Feuille 5

Ex. 1. $\int_0^1 x^3\,dx = \left[\frac{x^4}{4} \right]_0^1 = \frac{1}{4}$

$$\int_1^4 \frac{1}{x^2} = \left[-\frac{1}{x} \right]_1^4 = -\frac{1}{4} - (-1) = \frac{3}{4}$$

$\int_0^1 \frac{1}{\sqrt{x}}\,dx =$ attention : $\frac{1}{\sqrt{x}}$ n'est pas défini en $x=0$, donc c'est une intégrale impropre ou généralisée

$$= \lim_{c \to 0^+} \int_c^1 \frac{1}{\sqrt{x}}\,dx = \lim_{c \to 0^+} \left[2\sqrt{x} \right]_c^1 = \lim_{c \to 0^+} \left(2 - 2\sqrt{c} \right) = 2$$

$$\int_1^4 \frac{1}{x\sqrt{x}}\,dx = \int_1^4 x^{-\frac{3}{2}}\,dx = \left[\frac{x^{-\frac{1}{2}}}{-\frac{1}{2}} \right]_1^4 = \left[\frac{-2}{\sqrt{x}} \right] = -\frac{2}{2} - \left(-\frac{2}{1} \right) = -1+2 = 1$$

Ex. 2. $\int \cos(3x-5)\,dx = \int \cos u \cdot \frac{1}{3}\,du = \frac{1}{3}\int \cos u\,du = \frac{1}{3}\sin u + \cancel{\text{cst}}\ C$

$= \frac{1}{3}\sin(3x-5) + C$

$3x-5 = u$

$x = \frac{u+5}{3} \quad dx = \frac{1}{3}\,du$

soit directement, si vous le voyez :

$$\int \cos(3x-5)\,dx = \frac{1}{3}\int 3\cos(3x-5)\,dx = \frac{1}{3}\int \cos(3x-5)\,(3x-5)'\,dx = \frac{1}{3}\sin(3x-5) + C$$

intervalle maximale de définition : $]-\infty, +\infty[$

$$\int \frac{x^2-3x+4}{x}\,dx = \int x\,dx - 3\int dx + 4\int \frac{1}{x}\,dx$$

pour $x \neq 0$ $\quad = \frac{x^2}{2} - 3x + 4\ln|x| + C$

Int. max : $]-\infty, 0[$ ou $]0, +\infty[$

$\int \frac{1}{x-2}\,dx = \ln|x-2| + C$ pour $x \neq 2$. $\quad]-\infty, 2[$ ou $]2, +\infty[$

Ex. 3. $\int_0^3 \sqrt{9-x^2}\,dx = \frac{9\pi}{4}$ est connue.

Alors $A = \int_0^3 \left(\sqrt{9-x^2} - 3 \right) dx = \int_0^3 \sqrt{9-x^2}\,dx - 3\int_0^3 dx = \frac{9}{4}\pi - 3\left[x \right]_0^3 = \frac{9}{4}\pi - 9.$

$$A+B = \int_0^3 \frac{x^2}{\sqrt{9-x^2}+3} + \int_0^3 \sqrt{9-x^2} - 3 = \int_0^3 \frac{\cancel{x^2} + \cancel{9} - \cancel{x^2} - \cancel{9}}{\sqrt{9-x^2}+3}\,dx = 0$$

donc $B = (A+B) - A = 0 - \frac{9}{4}\pi + 9 = -\frac{9}{4}\pi + 9$

Ex. 4.

$$\int \underbrace{x}_{u}\underbrace{\sin x}_{v'}\,dx = \underbrace{(-\cos x)}_{v}\underbrace{x}_{u} - \int \underbrace{(-\cos x)}_{v}\cdot\underbrace{1}_{u'}\,dx$$

$$= -x\cos x + \int \cos x\,dx = -x\cos x + \sin x + C$$

$$\int 2x\,e^{-x}\,dx = 2x\left(-e^{-x}\right) - \int\left(-e^{-x}\right)\cdot 2\,dx$$

$$= -2x\,e^{-x} + 2\int e^{-x}\,dx = -2x\,e^{-x} - 2e^{-x} + C = -2(x+1)\,e^{-x} + C$$

$$\int \ln(1+x)\,dx = x\ln(1+x) - \int x\cdot\frac{1}{1+x}\,dx$$

$(x>-1)$

$$= x\ln(1+x) - \int \frac{x+1-1}{x+1}\,dx$$

$$= x\ln(1+x) - \int 1\,dx + \int\frac{1}{x+1}\,dx$$

$$= x\ln(1+x) - x + \ln(1+x)$$

$$\int 2x\ln(x-5)\,dx = x^2\ln(x-5) - \int x^2\cdot\frac{1}{x-5}\,dx$$

$(x>5)$

$$= x^2\ln(x-5) - \int\left(x+5+\frac{25}{x-5}\right)dx$$

$$= x^2\ln(x-5) - \frac{x^2}{2} - 5x + 25\ln(x-5) + C$$

$$\frac{x^2}{x-5} = x+5+\frac{25}{x-5}$$

$$\begin{array}{r|l} x^2 & x-5 \\ \cline{2-2} -x^2+5x & x+5 \\ \hline \cancel{5x} & \\ -5x+25 & \\ \hline 25 & \end{array}$$

$$\int x\log^2(5x)\,dx = \frac{x^2}{2}\log^2(5x) - \int \frac{x^{\cancel{2}}}{\cancel{2}}\cdot \cancel{2}\log 5x\cdot\frac{1}{\cancel{5x}}\cdot\cancel{5}\,dx$$

$(x>0)$

$$= \frac{x^2}{2}\log^2(5x) - \int x\log 5x\,dx$$

$$= \frac{x^2}{2}\log^2(5x) - \left[\frac{x^2}{2}\log 5x - \int\frac{x^{\cancel{2}}}{2}\cdot\frac{1}{\cancel{5x}}\cdot\cancel{5}\,dx\right]$$

$$= \frac{x^2}{2}\log^2(5x) - \frac{x^2}{2}\log 5x + \frac{1}{2}\int x\,dx$$

$$= \frac{x^2}{2}\log^2(5x) - \frac{x^2}{2}\log 5x + \frac{x^2}{4} + C$$

$$\int (x+1)^2\cos x\,dx = \sin x\,(x+1)^2 - \int \sin x\cdot 2(x+1)\,dx$$

$$= \sin x\,(x+1)^2 - \left[(-\cos x)\cdot 2(x+1) - \int(-\cos x)\cdot 2\,dx\right]$$

$$= \sin x\,(x+1)^2 + 2(x+1)\cos x - 2\int\cos x\,dx$$

$$= \sin x\,(x+1)^2 + 2(x+1)\cos x - 2\sin x + C.$$

$$\int \frac{dx}{\sqrt{3+x}}\, dx = \qquad \sqrt{3+x} = t,\quad 3+x = t^2,\quad x = t^2-3,\quad dx = 2t\,dt$$

$$= \int \frac{1}{t}\cdot 2t\,dt$$

$$= \int 2\,dt = 2t + C = 2\sqrt{3+x} + C$$

$$\int \frac{x}{\sqrt{x-1}}\, dx = \qquad \sqrt{x-1} = t,\quad x-1 = t^2,\quad x = t^2+1,\quad dx = 2t\,dt$$

$$= \int \frac{t^2+1}{t}\cdot 2t\,dt$$

$$= \int (t^2+1)\,dt = \frac{t^3}{3} + t + C = \frac{1}{3}(x-1)\sqrt{x-1} + \sqrt{x-1} + C$$

<u>Ex. 6.</u> $\int_0^1 e^{-x}\,dx = \left[-e^{-x}\right]_0^1 = -\frac{1}{e} - (-1) = 1 - \frac{1}{e}$

$$\int_0^1 x e^{2x}\,dx = \left[\frac{1}{2}e^{2x}\cdot x\right]_0^1 - \int_0^1 \frac{1}{2}e^{2x}\,dx$$

$$= \frac{1}{2}e^2 - 0 - \left[\frac{1}{4}e^{2x}\right]_0^1 = \frac{1}{2}e^2 - \left(\frac{1}{4}e^2 - \frac{1}{4}\cdot 1\right) = \frac{1}{2}e^2 - \frac{1}{4}e^2 + \frac{1}{4}$$

$$= \frac{1}{4}e^2 + \frac{1}{4}$$

$$\int_0^1 2x e^{x^2}\,dx = \int_0^1 e^{x^2}\cdot (x^2)'\,dx = \left[e^{x^2}\right]_0^1 = e^1 - 1 = e - 1.$$

$$\int_0^1 e^x\sqrt{e^x+3}\,dx = \int_0^1 \sqrt{e^x+3}\cdot(e^x+3)'\,dx = \left[\frac{(e^x+3)^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^1 =$$

$$= \frac{2}{3}(e+3)^{\frac{3}{2}} - \frac{2}{3}(4+3)^{\frac{3}{2}}.$$

<u>Ex. 7.</u> $\int_2^3 x\sin x^2\,dx = \frac{1}{2}\int_2^3 \sin x^2\cdot 2x\,dx = \frac{1}{2}\int_2^3 \sin x^2\cdot (x^2)'\,dx =$

$$= \frac{1}{2}\left[-\cos(x^2)\right]_2^3 = \frac{1}{2}\left(-\cos 9 + \cos 4\right)$$

$$\int_2^3 \frac{x}{x^2-3}\,dx =$$

$$= \frac{1}{2}\int_2^3 \frac{2x}{x^2-3}\,dx = \frac{1}{2}\int_2^3 \frac{(x^2-3)'}{x^2-3}\,dx = \frac{1}{2}\left[\ln|x^2-3|\right]_2^3 = \frac{1}{2}\left(\ln 6 - \underbrace{\ln 1}_{0}\right)$$

$$= \frac{1}{2}\ln 6.$$

$$\int 2x \arctan x \, dx = x^2 \arctan x - \int x^2 \cdot \frac{1}{1+x^2}\, dx$$

$$= x^2 \arctan x - \int \frac{x^2+1-1}{x^2+1}\, dx$$

$$= x^2 \arctan x - \int 1 - \frac{1}{x^2+1}\, dx$$

$$= x^2 \arctan x - x + \arctan x + C$$

$$\int e^x \sin x \, dx = e^x \sin x - \int e^x \cos x \, dx$$ (double tour par parties)

$$= e^x \sin x - \left[e^x \cos x - \int e^x (-\sin x)\, dx \right]$$

$$= e^x \sin x - e^x \cos x - \int e^x \sin x \, dx$$

donc $2 \int e^x \sin x \, dx = e^x(\sin x - \cos x) + C \Rightarrow \int e^x \sin x \, dx = \frac{1}{2} e^x (\sin x - \cos x) + C.$

$$\int \sqrt{1-x^2}\, dx = x\sqrt{1-x^2} - \int x \cdot \frac{-\cancel{2}x}{\cancel{2}\sqrt{1-x^2}}\, dx$$

$$= x\sqrt{1-x^2} - \int \frac{-x^2+1-1}{\sqrt{1-x^2}}\, dx$$

$$= x\sqrt{1-x^2} - \int \left(\sqrt{1-x^2} - \frac{1}{\sqrt{1-x^2}} \right) dx$$

$$= x\sqrt{1-x^2} - \int \sqrt{1-x^2}\, dx - \arccos x$$

donc $2\int \sqrt{1-x^2}\, dx = x\sqrt{1-x^2} - \arccos x + C \Rightarrow \int \sqrt{1-x^2}\, dx = \frac{1}{2}x\sqrt{1-x^2} - \frac{1}{2}\arccos x + C.$

<u>Ex. 5.</u> $\int \sqrt{1-x^2}\, dx =$ $x = \cos t \Rightarrow t = \arccos x$

$dx = -\sin t \, dt$

$$= \int \sqrt{1-\cos^2 t}\,(-\sin t)\, dt = \int -\sin^2 t \, dt = -\int \frac{1-\cos 2t}{2}\, dt$$

$$= -\frac{1}{2} t + \frac{1}{4}\sin 2t + C$$

$$= -\frac{1}{2}\arccos x + \frac{1}{4}\sin(2\arccos x) + C$$

$$= -\frac{1}{2}\arccos x + \frac{1}{4}\cdot 2\sin(\arccos x)\cos(\arccos x) + C$$

$$= -\frac{1}{2}\arccos x + \frac{1}{2}\sqrt{1-x^2}\cdot x + C$$ (comme dessus)

$$\int_{\sqrt{2}}^{\sqrt{3}} \frac{x}{\sqrt{4-x^2}}\,dx = -\int_{\sqrt{2}}^{\sqrt{3}} \frac{-2x}{2\sqrt{4-x^2}}\,dx = -\left[\sqrt{4-x^2}\right]_{\sqrt{2}}^{\sqrt{3}}$$

$$= -(1-0) = -1$$

$$\int_0^{\frac{\pi}{3}} \frac{\cos x}{1-\sin x}\,dx =$$

$\sin x = t$

$x = \arcsin t$ sur $\left[0, \frac{\pi}{3}\right]$

$\cos x = \sqrt{1-t^2}$ $\quad dx = \frac{1}{\sqrt{1-t^2}}\,dt$

$$= \int_{t=0}^{t=\frac{\sqrt{3}}{2}} \frac{\sqrt{1-t^2}}{1-t} \cdot \frac{1}{\sqrt{1-t^2}}\,dt$$

$$= \int_0^{\frac{\sqrt{3}}{2}} \frac{1}{1-t}\,dt = \left[-\ln(1-t)\right]_0^{\frac{\sqrt{3}}{2}} = -\ln\left(1-\frac{\sqrt{3}}{2}\right) + \underbrace{\ln 1}_{=0} = -\ln\left(1-\frac{\sqrt{3}}{2}\right)$$

$$\int_0^1 x^2\sqrt{x^3+1}\,dx = \frac{1}{3}\int_0^1 \sqrt{x^3+1}\cdot(3x^2)\,dx = \frac{1}{3}\left[\frac{(x^3+1)^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^1 = \frac{2}{9}\left(2^{\frac{3}{2}} - 1\right)$$

$$\int_0^{\frac{\pi}{2}} \sin x\cos x\,dx = \int_0^{\frac{\pi}{2}} \frac{1}{2}\sin 2x\,dx = \frac{1}{2}\left[-\frac{\cos 2x}{2}\right]_0^{\frac{\pi}{2}} = \frac{1}{2}\left(-\frac{1}{2}\overset{-1}{\cos\pi} + \frac{1}{2}\cos 0\right)$$

$$= \frac{1}{2}\left(\frac{1}{2}+\frac{1}{2}\right) = \frac{1}{2}$$

$$\lim_{x\to+\infty} \int_0^x \frac{1}{1+t^2}\,dt = \lim_{x\to+\infty} \left[\arctan t\right]_0^x = \lim_{x\to+\infty}\left(\arctan x - \underbrace{\arctan 0}_{=0}\right) = \frac{\pi}{2}$$

$$\int_0^{\pi} \frac{\sin(\sqrt{x})}{\sqrt{x}}\,dx = 2\int_0^{\pi} \frac{\sin\sqrt{x}}{(2\sqrt{x})}\,dx = 2\left[-\cos\sqrt{x}\right]_0^{\pi} = 2\left(-\cos\sqrt{\pi} + 1\right)$$

Ex. 8. $\displaystyle\int \frac{\log x}{x}\,dx = \int \log x \cdot \frac{1}{x}\,dx = \int \log x \cdot (\log x)'\,dx = \frac{1}{2}\log^2 x + C$

ou $\displaystyle\int \frac{\log x}{x}\,dx = \log x \cdot \log x - \int \log x \cdot \frac{1}{x} \implies 2\int \frac{\log x}{x}\,dx = \log^2 x + C$ etc.

Ex. 9. $\displaystyle\frac{1}{x^2-4x-5} = \frac{a}{x+1} + \frac{b}{x-5} = \frac{a(x-5)+b(x+1)}{(x+1)(x-5)} = \frac{x(a+b) - 5a+b}{x^2-4x-5}$

$$\begin{cases} a+b=0 \\ -5a+b=1 \end{cases} \quad \begin{cases} a=-b \\ 5b+b=1 \end{cases} \quad \begin{cases} a=-\frac{1}{6} \\ b=\frac{1}{6} \end{cases}$$

$$\implies \int_0^2 \frac{1}{x^2-4x-5}\,dx = -\frac{1}{6}\int_0^2 \frac{1}{x+1}\,dx + \frac{1}{6}\int_0^2 \frac{1}{x+5}\,dx$$

$$= -\frac{1}{6}\left[\ln|x+1|\right]_0^2 + \frac{1}{6}\left[\ln|x-5|\right]_0^2 = -\frac{1}{6}\ln 3 + \frac{1}{6}(\ln 3 - \ln 5)$$

$$= -\frac{1}{6}\ln 5$$

Ex.10. $\displaystyle\int_2^3 \frac{x}{x^2-3}\,dx = \frac12\int_2^3 \frac{2x}{x^2-3}\,dx = \frac12\Big[\ln|x^2-3|\Big]_2^3 = \frac12\ln 6 - \cancel{\frac12\ln 1} = \frac12\ln 6$

$$\int_1^2 \frac{x}{\sqrt{5-x^2}}\,dx = -\int_1^2 \frac{-2x}{2\sqrt{5-x^2}}\,dx = -\Big[\sqrt{5-x^2}\Big]_1^2 = -(1-2) = 1.$$

$$\int_0^1 \frac{\cos x}{1-\sin^2 x}\,dx =$$

il suffit de remarquer que $1-\sin^2 x = \cos^2 x$

$$= \int_0^1 \frac{\cos x}{\cos^2 x}\,dx = \int_0^1 \frac{1}{\cos x}\,dx = \Big[\ln\Big|\tan\Big(\frac{x}{2}+\frac{\pi}{4}\Big)\Big|\Big]_0^1 =$$

$$= \ln\Big(\tan\Big(\frac12+\frac{\pi}{4}\Big)\Big) - \underbrace{\ln \underbrace{\tan\Big(\frac{\pi}{4}\Big)}_{=1}}_{=0} = \ln\Big(\tan\Big(\frac{\pi}{4}+\frac12\Big)\Big)$$

$$= \ln\frac{\sin(\frac{\pi}{4}+\frac12)}{\cos(\frac{\pi}{4}+\frac12)} = \ln\frac{\frac{1}{\sqrt2}\cos\frac12+\frac{1}{\sqrt2}\sin\frac12}{\frac{1}{\sqrt2}\cos\frac12-\frac{1}{\sqrt2}\sin\frac12} = \ln\frac{\cos\frac12+\sin\frac12}{\cos\frac12-\sin\frac12}$$

On peut le calculer aussi en posant $\sin x = t$.

Ex. 11. $\displaystyle\int x^2\sqrt{x^3+1}\,dx$ voir ex. 7

$$\int \frac{x+1}{x^2+2x+2}\,dx = \frac12\int\frac{2x+2}{x^2+2x+2}\,dx = \frac12\int\frac{(x^2+2x+2)'}{x^2+2x+2}\,dx = \frac12\ln|x^2+2x+2| + C$$

$$\int \sin x\cos x\,dx = \int \sin x\cdot(\sin x)'\,dx = \frac{(\sin x)^2}{2}+C$$

$$\int \sin x\cos x\,dx = \frac12\int \sin 2x\,dx = -\frac14\cos 2x + C$$

Remarquez que $\frac{(\sin x)^2}{2}$ et $-\frac14\cos 2x$ ont la même dérivée, donc elles diffèrent d'une constante :

$$\frac{(\sin x)^2}{2} = \frac{1-\cos 2x}{4} = \frac14 - \frac14\cos 2x.$$

Ex. 12. $\alpha, T > 0$

$$I(T) = \int_0^T \alpha e^{-\alpha t}\, dt = \left[-e^{-\alpha t}\right]_0^T = \left(-e^{-\alpha T} + 1\right) = 1 - e^{-\alpha T} \xrightarrow[T \to +\infty]{} 1$$

$$E(T) = \int_0^T t\alpha e^{-\alpha t}\, dt = \left[-e^{-\alpha t}\cdot t\right]_0^T + \int_0^T e^{-\alpha t}\cdot dt = -e^{-\alpha T}\cdot T + \left[-\frac{1}{\alpha} e^{-\alpha t}\right]_0^T$$

$$= \underbrace{-e^{-\alpha T}\cdot T}_{\downarrow \, 0} - \underbrace{\frac{1}{\alpha} e^{-\alpha T}}_{\downarrow \, 0} + \frac{1}{\alpha} \xrightarrow[T \to +\infty]{} \frac{1}{\alpha}$$

Ex. 13. $$\int \frac{1}{\sin x}\, dx = \int \frac{1}{\sin\left(2\,\frac{x}{2}\right)}\, dx = \int \frac{1}{2 \sin\frac{x}{2} \cos\frac{x}{2}}\, dx$$

$$= \int \frac{1}{2} \cdot \frac{\cos\frac{x}{2}}{\sin\frac{x}{2}} \cdot \frac{1}{\cos^2\frac{x}{2}}\, dx = \int \frac{1}{2} \cdot \frac{1}{\tan\frac{x}{2}} \cdot \frac{1}{\cos^2\frac{x}{2}}\, dx = \ln\left|\tan\frac{x}{2}\right| + C$$

$$\int \frac{1}{x \ln x \ln(\ln x)}\, dx = \int \frac{1}{\ln \ln x} \cdot \frac{1}{\ln x} \cdot \frac{1}{x}\, dx = \ln\left(\ln\left(\ln x\right)\right) + C$$