if } x < \lambda, \\ x, & \text{if } \lambda \le x \le 1-\lambda, \\ 1, & \text{if } x > 1-\lambda, \end{cases}$$

then $b_0 = 2$ *for* $\lambda > 1/3$ *and* $b_0 = 2\lambda + 1$ *for* $\lambda \le 1/3$.

3. If $g \equiv f_{BJ,1}(\cdot; C) : [0, 1] \to [0, 1]$*, proposed in [2], with* $C \ge 1$*, i.e.,*

$$g(x) = \frac{1}{\lfloor C \rfloor} * \left(\lfloor Cx \rfloor + \max\left\{0, \frac{\mathsf{frac}(Cx) - \mathsf{frac}(C)}{1 - \mathsf{frac}(C)}\right\}\right),$$

then $b_0 = 1 + \min\{1, \frac{\mathsf{frac}(C)}{1-\mathsf{frac}(C)}\}/\lfloor C \rfloor$.

4. If $g \equiv f_{CCM,1}(\cdot; C) : [0, 1] \to [0, 1]$*, proposed in [3], with* $C \ge 1$*, i.e.,*

$$g(x) = \begin{cases} \lfloor Cx \rfloor / \lfloor C \rfloor, & \text{if } x < 1/2, \\ 1/2, & \text{if } x = 1/2, \\ 1 - g(1-x), & \text{if } x > 1/2, \end{cases}$$

then $b_0 = 2$ *if* $C < 3$ *and* $b_0 = 1 + 2/\lfloor C \rfloor$ *if* $C \ge 3$.

Proof.
1. Let $u := 1 - y$ and $v := x + y - 1$. That implies

$$x, y \in [0, 1] \;\wedge\; x + y \ge 1 \iff u, v \in [0, 1] \wedge u + v \le 1.$$

The symmetry of g yields $g(x)+g(y)-g(x+y-1) = g(x)+1-g(1-y)-g(v) = 1+g(u+v)-g(u)-g(v)$, and hence $b_0 = 1+\sup\{h(u, v) : u, v \in [0, 1], u+v \le 1\}$. It has to be shown that the supremum expression is the same as $\sup\{h(x, y) : 0 < x \le y < 1/2 \wedge x + y \le 2/3\}$ with the apparent stronger restrictions to the bound variables. That can be done like in Theorem 2 of [9] or part (d) of Theorem 1. It will be shown step by step, how to get the stronger restrictions. If one of them is violated, then the supremum expression is replaced by an equivalent one, where the special restriction holds.

Assume without loss of generality that $u \le v$. Since $u, v \in [0, 1]$ and $u + v \le 1$, it follows that $0 \le u \le v \le 1 - u$, and hence $u \le 1/2$. Since g is a MDFF, the following holds.

- $h(0, v) = g(v) - g(0) - g(v) = 0 \le h(u, v)$ for any feasible u, v due to the superadditivity of g. Therefore, the constraint $u, v \in [0, 1] \wedge u + v \le 1$ may be sharpened to $0 < u \le v \le 1 - u$, *i.e.*,

$$\sup\{h(u, v) : u, v \in [0, 1], u + v \le 1\} = \sup\{h(u, v) : 0 < u \le v \le 1 - u\}.$$

- If $1/2 \le v \le 1$ and $u > 0$, then $1 - u - v < 1/2$. Therefore, $h(u, v) = g(u + v) - g(u) - g(v) = 1 - g(1 - u - v) - g(u) - 1 + g(1 - v) = h(u, 1 - u - v)$ implies $\sup\{h(u, v) : 0 < u \le v \le 1 - u\} = \sup\{h(u, v) : 0 < u \le v < 1/2\}$.

- If $0 < u \le v < 1/2$ and $u + v > 2/3$, then $1 - u - v < 1/3$ and $1 - v < 2/3$ due to $u \le v$ and $u+v > 2/3$. One has again $h(u, v) = h(u, 1-u-v)$, such that replacing v by $1-u-v$ and eventually exchanging u and v after that leads to the desired restrictions in the supremum expression.

Renaming u, v to x, y yields the proposition.
2. If $\lambda > 1/3$, then $h(\frac{1}{3}, \frac{1}{3}) = 1$. If $\lambda \leq 1/3$, then one has for $\varepsilon \in (0, \lambda/2)$ that $\lambda < 2\lambda - 2\varepsilon < 1 - \lambda$, and hence

$$h(\lambda - \varepsilon, \lambda - \varepsilon) = g(2\lambda - 2\varepsilon) - 2g(\lambda - \varepsilon) = 2\lambda - 2\varepsilon - 2 * 0 \to 2\lambda,$$

for $\varepsilon \downarrow 0$. Suppose according to point **1.** that there are x, y with $0 < x \leq y < 1/2$, $x + y \leq 2/3$ and $h(x, y) > 2\lambda$, and hence, $g(y) < g(x + y) \leq x + y$. If $g(y) = 0$, then $y < \lambda$. In that case, we have $x + y < 2\lambda$, which leads to a contradiction. Therefore, $\lambda \leq y < 1/2$, $g(y) = y$ and $h(x, y) \leq x - g(x) < \lambda$, which is again a contradiction. Hence, $b_0 = 1 + 2\lambda$.
3. Since $\mathsf{frac}(C) < \lfloor C \rfloor$, it follows that $2 * \mathsf{frac}(C) < C$ and

$$\begin{aligned} h(\mathsf{frac}(C)/C, \mathsf{frac}(C)/C) &= g(2 * \mathsf{frac}(C)/C) - 2 * 0 \\ &= \left(\lfloor 2 * \mathsf{frac}(C) \rfloor + \max\left\{0, \frac{\mathsf{frac}(2 * \mathsf{frac}(C)) - \mathsf{frac}(C)}{1 - \mathsf{frac}(C)}\right\} \right) / \lfloor C \rfloor \\ &= \min\left\{1, \frac{\mathsf{frac}(C)}{1 - \mathsf{frac}(C)}\right\} / \lfloor C \rfloor. \end{aligned}$$

To verify the last equation, the two cases $0 \leq \mathsf{frac}(C) < 1/2$ and $1/2 \leq \mathsf{frac}(C) < 1$ must be distinguished.

One has $g(z + p/C) = g(z) + p/\lfloor C \rfloor$ for all $p \in \mathbb{Z}$ and $z \in [0, 1]$ with $z + p/C \in [0, 1]$, and hence, $h(x, y)$ does not change if x or y is changed inside the domain of h by p/C with $p \in \mathbb{Z}$. Therefore, it is enough to discuss h in an environment of $x = y = \mathsf{frac}(C)/C$. If x or y is decreased by not more than $\mathsf{frac}(C)/C$, then $g(x)$ or $g(y)$ will not change, but $g(x + y)$ may fall. If x or y is increased a bit then $g(x)$ or $g(y)$ raises, but $g(x + y)$ will not increase more than $g(x)$ or $g(y)$. Therefore, the maximum value of $h(x, y)$ is obtained at $x = y = \mathsf{frac}(C)/C$.
4. If $C < 3$, then $h(\frac{1}{3}, \frac{1}{3}) = 1$, and hence, $b_0 = 2$. Therefore, assume $C \geq 3$ in the following. According to point **1.**, let $0 < x \leq y < 1/2$. Three cases remain (recall that $\lfloor a + b \rfloor - \lfloor a \rfloor - \lfloor b \rfloor \in \{0, 1\}$ for all $a, b \in \mathbb{R}$).

1. $x + y < 1/2$: $h(x, y) = (\lfloor Cx + Cy \rfloor - \lfloor Cx \rfloor - \lfloor Cy \rfloor)/\lfloor C \rfloor \in \{0, 1/\lfloor C \rfloor\}$.
2. $x + y = 1/2$: $h(x, y) = 1/2 - (\lfloor Cx \rfloor + \lfloor Cy \rfloor)/\lfloor C \rfloor \in \{1/2 - \lfloor C/2 \rfloor/\lfloor C \rfloor, 1/2 - \lfloor C/2 - 1 \rfloor/\lfloor C \rfloor\}$.
3. $x + y > 1/2$: $h(x, y) = 1 - (\lfloor C - Cx - Cy \rfloor + \lfloor Cx \rfloor + \lfloor Cy \rfloor)/\lfloor C \rfloor \in \{0, 1/\lfloor C \rfloor, 2/\lfloor C \rfloor\}$.

An upper bound for $h(x, y)$ is $\max\{2/\lfloor C \rfloor, 1/2 - \lfloor C/2 - 1 \rfloor/\lfloor C \rfloor\}$. Since $C \geq 3$, the last partial expression can be estimated to be not above

$$\frac{1}{2} - \frac{\lfloor C/2 \rfloor - 1}{2\lfloor C/2 \rfloor + 1} = \frac{3}{4\lfloor C/2 \rfloor + 2} \leq \frac{3}{2\lfloor C \rfloor} < \frac{2}{\lfloor C \rfloor}.$$

It remains to show the sharpness of this bound. Choose for this purpose an $\varepsilon \in (0, \frac{1 - \mathsf{frac}(C)}{4})$ and $x := y := \frac{\lfloor C/2 \rfloor - \varepsilon}{C} < \frac{1}{2}$. Then, $x, y > 1/4$, because if $3 \leq C < 4$, then $x = \frac{1 - \varepsilon}{C} > \frac{3 + \mathsf{frac}(C)}{4C} = \frac{1}{4}$. If $4 \leq C < 5$, then $x = \frac{2 - \varepsilon}{C} > \frac{7}{20}$. Finally, if $C \geq 5$, then $x > \frac{C/2 - 1 - \varepsilon}{C} > \frac{2C - 4 - 1 + \mathsf{frac}(C)}{4C} \geq \frac{C}{4C}$. One has $x + y > 1/2$ and

$$\begin{aligned} h(x, x) &= 1 - (\lfloor C - 2Cx \rfloor + 2\lfloor Cx \rfloor)/\lfloor C \rfloor \\ &= 1 - (\lfloor C - 2\lfloor C/2 \rfloor + 2\varepsilon \rfloor + 2\lfloor \lfloor C/2 \rfloor - \varepsilon \rfloor)/\lfloor C \rfloor \\ &= 1 - (\lfloor C + 2\varepsilon \rfloor - 2)/\lfloor C \rfloor = 2/\lfloor C \rfloor. \end{aligned}$$

□

4.2 From principle II

The following general MDFFs $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ are based on Proposition 2 by using $g(x) \equiv x$ and $p \in \{1, t\}$.

$$f_1(x) = \begin{cases} tx, & \text{if } x < 0, \\ tx + 1 - t, & \text{if } x > 1, \\ x, & \text{otherwise;} \end{cases} \qquad f_2(x) = \begin{cases} tx + 1 - t, & \text{if } x < 0, \\ tx, & \text{if } x > 1, \\ x, & \text{otherwise.} \end{cases}$$

We proved in [13] that $t := \sup\limits_{x>0} \frac{f(x)}{x}$ yields $\lim\limits_{x\to\infty} \frac{f(x)}{x} = t \leq -f(-1)$ and $f(x) \leq tx$ for any general MDFF $f : \mathbb{R} \to \mathbb{R}$ and $x \in \mathbb{R}$. These inequalities may be strict, as the above examples show for $t > 1$, namely $f_1(x)/x < \sup\{f_1(y)/y : y > 0\}$ for all $x > 0$ and $t < -f_2(-1)$.

Proposition 2 can be used with the aforementioned function $f_{BJ,1}(\cdot; C)$, because this function is Lipschitz-continuous for all $C \geq 1$. The function $f_{BJ,1}(\cdot; C)$ is also one of the strongest standard dual-feasible functions as shown in [11]. The smallest valid Lipschitz-constant for this function is its largest slope, namely $t_1 := \frac{C}{\lfloor C \rfloor * (1 - \mathsf{frac}(C))}$. Therefore, a general MDFF $f : \mathbb{R} \to \mathbb{R}$ is obtained for all $t \geq t_1$ and $p \in [1, t]$ by

$$f(x) := \begin{cases} tx + 1 - p, & \text{if } x < 0, \\ tx + p - t, & \text{if } x > 1, \\ \left(\lfloor Cx \rfloor + \max\left\{0; \frac{\mathsf{frac}(Cx) - \mathsf{frac}(C)}{1 - \mathsf{frac}(C)}\right\} \right) / \lfloor C \rfloor, & \text{otherwise.} \end{cases}$$

4.3 From principle III

The weaker prerequisites of Proposition 3 allow the application to any MDFF $g : [0, 1] \to [0, 1]$ for all $t \geq t_0 := \sup\{g(x)/x : 0 < x < 1\}$. The value of t_0 is calculated in the sequel for some MDFFs $g : [0, 1] \to [0, 1]$ mentioned above.

Proposition 5. *Let $g : [0, 1] \to [0, 1]$ be a MDFF and $t_0 := \sup\{g(x)/x : 0 < x < 1\}$. Then the following is true:*

1. *If $g \equiv f_{BJ,1}(\cdot; C)$ with $C \geq 1$, then $t_0 = C/\lfloor C \rfloor = 1 + \mathsf{frac}(C)/\lfloor C \rfloor$.*
2. *If $g \equiv f_{MT,0}(\cdot; \lambda)$ with $\lambda \in (0, 1/2)$, then $t_0 = \frac{1}{1-\lambda}$.*
3. *If $g \equiv f_{CCM,1}(\cdot; C)$ with $C \geq 1$, then $t_0 = \max\left\{ \frac{C}{\lfloor C \rfloor} * \frac{\lfloor C \rfloor + 2 - \lceil C/2 \rceil}{C + 1 - \lceil C/2 \rceil}, 2 + \frac{2}{\lfloor C \rfloor} * \left(1 - \lceil \frac{C}{2} \rceil\right) \right\}$.*

Proof.
1. In the intervals where g is constant, only the limit of $g(x)/x$ at the left side may play a role for t_0. In the other intervals, g has a constant slope $\frac{C}{\lfloor C \rfloor * (1 - \mathsf{frac}(C))} \geq t_0$. Therefore, in these intervals only the limit of $g(x)/x$ at the right side needs to be checked. The interesting points are therefore $x_k = k/C$ with $k \in \mathbb{N}$, $0 < k \leq C$. One obtains $g(x_k)/x_k = \frac{k}{\lfloor C \rfloor * x_k} = C/\lfloor C \rfloor$.
2. One has $g(x)/x = 0$ for $x < \lambda$, $g(x)/x = 1$ for $\lambda \leq x \leq 1 - \lambda$ and $g(x)/x = 1/x$ for $x > 1 - \lambda$.
3. Since g is a staircase function, only the right limits of $g(x)/x$ at the points with discontinuity need to be explored. For $x < 1/2$, one gets $f_{CCM,1}(x)/x \leq C/\lfloor C \rfloor$. For $x > 1/2$, one gets $g(x) = 1 - \lfloor C - Cx \rfloor / \lfloor C \rfloor$. If $C \leq 2$, then $g(x) = 1$ for all $x > 1/2$, and hence, $t_0 = 2$, which also fits the formula in the proposition. Therefore, assume $C > 2$ for the rest of the proof. Discontinuities arise at $x_k = 1 - k/C$ with $k \in \mathbb{N}$, $0 < k < C/2$. It follows that

$$\lim_{x \downarrow x_k} \frac{g(x)}{x} = \left(1 - \frac{k-1}{\lfloor C \rfloor}\right) / x_k = \frac{C}{\lfloor C \rfloor} * \frac{\lfloor C \rfloor + 1 - k}{C - k} = \frac{C}{\lfloor C \rfloor} * \left(1 + \frac{1 - \mathsf{frac}(C)}{C - k}\right).$$

The last expression is greater than in the case $x < 1/2$, and it reaches its maximum for the largest possible k, *i.e.*, for $k = \lceil \frac{C}{2} \rceil - 1$. The environment of $1/2$ yields $\lim\limits_{x \downarrow 1/2} \frac{g(x)}{x} = (1 - \frac{\lceil C/2 \rceil - 1}{\lfloor C \rfloor}) * 2$. □

5 Conclusions

In this paper, we described the first general methods to generate dual-feasible functions with domain and range $\mathbb{R}$. Despite their effectiveness for deriving lower bounds and valid inequalities for integer linear optimization problems with knapsack constraints, the applicability of current dual-feasible functions is very much limited given the restricted domains for which they are defined. By extending the concept of dual-feasible function to the domain and range $\mathbb{R}$, and by providing new functions on these general domain and range, we provide functions that can be used for problems with knapsack constraints and unrestricted coefficients, thus increasing significantly the applicability of these tools.

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