Exercices sur l'intégration en deux variables

Ex. $\displaystyle\int_{x=1}^{2}\int_{y=0}^{1}\frac{e^{-y}}{x\sqrt{x}}\,dx\,dy$ $\qquad D=[1,2]\times[0,1]$

$$=\int_{x=1}^{2}\frac{1}{x\sqrt{x}}\left(\int_{y=0}^{1}e^{-y}\,dy\right)dx$$

$$=\int_{x=1}^{2}\frac{1}{x\sqrt{x}}\left[-e^{-y}\right]_0^1 dx=\int_{x=1}^{2}\frac{1}{x\sqrt{x}}\left(-e^{-1}+1\right)dx=\left(1-\frac{1}{e}\right)\left[\frac{x^{-\frac{1}{2}}}{-\frac{1}{2}}\right]_1^2=$$

$$=\left(1-\frac{1}{e}\right)\cancel{(\ldots)}\cdot(-2)\left(\frac{1}{\sqrt{2}}-1\right)=\left(1-\frac{1}{e}\right)\left(-\sqrt{2}+1\right)$$

Ex. $\displaystyle\iint_D x^2(2y+1)\,dx\,dy=$ $\qquad D=[-1,1]\times[0,1]$

$$=\int_{x=-1}^{1}x^2\,dx\cdot\int_{y=0}^{1}(2y+1)\,dy=\left[\frac{x^3}{3}\right]_{-1}^{1}\cdot\left[2\frac{y^2}{2}+y\right]_{y=0}^{1}=\left(\frac{1}{3}+\frac{1}{3}\right)\left(1+1-0\right)=\frac{4}{3}.$$

Ex. $\displaystyle\int_D\sqrt{x+y}\,dx\,dy=$ $\qquad D=[0,1]\times[0,1]$

$$=\int_{y=0}^{1}\left(\int_{x=0}^{1}\sqrt{x+y}\,dx\right)dy=\int_{y=0}^{1}\left[\frac{2}{3}(x+y)\sqrt{x+y}\right]_0^1 dy=$$

$$=\int_{y=0}^{1}\left(\frac{2}{3}(y+1)^{\frac{3}{2}}-\frac{2}{3}(y)^{\frac{3}{2}}\right)dy=\frac{2}{3}\int_{y=0}^{1}\left((y+1)^{\frac{3}{2}}-y^{\frac{3}{2}}\right)dy=\frac{2}{3}\left[\frac{2}{5}\left((y+1)^{\frac{5}{2}}-y^{\frac{5}{2}}\right)\right]_{y=0}^{1}=$$

$$=\frac{2}{3}\cdot\frac{2}{5}\left(2^{\frac{5}{2}}-1-(1)\right)=\frac{4}{15}\cdot\left(\sqrt{2}\cdot 4-2\right)$$

Ex. $\displaystyle\int_D y\cos(xy)\,dx\,dy=$ $\qquad D=[1,2]\times[1,2]$

$$=\int_{y=1}^{2}\left(\int_{x=1}^{2}y\cos(xy)\,dx\right)dy=\int_{y=1}^{2}\left[y\frac{\sin(xy)}{y}\right]_{x=1}^{2}dy=\int_{y=1}^{2}\left[\sin(xy)\right]_{x=1}^{2}dy=$$

$$=\int_{y=1}^{2}\left(\sin 2y-\sin y\right)dy=\left[\frac{-\cos 2y}{2}+\cos y\right]_{y=1}^{2}=-\frac{\cos 4}{2}+\cos 2-\left(-\frac{\cos 2}{2}+\cos 1\right)=$$

$$=-\cos 1+\frac{1}{2}\cos 2+\cos 2-\frac{1}{2}\cos 4=-\cos 1+\frac{3}{2}\cos 2-\frac{1}{2}\cos 4.$$

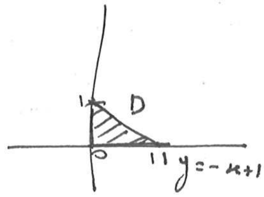
Ex. $\displaystyle\int_D e^x \cos y \, dx\,dy =$ $\qquad D = [0,1] \times [0,3]$

$$= \int_{x=0}^{1} e^x\,dx \cdot \int_{y=0}^{3} \cos y\,dy = \left[e^x\right]_0^1 \cdot \left[\sin y\right]_0^3 = (e-1)(\sin 3)$$

Ex. $\displaystyle\int_D xy \, dx\,dy =$ $\qquad D = \{(x,y) \in \mathbb{R}^2 : x,y \geq 0,\ x+y \leq 1\}$

$x+y \leq 1$

$y \leq -x+1$

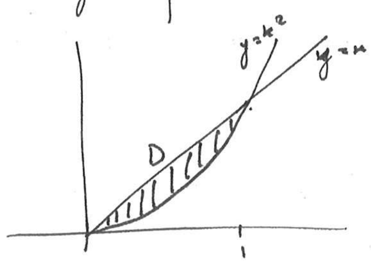


$$= \int_{x=0}^{1} \left(\int_{y=0}^{-x+1} xy\,dy \right) dx$$

$$= \int_{x=0}^{1} x \left[\frac{y^2}{2}\right]_{y=0}^{y=-x+1} dx = \int_{x=0}^{1} x \cdot \left(\frac{(-x+1)^2}{2}\right) dx = \frac{1}{2}\int_{x=0} \left(x^3 - 2x^2 + x\right) dx =$$

$$= \frac{1}{2}\left[\frac{x^4}{4} - 2\frac{x^3}{3} + \frac{x^2}{2}\right]_0^1 = \frac{1}{2}\left(\frac{1}{4} - \frac{2}{3} + \frac{1}{2}\right) = \frac{1}{2} \cdot \frac{3-8+6}{12} = \frac{1}{24}$$

Ex. $\displaystyle\int_D x^2 \, dx\,dy =$ $\qquad D = \{(x,y) \in \mathbb{R}^2 : x \in [0,1],\ x^2 \leq y \leq x\}$

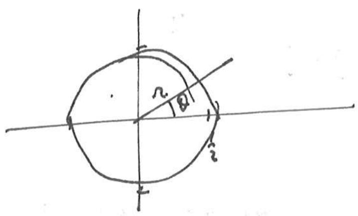


$$= \int_{x=0}^{1} \left(\int_{y=x^2}^{y=x} x^2\,dy \right) dx$$

$$= \int_{x=0}^{1} x^2 \left[y\right]_{y=x^2}^{y=x} dx = \int_{x=0}^{1} x^2 (x - x^2)\,dx = \int_{x=0}^{1} (x^3 - x^4)\,dx = \left[\frac{x^4}{4} - \frac{x^5}{5}\right]_{x=0}^{1} = \left(\frac{1}{4} - \frac{1}{5}\right) = \frac{1}{20}$$

Ex. $\displaystyle\int_D x \, dx\,dy =$ $\qquad D = \{(x,y) \in \mathbb{R} : x^2 + y^2 - x \leq 0\}$

$$\left(x - \frac{1}{2}\right)^2 + y^2 \leq \frac{1}{4}$$



$x = r\cos\theta$

$y = r\sin\theta$

coordonnées polaires : r, θ

$\Delta : \quad 0 \leq r \leq \frac{1}{2}, \quad 0 \leq \theta \leq 2\pi$

$$= \int_\Delta r\cos\theta \cdot r\,dr\,d\theta$$

$$= \int_0^{2\pi} \cos\theta\,d\theta \cdot \int_0^{\frac{1}{2}} r^2\,dr = \left[\sin\theta\right]_0^{2\pi} \cdot \int_0^{\frac{1}{2}} r^2\,dr = (\underbrace{\sin 2\pi}_{=0} - \sin 0) \cdot \int \ldots = 0$$