Feuille 1

Exercices sur les systèmes linéaires

Ex. 1

i) $\begin{cases} 3x + 5y = 11 \\ 2x + 3y = 7 \end{cases}$ $\begin{cases} x = \dfrac{11-5y}{3} \\ 2\cdot\dfrac{11-5y}{3} + 3y = 7 \end{cases}$ $\begin{cases} x = \dfrac{11-5y}{3} \\ \dfrac{22-10y}{3} + 3y = 7 \end{cases}$

$\begin{cases} x = \dfrac{11-5y}{3} \\ 22 - 10y + 9y = 21 \end{cases}$ $\begin{cases} x = \dfrac{11-5y}{3} \\ -y = -1 \end{cases}$ $\begin{cases} y = 1 \\ x = 2 \end{cases}$ par substitution

ii) $\begin{cases} 2x + 5y = 10 & (\mathrm{I}) \\ 2x + 3y = 8 & (\mathrm{II} - \mathrm{I}) \end{cases}$ $\begin{cases} 2x + 5y = 10 \\ -2y = -2 \end{cases}$ $\begin{cases} y = 1 \\ 2x = 10 - 5 = 5 \end{cases}$ $\begin{cases} y = 1 \\ x = \dfrac{5}{2} \end{cases}$

combinaison linéaire

iii) $\begin{cases} 6x + 12y = 30 & \frac{1}{6}\,\mathrm{I} \\ 3x + 3y = 9 & \frac{1}{3}\,\mathrm{II} \end{cases}$ $\begin{cases} x + 2y = 5 & (\mathrm{I}) \\ x + y = 3 & (\mathrm{II} - \mathrm{I}) \end{cases}$ $\begin{cases} x + 2y = 5 \\ -y = -2 \end{cases}$ $\begin{cases} y = 2 \\ x = 1 \end{cases}$

~~iv)~~ **Ex. 2.**

$\begin{cases} 2x + 3y = 4 \\ 3x + 7y = 0 & 2\mathrm{II} - 3\mathrm{I} \\ 4x + 7y = 1 \end{cases}$ $\begin{cases} 2x + 3y = 4 \\ 5y = -12 \\ y = -7 \end{cases}$ $\Longrightarrow$ aucune solution

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Ex. 3.

i) $\begin{cases} x + 2y = \lambda^2 & \mathrm{I} \\ 4x + 3y = 1 & (\mathrm{II} - 4\mathrm{I}) \end{cases}$ $\begin{cases} x + 2y = \lambda^2 \\ -5y = -4\lambda^2 + 1 \end{cases}$ $\begin{cases} y = \dfrac{4\lambda^2 + 1}{5} \quad (*) \\ x = \lambda^2 - 2\cdot\dfrac{4\lambda^2+1}{5} \\ \quad = \dfrac{1}{5}\left(-3\lambda^2 - 2\right) \end{cases}$

donc ~~tout~~ la solution existe toujours,
elle est unique et elle est donnée par (*) en fonction de λ.

ii) $\begin{cases} 2dx + 9y = 21 & (II) \\ 8x + dy = 14 & (I) \end{cases}$ $\begin{cases} 8x + dy = 14 \\ 2dx + 9y = 21 \end{cases}$ $\begin{cases} 8x + dy = 14 \\ 2d\left(\dfrac{14-dy}{8_4}\right) + 9y = 21 \end{cases}$ $\begin{cases} 8x+dy = 14 \\ 14d - d^2 y + 36y = 84 \end{cases}$

$\begin{cases} 8x + dy = 14 \\ y(-d^2+36) = 84 - 14d \end{cases}$ $\begin{cases} 8x + dy = 14 \\ (d^2-36)y = 14d - 84 \end{cases}$

si $d^2 \neq 36$ c-à-d $d \neq \pm 6$ $\Rightarrow$ $y = \dfrac{14d-84}{d^2-36}$

et $x = \dfrac{1}{8}\left(14 - d\,\dfrac{14d-84}{d^2-36}\right)$ solution unique

si $d = 6$: $\begin{cases} 8x + 6y = 14 \\ 0 = 84 - 84 = 0 \end{cases}$ la deuxième est tjrs vérifiée, la première donne

toute la droite $8x + 6y = 14$ comme solution, c-à-d: $y = -\frac{4}{3}x + \frac{7}{3}$

$d = -6$: $\begin{cases} 8x - 6y = 14 \\ 0 = 84 + 84 \end{cases}$ la deuxième equation n'est jamais satisfaite $\Rightarrow$ aucune solution.

Ex. 4. i) $\begin{cases} x - 3y + 2z = 8 & I \\ -x + 3y - 4z = -16 & II+I \end{cases}$ $\begin{cases} x - 3y + 2z = 8 \\ -2z = -8 & -\frac{1}{2}II \end{cases}$ $I \begin{cases} \boxed{x} - 3y + 2z = 8 \\ \boxed{z} = 4 \end{cases}$ libre

$I - 2II$ $\begin{cases} x - 3y = 0 \\ z = 4 \end{cases}$ $\begin{cases} x = 3y \\ z = 4 \end{cases}$ pivots (équations système)

c-à-d $\begin{cases} y = t \\ x = 3t \\ z = 4 \end{cases}$ avec le (paramètre) c-à-d $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix} + t \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$

point fixe + direction = droite

ii) $\begin{cases} 6x + 3y + 1 = 10 \\ 6x + 3y + 3 = 12 \end{cases}$ $\begin{cases} 6x + 3y = 9 \\ 6x + 3y = 9 \end{cases}$ $\begin{cases} 6x + 3y = 9 \\ 0 = 0 \end{cases}$ $\Rightarrow$ (Attention: le système est 2×2)

$\Rightarrow$ $2x + y = 3$ soit $y = -2x + 3$ est ~~la droite~~ la droite des solutions

iii) $\left\{\begin{array}{lr} \boxed{x}+2y-4z=-1 & \mathrm{I} \\ 3x+y+2z=-2 & \mathrm{II}-3\mathrm{I} \end{array}\right. \quad \left\{\begin{array}{l} \boxed{x}+2y-4z=-1 \\ \boxed{-5y}+14z=1 \end{array}\right.$

$z \longrightarrow$ libre

pivots

on exprime x et y en fonction de z

$$\left\{\begin{array}{l} x=-2\left(\frac{14}{5}z-\frac{1}{5}\right)-4z-1 \\ y=\frac{14}{5}z-\frac{1}{5} \end{array}\right.$$

$$\left\{\begin{array}{l} x=-\frac{48}{5}z-\frac{3}{5} \\ y=\frac{14}{5}z-\frac{1}{5} \end{array}\right. \text{ équations } \iff \left\{\begin{array}{l} x=-\frac{48}{5}t-\frac{3}{5} \\ y=\frac{14}{5}t-\frac{1}{5} \\ z=t \end{array}\right. \quad t\in\mathbb{R} \text{ paramètre}$$

$$\iff \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -\frac{3}{5} \\ -\frac{1}{5} \\ 0 \end{pmatrix} + t\begin{pmatrix} -\frac{48}{5} \\ \frac{14}{5} \\ 1 \end{pmatrix}$$

point + direction = droite des solutions.

Ex. 5

$$\left\{\begin{array}{lr} y+3z=3 & \mathrm{II} \\ x-y+z=0 & \mathrm{I} \\ x+y=-1 & \mathrm{III}-\mathrm{I} \end{array}\right. \quad \left\{\begin{array}{lr} x-y+z=0 & \\ y+3z=3 & \\ 2y-z=-1 & \mathrm{III}-2\mathrm{II} \end{array}\right. \quad \left\{\begin{array}{l} \boxed{x}-y+z=0 \\ \boxed{y}+3z=3 \\ \boxed{-7z}=-7 \end{array}\right.$$

$$\left\{\begin{array}{l} x=-1 \\ y=0 \\ z=1 \end{array}\right. \quad \text{la solution est le point } \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

Ex. 6.

$$\left\{\begin{array}{lr} x+y=-1 & (\mathrm{I}) \\ x-y+z=0 & (\mathrm{II}-\mathrm{I}) \\ 2x+z=-1 & (\mathrm{III}-2\mathrm{I}) \end{array}\right.$$

voir à la fin

~~aucune solution~~

Ex. 7.

$$\left\{\begin{array}{lr} \boxed{x}+y-z=2 & \\ x+y=3 & (\mathrm{II}-\mathrm{I}) \\ x+y-2z=1 & (\mathrm{III}-\mathrm{I}) \end{array}\right. \quad \left\{\begin{array}{l} x+y-z=2 \\ z=1 \\ -z=-1 \end{array}\right. \quad \left\{\begin{array}{l} \boxed{x}+y-z=2 \\ \boxed{z}=1 \end{array}\right. \quad \left\{\begin{array}{l} x=-y+3 \\ z=1 \end{array}\right. \text{ équations}$$

$y \longrightarrow$ libre

pivots

$$\left\{\begin{array}{l} x=-t+3 \\ y=t \\ z=1 \end{array}\right. \text{ paramètres} \quad \text{c.-à-d.} \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} + t\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

point fixe + direction

Ex. 12.

$$\begin{cases} x+y=2 & \mathrm{I} \\ \alpha x - y = 0 & \mathrm{II}-\alpha\mathrm{I} \\ x-\alpha y = 1 & \mathrm{III}-\mathrm{I} \end{cases} \quad \begin{cases} \boxed{x}+y=2 \\ (-\alpha-1)y = -2\alpha \\ (-1-\alpha)y = -1 \end{cases} \quad \begin{cases} x+y=2 \\ (\alpha+1)y = 2\alpha \\ (\alpha+1)y=1 \end{cases}$$

si $\alpha \neq -1$ $\begin{cases} x+y=2 \\ y = \dfrac{2\alpha}{\alpha+1} \\ y = \dfrac{1}{\alpha+1} \end{cases}$ sol. si $\dfrac{2\alpha}{\alpha+1} = \dfrac{1}{\alpha+1}$

$\iff 2\alpha = 1 \iff \boxed{\alpha = \frac{1}{2}}$

et dans ce cas $\begin{cases} x+y=2 \\ y = \dfrac{1}{\frac{3}{2}} = \dfrac{2}{3} \end{cases}$ $\begin{cases} x = \dfrac{4}{3} \\ y = \dfrac{2}{3} \end{cases}$

si $\alpha = -1$: $\begin{cases} x+y=2 \\ 0=-2 \\ 0=1 \end{cases} \Rightarrow$ aucune solution

en conclusion, il y a ~~une~~ unique sol. ~~…~~ seulement pour $\alpha = \frac{1}{2}$.

Ex. ~~7~~.

$$\begin{cases} x+y+z=2 & (\mathrm{I}) \\ x+y=3 & (\mathrm{II}-\mathrm{I}) \\ x+y-2z=1 & (\mathrm{III}-\mathrm{I}) \end{cases} \quad \begin{cases} x+y+z=2 \\ z=1 \\ -z=-1 \end{cases} \quad \begin{cases} x+y-z=2 \\ z=1 \end{cases} \quad \begin{cases} \boxed{x}+y=3 \\ \boxed{z}=1 \end{cases}$$

libres

pivots

$\begin{cases} x=-y+3 \\ z=1 \end{cases}$ c.à.d $\begin{cases} x=-t+3 \\ y=t \\ z=1 \end{cases}$ c-à-d $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} + t\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$

Ex. 8.

$$\begin{cases} x+y+z=2 & \mathrm{I} \\ x-y-z=0 & \mathrm{II}-\mathrm{I} \\ 2x-2z=1 & \mathrm{III}-2\mathrm{I} \end{cases} \quad \begin{cases} x+y-z=2 \\ -2y=-2 \\ -2y=-3 \end{cases} \Rightarrow \text{aucune solution.}$$

Ex. 9.

$$\begin{cases} 3x+y-z=3 & \mathrm{III} \\ 2x-3y+2z=3 & \mathrm{I}-3\mathrm{III} \\ x+4y-3z=0 & \mathrm{II}-2\mathrm{III} \end{cases} \quad \begin{cases} x+4y-3z=0 \\ -11y+8z=3 \\ -11y+8z=3 \end{cases} \quad \begin{cases} \boxed{x}+4y-3z=0 \\ \boxed{11y} - 8z = -3 \end{cases}$$

pivots libre

$\begin{cases} x=-4y+3z \\ ~~x~~\ y = \dfrac{8z-3}{11} \end{cases}$ $\begin{cases} y = \dfrac{8}{11}z - \dfrac{3}{11} \\ x = -4\left(\dfrac{8}{11}z - \dfrac{3}{11}\right) + 3z = \dfrac{1}{11}z + \dfrac{12}{11} \end{cases}$

donc $\begin{cases} x = \dfrac{1}{11}t + \dfrac{12}{11} \\ y = \dfrac{8}{11}t - \dfrac{3}{11} \end{cases}$ et $z=t$ $\quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{12}{11} \\ -\frac{3}{11} \\ 0 \end{pmatrix} + t\begin{pmatrix} \frac{1}{11} \\ \frac{8}{11} \\ 1 \end{pmatrix}$

Ex. 9) iii)

$$\begin{cases} 2x+y+z=2 \\ x-y+z=5 \\ 3x+y-2z=-12 \end{cases} \quad \begin{matrix} \mathrm{II} \\ \mathrm{I}-2\mathrm{II} \\ \mathrm{III}-3\mathrm{II} \end{matrix} \begin{cases} \boxed{x}-y+z=5 \\ 3y-z=-8 \\ 4y-5z=-27 \end{cases} \quad \begin{cases} x-y+z=5 \\ 3y-z=-8 \\ -11z=-49 \end{cases} \Rightarrow z=\frac{49}{11} \text{ etc.}$$

$3\mathrm{III}-4\mathrm{II}$

Ex. 10.

$$\begin{cases} x +2z=0 \\ \lambda x+\lambda y+2z=1 \\ -2x-2\lambda y+2z=3 \end{cases} \quad \begin{matrix} \\ \mathrm{II}-\lambda\mathrm{I} \\ \mathrm{III}+2\mathrm{I} \end{matrix} \begin{cases} x +2z=0 \\ \lambda y+(2-2\lambda)z=1 \\ -2\lambda y+6z=3 \end{cases}$$

$$\begin{cases} x +2z=0 \\ \lambda y+(2-2\lambda)z=1 \\ (10-4\lambda)z=5 \end{cases} \quad \mathrm{III}+2\mathrm{II}$$

$\lambda\neq 0$ et $\lambda\neq\frac{5}{2}$ $\Rightarrow$ une solution $\begin{cases} z=\dfrac{5}{10-4\lambda} \\ y=\dfrac{1}{\lambda}\left(1-\dfrac{(2-2\lambda)5}{10-4\lambda}\right) \\ x=-2\cdot\dfrac{5}{10-4\lambda}=\dfrac{5}{2\lambda-5} \end{cases}$

$\lambda=\frac{5}{2}$: III devient $0=5$ $\Rightarrow$ aucune sol.

$\lambda=0$: $\begin{cases} x+2z=0 \\ 2z=1 \\ 10z=5 \end{cases} \quad \begin{cases} x+2z=0 \\ z=\frac{1}{2} \\ z=\frac{1}{2} \end{cases} \quad \begin{cases} x=-1 \\ z=\frac{1}{2} \end{cases}$

c-à-d $\begin{cases} x=-1 \\ y=t \\ z=\frac{1}{2} \end{cases}$ c-à-d $\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} -1 \\ 0 \\ \frac{1}{2} \end{pmatrix}+t\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

Ex. 11.

$$\begin{cases} \lambda x+y+z=3 \\ -\lambda^2 x=-2\lambda+1 \\ \lambda x+(1-\lambda)y-z=-\lambda+2 \end{cases} \quad \begin{matrix} \\ \mathrm{I}+\lambda\mathrm{I} \\ \mathrm{III}-\mathrm{I} \end{matrix} \begin{cases} \lambda x+y+z=3 \\ \lambda y+\lambda z=\lambda+1 \\ -\lambda y-\cancel{2z}=-\lambda-1 \end{cases} \quad \mathrm{III}+\mathrm{II} \quad \begin{cases} \lambda x+y+z=3 \\ \lambda y+\lambda z=\lambda+1 \\ (\lambda-2)z=0 \end{cases}$$

$\lambda\neq 2,\ \lambda\neq 0$: $\begin{cases} z=0 \\ \lambda y=\lambda+1 \\ \lambda x+y=3 \end{cases} \quad \begin{cases} z=0 \\ y=\frac{\lambda+1}{\lambda} \\ x=\frac{1}{\lambda}\left(-\frac{\lambda+1}{\lambda}+3\right) \end{cases}$ solution unique

$\lambda=2$: $\begin{cases} \boxed{2x}+y+z=3 \\ \boxed{2y}+2z=3 \end{cases} \quad \begin{cases} x=\frac{3}{2}-\frac{1}{2}z-\frac{1}{2}y \\ y=-z+\frac{3}{2} \end{cases} \quad \begin{cases} x=-\frac{1}{2}z+\frac{3}{2}+\frac{1}{2}z-\frac{3}{4} \\ y=-z+\frac{3}{2} \end{cases}$ etc.

une droite de solutions.

$d=0$ $\quad \begin{cases} y+z=3 \\ 0=1 \\ -2z=0 \end{cases}$ aucune sol.

<u>Ex. 13.</u> p = pirates $\quad v$ = vikings :

$\begin{array}{l|l} \text{poulets} & 4p+3v=65 \\ \text{bières} & 5p+7v=117 \end{array}$ etc.

<u>Ex. 14.</u> l = lapins $\quad p$ = poulets

$\begin{array}{l|l} \text{animaux} & l+p=27 \\ \text{pattes} & 4l+2p=72 \end{array}$ etc.

<u>Ex. 15.</u> m = temps de montée

d = temps de descente

$\begin{array}{l|l} \text{durée} & m+d=6 \\ \text{espace} & 3m=5d \end{array}$ $\quad \begin{cases} m=\frac{5}{3}d \\ \frac{5}{3}d+d=6 \end{cases}$ $\quad \begin{cases} m=\frac{5}{3}d \\ \frac{8}{3}d=6 \end{cases}$ $\quad \begin{cases} m=\frac{5}{\cancel{3}_1}\cdot\frac{\cancel{9}^3}{4}=\frac{15}{4} \\ d=\frac{18}{8}=\frac{9}{4} \end{cases}$

le chemin est le même

$m=\frac{15}{4}=3h\,\frac{3}{4}$

donc si on part à 8h on est au sommet à 11h 45

<u>Ex. 6.</u> $\begin{cases} x+y=-1 & \mathrm{I} \\ x-y+z=0 & \mathrm{II}-\mathrm{I} \\ 2x+z=-1 & \mathrm{III}-2\mathrm{I} \end{cases}$ $\begin{cases} x+y=-1 \\ -2y+z=1 \\ -2y+z=1 \end{cases}$ $\begin{cases} x+y=-1 \\ -2y+z=1 \end{cases}$ $\begin{cases} x=-1-y=-\frac{1}{2}-\frac{1}{2}z \\ y=-\frac{1}{2}+\frac{1}{2}z \end{cases}$ (équations)

infinité de solutions

$\begin{cases} x=-\frac{1}{2}-\frac{1}{2}t \\ y=-\frac{1}{2}+\frac{1}{2}t \\ z=t \end{cases}$ c-à-d $\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\underbrace{\begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 0 \end{pmatrix}}_{\text{point fixe}}+t\underbrace{\begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \end{pmatrix}}_{\text{direction}}$

(paramètres)