Feuille 6

Ex. 1. $f(x)=e^x \quad f'(x)=e^x \quad f''(x)=e^x$

$f(0)=1 \quad f'(0)=1 \quad f''(0)=1$

$\Rightarrow f(x)=f(0)+f'(0)\,x+\frac{f''(0)}{2}x^2+o(x^2) = 1+x+\frac{x^2}{2}+o(x^2)$

$f(x)=\sin x \quad f'(x)=\cos x \quad f''(x)=-\sin x$

$f(0)=0 \quad f'(0)=1 \quad f''(0)=0$

$\Rightarrow f(x)=0+x+0+o(x^2)=x+o(x^2)$

$\cos x$ et $\tan x$: même travail.

$f(x)=\frac{1}{1-x} \quad f'(x)=\frac{1}{(1-x)^2} \quad f''(x)=(-2)\frac{-1}{(1-x)^3}=\frac{2}{(1-x)^3}$

$f(0)=1 \quad f'(0)=1 \quad f''(0)=2$

$\Rightarrow \frac{1}{1-x}=1+x+\frac{1}{2}\cdot 2x^2+o(x^2)=1+x+x^2+o(x^2)$ etc.

Ex. 2. $f(x)=\arctan x \quad f'(x)=\frac{1}{1+x^2}$

$f(0)=0 \quad f'(0)=1$

$\Rightarrow \arctan x = 0+x+o(x)=x+o(x)$

$f(x)=\frac{1+3x-2x^2}{(x-1)^2} \quad f'(x)=\frac{(3-4x)(x-1)^2-(1+3x-2x^2)\cdot 2(x-1)}{(x-1)^4}$

$f(0)=\frac{1}{1}=1 \quad f'(0)=\frac{3\cdot 1-(1)\cdot 2\cdot(-1)}{1}=\frac{5}{1}=5$

$\Rightarrow f(x)=1+5x+o(x)$

$x \xmapsto{f} \frac{\sin(e^x-1)}{x}$: on remarque que

si $g(x)=\sin(e^x-1) \Rightarrow g'(x)=\cos(e^x-1)\,e^x \quad g''(x)=-\sin(e^x-1)e^x\cdot e^x+\cos(e^x-1)\,e^x$

$g(0)=0 \quad g'(0)=1 \quad g''(x)=1$

$g(x)=x+\frac{1}{2}x^2+o(x^2) \Rightarrow f(x)=\frac{g(x)}{x}=1+\frac{1}{2}x+o(x)$

$f(x)=\frac{\sin x}{x}=\frac{x+o(x^2)}{x}=1+o(x)$

Rq. $\frac{o(x^n)}{x^m}=o(x^{n-m}) \quad m\le n$

Ex. 3. facile.

$$x \longmapsto \frac{\ln(1+x^2)}{x}$$

Soit $g(x) = \ln(1+x^2)$ $\quad g'(x) = \frac{2x}{1+x^2}$ $\quad g''(x) = \frac{2(1+x^2) - 2x(2x)}{(1+x^2)^2} = \frac{2-2x^2}{(1+x^2)^2}$

$g(0) = 0$ $\quad g'(0) = 0$ $\quad g''(0) = 2$ $\quad g'''(x) = \frac{(-4x)(1+x^2)^2 - (2-2x^2)2(1+x^2)(2x)}{(1+x^2)^4}$

$$g'''(0) = 0$$

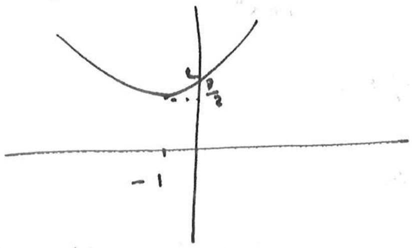
$\Rightarrow g(x) = \frac{1}{2}\cdot 2x^2 + o(x^3)$ et $\frac{\ln(1+x^2)}{x} = x + o(x^2)$

Ex. 4. $f(x) = \frac{1}{1-x} + \cos x$.

DL de f à l'ordre 2 en 0 : $\frac{1}{1-x} = 1 + x + x^2 + o(x^2)$ $\quad \cos x = 1 - \frac{x^2}{2} + o(x^3)$

$\Rightarrow f(x) = 2 + x + \frac{x^2}{2} + o(x^2)$ et dans le voisinage de $x_0 = 0$, on peut approximer le graphe de f par le graphe de son polynôme de Taylor :

$p(x) = \frac{x^2}{2} + x + 2$



$f(x) = \arctan x$ $\quad f'(x) = \frac{1}{1+x^2}$ $\quad f''(x) = \frac{-2x}{(1+x^2)^2}$

$f(0) = 0$ $\quad f'(0) = 1$ $\quad f''(0) = 0$ $\qquad \arctan x = x + o(x^2)$

$f(1) = \frac{\pi}{4}$ $\quad f'(1) = \frac{1}{2}$ $\quad f''(1) = \frac{-2}{4} = -\frac{1}{2}$

$$\arctan x = \frac{\pi}{4} + \frac{1}{2}(x-1) + \frac{1}{2}\left(-\frac{1}{2}\right)(x-1)^2 + o\left((x-1)^2\right).$$

Ex. 5. $\lim\limits_{x\to 0} \frac{\sin x}{x} = \lim\limits_{x\to 0} \frac{x + o(x)}{x} = \lim\limits_{x\to 0} \frac{\cancel{x}\left(1 + \frac{o(x)}{x}\right)}{\cancel{x}} = 1$

$\lim\limits_{x\to 0} \frac{\ln(1+3x)}{x} = \lim\limits_{x\to 0} \frac{3x + o(x)}{x} = 3$

$\lim\limits_{x\to 0} \frac{\sqrt{x+1} - 1}{x} = \lim\limits_{x\to 0} \frac{1 + \frac{1}{2}x + o(x) - 1}{x} = \lim\limits_{x\to 0} \frac{\frac{1}{2}x + o(x)}{x} = \frac{1}{2}$

$\lim\limits_{x\to \pi} \frac{\sin x}{x-\pi} \overset{(H)}{=} \lim\limits_{x\to\pi} \frac{\cos x}{1} = -1$ $\qquad$ (la règle de l'Hôpital est un cas particulier de la technique avec les DL)

$$\lim_{x\to 0} \frac{(1-e^x)\sin x}{x^2+x^3} = \lim_{x\to 0} \frac{\left(1-\left(1+x+\frac{x^2}{2}+o(x^2)\right)\right)(x+o(x^2))}{x^2+x^3}$$

$$= \lim_{x\to 0} \frac{\left(-x-\frac{x^2}{2}+o(x^2)\right)(x+o(x^2))}{x^2+o(x^2)}$$

$$= \lim_{x\to 0} \frac{-x^2+o(x^2)}{x^2+o(x^2)} = -1$$

ou bien avec la règle de l'Hôpital :

$$\lim_{x\to 0} \frac{(1-e^x)\sin x}{x^2+x^3} \overset{(H)}{=} \lim_{x\to 0} \frac{-e^x \sin x + (1-e^x)\cos x}{2x+3x^2} \overset{(H)}{=} \lim_{x\to 0} \frac{-\overset{\to 0}{e^x\sin x} - \overset{\to 1}{e^x\cos x} + \overset{\to 1}{(-e^x)\cos x} + \overset{\to 0}{(1-e^x)(-\sin x)}}{2} = \frac{-2}{2} = -1 .$$

Ex-6. $f(x) = \sin x^2 \Rightarrow f'(x) = \cos x^2 \cdot 2x \qquad f''(x) = -\sin x^2 \cdot 2x \cdot 2x + \cos x^2 \cdot 2$

$f(0) = 0 \qquad f'(0) = 0 \qquad f''(0) = 2$

$$\sin x^2 = \frac{1}{2} \cdot 2x^2 + o(x^2) = x^2 + o(x^2)$$

$g(x) = 1-\cos x \qquad g'(x) = \sin x \qquad g''(x) = \cos x$

$g(0) = 0 \qquad g'(0) = 0 \qquad g''(0) = 1$

$$1-\cos x = \frac{1}{2}x^2 + o(x^2)$$

$$\Rightarrow \lim_{x\to 0} \frac{\sin x^2}{1-\cos x} = \lim_{x\to 0} \frac{x^2+o(x^2)}{\frac{1}{2}x^2+o(x^2)} = \frac{1}{\frac{1}{2}} = 2 .$$

$$f(x) = \sqrt{x+1} - 1 - \frac{x}{2} \qquad \sqrt{x+1} = 1 + \frac{1}{2}x \cancel{+} - \frac{1}{8}x^2 + o(x^2)$$

$$= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + o(x^2) - 1 - \frac{x}{2} = -\frac{1}{8}x^2 + o(x^2)$$

$$\lim_{x\to 0} \frac{f(x)}{x^2} = \lim_{x\to 0} \frac{-\frac{1}{8}x^2 + o(x^2)}{x^2} = -\frac{1}{8} .$$

Pareil pour les autres limites.