A VARIATIONAL APPROACH TO REMOVE MULTIPLICATIVE NOISE

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Abstract. This paper focuses on the problem of multiplicative noise removal. We draw our inspiration from the modeling of speckle noise. By using a MAP estimator, we can derive a functional whose minimizer corresponds to the denoised image we want to recover. Although the functional is not convex, we prove the existence of a minimizer and we show the capability of our model on some numerical examples. We study the associated evolution problem, for which we derive existence and uniqueness results for the solution. We prove the convergence of an implicit scheme to compute the solution.

Key words. Calculus of variation, functional analysis, BV, variational approach, multiplicative noise, speckle noise, image restoration.

AMS subject classifications. 68U10, 94A08, 49J40, 35A15, 35B45, 35B50.

1. Introduction. Image denoising is a widely studied problem in the applied mathematics community. We refer the reader to [4, 14] and references herein for an overview of the subject. Most of the literature deals with the additive noise model: given an original image u, it is assumed that it has been corrupted by some additive noise v. The problem is then to recover u from the data $f = u + v$. Many approaches have been proposed. Among the most famous ones are wavelets approaches [17], stochastic approaches [21], and variational approaches [37, 31].

In this paper, we are concerned with a different denoising problem. The assumption is that the original image u has been corrupted by some multiplicative noise v: the goal is then to recover u from the data $f = uv$. Multiplicative noise occurs as soon as one deals with active imaging system: laser images, microscope images, SAR images, ... As far as we know, the only variational approach devoted to multiplicative noise is the one by Rudin et al [36] as used for instance in [33, 28, 29, 38]. The goal of this paper is to go further and to propose a functional well-adapted to the removing of multiplicative noise. Inspired from the modeling of active imaging systems, this functional is:

$$E(u) = \int |Du| + \int \left(\log u + \frac{f}{u} \right)$$

where f is the original corrupted image and $\int |Du|$ stands for the total variation of u.

From a mathematical point of view, part of the difficulty comes from the fact that, contrary to the additive case, the proposed model is nonconvex, which causes uniqueness problems, as well as the issue of convergence of the algorithms. Another mathematical issue comes from the fact that we deal with linear growth functional. The natural space in which we compute a solution is BV, the space of functions with bounded variations. But contrary to what happens with classical Sobolev spaces, the minimum of the functional does not verify an associated Euler-Lagrange equation (see [3] and [2] where this problem is studied) but a differential inclusion involving the subdifferential of the energy.

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The paper is organized as follows. We draw our inspiration from the modeling of active imaging systems, which we remind to the reader in Section 2. We use the classical MAP estimator to derive a new model to denoise non textured SAR images in Section 3. We then consider this model from a variational point of view in Section 4 and we carry out the mathematical analysis of the functional in the continuous setting. In Section 5 we illustrate our model by displaying some numerical examples. We also compare it with other ones. We then study in Section 6 the evolution equation associated to the problem. To prove the existence and the uniqueness of a solution to the evolution problem we first consider a semi-implicit discretization scheme and then we let the discretization time step goes to zero. The proofs are rather technical and we give them in an appendix.

2. Speckle noise modeling. Synthetic Aperture Radar (SAR) images are strongly corrupted by a noise called speckle. A radar sends a coherent wave which is reflected on the ground, and then registered by the radar sensor [30, 26]. If the coherent wave is reflected on a coarse surface (compared to the radar wavelength), then the image processed by the radar is degraded by a noise with large amplitude: this gives a speckled aspect to the image, and this is the reason why such a noise is called speckle [24]. To illustrate the difficulty of speckle noise removal, Figure 2.1 shows a 1 Dimensional noise free signal, and the corresponding speckled signal (the noise free signal has been multiplied by a speckle noise of mean 1). It can be seen that almost all the information has disappeared (notice in particular that the vertical scale goes from 40 to 120 for the noise free signal presented in (a), whereas it goes from 0 to 600 on the speckled signal presented in (b)). As a comparison, Figure 2.1 (c) shows the 1D signal of (a) once it has been multiplied by a Gaussian noise of mean 1 and standard deviation 0.2 (as used for instance in [36]), and Figure 2.1 (d) shows the 1D signal of (a) once it has been added a Gaussian noise of zero mean and standard deviation $\sigma = 15$ (notice that for both (c) and (d), the vertical scale goes from 20 to 140).

If we denote by I the image intensity considered as a random variable, then I follows a negative exponential law. The density function is: $g_I(x) = \frac{1}{\mu_I} e^{-\frac{x}{\mu_I}} \mathbf{1}_{\{x \geq 0\}}$, where μ_I is both the mean and the standard deviation of I. In general the image is obtained as the summation of L different images (this is very classical with satellite images). If we assume that the variables I_k, $1 \leq k \leq L$ are independent, and have the same mean μ_I, then the intensity $J = \frac{1}{L} \sum_{k=1}^{L} I_k$ follows a gamma law, with density function: $g_J(x) = \left(\frac{L}{\mu_I}\right)^L \frac{1}{\Gamma(L)} x^{L-1} \exp\left(-\frac{Lx}{\mu_I}\right) \mathbf{1}_{\{x \geq 0\}}$, where $\Gamma(L) = (L-1)!$. Moreover, μ_I is the mean of J, and $\frac{\mu_I}{\sqrt{L}}$ its standard deviation.

The classical modeling [41] for SAR image is: $I = RS$, where I is the intensity of the observed image, R the reflectance of the scene (which is to be recovered), and S the speckle noise. S is assumed to follow a gamma law with mean equal to one: $g_S(s) = \frac{L^L}{\Gamma(L)} s^{L-1} \exp(-Ls) \mathbf{1}_{\{s \geq 0\}}$. In the rest of the paper, we will assume that the image to recover has been corrupted by some multiplicative gamma noise.

Speckle removal methods have been proposed in the literature. There are geometric filters, such as Crimmins filter [15] based on the application of convex hull algorithms. There are adaptive filters, such as Lee filter, Kuan filter, or its improvement proposed by Wu and Maitre [42]: first and second order statistic computed in local windows are incorporated in the filtering process. Adaptive filters with some modeling of the scene, such as Frost filter have been proposed. The criterion is based on a MAP estimator, and Markov random fields can be used such as in [40, 16]. Another class of filters are multi-temporal ones, such as Bruniquel filter [10]: by

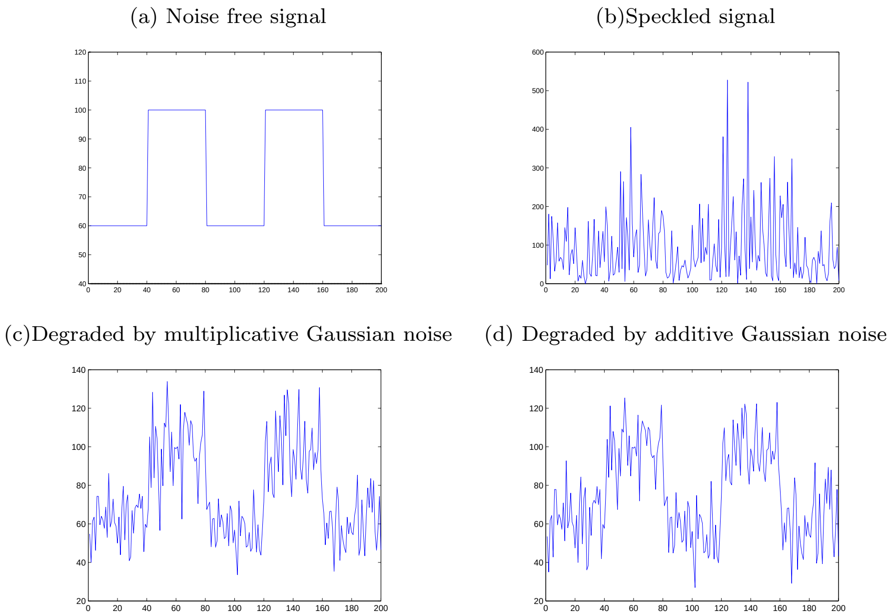

FIG. 2.1. *Speckle noise in 1D: notice that the vertical scale is not the same on the different images (scale between 40 and 120 on (a), 0 and 600 on (b), 20 and 140 on (c), 20 and 140 on (d)) (a) 1D signal f; (b) f degraded by speckle noise of mean 1; (c) f degraded by a multiplicative Gaussian noise ($\sigma = 0.2$); (d) f degraded by an additive Gaussian noise ($\sigma = 15$). Speckle noise is much stronger than classical additive Gaussian noise [37] or classical multiplicative Gaussian noise [36].*

computing barycentric means, the standard deviation of the noise can be reduced (provided that several different images of the same scene are available). A last class of methods are variational ones, such as [37, 36, 6], where the solution is computed with PDEs.

3. A variational multiplicative denoising model. The goal of this section is to propose a new variational model for denoising images corrupted by multiplicative noise and in particular for SAR images. We start from the following multiplicative model: $f = uv$, where f is the observed image, $u > 0$ the image to recover, and v the noise. We consider that f, u, and v are instances of some random variables F, U and V. In the following, if X is a random variable, we denote by g_X its density function. We refer the interested reader to [25] for further details about random variables. In this section, we consider discretized images. We denote by $\mathcal{S}$ the set of the pixels of the image. Moreover, we assume that the samples of the noise on each pixel $s \in S$ are mutually independent and identically distributed (i.i.d) with density function g_V.

3.1. Density laws with a multiplicative model. Our goal is to maximize $P(U|F)$ thus thanks to Bayes rule we need to know $P(F|U)$ and $g_{F|U}$.

PROPOSITION 3.1. *Assume U and V are independent random variables, with continuous density functions g_U and g_V. Denote by $F = UV$. Then we have for $u > 0$:*

$$g_V\left(\frac{f}{u}\right)\frac{1}{u} = g_{F|U}(f|u) \tag{3.1}$$

Proof. It is a standard result (see [25]) for instance). We give the proof here for the sake of completeness.

Let $\mathcal{A}$ an open subset in $\mathbb{R}$. We have:
$\int_{\mathbb{R}} g_{F|U}(f|u)\mathbf{1}_{\{f\in\mathcal{A}\}} = P(F\in\mathcal{A}|U) = \frac{P(F\in\mathcal{A},U)}{P(U)} = \frac{P\left(\left(V=\frac{F}{U}\right)\in\frac{\mathcal{A}}{U},U\right)}{P(U)}$. Using the fact that U and V are independent, we have:

$$\frac{P\left(\left(V=\frac{F}{U}\right)\in\frac{\mathcal{A}}{U},U\right)}{P(U)} = P\left(\left(V=\frac{F}{U}\right)\in\frac{\mathcal{A}}{U}\right) = \int_{\mathbb{R}} g_V(v)\mathbf{1}_{\left\{v\in\frac{\mathcal{A}}{u}\right\}}\,dv = \int_{\mathbb{R}} g_V(f/u)\mathbf{1}_{\{f\in\mathcal{A}\}}\,\frac{df}{u}.$$

□

3.2. Our model via the MAP estimator. We assume the following multiplicative model: $f = uv$, where f is the observed image, u the image to recover, and v the noise. We assume that v follows a gamma law with mean one, and with density function:

$$g_V(v) = \frac{L^L}{\Gamma(L)}v^{L-1}e^{-Lv}\,\mathbf{1}_{\{v\geq 0\}} \tag{3.2}$$

Using Proposition 3.1, we therefore get:

$$g_{F|U}(f|u) = \frac{L^L}{u^L\Gamma(L)}f^{L-1}e^{-\frac{Lf}{u}} \tag{3.3}$$

We also assume that U follows a Gibbs prior:

$$g_U(u) = \frac{1}{Z}\exp\left(-\gamma\phi(u)\right) \tag{3.4}$$

where Z is a normalizing constant, and ϕ a non negative given function. We aim at maximizing $P(U|F)$. This will lead us to the classical Maximum a Posteriori estimator. From Bayes rule, we have: $P(U|F) = \frac{P(F|U)\,P(U)}{P(F)}$. Maximizing $P(U|F)$ amounts to minimizing the log-likelihood:

$$-\log(P(U|F) = -\log(P(F|U)) - \log(P(U)) + \log(P(F)) \tag{3.5}$$

Let us remind the reader that the image is discretized. We denote by $\mathcal{S}$ the set of the pixels of the image. Moreover, we assume that the samples of the noise on each pixel $s\in\mathcal{S}$ are mutually independent and identically distributed (i.i.d) with density g_V. We therefore have: $P(F|U) = \prod_{s\in\mathcal{S}} P(F(s)|U(s))$, where $F(s)$ (resp. $U(s)$) is the instance of the variable F (resp. U) at pixel s. Since $\log(P(F))$ is a constant, we just need to minimize:

$$-\log\left(P(F|U)\right) - \log(P(U)) = -\sum_{s\in\mathcal{S}}\left(\log\left(P(F(s)|U(s))\right) - \log(P(U(s)))\right) \tag{3.6}$$

Using (3.3), and since Z is a constant, we eventually see that minimizing $-\log\left(P(F|U)\right)$ amounts to minimizing:

$$\sum_{s\in\mathcal{S}}\left(L\left(\log U(s) + \frac{F(s)}{U(s)}\right) + \gamma\phi(U(s))\right) \tag{3.7}$$

The previous computation leads us to propose the following functional for restoring images corrupted with gamma noise:

$$\int \left(\log u + \frac{f}{u} \right) dx + \frac{\gamma}{L} \int \phi(u)\, dx \tag{3.8}$$

Remarks: *1)* It is easy to check that the function $u \to \log u + \frac{f}{u}$ reaches its minimum value $1 + \log f$ over $\mathbb{R}_*^+$ for $u = f$.

2) Multiplicative Gaussian noise: in the additive noise case, the most classical assumption is to assume that the noise is a white Gaussian noise. However, this can no longer be the case when dealing with multiplicative noise, except in the case of tiny noise. Indeed, if the model is $f = uv$ where v is a Gaussian noise with mean 1, then some instances of v are negative. Since the data f is assumed positive, this implies that the restored image u has some negative values which is of course impossible. Nevertheless, numerically, if the standard deviation of the noise is smaller than 0.2 (i.e. in the case of tiny noise), then it is very unlikely that v takes some negative values. See also [32] where some limitations of Bayesian estimators approach are investigated.

4. Mathematical study of the Variational model. In this section, we propose a nonconvex model to remove multiplicative noise, for which we prove the existence of a solution.

4.1. Preliminaries. Throughout our study, we will use the following classical distributional spaces. $\Omega \subset R^2$ will denote an open bounded set of R^2 with Lipschitz boundary.

• $\mathcal{D}(\Omega) = C_0^\infty(\Omega)$ is the set of functions in $C^\infty(\Omega)$ with compact support in Ω. We denote by $\mathcal{D}'(\Omega)$ the dual space of $\mathcal{D}(\Omega)$, i.e. the space of distributions on Ω.

• $W^{m,p}(\Omega)$ denotes the space of functions in $L^p(\Omega)$ whose distributional derivatives $D^\alpha u$ are in $L^p(\Omega)$, $p \in [1, +\infty)$, $m \geq 1$, $m \in \mathbb{N}$, $|\alpha| \leq m$. For further details on these spaces, we refer the reader to [19, 20].

• $BV(\Omega)$ is the subspace of functions $u \in L^1(\Omega)$ such that the following quantity is finite:

$$J(u) = \sup \left\{ \int_\Omega u(x) \mathrm{div}\,(\xi(x)) dx / \xi \in C_0^\infty(\Omega, \mathbb{R}^2), \|\xi\|_{L^\infty(\Omega, \mathbb{R}^N)} \leq 1 \right\} \tag{4.1}$$

$BV(\Omega)$ endowed with the norm $\|u\|_{BV} = \|u\|_{L^1} + J(u)$ is a Banach space. If $u \in BV(\Omega)$, the distributional derivative Du is a bounded Radon measure and (4.1) corresponds to the total variation, i.e. $J(u) = \int_\Omega |Du|$.

For $\Omega \subset R^2$, if $1 \leq p \leq 2$, we have: $BV(\Omega) \subset L^p(\Omega)$. Moreover, for $1 \leq p < 2$, this embedding is compact. For further details on $BV(\Omega)$, we refer the reader to [1].

• Since $BV(\Omega) \subset L^2(\Omega)$, we can extend the functional J (which we still denote by J) over $L^2(\Omega)$:

$$J(u) = \begin{cases} \int_\Omega |Du| \text{ if } u \in BV(\Omega) \\ +\infty \text{ if } u \in L^2(\Omega) \backslash BV(\Omega) \end{cases} \tag{4.2}$$

We can then define the subdifferential ∂J of J [35]: $v \in \partial J(u)$ iff for all $w \in L^2(\Omega)$, we have $J(u + w) \geq J(u) + \langle v, w \rangle_{L^2(\Omega)}$ where $\langle ., . \rangle_{L^2(\Omega)}$ denotes the usual inner product in $L^2(\Omega)$.

• *Decomposability of $BV(\Omega)$:* If u in $BV(\Omega)$, then $Du = \nabla u\, dx + D_s u$, where $\nabla u \in L^1(\Omega)$ and $D_s u \perp dx$. ∇u is called the regular part of Du.

• *Weak -* topology on* $BV(\Omega)$: If (u_n) is a bounded sequence in $BV(\Omega)$, then up to a subsequence, there exists $u \in BV(\Omega)$ such that: $u_n \to u$ in $L^1(\Omega)$ strong, and $Du_n \to Du$ in the sense of measure, i.e. $\langle Du_n, \phi \rangle \to \langle Du, \phi \rangle$ for all ϕ in $(C_0^\infty(\Omega))^2$.
• *Approximation by smooth functions*: If u belongs to $BV(\Omega)$, then there exits a sequence u_n in $C^\infty(\Omega) \bigcap BV(\Omega)$ such that $u_n \to u$ in $L^1(\Omega)$ and $J(u_n) \to J(u)$ as $n \to +\infty$.
• In this paper, if a function f belongs to $L^\infty(\Omega)$, we denote by $\sup_\Omega f$ (resp. $\inf_\Omega f$) the supess of f (resp. the infess of f). We recall that supess $f = \inf\{C \in \mathbb{R};\ f(x) \leq C\, a.e.\}$ and infess $f = \sup\{C \in \mathbb{R};\ f(x) \geq C\, a.e.\}$.

4.2. The variational Model. The application we have in mind is the denoising of non textured SAR images. Inspired by the works of Rudin et al [37, 36], we decide to choose $\phi(u) = J(u)$.

We thus propose the following restoration model (λ being a regularization parameter):

$$\inf_{u \in S(\Omega)} J(u) + \lambda \int_\Omega \left(\log u + \frac{f}{u} \right) \tag{4.3}$$

where $S(\Omega) = \{u \in BV(\Omega), u > 0\}$, and $f > 0$ in $L^\infty(\Omega)$ the given data.
From now on, without loss of generality, we assume that $\lambda = 1$.

4.3. Existence of a minimizer. In this subsection, we show that Problem (4.3) has at least one solution.

Theorem 4.1. *Let f be in $L^\infty(\Omega)$ with $\inf_\Omega f > 0$, then problem (4.3) has at least one solution u in $BV(\Omega)$ satisfying:*

$$0 < \inf_\Omega f \leq u \leq \sup_\Omega f \tag{4.4}$$

Proof.
Let us denote by $\alpha = \inf f$ and $\beta = \sup f$. Let us consider a minimizing sequence $(u_n) \in S(\Omega)$ for Problem (4.3). Let us denote by

$$E(u) = J(u) + \int_\Omega \left(\log u + \frac{f}{u} \right) \tag{4.5}$$

We split the proof in two parts.
First part: we first show that we can assume without restriction that $\alpha \leq u_n \leq \beta$.

We remark that $x \mapsto \log x + \frac{f}{x}$ is decreasing if $x \in (0, f)$ and increasing if $x \in (f, +\infty)$. Therefore, if $M \geq f$, one always has:

$$\left(\log(\min(x, M)) + \frac{f}{\min(x, M)} \right) \leq \log x + \frac{f}{x} \tag{4.6}$$

Hence, if we let $M = \beta = \sup f$, we find that:

$$\int_\Omega \left(\log \inf(u, \beta) + \frac{f}{\inf(u, \beta)} \right) \leq \int_\Omega \left(\log u + \frac{f}{u} \right) \tag{4.7}$$

Moreover, we have that (see Lemma 1 in section 4.3 of [27] for instance): $J(\inf(u, \beta)) \leq J(u)$. We thus deduce that:

$$E(\inf(u, \beta)) \leq E(u) \tag{4.8}$$

And we get in the same way that $E(\sup(u,\alpha)) \leq E(u)$ where $\alpha = \inf f$.
Second part: From the first part of the proof, we know that we can assume that $\alpha \leq u_n \leq \beta$. This implies in particular that u_n is bounded in $L^1(\Omega)$.

By definition of (u_n), the sequence $E(u_n)$ is bounded, i.e. there exists a constant C such that $J(u_n) + \int_\Omega \left(\log u_n + \frac{f}{u_n}\right) \leq C$. Moreover, standard computations show that $\int_\Omega \left(\log u_n + \frac{f}{u_n}\right)$ reaches its minimum value $\int_\Omega (1+\log f)$ when $u=f$, and thus we deduce that $J(u_n)$ is bounded.

Therefore we get that u_n is bounded in $BV(\Omega)$ and there exists u in $BV(\Omega)$ such that up to a subsequence, $u_{_n} \to u$ in $BV(\Omega)$-weak * and $u_n \to u$ in $L^1(\Omega)$-strong. Necessarily, we have $0 \leq \alpha \leq u \leq \beta$, and thanks to the lower semi-continuity of the total variation and Fatou's lemma, we get that u is a solution of problem (4.3).
□

4.4. Uniqueness and comparison principle. In this subsection, we adress the problem of the uniqueness of a solution of Problem (4.3). The question remains open in general, but we prove two results: we give a sufficient condition ensuring uniqueness and we show that a comparison principle holds.

PROPOSITION 4.2. *Let $f > 0$ be in $L^\infty(\Omega)$, then problem (4.3) has at most one solution $\hat{u}$ such that $0 < \hat{u} < 2f$.*

Proof. Let us denote by

$$h(u) = \log u + \frac{f}{u} \tag{4.9}$$

We have $h^{'}(u) = \frac{1}{u} - \frac{f}{u^2} = \frac{u-f}{u^2}$, and $h^{''}(u) = -\frac{1}{u^2} + 2\frac{f}{u^3} = \frac{2f-u}{u^3}$. We deduce that if $0 < u < 2f$, then h is stricty convex implying the uniqueness of a minimizer. □

We now state a comparison principle.

PROPOSITION 4.3. *Let f_1 and f_2 be in $L^\infty(\Omega)$ with $\inf_\Omega f_1 > 0$ and $\inf_\Omega f_2 > 0$. Let us assume that $f_1 < f_2$. We denote by u_1 (resp. u_2) a solution of (4.3) for $f = f_1$ (resp. $f = f_2$). Then we have $u_1 \leq u_2$.*

Proof. We use here the following classical notations: $u \vee v = \sup(u,v)$, and $u \wedge v = \inf(u,v)$.

From Theorem 4.1, we know that u_1 and u_2 do exist. We have since u_i is a minimizer with data f_i:

$$J(u_1 \wedge u_2) + \int_\Omega \left(\log(u_1 \wedge u_2) + \frac{f_1}{u_1 \wedge u_2}\right) \geq J(u_1) + \int_\Omega \left(\log u_1 + \frac{f_1}{u_1}\right) \tag{4.10}$$

and:

$$J(u_1 \vee u_2) + \int_\Omega \left(\log(u_1 \vee u_2) + \frac{f_2}{u_1 \vee u_2}\right) \geq J(u_2) + \int_\Omega \left(\log u_2 + \frac{f_2}{u_2}\right) \tag{4.11}$$

Adding these two inequalities, and using the fact that $J(u_1 \wedge u_2) + J(u_1 \vee u_2) \leq J(u_1) + J(u_2)$ [12, 23], we get:

$$\int_\Omega \left(\log(u_1 \wedge u_2) + \frac{f_1}{u_1 \wedge u_2} - \log u_1 - \frac{f_1}{u_1} + \log(u_1 \vee u_2) + \frac{f_2}{u_1 \vee u_2} - \log u_2 - \frac{f_2}{u_2}\right) \geq 0 \tag{4.12}$$

Writing $\Omega = \{u_1 > u_2\} \cup \{u_1 \leq u_2\}$, we easily deduce that:

$$\int_{\{u_1>u_2\}} (f_1 - f_2)\frac{u_1 - u_2}{u_1 u_2} \geq 0 \tag{4.13}$$

Since $f_1 < f_2$, we thus deduce that $\{u_1 > u_2\}$ has a zero Lebesgue measure, i.e. $u_1 \leq u_2$ a.e. in Ω.

□

4.5. Euler-Lagrange equation associated to Problem (4.3):. Let us now write an "Euler-Lagrange" equation for any solution of problem (4.3), the difficulty being that the ambient space is $BV(\Omega)$.

PROPOSITION 4.4. *Let f be in $L^\infty(\Omega)$ with $\inf_\Omega f > 0$. If u in $BV(\Omega)$ is a solution of Problem (4.3), then we have:*

$$-h^{'}(u) \in \partial J(u) \tag{4.14}$$

Proof. This is a consequence of the maximum principle (4.4) of Theorem 4.1. Indeed, h can be replaced below $\inf_\Omega f$ by its C^1- quadratic extension and this change does not alter the set of minimizers. The new functional has a Lipschitz derivative, and then standard results can be used to get (4.14).

□

To give more insight to equation (4.14), we state the following result (see Proposition 1.10 in [2] for further details):

PROPOSITION 4.5. *Let (u, v) in $L^2(\Omega)$ with u in $BV(\Omega)$. The following assertions are equivalent:*

(i) $v \in \partial J(u)$.

(ii) Denoting by $X(\Omega)_2 = \{z \in L^\infty(\Omega, \mathbb{R}^2) \ : \ \mathrm{div}\,(z) \in L^2(\Omega)\}$, *we have:*

$$\int_\Omega vu\,dx = J(u) \tag{4.15}$$

and

$$\begin{array}{c} \exists z \in X(\Omega)_2\ ,\ \|z\|_\infty \leq 1\ , z.N = 0\ ,\ \textit{on } \partial\Omega \\ \textit{such that } v = -\mathrm{div}\,(z) \ \textit{in } \mathcal{D}'(\Omega) \end{array} \tag{4.16}$$

(iii) (4.16) holds and:

$$\int_\Omega (z, Du) = \int_\Omega |Du| \tag{4.17}$$

From this proposition, we see that (4.14) means: $-h^{'}(u) = \mathrm{div}\, z$,
with z satisfying (4.16) and (4.17). This is a rigorous way to write $-\mathrm{div}\left(\frac{\nabla u}{|\nabla u}\right) + h^{'}(u) = 0$.

5. Numerical results. We present in this section some numerical examples illustrating the capability of our model. We also compare it with some existing other ones.

5.1. Algorithm. To numerically compute a solution to Problem (4.3), we use the equation $-\text{div}\left(\frac{\nabla u}{|\nabla u}\right) + h'(u) = 0$ and as it is classically done in image analysis we embed it into a dynamical equation which we drive to a steady state:

$$\frac{\partial u}{\partial t} = \text{div}\left(\frac{\nabla u}{|\nabla u|}\right) + \lambda \frac{f-u}{u^2} \tag{5.1}$$

with initial data $u(x,0) = \frac{1}{|\Omega|}\int_\Omega f$. We denote this model as the AA model. We use the following explicit scheme, with finite differences (we have checked numerically that for $\delta t > 0$ small enough, the sequence (u_n) satisfies a maximum principle):

$$\frac{u_{n+1}-u_n}{\delta t} = \left(\text{div}\left(\frac{\nabla u_n}{\sqrt{|\nabla u_n|^2+\beta^2}}\right) - \lambda h'(u_n)\right) \tag{5.2}$$

with β a small fixed parameter.

5.2. Other models. We have compared our results with some other classical variational denoising models.

*Additive model (*log*).* A natural way to turn a multiplicative model into an additive one is to use the logarithm transform(see [5, 22] for instance). Nevertheless, as can be seen on the numerical results, such a straightforward method does not lead satisfactory results. In the numerical results presented in this paper, we refer to this model as the log model. We first take the logarithm of the original image f. We then denoise $\log(f)$ by using the ROF model [37], with the following functional $\inf_y \left(J(y) + \frac{1}{2\lambda}\|x-z\|^2_{L^2}\right)$. We finally take the exponential to obtain the restored image. As can be seen on Figures 5.1 and 5.2, there is no maximum principle for this algorithm. In particular, the mean of the restored image is much smaller than the one of the original image. In fact, in such an approach, the assumptions are not consistent with the modelling, as explained hereafter.

The original considered model is the following: $f = uv$, under the assumptions that u and v are independent, and $E(v) = 1$ (i.e. v is of mean one). Hence $E(f) = E(u)$.

Now, if we take the logarithm, denoting by $x = \log(f)$, $y = \log(u)$, and $z = \log(v)$, we get the additive model $x = y + z$. To recover y from x, the classical assumption is $E(z) = 0$: this is the basic assumption in all the classical additive image restoration methods [11, 4] (total variation minimization, nonlinear diffusion, wavevelet shrinkage, non local means, heat equation, ...).

But, from Jensen inequality, we have: $\exp(E(z)) \leq E(\exp(z))$, i.e. $1 \leq E(v)$. As soon as there is some noise, we no longer are in the case of equality in Jensen inequality, which implies $E(v) > 1$. As a consequence, $E(u) < E(f)$ (in the numerical examples presented in Figure 5.1 and 5.2, we obtain $E(u) \approx E(f)/2$).

As a conclusion, if one wants to use the logarithm to get an additive model, then one cannot directly apply a standard additive noise removal algorithm. One needs to be more careful.

RLO model. The second model we use is a multiplicative version of the ROF model: it is a constrained minimization problem proposed by Rudin, Lions, and Osher in [36, 34], and we will call it the RLO model. In this approach, the model considered is $f = uv$, under the constraints that $\int_\Omega v = 1$ (mean one), and $\int_\Omega (v-1)^2 = \sigma^2$ (given variance). The goal is then to minimize $\int_\Omega |Du|$ under the two previous constraints. The gradient projection method leads to:

$$\frac{\partial u}{\partial t} = \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) - \lambda \frac{f^2}{u^3} - \mu \frac{f}{u^2} \tag{5.3}$$

The two lagrange multipliers λ and μ are dynamically updated to satisfy the constraints (as explained in [36]). With this algorithm, there is no regularization parameter to tune: the parameter to tune is here the number of iterations (since the considered flow is not associated to any functional). In practice, it appears that the Lagrange multipliers λ and μ are almost always of opposite signs.

Notice that the model proposed in this paper (AA) is specifically devoted to the denoising of images corrupted by gamma noise. The RLO model does not make such an assumption on the noise, and therefore cannot be expected to perform as well as the AA model for speckle removal. Notice also that in the case of small Gaussian multplicative noise, both RLO and AA models give very good results, as can be seen on Figure 5.4.

5.3. Deblurring. It is possible to modify our model to incorporate a linear blurring operator K. u being the image to recover, we assume that the observed image f is obtained as: $f = (Ku).v$. The functional to minimize in this case becomes:

$$\inf_u \left(J(u) + \lambda \int_\Omega \left(\frac{f}{Ku} + \log(Ku) \right) \right) \tag{5.4}$$

The associated Euler-Lagrange equation is (denoting by K^T the transpose of K):

$$0 \in \partial J(u) + \lambda K^T \left(\frac{-f}{(Ku)^2} + \frac{1}{Ku} \right) \tag{5.5}$$

Numerically, we use a steepest gradient descent approach by solving:

$$\frac{\partial u}{\partial t} = \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) + \lambda K^T \left(\frac{f - Ku}{(Ku)^2} \right) \tag{5.6}$$

5.4. Results. On Figure 5.1, we show a first example. The original synthetic image is corrupted by some multiplicative noise with gamma law of mean one (see (3.2)). We display the denoising results obtained by our approach (AA), as well as with the RLO model. Due to the very strong noise, the RLO model has some difficulties to bring back in the range of the image some isolated points (white and black points on the denoised image) and in the same time keep sharp edges: to remove these artefacts, one needs to regularize more, and therefore some part of the edges are lost. Moreover, the mean of the original image is not preserved (the mean of the restored image is quite larger than the one of the original image): this is the reason why the SNR is not much improved, and also why the restored image with the RLO model looks lighter. We also display the results obtained with the log model: as explained before, this model gives bad results, due to the fact that the mean is not preserved (with the log model, the mean is much reduced). This is the reason why the restored image with the log model is much darker.

On Figure 5.2, we show how our model behaves with a complicated geometrical image. We also give the results with the RLO model and the log model (which have the same drawbacks as on Figure 5.1).

On Figure 5.3, we show the result we get on a SAR image provided by the CNES (French space agency: http://www.cnes.fr/index_v3.htm). The reference image (also furnished by the CNES) has been obtained by amplitude summation.

On Figure 5.4, we show how our model behave with multiplicative Gaussian noise. We have used the same parameters for the Gaussian noise as in [36], i.e. a standard deviation of 0.2 (and a mean equal to 1). The original image is displayed on Figure 5.2. In this case, we see that we get a very good restoration result. Notice that such a multiplicative Gaussian noise is much easier to handle than the speckle noise which was tackled on Figures 5.1 to 5.3. But, as far as we know, this is the type of multiplicative noise which was considered in all the variational approaches inspired by [36] (as used for instance in [33, 28, 29, 38]). We also show the results with the RLO model and the log model. Notice that in this case all the models perform very well, even the log model: indeed, since the noise is small, the Jensen inequality is almost an equality.

On Figure 5.5, we finally show a deblurring example with our model (5.4). The original image (displayed on Figure 5.2) has been convolved with a Gaussian kernel of standard deviation $\sqrt{2}$ and then multiplied by a Gaussian noise of standard deviation 0.2 and mean 1 (we use the same parameters as in [36]). Even though the restored image is not as good as in the denoising case presented on Figure 5.4, we see that our model works well for deblurring.

6. Evolution equation. In this section we study the evolution equation associated to (4.14). The motivation is that when searching for a numerical solution of (4.14) it is, in general, easier to compute a solution of the associated evolution equation (by using for example explicit or semi-implicit schemes) and then studying the asymptotic behaviour of the process to get a solution of the stationary equation.

We first consider a semi-discrete version of the problem: the space Ω is still included in R^2, but we discretize the time variable. We consider the case of a regular time discretization, (t_n), with t_0 given, and $t_{n+1} - t_n = \delta t$ in $\mathbb{R}_+^*$ (in this section, δt is fixed). We define $u_n = u(.,t_n)$, and we consider the following implicit scheme.

$$0 \in \frac{u_{n+1} - u_n}{\delta t} + \partial J(u_{n+1}) + h^{'}(u_{n+1}) \tag{6.1}$$

where J is the extended total variation as defined in (4.2). We first need to check that indeed (6.1) defines a sequence (u_n). To this end, we intend to study the following functional:

$$\inf_{u \in BV(\Omega),\, u>0} F(u, u_n) \tag{6.2}$$

with:

$$F(u, u_n) = \int_\Omega \frac{u^2}{2}\, dx - \int_\Omega u_n u\, dx + \delta t \left(J(u) + \int_\Omega h(u)\, dx \right) \tag{6.3}$$

We want to define u_{n+1} as:

$$u_{n+1} = \operatorname*{argmin}_{\{u \in BV(\Omega),\, u>0\}} F(u, u_n) \tag{6.4}$$

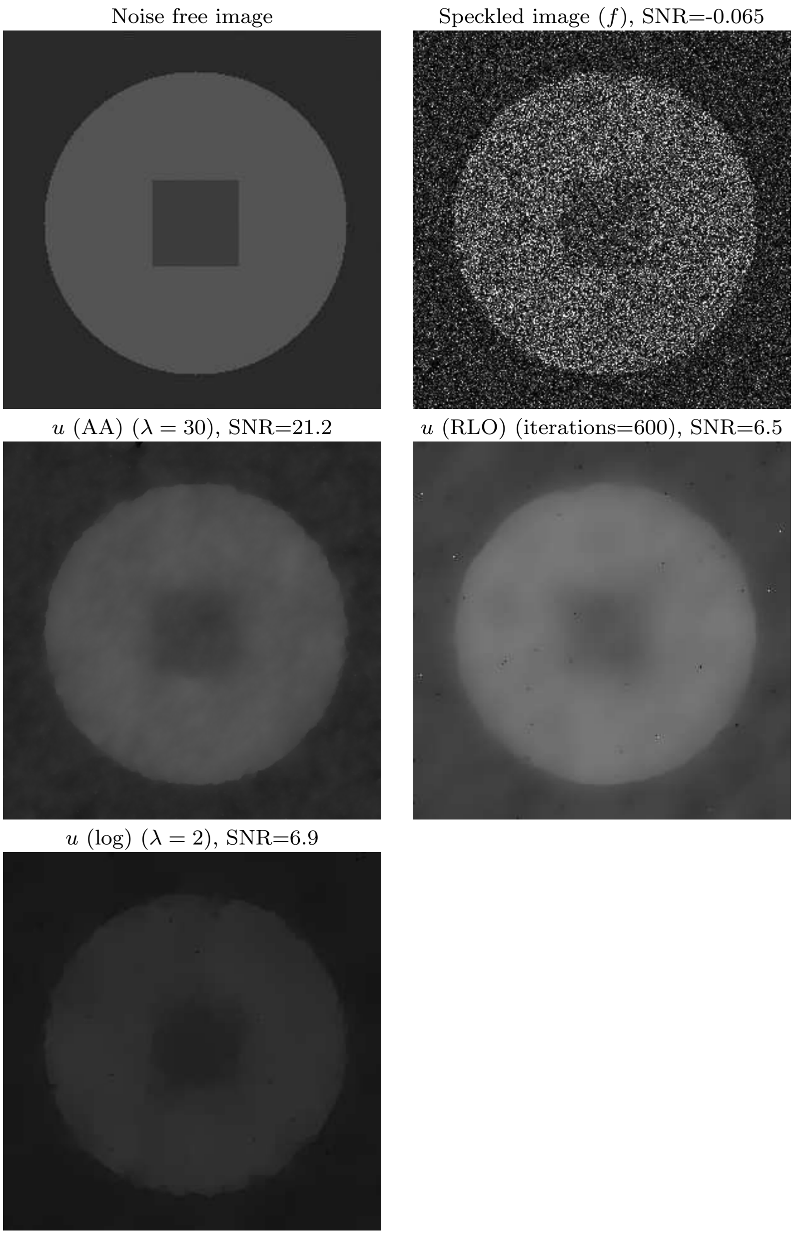

FIG. 5.1. *Denoising of a synthetic image with gamma noise. f has been corrupted by some multiplicative noise with gamma law of mean one. u is the denoised image.*

6.1. Existence and uniqueness of the sequence (u_n). We first need to check that indeed the sequence (u_n) is well-defined.

PROPOSITION 6.1. *Let f be in $L^\infty(\Omega)$ with $\inf_\Omega f > 0$. Let (u_n) be in $BV(\Omega)$ such that $\inf_\Omega f \leq u_n \leq \sup_\Omega f$. If $\delta t < 27(\inf_\Omega f)^2$, then there exists a unique u_{n+1}*

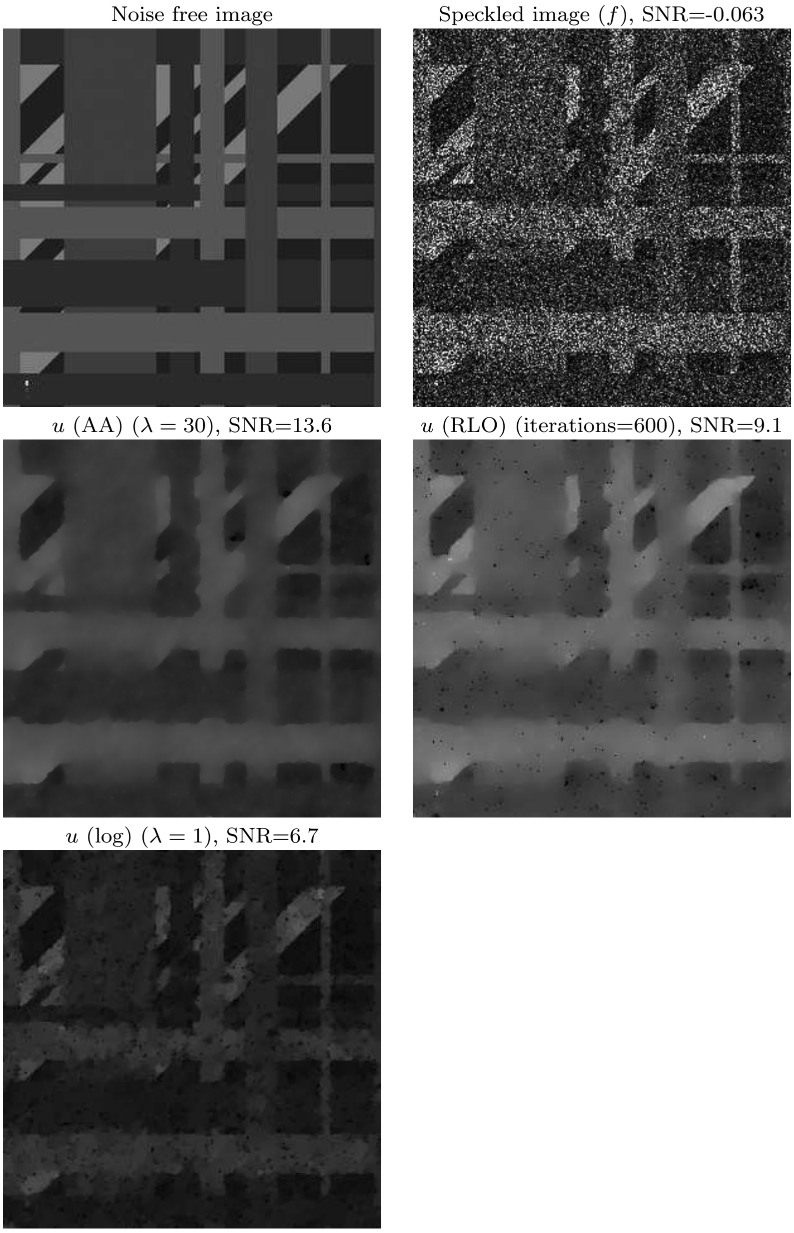

FIG. 5.2. *Denoising of a synthetic image with gamma noise. f has been corrupted by some multiplicative noise with gamma law of mean one.*

in $BV(\Omega)$ satisfying (6.4). Moreover, we have:

$$\inf\left(\inf_{\Omega} f\,,\,\inf_{\Omega} u_0\right) \leq u_n \leq \sup\left(\sup_{\Omega} f\,,\,\sup_{\Omega} u_0\right) \tag{6.5}$$

Proof. We split the proof in two parts.
First part: we first show the existence and uniqueness of u^{n+1}. We consider: $g(u) =$

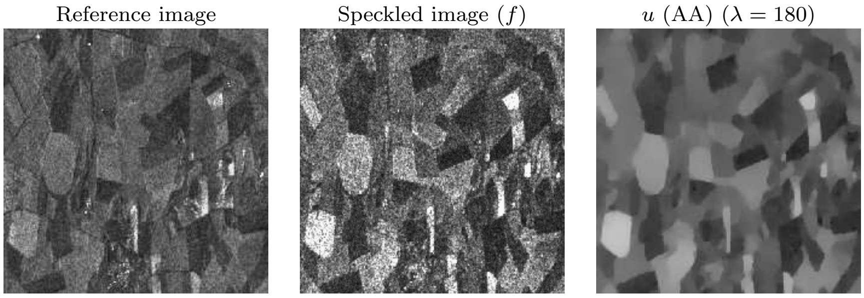

Fig. 5.3. *Denoising of a SAR image provided by the CNES (French space agency).*

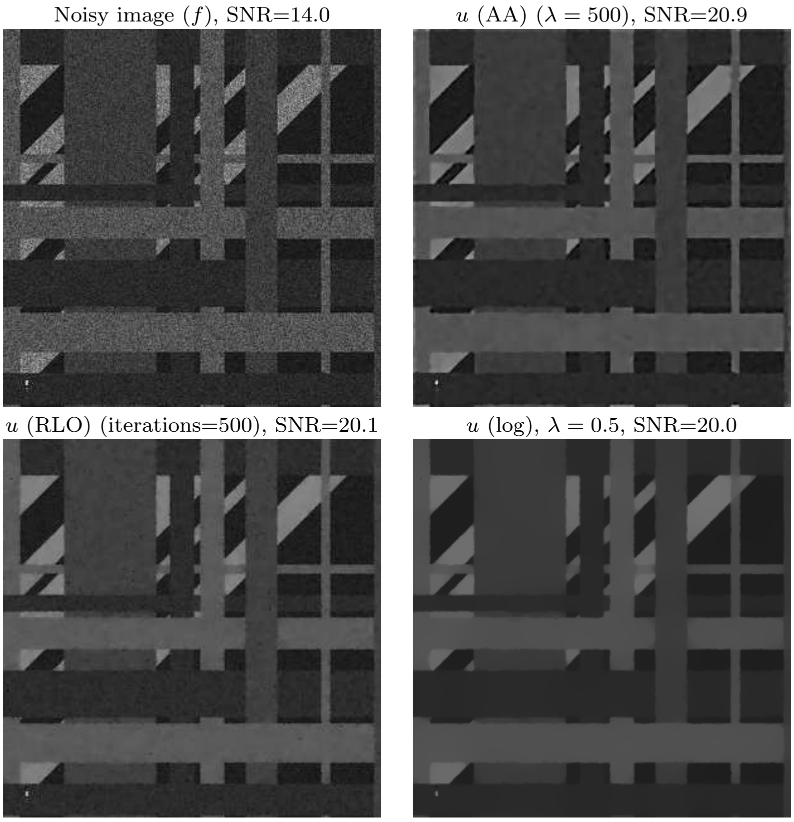

Fig. 5.4. *Denoising of a synthetic image degraded by multiplicative Gaussian noise with* $\sigma = 0.2$. *The original noise free image is shown in Figure 5.2.*

$\delta t h(u) + u^2/2 - u_n u$. We have: $g^{''}(u) = 1 + \delta t \frac{f-u}{u^2} = \frac{u^3 - \delta t u + 2\delta t f}{u^3}$. A simple computation shows that if $\delta t < 27(\inf_\Omega f)^2$, then $g^{''}(u) > 0$ for all $u > 0$, i.e. g strictly convex on $\mathbb{R}_+^*$. It is then standard to deduce the existence and uniqueness of u^{n+1}.
Second part: As in the proof of theorem 4.1, we have:

$$\int_\Omega \left(\log \inf(u,\beta) + \frac{f}{\inf(u,\beta)} \right) \leq \int_\Omega \left(\log u + \frac{f}{u} \right) \tag{6.6}$$

and $J(\inf(u,\beta)) \leq J(u)$.

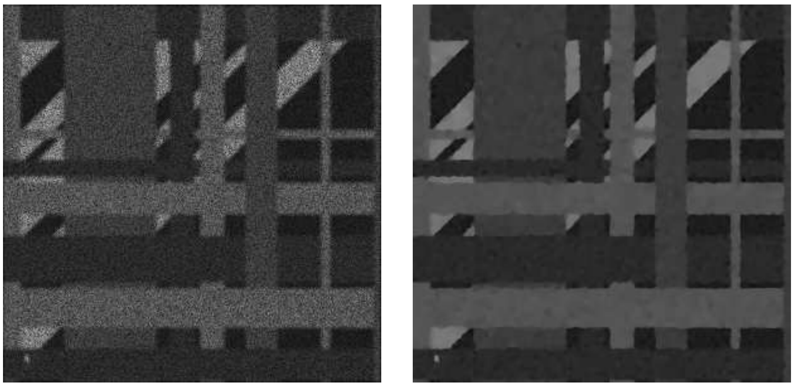

FIG. 5.5. *Deblurring of the synthetic image of Figure 5.2 (the original image, which is shown in Figure 5.2, has first been convolved with a Gaussian kernel of standard deviation $\sigma = \sqrt{2}$, and then multiplied by some Gaussian noise of mean 1 and standard deviation $\sigma = 0.2$).*

We remark that $x \mapsto x^2/2 - xu_n$ is decreasing if $x \in (0, u_n)$ and increasing if $x \in (u_n, +\infty)$. Therefore, proceeding as in the proof of theorem 4.1, we get:

$$\int_\Omega \frac{(\inf(u\,,\,\sup u_n))^2}{2} - u_n \inf(u\,,\,\sup u_n) \leq \int_\Omega \frac{u^2}{2} - uu_n \tag{6.7}$$

Thus the truncation procedure makes decrease the energy and we deduce the right-hand side inequality in (6.5). We get the other one in the same way.
□

We can thus derive the following theorem:

THEOREM 6.2. *Let f be in $L^\infty(\Omega)$ with $\inf_\Omega f > 0$, and u_0 in $L^\infty(\Omega) \bigcap BV(\Omega)$ with $\inf_\Omega u_0 > 0$ be given. If $\delta t < 27(\inf_\Omega f)^2$, then there exists a unique sequence (u_n) in $BV(\Omega)$ satisfying (6.4). Moreover, the following estimates hold:*

$$\inf\left(\inf_\Omega f\,,\,\inf_\Omega u_0\right) = \alpha \leq u_n \leq \beta = \sup\left(\sup_\Omega f\,,\,\sup_\Omega u_0\right) \tag{6.8}$$

and

$$J(u_n) \leq J(u_0) + \int_\Omega h(u_0)\,dx - \int_\Omega (1 + \log f) \tag{6.9}$$

Proof. This theorem is just a consequence (by induction) of proposition 6.1, except for estimate (6.9) which we prove now:

From (6.4), we have: $F(u_{n+1}, u_n) \leq F(u_n, u_n)$, which means:

$$\begin{aligned} &\delta t \left(J(u_{n+1}) - J(u_n) + \int_\Omega h(u_{n+1}) - \int_\Omega h(u_n) \right) \\ &+ \frac{1}{2} \int_\Omega (u_{n+1} - u_n)^2 \leq 0 \end{aligned} \tag{6.10}$$

This implies:

$$J(u_{n+1}) - J(u_n) + \int_\Omega h(u_{n+1}) - \int_\Omega h(u_n) \leq 0 \tag{6.11}$$

By summation, we obtain:

$$J(u_{n+1}) \leq -\int_\Omega h(u_{n+1}) + \int_\Omega h(u_0) + J(u_0) \tag{6.12}$$

Standard computations show that $\int_\Omega h(u_{n+1}) \geq \int_\Omega (1+\log f)\,dx$, from which we deduce (6.9). □

6.2. Euler-Lagrange equation. We have the following "Euler-Lagrange" equation:

PROPOSITION 6.3. *The sequence (u_n) satisfying (6.4) is such that:*

$$0 \in \frac{u_{n+1}-u_n}{\delta t} + \left(\partial J(u_{n+1}) + h^{'}(u_{n+1})\right) \tag{6.13}$$

Proof. The proof is similar to the one of Proposition 4.4. □

6.3. Convergence of the sequence u_n. The following convergence result holds:

PROPOSITION 6.4. *Let f be in $L^\infty(\Omega)$ with $\inf_\Omega f > 0$, and u_0 in $L^\infty(\Omega)\bigcap BV(\Omega)$ with $\inf_\Omega u_0 > 0$ be fixed. Let $\delta t < 27(\inf_\Omega f)^2$. The sequence (u_n) defined by equation (6.1) is such that there exists u in $BV(\Omega)$ with $u_n \rightharpoonup u$ (up to a subsequence) for the $BV(\Omega)$ weak * topology, and u is solution of*

$$0 \in \partial J(u) + h^{'}(u) \tag{6.14}$$

in the distributional sense.

Proof. As in the proof of Theorem 6.2, we get the same kind of equation as (6.10):

$$\frac{1}{2}\int_\Omega (u_{n+1}-u_n)^2 \leq \delta t\left(J(u_n) - J(u_{n+1}) + \int_\Omega h(u_n) - \int_\Omega h(u_{n+1})\right) \tag{6.15}$$

By summation, we obtain:

$$\begin{aligned}\frac{1}{2}\sum_{n=0}^{N-1}\int_\Omega (u_{n+1}-u_n)^2 &\leq \delta t\left(J(u_0) - J(u_N) + \int_\Omega h(u_0) - \int_\Omega h(u_N)\right)\\ &\leq \delta t\left(J(u_0) + \int_\Omega h(u_0) - \int_\Omega h(f)\right) < +\infty\end{aligned}$$

(since $\int_\Omega h(u_N \geq \int_\Omega h(f))$. In particular, this implies that $u_{n+1} - u_n \to 0$ in $L^2(\Omega)$ strong.

From estimate (6.9), we know that there exists u in $BV(\Omega)$ such that up to a subsequence $u_n \rightharpoonup u$ for the $BV(\Omega)$ weak * topology. Moreover, $u_n \to u$ in $L^1(\Omega)$ strong. Let $v \in L^2(\Omega)$. From (6.13), we have:

$$J(v) \geq J(u_{n+1}) + \left\langle v - u_{n+1}, -\frac{u_{n+1}-u_n}{\delta t} - h^{'}(u_{n+1})\right\rangle_{L^2(\Omega)} \tag{6.16}$$

Using estimate (6.8) and the fact that $u_n \to u$ in $L^1(\Omega)$ strong, we deduce from Lebesgues dominated convergence theorem that (up to a subsequence) $u_n \to u$ in $L^2(\Omega)$ strong. Moreover, since $u_{n+1} - u_n \to 0$ in $L^2(\Omega)$ strong, and thanks to the lower semi-continuity of the total variation, we get: $J(v) \geq J(u) + \langle v-u, -h^{'}(u)\rangle_{L^2(\Omega)}$. Hence (6.14) holds.

□

6.4. Continuous setting. Let us consider the following evolution equation

$$\frac{\partial u}{\partial t} \in -\partial J(u) - h'(u) \tag{6.17}$$

with the initial condition $u(0) = u_0$ and with $h(u) = \frac{f}{u} + \log u$, i.e. $h^{'}(u) = \frac{u-f}{u^2}$. $J(u)$ still denotes the extended total variation of u with respect to the space variable x.

To show the existence and uniqueness of a solution for (6.17), we could apply the theory of maximal monotone operator [9, 8, 2]. This theory works provided $h^{'}$ is Lipschitz. One only needs to replace h by its C^1- quadratic extension below $\inf_\Omega$. This would yield a solution in $L^2(\Omega)$. Here, we derive sharper bounds with the next result, whose proof is given in Appendix A.

THEOREM 6.5. *Let f be in $L^\infty(\Omega)$ with $\inf_\Omega f > 0$, and u_0 in $L^\infty(\Omega) \bigcap BV(\Omega)$ with $\inf_\Omega u_0 > 0$. Then problem (6.17) has exactly one solution u in $L^\infty_w((0,T); BV(\Omega)) \bigcap W^{1,2}((0,T); L^2(\Omega))$.*

Remark. u belongs to $L^\infty_w((0,T); BV(\Omega))$ means that u belongs to $L^\infty((0,T)\times\Omega)$ and Du belongs to $L^\infty_w((0,T); \mathcal{M}_b(\Omega))$. $L^\infty_w((0,T); \mathcal{M}_b(\Omega))$ is the space of equivalent classes of weak* measurable mappings μ that are essentially bounded, i.e. $\sup\mathrm{ess}_{x\in\Omega}\|\mu(x)\| < +\infty$ (we say that μ is weak* measurable if $\langle \mu(x), f\rangle_{\mathcal{M}_b(\Omega)\times C_0(\Omega;\mathbb{R}^2)}$ is measurable with respect to x for every f in $C_0(\Omega;\mathbb{R}^2)$, see Lemma A.5 and [7] for further details).

Appendix A. Evolution equation: continuous setting.

To show that problem(6.17) has a solution, we start from the semi-discrete problem we have studied in the previous section. We therefore consider a sequence (u_n) satisfying (6.4). From Proposition 6.3, we know that (u_n) satisfies:

$$0 \in \frac{u_{n+1} - u_n}{\delta t} + \Big(\partial J(u_{n+1}) + h^{'}(u_{n+1})\Big) \tag{A.1}$$

and u_{n+1} satisfies Neumann boundary conditions $\frac{\partial u_{n+1}}{\partial N} = 0$ on the boundary of Ω. From Theorem 6.2, we know that the sequence (u_n) exists and is unique provided $\delta t < 27(\inf_\Omega f)^2$.

A.1. Definitions of interpolate functions. We classically introduce two functions defined on $\Omega \times \mathbb{R}^+$. We assume that $t_0 = 0$, and $t_n = n\delta t$.

$$\tilde{u}_{\delta t}(t,x) = u_{[t/\delta t]+1}(x) = u_{n+1}(x) \text{ if } t_n < t \leq t_{n+1} \tag{A.2}$$

where $[t/\delta t]$ is the integer part of $t/\delta t$. $\tilde{u}_{\delta t}(.,x)$ is thus piecewise constant. We also introduce:

$$\hat{u}_{\delta t}(t,x) = (t - t_n)\frac{u_{n+1}(x) - u_n(x)}{\delta t} + u_n(x) \tag{A.3}$$

with $n = [t/\delta t]$. $\hat{u}_{\delta t}(.,x)$ is piecewise affine, continuous, and we have:

$$\frac{\partial \hat{u}_{\delta t}}{\partial t}(t,x) = \frac{u_{n+1}(x) - u_n(x)}{\delta t} \ , \ t_n < t < t_{n+1} \tag{A.4}$$

With these notations, we can rewrite (A.1) as:

$$\frac{\tilde{u}_{\delta t}(t,x)) - \tilde{u}_{\delta t}(t - \delta t, x)}{\delta t} \in -\partial J(\tilde{u}_{\delta t}(t,x)) - h^{'}(\tilde{u}_{\delta t}(t,x)) \tag{A.5}$$

i.e.:

$$\frac{\partial \hat{u}_{\delta t}}{\partial t}(t,x) \in -\partial J(\tilde{u}_{\delta t}(t,x)) - h^{'}(\tilde{u}_{\delta t}(t,x)) \tag{A.6}$$

A.2. A priori estimates. We first need to show some a priori estimates.

PROPOSITION A.1. *Let $T>0$ be fixed, f in $L^\infty(\Omega)$ with $\inf_\Omega f > 0$, and u_0 in $L^\infty(\Omega)\bigcap BV(\Omega)$ with $\inf_\Omega u_0 > 0$. Then, if $0 \leq t \leq T$:*

$$\inf\left(\inf_\Omega f\ ,\ \inf_\Omega u_0\right) = \alpha \leq \tilde{u}_{\delta t}, \hat{u}_{\delta t} \leq \beta = \sup\left(\sup_\Omega f\ ,\ \sup_\Omega u_0\right) \tag{A.7}$$

and:

$$\sup_{t\in(0,T)} \{J(\tilde{u}_{\delta t}), J(\hat{u}_{\delta t})\} \leq J(u_0) + \int_\Omega h(u_0) - \int_\Omega h(f) \tag{A.8}$$

Proof.
(A.7) for $\tilde{u}_{\delta t}$ comes from (6.8) in Theorem 6.2, and (A.8) comes from (6.9). We then get the estimates for $\hat{u}_{\delta t}$ from (A.3).
□

PROPOSITION A.2. *Let $T>0$ be fixed. There exists a constant $C>0$, which does not depend on δt, such that:*

$$\int_0^T \left\| \frac{\partial \hat{u}_{\delta t}}{\partial t} \right\|^2_{L^2(\Omega)} \leq C \tag{A.9}$$

Proof. Let us denote by $N = [t/\delta t]$. We have:

$$\int_{t_n}^{t_{n+1}} \left\| \frac{\partial \hat{u}_{\delta t}}{\partial t} \right\|^2_{L^2(\Omega)} = \delta t \int_\Omega \left| \frac{u_{n+1}(x) - u_n(x)}{\delta t} \right|^2 \, dx \tag{A.10}$$

By using (6.15), we get:

$$\int_{t_n}^{t_{n+1}} \left\| \frac{\partial \hat{u}_{\delta t}}{\partial t} \right\|^2_{L^2(\Omega)} \leq 2\left(J(u_n) - J(u_{n+1}) + \int_\Omega h(u_n) - \int_\Omega h(u_{n+1}) \right) \tag{A.11}$$

Hence:

$$\begin{aligned}\sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \left\| \frac{\partial \hat{u}_{\delta t}}{\partial t} \right\|^2_{L^2(\Omega)} &\leq 2\left(J(u_0) - J(u_N) + \int_\Omega h(u_0) - \int_\Omega h(u_N) \right) \\ &\leq 2\left(J(u_0) + \int_\Omega h(u_0) - \int_\Omega h(f) \right)\end{aligned}$$

We thus deduce that:

$$\int_0^T \left\| \frac{\partial \hat{u}_{\delta t}}{\partial t} \right\|^2_{L^2(\Omega)} dt \leq 2T\left(J(u_0) + \int_\Omega h(u_0) - \int_\Omega h(f) \right) + \int_{t_N}^T \left\| \frac{\partial \hat{u}_{\delta t}}{\partial t} \right\|^2_{L^2(\Omega)} dt \tag{A.12}$$

But, by using (6.15), we have:

$$\int_{t_N}^{T} \left\| \frac{\partial \hat{u}_{\delta t}}{\partial t} \right\|^2_{L^2(\Omega)} dt \leq 2\frac{T-t_n}{\delta t}\left(J(u_N) - J(u_{N+1}) + \int_\Omega h(u_N) - \int_\Omega h(u_{N+1}) \right)$$
$$\leq 2\left(J(u_0) - J(u_{N+1}) + \int_\Omega h(u_N) - \int_\Omega h(u_{N+1}) \right)$$

We then get from (6.9) and (6.8) that there exists $B > 0$ which does not depend on N and δt such that: $\int_{t_N}^{T} \left\| \frac{\partial \hat{u}_{\delta t}}{\partial t} \right\|^2_{L^2(\Omega)} dt \leq B$. We then conclude thanks to (A.12).
□

Corollary A.3. *Let $T > 0$ be fixed. Then:*

$$\lim_{\delta t \to 0} \int_0^T \| \hat{u}_{\delta t} - \tilde{u}_{\delta t} \|^2_{L^2(\Omega)} dt = 0 \tag{A.13}$$

Proof. Let us denote by $N = [t/\delta t]$. We have:

$$\int_0^T \| \hat{u}_{\delta t} - \tilde{u}_{\delta t} \|^2_{L^2(\Omega)} dt = \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \| \hat{u}_{\delta t} - \tilde{u}_{\delta t} \|^2_{L^2(\Omega)} dt + \int_{t_N}^{T} \| \hat{u}_{\delta t} - \tilde{u}_{\delta t} \|^2_{L^2(\Omega)} dt \tag{A.14}$$

But:

$$\sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \| \hat{u}_{\delta t} - \tilde{u}_{\delta t} \|^2_{L^2(\Omega)} dt = \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \| (t - t_n - \delta t)(u_{n+1} - u_n) \|^2_{L^2(\Omega)} dt \tag{A.15}$$

We then deduce from (A.4) that:

$$\sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \| \hat{u}_{\delta t} - \tilde{u}_{\delta t} \|^2_{L^2(\Omega)} dt \leq \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \left\| \delta t \frac{\partial \hat{u}_{\delta t}}{\partial t} \right\|_{L^2(\Omega)} dt$$
$$\leq \underbrace{(\delta t)^2 \int_0^T \left\| \frac{\partial \hat{u}_{\delta t}}{\partial t} \right\|^2_{L^2(\Omega)} dt}_{\to 0 \text{ as } \delta t \to 0}$$

And:

$$\int_{t_N}^{T} \| \hat{u}_{\delta t} - \tilde{u}_{\delta t} \|^2_{L^2(\Omega)} dt \leq \underbrace{(\delta t)^3 \left\| \frac{u_{N+1} - u_N}{\delta t} \right\|^2_{L^2(\Omega)} dt}_{\to 0 \text{ as } \delta t \to 0} \tag{A.16}$$

□

We summarize the a priori estimates we have proved in the following corollary:
Corollary A.4. *Let $T > 0$ be fixed. There exists a constant $C > 0$ such that:*

$$\sup \left\{ \sup_{t \in (0,T)} \| \tilde{u}_{\delta t} \|_{L^\infty(\Omega)}, \sup_{t \in (0,T)} \| \hat{u}_{\delta t} \|_{L^\infty(\Omega)} \right\} \leq C \tag{A.17}$$

$$\sup\left\{\sup_{t\in(0,T)} J(\tilde{u}_{\delta t})\,,\ \sup_{t\in(0,T)} J(\hat{u}_{\delta t})\right\} \leq C \tag{A.18}$$

$$\int_0^T \left\|\frac{\partial \hat{u}_{\delta t}}{\partial t}\right\|^2_{L^2(\Omega)} \leq C \tag{A.19}$$

$$\lim_{\delta t\to 0}\int_0^T \|\hat{u}_{\delta t}-\tilde{u}_{\delta t}\|^2_{L^2(\Omega)}\, dt = 0 \tag{A.20}$$

A.3. Existence of a solution. We can now prove Theorem 6.5.

Proof. The uniqueness of u will come from Proposition A.6. Here we just show the existence of u.

We first remark that, from (A.17) and(A.19), $\hat{u}_{\delta t}$ is uniformly bounded in $W^{1,2}((0,T);L^2(\Omega))$. Thus, up to a subsequence, there exists u in $W^{1,2}((0,T);L^2(\Omega))$ such that $\hat{u}_{\delta t} \rightharpoonup u$ in $W^{1,2}((0,T);L^2(\Omega))$ weak. Since $W^{1,2}((0,T);L^2(\Omega))$ is compactly embedded in $L^2((0,T);L^2(\Omega))$ (see [39], Theorem 2.1, chapter 3), $\hat{u}_{\delta t} \to u$ strongly in $L^2((0,T);L^2(\Omega))$.

Since (A.17) and (A.18) hold, we can apply Lemma A.5 (stated below) with $(\tilde{u}_{\delta t})$. Thus, up to a subsequence, there exists $\tilde{u}$ in $L^\infty_w((0,T);BV(\Omega))$ such that $\tilde{u}_{\delta t} \rightharpoonup \tilde{u}$ in $L^\infty(\Omega\times(0,T))$ weak * and $D_x\tilde{u}_{\delta t} \rightharpoonup D_x\tilde{u}$ in $L^\infty_w((0,T);\mathcal{M}_b(\Omega))$ weak *. From (A.20), we have that $\tilde{u}_{\delta t} \to u$ strongly in $L^2((0,T);L^2(\Omega))$, and we thus deduce that $\tilde{u}=u$.

The semi-discrete implicit scheme writes for a.e. $t\in(0,T)$:

$$-\frac{\partial \hat{u}_{\delta t}}{\partial t} - h^{'}(\tilde{u}_{\delta t}) \in \partial J(\tilde{u}_{\delta t}) \tag{A.21}$$

i.e., for all v in $BV(\Omega)$, $v>0$, and a.e. $t\in(0,T)$:

$$J(v) \geq J(\tilde{u}_{\delta t}) + \langle v-\tilde{u}_{\delta t}, -\frac{\partial \hat{u}_{\delta t}}{\partial t} - h^{'}(\tilde{u}_{\delta t})\rangle_{L^2(\Omega)\times L^2(\Omega)} \tag{A.22}$$

Let ϕ in $C^0_c(0,T)$ a test function, $\phi\geq 0$. We multiply (A.22) by ϕ and integrate on $(0,T)$:

$$\int_0^T J(v)\phi(t)\,dt \geq \int_0^T J(\tilde{u}_{\delta t})\phi(t)\,dt + \int_0^T\int_\Omega (v-\tilde{u}_{\delta t})\left(-\frac{\partial \hat{u}_{\delta t}}{\partial t} - h^{'}(\tilde{u}_{\delta t})\right)\phi(t)\,dtdx \tag{A.23}$$

We want to let $\delta t\to 0$ in (A.23). By convexity, we have:

$$\liminf \int_0^T J(\tilde{u}_{\delta t})\phi(t)\,dt \geq \int_0^T J(u)\phi(t)\,dt \tag{A.24}$$

Now, since $\tilde{u}_{\delta t}\to u$ strongly in $L^2((0,T);L^2(\Omega))$, $\frac{\partial \hat{u}_{\delta t}}{\partial t} \rightharpoonup \frac{\partial u}{\partial t}$ in $L^2((0,T);L^2(\Omega))$ weak, and $h^{'}$ bounded on the interval $[\alpha,\beta]$, the second term on the right hand-side of (A.23) tends to

$$\int_0^T\int_\Omega (v-u)\left(-\frac{\partial u}{\partial t} - h^{'}(u)\right)\phi(t)\,dtdx$$

We thus get:

$$\int_0^T J(v)\phi(t)\,dt \geq \int_0^T J(u)\phi(t)\,dt + \int_0^T \int_\Omega (v-u)\left(-\frac{\partial u}{\partial t} - h^{'}(u)\right)\phi(t)\,dtdx \quad (A.25)$$

This inequality holds for all $\phi \geq 0$, we deduce that for a.e. t in $(0,T)$:

$$J(v) \geq J(u) + \int_\Omega (v-u)\left(-\frac{\partial u}{\partial t} - h^{'}(u)\right)dx \quad (A.26)$$

i.e.: $-\frac{\partial u}{\partial t} \in \partial J(u) + h^{'}(u)$. Hence we deduce that u is a solution of (6.17) in the distributional sense.
□

In the above proof, we have used the following lemma:

LEMMA A.5. *Let (u_n) be a bounded sequence in $L^\infty_w(\Omega\times(0,T))$, such that $(D_x u_n)$ be a bounded sequence in $L^\infty_w((0,T);\mathcal{M}_b(\Omega))$. Then, up to a subsequence, there exists u in $L^\infty_w((0,T);BV(\Omega))$ such that $u_n \rightharpoonup u$ in $L^\infty(\Omega\times(0,T))$ weak *, and $D_x u_n \rightharpoonup D_x u$ in $L^\infty_w((0,T);\mathcal{M}_b(\Omega))$ weak *, i.e. for all ψ in $L^1((0,T);C_0(\Omega))$:*

$$\int_0^T \langle Du_n, \psi\rangle_{\mathcal{M}_b(\Omega)\times C_0(\Omega;\mathbb{R}^2)}\,dt \to \int_0^T \langle Du, \psi\rangle_{\mathcal{M}_b(\Omega)\times C_0(\Omega;\mathbb{R}^2)}\,dt \quad (A.27)$$

where $\langle .,.\rangle_{\mathcal{M}_b(\Omega)\times C_0(\Omega)}$ denotes the duality product between bounded measures on Ω and $C_0(\Omega;\mathbb{R}^2)$ the space of continuous functions on Ω and vanishing in $\partial\Omega$.

Proof. From the Riesz representation theorem [1, 20], there is an isometric isomorphism between $\mathcal{M}_b(\Omega)$ and the dual space of $C_0(\Omega)$. Moreover, since $C_0(\Omega)$ is separable, there is an isometric isomorphism between $L^\infty_w((0,T);\mathcal{M}_b(\Omega))$ and the dual space of $L^1((0,T);C_0(\Omega))$ (see [7] or [18] page 588). Up to a subsequence, there exists u in $L^\infty(\Omega\times(0,T))$ and v in $L^\infty_w((0,T);\mathcal{M}_b(\Omega))$ such that $u_n \rightharpoonup u$ in $L^\infty(\Omega\times(0,T))$ weak * , and $D_x u_n \rightharpoonup v$ in $L^\infty_w((0,T);\mathcal{M}_b(\Omega))$ weak * . We therefore have for all ψ in $L^1((0,T);C_0(\Omega))$:

$$\int_0^T \langle Du_n, \psi\rangle_{\mathcal{M}_b(\Omega)\times C_0(\Omega;\mathbb{R}^2)}\,dt \to \int_0^T \langle v, \psi\rangle_{\mathcal{M}_b(\Omega)\times C_0(\Omega;\mathbb{R}^2)}\,dt \quad (A.28)$$

Moreover, we have $D_x u_n \to D_x u$ in $\mathcal{D}^{'}(\Omega\times(0,T))$ and $D_x u_n \to v$ in $\mathcal{D}^{'}(\Omega\times(0,T))$: this implies that $D_x u = v$. □

A.4. Uniqueness of the solution. A uniqueness result holds.

PROPOSITION A.6. *Let f be in $L^\infty(\Omega)$ with $\inf_\Omega f > 0$, and u_0 in $L^\infty(\Omega)\bigcap BV(\Omega)$ with $\inf_\Omega u_0 > 0$. Then Problem (6.17) has at most one solution u such that $0 < \alpha \leq u \leq \beta$.*

Proof. This is a standard result. It is based on the convexity of J, the fact that h' is Lipshitz on $[\inf_\Omega f, +\infty)$, and Gronwall inequality. □

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