MOSE, Contrôle 3, corrigé

Ex. 1. $f(x,y) = x^3 - 3xy + 2y^2$

$\partial_x f = 3x^2 - 3y$

$\partial_y f = -3x + 4y$

$$\begin{cases} 3x^2 - 3y = 0 \\ -3x + 4y = 0 \end{cases} \quad \begin{cases} x^2 - y = 0 \\ y = \frac{3}{4}x \end{cases} \quad \begin{cases} x^2 - \frac{3}{4}x = 0 \\ y = \frac{3}{4}x \end{cases}$$

$$\begin{cases} x\left(x - \frac{3}{4}\right) = 0 \\ y = \frac{3}{4}x \end{cases}$$

$$\begin{cases} x = 0 \\ y = 0 \end{cases} \quad P_1 = (0,0)$$

$$\begin{cases} x = \frac{3}{4} \\ y = \frac{9}{16} \end{cases} \quad P_2 = \left(\frac{3}{4}, \frac{9}{16}\right)$$

$\partial_{xx} f = 6x$

$\partial_{xy} f = -3$

$\partial_{yy} f = 4$

Pour P_1 : $\operatorname{Hess} f(0,0) = \begin{pmatrix} 0 & -3 \\ -3 & 4 \end{pmatrix}$ $\det \operatorname{Hess} = -9 < 0 \Rightarrow$ P_1 point selle.

Pour P_2 : $\operatorname{Hess} f\left(\frac{3}{4}, \frac{9}{16}\right) = \begin{pmatrix} \frac{9}{2} & -3 \\ -3 & 4 \end{pmatrix}$ $\det = 18 - 9 = 9 > 0$ et $\partial_{xx} f = \frac{9}{2} > 0 \Rightarrow$ min.

P_2 point de minimum.

Ex. 2. $$\int_D (x\cos y + 2)\,dx\,dy = \int_D x\cos y\,dx\,dy + \int_D 2\,dx\,dy =$$

$$= \int_{x=0}^{2}\left(\int_{y=0}^{\frac{\pi}{2}} x\cos y\,dy\right)dx + 2\int_{x=0}^{2}\left(\int_{y=0}^{\frac{\pi}{2}} 1\cdot dy\right)dx$$

$$= \int_{x=0}^{2} x\,\left[\sin y\right]_0^{\frac{\pi}{2}}\,dx + 2\int_{x=0}^{2}\left[y\right]_0^{\frac{\pi}{2}}\,dx$$

$$= \int_{x=0}^{2} x\,(1-0)\,dx + 2\int_{x=0}^{2}\frac{\pi}{2}\,dx$$

$$= \int_0^2 x\,dx + \pi\int_0^2 dx = \left[\frac{x^2}{2}\right]_0^2 + \pi\left[x\right]_0^2 = \frac{4}{2} + 2\pi = 2 + 2\pi$$

Ex. 3. $ty' + y - t^2 = 0$

(E) $y' = -\frac{1}{t}y + t$ $\quad t \in [1, +\infty[$

(E_0) $y' = -\frac{1}{t}y$ $\quad a(t) = -\frac{1}{t}$ $\quad A(t) = -\ln t$

donc $y_h(t) = C e^{-\ln t} = C \cdot \frac{1}{e^{\ln t}} = C \cdot \frac{1}{t}$ $\quad C \in \mathbb{R}$.

$y_0(t) = \frac{1}{t}$ $\quad y_p(t) = C(t)\, y_0(t)$ avec

$$C(t) = \int \frac{b(t)}{y_0(t)}\,dt = \int \frac{t}{\frac{1}{t}}\,dt = \int t^2\,dt = \frac{t^3}{3}$$

donc $y_p(t) = \frac{t^3}{3} \cdot \frac{1}{t} = \frac{t^2}{3}$

$\Rightarrow y(t) = y_p(t) + y_h(t) = \boxed{\frac{t^2}{3} + C \cdot \frac{1}{t}}$ $\quad C \in \mathbb{R}$.

j'impose ~~j'ajoute~~ $y(1) = 0$:

$0 = \frac{1}{3} + C \cdot 1 \Rightarrow C = -\frac{1}{3}$

$\Rightarrow \boxed{y(t) = \frac{t^2}{3} - \frac{1}{3} \cdot \frac{1}{t}}$.

Ex. 4. $y'' - 2y' + 5y = 2$ $\quad$ (E)

(E_0) $y'' - 2y' + 5y = 0$ $\quad$ (EC) $\lambda^2 - 2\lambda + 5 = 0$

$\lambda = 1 \pm \sqrt{1-5} = 1 \pm \sqrt{-4} = 1 \pm 2i$ $\quad \alpha = 1 \quad \beta = 2$

$\Rightarrow y_h(t) = e^t \left[C_1 \cos(2t) + C_2 \sin(2t) \right]$. $\quad C_1, C_2 \in \mathbb{R}$.

$b(t) = 2$ constante $\Rightarrow$ on essaye $y_p(t) = A$ constante :

il faut que $y_p'' - 2y_p' + 5y_p = 2$ donc $0 - 2 \cdot 0 + 5A = 2 \Leftrightarrow A = \frac{2}{5}$

donc $y_p(t) = \frac{2}{5}$ et $\boxed{y(t) = y_p(t) + y_h(t) = \frac{2}{5} + e^t \left[C_1 \cos(2t) + C_2 \sin(2t) \right]}$

j'impose $y(0) = 0$: $\quad 0 = \frac{2}{5} + 1 \cdot [C_1 + 0] \Rightarrow C_1 = -\frac{2}{5}$

je dérive : $y'(t) = e^t \left[C_1 \cos(2t) + C_2 \sin(2t) \right] + e^t \left[-2C_1 \sin(2t) + 2C_2 \cos(2t) \right]$

j'impose $y'(0) = 1$ $\quad 1 = C_1 + 0 + 0 + 2C_2$ $\quad$ donc $1 = -\frac{2}{5} + 2C_2$ d'où

$2C_2 = \frac{7}{5} \Rightarrow C_2 = \frac{7}{10}$ $\quad \Rightarrow \boxed{y(t) = \frac{2}{5} + e^t \left[-\frac{2}{5} \cos(2t) + \frac{7}{10} \sin(2t) \right]}$.