Ex. 1.

Feuille de TD 3

$$A = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$$

- le polynôme caractéristique est

$$P_A(x) = \det(A - x I_2) = \det\left(\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} - x \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = \det \begin{pmatrix} 1-x & 3 \\ 3 & 1-x \end{pmatrix}$$

$$= (1-x)^2 - 9$$

$$= x^2 - 2x - 8$$

- on résout $P_A(x) = 0$ pour trouver les valeurs propres :

$$x^2 - 2x - 8 = 0 \iff x = 1 \pm \sqrt{1+8} = 1 \pm 3 = \begin{cases} -2 \\ 4 \end{cases}$$

donc $\lambda_1 = -2$ et $\lambda_2 = 4$.

- A est diagonalisable, car $\lambda_1 \neq \lambda_2$, et sa forme diagonale est $\Delta = \begin{pmatrix} -2 & 0 \\ 0 & 4 \end{pmatrix}$.

- On résout $(A + 2I)X = 0$

$$\begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0 \quad \begin{cases} 3x + 3y = 0 \\ 3x + 3y = 0 \end{cases} \iff 3x + 3y = 0 \iff x + y = 0$$

- un vecteur propre v_1 pour A de valeur propre -2 est p. ex. $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

- On résout $(A - 4I)X = 0$

$$\begin{pmatrix} -3 & 3 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0 \quad \begin{cases} -3x + 3y = 0 \\ 3x - 3y = 0 \end{cases} \iff x - y = 0$$

- un vecteur propre v_2 pour A de valeur propre 4 est $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

La matrice P t.q. $P^{-1}AP = \Delta$ est $P = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$

On a $P^{-1} = \dfrac{1}{\det P}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \dfrac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

$$P^{-1}AP = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -2 & 2 \\ 4 & 4 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -4 & 0 \\ 0 & 8 \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 0 & 4 \end{pmatrix} \checkmark$$

Enfin, les puissances de A :

si $P^{-1}AP = \Delta$, alors $A = P\Delta P^{-1}$ et

$$A^5 = P\Delta^5 P^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} -32 & 0 \\ 0 & 4^5 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -32 & 4^5 \\ 32 & 4^5 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \frac{1}{2}\{ \text{ etc.}$$

et

$$A^n = P\Delta^n P^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} (-2)^n & 0 \\ 0 & 4^n \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \frac{2^n}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} (-1)^n & 0 \\ 0 & 2^n \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = 2^{n-1}\begin{pmatrix} (-1)^n + 2^n & (-1)^{n+1} + 2^n \\ (-1)^{n+1} + 2^n & (-1)^n + 2^n \end{pmatrix}$$

Ex. 2. $A = \begin{pmatrix}1&1\\1&1\end{pmatrix}$

$P_A(x) = \det\begin{pmatrix}1-x&1\\1&1-x\end{pmatrix} = (1-x)^2-1 = x^2-2x$

$P_A(x) = 0 \iff x = \begin{cases}0\\2\end{cases} \Rightarrow A$ diagonalisable, $\Delta = \begin{pmatrix}0&0\\0&2\end{pmatrix}$

$\begin{pmatrix}1&1\\1&1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = 0 \iff x+y=0 \iff v_1 = \begin{pmatrix}1\\-1\end{pmatrix}$

$\begin{pmatrix}-1&1\\1&-1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = 0 \iff x-y=0 \iff v_2 = \begin{pmatrix}1\\1\end{pmatrix}$

$P = \begin{pmatrix}1&1\\-1&1\end{pmatrix} \qquad P^{-1} = \frac{1}{2}\begin{pmatrix}1&-1\\1&1\end{pmatrix} \qquad P^{-1}AP = \Delta$

et $A = P\Delta P^{-1}$, d'où $A^4 = P\Delta^4P^{-1} = \frac{1}{2}\begin{pmatrix}1&1\\-1&1\end{pmatrix}\begin{pmatrix}0&0\\0&2^4\end{pmatrix}\begin{pmatrix}1&-1\\1&1\end{pmatrix} =$

$= \frac{1}{2}\begin{pmatrix}0&2^4\\0&2^4\end{pmatrix}\begin{pmatrix}1&-1\\1&1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}2^4&2^4\\2^4&2^4\end{pmatrix} = \begin{pmatrix}2^3&2^3\\2^3&2^3\end{pmatrix} = \begin{pmatrix}8&8\\8&8\end{pmatrix}$

$B = \begin{pmatrix}1&2\\-1&4\end{pmatrix}$

$P_B(x) = \det\begin{pmatrix}1-x&2\\-1&4-x\end{pmatrix} = 4-x-4x+x^2+2 = x^2-5x+6$

$P_B(x) = 0 \iff x = \frac{5\pm\sqrt{25-24}}{2} = \frac{5\pm 1}{2} \Rightarrow A$ diagonalisable,

$\lambda_1 = 2 \quad \lambda_2 = 3 \qquad \Delta = \begin{pmatrix}2&0\\0&3\end{pmatrix}$

$\begin{pmatrix}-1&2\\-1&2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = 0 \iff -x+2y=0 \qquad v_1 = \begin{pmatrix}2\\1\end{pmatrix}$

$\begin{pmatrix}-2&2\\-1&1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = 0 \iff -x+y=0 \qquad v_2 = \begin{pmatrix}1\\1\end{pmatrix}$

$P = \begin{pmatrix}2&1\\1&1\end{pmatrix} \qquad P^{-1} = \frac{1}{1}\begin{pmatrix}1&-1\\-1&2\end{pmatrix} \qquad P^{-1}BP = \begin{pmatrix}2&0\\0&3\end{pmatrix}$

et $B = P\Delta P^{-1}$ d'où $B^4 = P\Delta^4P^{-1} = \begin{pmatrix}2&1\\1&1\end{pmatrix}\begin{pmatrix}2^4&0\\0&3^4\end{pmatrix}\begin{pmatrix}1&-1\\-1&2\end{pmatrix}$

$$C = \begin{pmatrix} 1 & 4 \\ 2 & -1 \end{pmatrix}$$

$$P_C(x) = \det\begin{pmatrix} 1-x & 4 \\ 2 & -1-x \end{pmatrix} = -1-x+x+x^2-8 = x^2-9$$

$P_C(x) = 0 \iff x = \pm 3$ $\quad$ C diagonalisable, $\Delta = \begin{pmatrix} -3 & 0 \\ 0 & 3 \end{pmatrix}$

$\begin{pmatrix} 4 & 4 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0 \iff x+y=0$ $\quad$ $v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\begin{pmatrix} -2 & 4 \\ 2 & -4 \end{pmatrix} = 0 \iff -x+2y=0$ $\quad$ $v_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$P = \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}$ $\quad$ $P^{-1} = \frac{1}{3}\begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}$, $\quad$ $P^{-1} C P = \Delta$ et $C = P \Delta P^{-1}$. etc.

$$D = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$P_D(x) = \det\begin{pmatrix} -x & 1 \\ 1 & -x \end{pmatrix} = x^2-1$$

$P_D(x) = 0 \iff x = \pm 1$ $\quad$ D diagonalisable, $\Delta = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0 \iff x+y=0$ $\quad$ $v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0 \iff x-y=0$ $\quad$ $v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$P = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ $\quad$ $P^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$, $\quad$ $P^{-1} D P = \Delta$ et $D = P \Delta P^{-1}$ etc.

<u>Ex. 3.</u> $\quad A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$

$$P_A(x) = \det\begin{pmatrix} 1-x & 1 \\ 0 & -x \end{pmatrix} = -x+x^2$$

$P_A(x) = 0 \iff x = \begin{cases} 0 \\ 1 \end{cases}$ $\quad$ A diagonalisable, $\Delta = \begin{pmatrix} 0 & \\ & 1 \end{pmatrix}$

$v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ est déjà un vecteur propre de valeur propre 1 :

$\begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0 \iff y = 0$

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0 \iff x+y=0 \qquad v_p = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$P = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \qquad P^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \qquad P^{-1}AP = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$P_B(x) = \det \begin{pmatrix} -x & 1 \\ 0 & -x \end{pmatrix} = x^2$$

$P_B(x) = 0 \iff x = 0$ donc $\lambda_1 = \lambda_2 = 0$.

B n'est pas déjà diagonale $\Rightarrow$ B n'est pas diagonalisable.

$$C = \begin{pmatrix} 1 & 4 \\ 4 & 1 \end{pmatrix}$$

$$P_C(x) = \det \begin{pmatrix} 1-x & 4 \\ 4 & 1-x \end{pmatrix} = 1 - 2x + x^2 - 16 = x^2 - 2x - 15$$

$$P_C(x) = 0 \iff x = 1 \pm \sqrt{1+15} = 1 \pm 4 = \begin{cases} -3 = \lambda_1 \\ 5 = \lambda_2 \end{cases}$$

C diagonalisable, car $\lambda_1 \neq \lambda_2$, et $\Delta = \begin{pmatrix} -3 & 0 \\ 0 & 5 \end{pmatrix}$ etc. comme avant.

$$D = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$ D est déjà diagonale.

Ex. 4.
$$\begin{cases} x_{n+1} = x_n + 2y_n \\ y_{n+1} = 2y_n + x_n \end{cases}$$

on écrit le problème en forme matricielle

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x_n \\ y_n \end{pmatrix} \quad \text{c-à-d} \quad \begin{pmatrix} x_n \\ y_n \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}}_{A}{}^{n}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

il faut donc calculer $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}^n$, et on le fait ~~en~~ calculant la forme diagonale de A.

$$p_A(x) = \det\begin{pmatrix} 1-x & 2 \\ 2 & 1-x \end{pmatrix} = 1-2x+x^2-4 = x^2-2x-3$$

$$x^2-2x-3=0 \iff x = 1 \pm \sqrt{1+3} \quad \begin{cases} \lambda_1 = -1 \\ \lambda_2 = 3 \end{cases}$$

$$\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0 \longrightarrow v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0 \longrightarrow v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \Delta = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}$$

$$P = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \qquad P^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$P^{-1}AP = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -1 & 1 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -2 & 0 \\ 0 & 6 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix} \text{ ok}$$

donc $A = P\Delta P^{-1}$, d'où

$$\begin{aligned} A^n &= P\Delta^n P^{-1} \\ &= \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} (-1)^n & 0 \\ 0 & 3^n \end{pmatrix}\frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} (-1)^n & 3^n \\ (-1)^{n+1} & 3^n \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} (-1)^n+3^n & (-1)^{n+1}+3^n \\ (-1)^{n+1}+3^n & (-1)^n+3^n \end{pmatrix} \end{aligned}$$

on en tire

$$\begin{cases} x_n = \dfrac{((-1)^n+3^n)x_0 + ((-1)^{n+1}+3^n)y_0}{2} \\[2ex] y_n = \dfrac{((-1)^{n+1}+3^n)x_0 + ((-1)^n+3^n)y_0}{2} \end{cases}$$

les deux autres ~~systèmes~~ récurrences se resolvent de la même façon.