SEVERAL RECENT RESULTS
IN NON LINEAR GEOMETRIC OPTICS

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1. Introduction.

Many recent works are devoted to the study of high frequency oscillatory nonlinear waves, and to nonlinear geometric optics. Typical questions are the existence, the propagation, the interaction and the reflection of waves of the form

$$u^\varepsilon(t,x) \;\sim\; \underline{u}(t,x) \;+\; \varepsilon^\alpha \sum_{n\geq 0} \varepsilon^n\, U_n(t,x,\varphi(t,x)/\varepsilon)\,. \tag{1.1}$$

The $U_n(t,x,\theta)$ are periodic (or quasi/almost periodic) in the θ variable and φ is a real phase function. In the summation, n runs in a countable set of nonnegative numbers. Typically, n is an integer, a half integer... . $\underline{u}$ is a a background state, ε is a wavelength and ε^α the magnitude of the oscillations (assuming that $\partial_\theta U_0 \neq 0$).

This is the natural generalization of phase-amplitude linear waves

$$u^\varepsilon(t,x) \;=\; a(t,x,\varepsilon)\; e^{i\,\varphi(t,x)/\varepsilon}\,, \quad a(t,x,\varepsilon) \;\sim\; \sum_{n\geq 0} \varepsilon^n\; a_n(t,x)\,, \tag{1.2}$$

which includes the phenomenon of generation of harmonics by nonlinear interaction.

The waves u^ε satisfy nonlinear equations of propagations, such as first order semilinear hyperbolic systems

$$\partial_t u \;+\; \sum_{j=1}^{d} A_j(t,x)\,\partial_{x_j} u \;+\; f(t,x,u) \;=\; 0\,, \tag{1.3}$$

or quasilinear first order hyperbolic systems,

$$\partial_t u \;+\; \sum_{j=1}^{d} A_j(t,x,u)\,\partial_{x_j} u \;+\; f(t,x,u) \;=\; 0\,, \tag{1.4}$$

or higher order equations such as semilinear wave equations

$$\Box u \;+\; f(t,x,u,\nabla u) \;=\; 0\,. \tag{1.5}$$

The choice of scales, both in the expansion (1.1) and in the equations, is very important. Solutions of (1.3) of the form (1.1), model waves of wavelength $O(\varepsilon)$ and amplitude $O(\varepsilon^\alpha)$, which propagate at speed $O(1)$ during times $O(1)$. When $\alpha > 0$ the equations for all the U_n are linear, and no nonlinear effect is revealed in the principal term U_0. For $\alpha < 0$, the problem is in general illposed : the domain of existence shrinks to $\{t = 0\}$ as $\varepsilon \to 0$. Thus the case $\alpha = 0$

$$u^\varepsilon(t,x) \;\sim\; \sum_{n\geq 0} \varepsilon^n \; U_n(t,x,\varphi(t,x)/\varepsilon)\,. \tag{1.6}$$

is the most interesting for general semilinear first order systems. In (1.6), the background state $\underline{u}$ is incorporated to U_0. For general semilinear systems (1.3), this is the typical range of scales (up to rescaling) for the validity of the nonlinear geometric optics approximations.

For quasilinear first order systems, the similar critical exponent is $\alpha = 1$. The *weakly* nonlinear geometric optics, deal with solutions of (1.4) of the form

$$u^\varepsilon(t,x) \;\sim\; \underline{u}(t,x) \;+\; \varepsilon \sum_{n\geq 0} \varepsilon^n \; U_n(t,x,\varphi(t,x)/\varepsilon)\,. \tag{1.7}$$

However, stronger waves with $\alpha = 0$ can be considered, when they are associated to a linearly degenerate mode, such as entropy waves in Euler's equations of fluid dynamics (see [Se 1],[Se 2] [He]).

For the semilinear wave equation (1.5), the critical exponent is $\alpha = 1$.

In the scalings above, the validity of the geometric optics approximations is in general limited to times $t = O(1)$. In general, solutions are shown to exist only for such times. For longer times, even if the solutions exist, other phenomena such as dissipation or dispersion can become important. The geometric optics approximations can be incorrect then, and other regimes such as paraxial approximations can occur (see [Do] [DR]).

This paper reviews several results about the justification of nonlinear geometric optics approximations. In §2, we discuss expansions (1.1) which model the propagation of a single wave. The nonlinear character of the equation models the interaction of the wave with the medium in which it propagates. Next, we consider the interaction of at least two oscillatory waves. The interesting phenomenon is the creation of a new outgoing oscillation of the same magnitude, from incoming oscillations. This happens when the phases φ_j of the incoming waves and the phase φ of the outgoing wave satisfy the *resonance relation*

$$\varphi \;=\; \sum_j \varphi_j\,, \tag{1.8}$$

or more generally

$$\alpha\,\varphi \;=\; \sum_j \alpha_j\,\varphi_j \quad \alpha,\alpha_j \in \mathbb{Z}\,, \tag{1.9}$$

when harmonics are present. For semilinear problems, the analysis is made through multiphase expansions

$$u^\varepsilon(t,x) \;\sim\; \sum_{n\geq 0} \varepsilon^n \; U_n(t,x,\psi_1(t,x)/\varepsilon,\ldots,\psi_m(t,x)/\varepsilon)\,, \tag{1.10}$$

where the $U_j(t,x,\theta_1,\ldots\theta_m)$ are periodic in the variables θ. The leading term U_0 is often sought as a superposition of incoming and outgoing waves

$$U_0(t,x,\psi_1(t,x)/\varepsilon,\ldots,\psi_m(t,x)/\varepsilon) \;=\; \sum_j \; S_j(t,x,\varphi_j(t,x)/\varepsilon)\,. \tag{1.11}$$

Similar expansions, with oscillations of amplitude $O(\varepsilon)$, are used for quasilinear systems, in the regime of weakly nonlinear geometric optics. The nonlinear interaction is seen in the coupling of the equations for the S_j. *This coupling strongly depends on the existence or not of resonances* among the linear combinations of the φ_j.

A formal derivation of the equations of nonlinear geometric optics, is given in [HK], [MR] and [HMR]. The justification of the asymptotic expansions for exact solutions is delicate. Simple examples show that complete expansions (1.10) are not true in general. Instead, one has to study a larger class of solutions which satisfy the weaker estimate

$$u^\varepsilon(t,x) \;=\; U_0(t,x,\psi_1(t,x)/\varepsilon,\dots,\psi_m(t,x)/\varepsilon) \;+\; o(1)\,, \tag{1.12}$$

or

$$u^\varepsilon(t,x) \;=\; \sum_j S_j(t,x,\varphi_j(t,x)/\varepsilon) \;+\; o(1)\,, \tag{1.13}$$

with error terms which are $o(1)$ and not $O(\varepsilon)$, in general.

In §3, we present the one space dimensional case where many questions are now understood. When the space dimension d is larger, an important difficulty arises from the *focusing* of oscillations. This phenomenon is already present for linear equations, but in the nonlinear case, its consequences can be much more dramatic. For example, it may imply that the domain of existence of the (weak) solutions u^ε, shrinks to the empty set as ε tends to zero (see [JMR 2] [JMR 7]). Examples of blow up, due to focusing of oscillations, are presented in §5.

A framework where all focusing is excluded, besides the one space dimensionnal case, occurs when one considers a *coherent* set of phases. A typical and important example is given by linear phases for a constant coefficient background system. Then, expansions like (1.10) or (1.12) can be justified, see [JMR 2] [JMR 3] [JMR 4] [Sch 2] [De]. Coherent multiphase expansions are discussed in §4.

As described above, focusing of nonlinear oscillations can lead to blow up and non existence. In the extreme opposite direction, when it is combined with strongly dissipative mechanisms, focusing can lead to a complete absorbtion of oscillations, in finite time. The oscillations disappear when they reach the caustic set. An example of such a behaviour is given in [JMR 5] and presented in §6 below. To understand an intermediate case, we have considered in [JMR 7] semilinear systems with uniformly Lipschitzian nonlinearities (thus with at most linear growth at infinity). In that case, global weak solutions always exist, and the question of the behaviour of the oscillations after crossing the caustic, makes sense. This problem is studied in [JMR 7] and the results are sketched §7 below. The first order geometric optic approximation is justified in L^2 for all finite time, and in particular after the crossing of the caustic. Remark that this provides examples of *noncoherent* multiphase expansions. However, no precise behaviour near the caustic is given. In particular, a challenging open question, is the validity of the formal expansions near the caustic, computed in [HK 2].

Further Results.

1. Another important problem is the reflection of oscillatory waves on boundaries. A formal approach is described in [M A]. The tranverse reflection at a boundary is studied in [Chi] for two speed equations, and in [Wi 1] under more general hypothesis. The possibility of glancing mode for flat boundaries and linear coherent phases, is studied in [Wi 2]. Oscillations near a diffractive point, for semilinear wave equations with globally Lipschitzian nonlinearity, are studied in [Che 4]

2. The equations of geometric optics expansions are simpler than the original full set of equations, and thus they give detailed descriptions of several mechanisms of nonlinear interaction.

Applications to the study of qualitative behavior of solutions of systems of conservation laws, are numerous. See e.g. [Hu], [MR], [MRS], [JMR 9].

3. The validity of geometric optics can be extended beyond the category of solutions of the form (1.1), to more general families of "oscillating" solutions u^ε. This is shown in [JMR 6] for semilinear systems either of size 3×3 or with quadratic interaction. This illustrates a link between geometric optics and compensated compactness, in the spirit of [DiP 1], [DiP 2], [T], [McLPT], [Wa].

4. The equations (1.3) (1.4) and (1.5) are dispersionless. The corresponding eiconal equations are homogeneous. When φ is a characteristic phase, all the harmonics $k\varphi$ are characteristic. Thus all the harmonics are expected in the leading term U_0 of the expansions, and they all propagate at the same group velocity.

For applications to nonlinear optics, it is important to consider dispersive media. For example, expansions (1.6) for solutions of

$$\varepsilon\, \partial_t u^\varepsilon \;+\; \sum_{j=1}^{d} \varepsilon\, A_j(t,x)\, \partial_{x_j} u^\varepsilon \;+\; E(t,x)\, u^\varepsilon \;+\; \varepsilon\;\; f(t,x,u^\varepsilon) \;=\; 0\,. \tag{1.14}$$

yield a nonhomogeneous the eiconal equation, since the zero-th order term Eu has no ε in front of it. For such models, harmonics are rarely present in the principal term U_0, as expected in nonlinear optics. We refer to [Do] and [DR] for details, and for examples and applications to Maxwell's equations and nonlinear optics.

2. One phase expansions.

We look for asymptotic solutions (1.6) of the first order system (1.3) which we assume here to be symmetric hyperbolic. Plugging the expansion into the equation leads to an infinite system of equations (see [CB])

$$L(t,x,d\varphi(t,x))\; \partial_\theta U_0 \;=\; 0\,, \tag{2.1}$$

and for $n \geq 0$

$$L(t,x,d\varphi(t,x))\, \partial_\theta U_{n+1} \;+\; L(t,x,\partial) U_n \;+\; F_n \;=\; 0\,. \tag{2.2}$$

Here $L(t,x,\partial) \;:=\; \partial_t \;+\; \sum\; A_j(t,x)\, \partial_{x_j}$, and F_n is a function of $(U_0, \ldots U_n)$.

To solve (2.1), one requires that φ satisfies the eiconal equation

$$\det L(t,x,d\varphi(t,x)) \;=\; 0\,, \tag{2.3}$$

and that the oscillation

$$U_0^*(t,x,\theta) := U_0(t,x,\theta) - \underline{U}_0\,, \quad \underline{U}_0(t,x) \;:=\; \frac{1}{2\pi}\int_0^{2\pi} U_0(t,x,\theta)\, d\theta\,, \tag{2.4}$$

satisfies the polarization condition

$$\Pi U_0^* \;=\; U_0^*\,, \tag{2.5}$$

where $\Pi(t,x)$ is the orthogonal projector onto the kernel of $L(t,x,d\varphi(t,x))$.

For $n \geq 0$, take the average of equation (2.2) and apply Π and $(I - \Pi)$ to its oscillation. This implies that (2.2) is equivalent to

$$L\underline{U}_n \;+\; \underline{F}_n \;=\; 0\,, \quad \Pi\, L\, \Pi\, U_n^* \;+\; \Pi\, F_n^* \;=\; -\,\Pi\, L\,(I - \Pi)\, U_n^*\,. \tag{2.6}$$

and

$$(I-\Pi)\,U^*_{n+1} \;=\; -\,\partial_\theta^{-1} Q\,(LU^*_n + F^*_n)\,. \tag{2.7}$$

The average and oscillations of U_n and F_n are defined as in (2.4). $Q(t,x)$ is the partial inverse of $L(t,x,d\varphi(t,x))$ such that $QL = LQ = I - \Pi$, $Q\Pi = 0$ and ∂_θ^{-1} is the inverse of ∂_θ acting on periodic functions with vanishing mean value.

Of particular importance are the equations for U_0. They are given by(2.5) and (2.6). With $F_0 := f(t,x,U_0(t,x,\theta))$, they read

$$\left\{\begin{aligned} L\,\underline{U}_0 \;+\; \underline{F}_0 \;&=\; 0\\ (I-\Pi)\,U_0^* \;&=\; 0\\ \Pi\,L\,\Pi U_0^* \;+\; \Pi\,F_0^* \;&=\; 0\end{aligned}\right. \tag{2.8}$$

To simplify matter, suppose that $\lambda(t,x,\xi)$ is, for $\xi \neq 0$, a real eigenvalue of constant multiplicity of $\sum \xi_j\,A_j(t,x)$. Then a real solution of (2.3) is obtained by solving

$$\partial_t\varphi(t,x) \;+\; \lambda(t,x,\partial_x\varphi(t,x)) \;=\; 0\,. \tag{2.9}$$

Given a smooth solution with $d\varphi \neq 0$, the matrices $\Pi(t,x)$ and $Q(t,x)$ are C^∞ functions. Moreover,

$$\Pi\,L\,\Pi \;=\; \partial_t \;+\; \sum_j \frac{\partial\lambda}{\partial\xi_j}(t,x,\partial_x\varphi)\,\partial_{x_j} \;+\; C(t,x)\,. \tag{2.10}$$

Thus (2.8) is a nonlinear coupled system of a hyperbolic equation for $\underline{U}_0$ and a transport equation for $U_0^* = \Pi U_0^*$. Because of the integral term $\underline{F}_0$, the coupling is nonlocal in θ. However, this system has the standard properties of symmetric hyperbolic systems, and the Cauchy problem has local smooth solutions.

The other terms of the expansion are found inductively. If $(U_0,\dots,U_n)$ are known, $(2.7)_n$ gives $(I-\Pi)\,U^*_{n+1}$. Then $(2.6)_{n+1}$ gives a coupled *linear* system for $(\underline{U}_{n+1}, \Pi U^*_{n+1})$ of the form

$$\left\{\begin{aligned} L\,\underline{U}_{n+1} \;+\; \underline{F}_{n+1} \;&=\; 0\\ \Pi\,L\,\Pi U^*_{n+1} \;+\; \Pi\,F^*_{n+1} \;&=\; \Pi G_n\\ F_{n+1} \;&=\; \partial_u f(t,x,U_0)\cdot U_{n+1} \;+\; H_n\end{aligned}\right. \tag{2.11}$$

where G_n and H_n only depend on $(U_0,\dots,U_n)$. The Cauchy problem. for (2.11) has smooth solutions. Moreover, since this problem is linear, the domain of existence of the solution only depends on the domain of existence for U_0.

Thus, given C^∞ Cauchy data for φ and U_0 with $\Pi U_0^* = 0$, one can solve the eikonal equation (2.9) and the nonlinear profile equation (2.8) on a suitable domain of determinacy Ω. Next, with arbitrary Cauchy data for $\underline{U}_n$ and ΠU_n^*, one can solve (2.11) on Ω. This provides formal solutions of the form (1.6).

THEOREM ([JR 1]) *Given such a formal solution, if*

$$u_0^\varepsilon(x) \;\sim\; \sum_{n\geq 0} \varepsilon^n\; U_n(0,x,\varphi(0,x)/\varepsilon))\,, \tag{2.12}$$

then the Cauchy problem for (1.3) with Cauchy data u_0^ε, has a unique solution u^ε for $\varepsilon \le 1$, defined and smooth on a domain $\Omega' \subset \Omega$ independent of ε, and satisfying

$$u^\varepsilon(t,x) \;\sim\; \sum_{n\ge 0} \varepsilon^n\, U_n(t,x,\varphi(t,x)/\varepsilon))\,. \tag{2.13}$$

Moreover, Ω' can be taken arbitrarily close to Ω if ε is restricted to be small enough.

Remark that the existence of solutions on a domain independent of ε is an important part of the statement, since the classical existence theorems for smooth solutions do not apply.

Further results.

1) Similar results are available for quasilinear equations (1.4), and expansions (1.7) ([G 2], [G 3]). The background state $\underline{u}$ is a solution of (1.4). φ satisfies the eiconal equation for the linearized operator on $\underline{u}$. The main novelty is that the transport operator (2.10) contains an additional nonlinear term in $U_0\,\partial_\theta$. In particular, the transport equation for U_0 looks like a Burger's equation and thus may develop shocks. The other equations for U_n, $n \ge 1$ remain linear. The expansions are justified, as long as the phase φ and the first profile U_0 exist and remain smooth.

2) Another interesting feature of [G 3] is the justification of expansions with *almost periodic profiles*. The construction sketched at the begining of this paragraph breaks down immediately, since ∂_θ^{-1} does not act in the space of almost periodic functions with vanishing mean value. This indicates that the second term U_1 cannot be defined in general. However, the equations (2.8) still make sense and define the profile U_0.

[G 3] provides a construction of solutions which satisfy

$$u^\varepsilon(t,x) \;=\; U_0(t,x,\varphi(t,x)/\varepsilon) \;+\; o(1)\,, \tag{2.14}$$

in place of (2.13). The error $o(1)$ tends to zero in an appropriate space, implying its convergence to 0 in L^∞.

3. One dimensional resonant expansions.

In this section, we consider expansions with several phases, in one space dimension. For simplicity, we consider strictly hyperbolic semilinear 3×3 systems, and refer the reader to [JMR 1] for higher order semi and quasi-linear systems. After a diagonalization and a change of dependent variables, the equations are

$$X_k u_k \;:=\; \partial_t u_k \;+ \lambda_k(t,x)\,\partial_x u_k \;=\; f_k(t,x,u_1,\ldots,u_N)\,, \quad k\in\{1,2,3\}\,. \tag{3.1}$$

The λ_k are real and $\lambda_1 < \lambda_2 < \lambda_3$. The f_k are smooth functions of the variables $(t,x,u) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}^3$.

The characterisitic set is the union of the 3 lines $\tau + \lambda_k(t,x)\,\xi = 0$ and characteristic phases φ satisfy one of the equation $X_k\,\varphi = 0$. The corresponding eigenvector is the k-th vector of the canonical basis in $\mathbb{R}^3$. The oscillatory wave trains associated to the j-th eigenvalue λ_j which are constructed in §2, are of the form

$$u_j^\varepsilon(t,x) \;=\; U_j(t,x,\varphi_j(t,x)/\varepsilon) \;+\; O(\varepsilon)\,, \quad u_k^\varepsilon \;=\; O(\varepsilon) \quad for\ k\ne j\,, \tag{3.2}$$

where φ_j satisfies

$$X_j\,\varphi_j \;=\; 0\,. \tag{3.3}$$

The interaction of such wave trains, is described by solution u^ε of (3.1) such that

$$u^\varepsilon_k(t,x) \;=\; U_k(t,x,\varphi_k(t,x)/\varepsilon) \;+\; o(1)\,, \quad k\in\{1,2,3\}\,, \tag{3.4}$$

The Cauchy data are assumed to be of the form

$$u^\varepsilon_k(0,x) \;=\; u^\varepsilon_{0,k}(x) \;:=\; A_k(x,\varphi_k(0,x)/\varepsilon) \;+\; o(1)\,, \quad k\in\{1,\dots,3\}\,. \tag{3.5}$$

For example, one can consider an initial phase $\psi(x)$, and oscillatory Cauchy data

$$u^\varepsilon_0 \;:=\; A(x,\psi(x)/\varepsilon)\,. \tag{3.6}$$

Then φ_j is the solution of (3.3) with initial data ψ and the A_k are the components of A. The equations of nonlinear geometric optics are of the form

$$\begin{cases} X_k\,U_k(t,x,\theta) &=& (\mathbf{E}_k F_k)\,(t,x,\theta) \\ U_k(0,x,\theta) &=& A_k(x,\theta) \end{cases} \qquad k\in\{1,2,3\}\,, \tag{3.7}$$

where

$$F_k(t,x,\theta_1,\theta_2,\theta_3) \;:=\; f_k\big(t,x,,U_1(t,x,\theta_1),\dots,U_3(t,x,\theta_3)\big) \tag{3.8}$$

and $\mathbf{E}_k$ is an integral operator whose structure depends on the existence or not of resonances among the phases φ_k.

Definition. *A resonance is an triple (s_1,s_2,s_3) of functions such that*

$$X_k\,s_k \;=\; 0 \quad \textit{for all } k\in\{1,2,3\}\,, \quad \textit{and} \quad \sum_{k=1}^{3} ds_k \;=\; 0\,. \tag{3.9}$$

The resonance is trivial when all the ds_k are identically zero.

Remark that the definition does not depend on the vector fields themselves but on the foliations they defined. The data of N foliations by curves in $\mathbb{R}^2$ is called an N-web in differential geometry, and resonances correspond to abelian relations on webs (see [P], [BB]). The existence of resonances is a rare phenomenon : *for a 3-web, the dimension of the space of resonances (modulo constants functions) is equal to 0 or 1.* Moreover, for "generic" vector fields X_k, there are no nontrivial resonances. Therefore, in order to have resonances, first the vector fields X_k must be suitably chosen and second, the phases φ_k are also to be chosen carefully. However, resonance is a very important phenomenon. Three constant coefficient vector fields $X_k = \partial_t - \lambda_k\partial_x$, are resonant. The space of resonance is the pace of (s_1,s_2,s_3) such that $s_k = \alpha_k(x-\lambda_k t)$, with $\sum\alpha_k = 0$ and $\sum\lambda_k\alpha_k = 0$. We refer to [JMR 1] for a more general discussion of resonances and more examples.

We now describe the averaging operators $\mathbf{E}_k$ in a simple case. Consider an initial phase $\psi(x)$ on an interval ω and Cauchy data of the form (3.6). We assume that $d\psi\neq 0$ almost everywhere. Let Ω be a domain of determinacy of ω. Let φ_k be the solution of (3.3) on Ω with initial data equal to ψ on ω. The idea is that the nonlinearities f_k will reveal oscillations associated to all the phases $\sum\alpha_k\varphi_k$, but that only the resonant combination will be propagated. Introduce

$$R \;:=\; \{\alpha\in\mathbb{R}^3 \mid \sum_k \alpha_k\varphi_k \equiv 0 \textit{ on } \Omega\}\,. \tag{3.10}$$

When $R = \{0\}$, the averaging operators are defined by

$$(3.11) \qquad (\mathbf{E}_1 F)(t,x,\theta_1) \ := \ \lim_{T\to+\infty} \frac{1}{T^2} \int_0^T\!\!\int_0^T F(\theta_1,\theta_2,\theta_3)\ d\theta_2\, d\theta_3\,,$$

with similar definitions for $\mathbf{E}_2$ and $\mathbf{E}_3$. On the other hand, when there exist $\alpha \in R\backslash\{0\}$,

$$(3.12) \qquad (\mathbf{E}_1 F)(t,x,\theta_1) \ := \ \lim_{T\to+\infty} \frac{1}{T} \int_0^T F\Big(\theta_1, -\frac{\alpha_1\theta_1 \ + \ \alpha_3\sigma}{\alpha_2}, \sigma\Big)\ d\sigma\,,$$

with similar formulas for $k = 2$ and $k = 3$. Since the vector fields are pairwise linearly independent, all three components α_k of $\alpha \in R\backslash\{0\}$ are not equal to 0. Thus the right hand side in (3.12) makes sense. Moreover, the dimension of R is one, therefore the definition does not depend on the choice of $\alpha \in R\backslash\{0\}$.,

The definition of $\mathbf{E}_k$ depends on the local structure of resonances among the linear combinations of the phases. To avoid difficulties, we assume that this structure is the same near all the points $(t,x) \in \Omega$. More precisely, we assume that

$$(3.13) \qquad \forall\alpha \notin R\,, \quad \sum_k \alpha_k\, \varphi_k \neq 0 \quad \textit{almost everywhere on } \Omega\,.$$

Theorem ([JMR 1]) *Suppose that* $A_k \in C^0(\omega \times \mathbb{R}/2\pi\mathbb{Z})$ *and that* $\{\, u^\varepsilon_{0,k} \,\}_{\varepsilon\in]0,1]}$ *is a bounded family in* $L^\infty(\omega)$ *such that*

$$(3.14) \qquad u^\varepsilon_{0,k}(\,.\,) \ - \ A_k(\,.\,,\psi(\,.\,)/\varepsilon) \ \to 0 \quad in \quad L^1(\omega)\ , \quad as \ \ \varepsilon \to 0\,.$$

Then there is $T > 0$ *such that :*

i) for all $\varepsilon \in]0,1]$*, the Cauchy problem for (3.1) with Cauchy data* $u^\varepsilon_{0,k}$ *as a unique solution* u^ε *in* $L^\infty(\Omega_T)$*, where* $\Omega_T := \Omega \cap \{0 < t < T\}$*,*

ii) the equations (3.7) *have a unique solution* $U \in C^0(\Omega_T \times \mathbb{R}/2\pi\mathbb{Z})$*,*

iii)

$$(3.15) \qquad u^\varepsilon_k(\,.\,) \ - \ U_k(\,.\,,\varphi_k(\,.\,)/\varepsilon) \ \to 0 \quad in \quad L^1(\Omega_T)\ , \quad as \ \ \varepsilon \to 0\,.$$

Remarks and further results

1. We refer the reader to [JMR 1] for the study of $N \times N$ systems, the justification of weakly nonlinear geometric optics for quasilinear systems, the introduction of almost periodic profiles and further results.

2. A formal derivation of the equations (3.7) is given in [HK], [MR] and [HMR]. Partial rigorous justifications were previously given by [T], [McLPT], [J], [Ka].

3. Even if the convergence in (3.14) holds in L^∞, the approximation (3.15) is not always true in L^∞. This is due to weak resonances, i.e. to linear combinations $\sum \alpha_k \varphi_k$ which are characteristic along isolated characteristic curves. They create new waves, of the form $V(t,\varphi/\varepsilon)$, with $V(t,\sigma)$ rapidly decreasing with respect to $\sigma \in \mathbb{R}$. These waves are small in L^1 but $O(1)$ in L^∞. This phenomenon is studied in detail in [G 1]. In the absence of weak resonances, L^∞ convergence is proved in [JMR 1].

4. The results quoted above concern continuous solutions of semilinear first order systems, or C^1 solutions in the quasilinear case. For weak solutions of quasilinear systems of conservation laws, which may present shocks, the validity of weakly nonlinear geometric optics is now proved in a general setting in [Che 2] (see also [Che 3]), generalizing a result of [Sch 1]. Special cases were previously treated in [DiPM], [Liu], [Che 1]. The interaction of (strong) shock waves or contact discontinuities and small amplitude oscillations is described in [Co 1] and [Co 2].

4. Coherent multiphase expansions.

For multidimensional problems, the description of resonances is much more complicated. To illustrate this difficulty, consider a semilinear system, with linear part L. Incoming oscillations with phases φ_k, produce as source terms, oscillations with the phases $\varphi := \sum \alpha_k \varphi_k$. To understand how these new oscillations are propagated by L, one has to study the behaviour of the solution u^ε of

$$Lu^\varepsilon \;=\; a\, e^{i\,\varphi/\varepsilon} . \tag{4.1}$$

When φ is everywhere characteristic or everywhere noncharacteristic, the response u^ε is given by a classical asymptotic expansion (see [La]). The difficulty comes from small sets where φ satisfies the eikonal equation, and from small sets where $d\varphi = 0$. In particular, u^ε is not necessarily $O(1)$. Combined with nonlinearities, this mechanism can force the solutions to blow up, even at arbitrarily small times (see [JMR 2], [JMR 7] and §5 below). However, an interesting situation occurs when the dichotomy between characteristic and noncharacteristic phases is clear.

Definition ([JMR 2]). *Let $L(t, , x, \partial_t, \partial_x)$ be a linear first order $N \times N$ system on Ω. A real vector space $\Phi \subset C^\infty(\Omega)$ is L-coherent when for all $\varphi \in \Phi\backslash\{0\}$, one of the following two condition holds :*

i) $\det L(t, x, d\varphi(t, x)) \;\equiv\; 0$ *and* $d\varphi(t, x) \neq 0$ *at every point* $(t, x) \in \Omega$,

ii) $\det L(t, x, d\varphi(t, x)) \;\neq 0$ *at every point* $(t, x) \in \Omega$.

An important example occurs when L has constant coefficients and Φ is the space of linear functions. For simplicity, we now consider only this case. Suppose that $L = L(\partial_t, \partial_x)$ is a constant coefficient symmetric hyperbolic first order system and consider a semilinear problem

$$Lu^\varepsilon \;=\; f(u^\varepsilon) . \tag{4.2}$$

We look for asymptotic solutions of the form

$$u^\varepsilon \;\sim\; \sum_{n\geq 0} \varepsilon^n\, U_n(t, x, t/\varepsilon, x/\varepsilon) \;\sim\; \sum_{n\geq 0} \varepsilon^n \sum_{\tau,\alpha} U_{n,\tau,\alpha}(t, x)\; e^{i\,(\tau t + \alpha x)/\varepsilon} . \tag{4.3}$$

The formal analysis is similar to the analysis sketched in §2. One obtains a sequence of equations

$$L(\partial_T, \partial_X)\, U_0 \;=\; 0 \tag{4.4}$$

and for $n \geq 0$

$$L(\partial_T, \partial_X)\, U_{n+1} \;+\; L(\partial_t, \partial_x)\, U_n \;=\; F_n(U_0, \ldots, U_n) . \tag{4.5}$$

Here, T and X denote the fast variables, and the profiles $U_n(t, x, T, X)$ are periodic in these variables. Introduce $\Pi_{\tau,\alpha}$ the orthogonal projector in $\mathbf{C}^N$ on the kernel of $L(\tau, \alpha)$ and $\mathbf{E}$, the operator defined on (almost) periodic functions by

$$\mathbf{E}\Big(a\; e^{i\,(\tau T + \alpha X)}\Big) \;= (\Pi_{\tau,\alpha}\, a)\; e^{i\,(\tau T + \alpha X)} . \tag{4.6}$$

Applying $\mathbf{E}$ to (4.5) with $n = 0$, yields the following set of equations for U_0 :

$$\mathbf{E}\, U_0 \;=\; U_0 \;, \quad \text{and} \quad \mathbf{E}\big(L(\partial_t, \partial_x)\, U_0\big) \;=\; \mathbf{E}\,\big(f(U_0)\big) . \tag{4.7}$$

Next, the Fourier coefficients of $(Id - \mathbf{E})\, U_1$ are determined by equations of the form

$$(Id - \Pi_{\tau,\alpha})\, L(\tau, \alpha)\, (Id - \Pi_{\tau,\alpha}) U_{1,\tau,\alpha} \;= g_{1,\tau,\alpha} \,, \tag{4.8}$$

where the $g_{1,\tau,\alpha}$ only depend on U_0. The summation of the Fourier series can produce small divisors problems. However, *generically*, the summation of the Fourier series can be performed, defining $(Id - \mathbf{E})U_1$ as a smooth function. Next, $\mathbf{E}U_1$ is determined by equations analogous to (4.7). This procedure can be carried to all orders n, and give a complete formal solution ([JMR 3]) (see also [De] for the case of the wave equation). Using a general theorem from [G 2], this implies the existence of exact solutions satisfying (4.3).

THEOREM ([JMR 3]). *Suppose that the* $U_n \in C^\infty(\Omega \times (\mathbb{R}/2\pi\mathbb{Z})^{1+d})$ *satisfy* (4.4) *and* (4.5), *where* Ω *is contained in the domain of determinacy of* $\omega \subset \mathbb{R}^d_x$. *Consider Cauchy data* u_0^ε *such that*

$$u_0^\varepsilon(x) \ \sim \ \sum_{n\geq 0} \varepsilon^n \, U_n(0,x,0,x/\varepsilon) \tag{4.9}$$

Then, there is $T>0$, *such that for all* $\varepsilon \in]0,1]$, *the Cauchy problem for* (4.2) *with Cauchy data* u_0^ε, *has a unique solution* u^ε *on* $\Omega_T := \Omega \cap \{0<t<T\}$. *Moreover,*

$$u_0^\varepsilon(t,x) \ \sim \ \sum_{n\geq 0} \varepsilon^n \, U_n(t,x,t/\varepsilon,x/\varepsilon)\,. \tag{4.10}$$

The meaning of (4.9) is that for all k and all $\nu \in \mathbb{Z}^d$

$$\partial_x^\nu\Big(u_0^\varepsilon(x) \ - \sum_{n=0}^{k} \ \varepsilon^n \, U_n(0,x,0,x/\varepsilon)\Big) \ = \ O(\varepsilon^{k+1-\nu}) \quad in \quad L^\infty(\omega)\,. \tag{4.9}$$

The meaning of (4.10) is similar.

Remarks.

1. No assumptions on the small divisors are necessary to define U_0. They are only needed to define U_1 and the higher order terms. When the small divisor assumptions are not satisfied, U_0 is still defined and one can look for solutions u^ε such that

$$u^\varepsilon(t,x) \ = \ U_0(t,x,t/\varepsilon,x/\varepsilon) \ + \ o(1)\,. \tag{4.12}$$

This program is worked out in different frameworks, in [JMR 2], [Sch 2] and [JMR 4].

2. To recover the more usual form (1.13)

$$u^\varepsilon(t,x) \ = \ \sum_j \ S_j(t,x,(\tau_j t \ + \ \alpha_j x)/\varepsilon) \ + \ o(1) \tag{4.13}$$

which describes u^ε as the superposition of wave trains associated with phases $\varphi_j(t,x) := \tau_j t \, + \, \alpha_j x$, one expands U_0 into its Fourier series

$$U_0(t,x,T,X) \ = \ \sum_{\tau,\alpha} U_{0,\tau,\alpha}(t,x) \ e^{i \ (\tau T \, + \, \alpha X)}\,, \tag{4.14}$$

and define S_j as the sum of terms when (τ,α) runs over the set $\mathbb{R}(\tau_j,\alpha_j)$. Here we assume that the directions of (τ_j,α_j) are pairwise different. In contrast with the one dimensional case, the number of characteristic directions (τ_j,α_j) is in general infinite and the form (4.12) contains an infinite number of oscillations propagating in an infinite number of directions. More surprisingly, three incoming phases can be sufficient to create by nonlinear interaction, an infinite number of oscillations (see [JR 2], [JMR 2], [JMR 8]).

3. The analysis of coherent oscillations, does not require L to be hyperbolic with constant multiplicity. In particular, the results mentioned above, apply to conical refraction. This example is developed in [JMR 4].

4. Suppose that L is a two speed system, which means that it has only two distinct eigenvalues. Suppose that Φ is generated by two phases which are everywhere linearly independent. Then Φ is L-coherent. Geometric optics expansions involving two phases, for two speed systems, was already considered in [JR 1].

5. Focusing of oscillations and blow up

Weakly nonlinear geometric optics for hyperbolic partial differential equations involve phases which are solutions of eikonal equations as in the linear theory. When the space has dimension at least equal to two, solutions of the eikonal equations exist only locally, the breakdown corresponding to focusing. The justifications of the weakly nonlinear geometric optics, quoted in the previous sections, are given on domains where focusing phenomena do not exist, either because the space dimension is equal to one, or thanks to the coherence assumption on the phases. However, it is an important issue to understand better the nonlinear effects of focusing.

When rays focus, amplitudes grow and, even in the linear case, one must change the asymptotic description. In the nonlinear case, the large amplitudes can be amplified by nonlinearities and therefore strongly nonlinear phenomena can occur. For example, the domain of existence of the solutions u^ε, may shrinks to the empty set as ε tends to zero (see [JMR 2] [JMR 7]). Focusing and blow up can be created by the principal oscillations themselves. This is called *direct focusing* in [JMR 2]. But nonlinear interactions make the problem much harder. Focusing and blow up can be created by phases not present in the principal term of the expansion, but which are generated after several interactions. This phenomenon is explored in detail in [JMR 2], where it is called *hidden focusing*.

In this section we give two examples of blow up caused by direct focusing of oscillations. The first one shows that blow up can occur before the first time of focusing. The second example shows solutions which cannot be extended, even in the weak sense, after focusing.

Example 1. Consider in space dimension $d = 3$, the equation

$$\Box u^\varepsilon \ - \ |\partial_t u^\varepsilon|^2 \, \partial_t u^\varepsilon \ = \ 0\,, \tag{5.1}$$

with oscillatory Cauchy data

$$u^\varepsilon(0,x) \ = \ \underline{u}_0(x) \ + \ \varepsilon\, U_0(x,|x|/\varepsilon) \quad , \quad \partial_t u^\varepsilon(0,x) \ = \ U_1(x,|x|/\varepsilon)\,, \tag{5.2}$$

where $\underline{u}_0$ is smooth and $U_0(x,\theta)$ and $U_1(x,\theta)$ are smooth, periodic in θ and vanish near $x = 0$. For small times, the solutions satisfy ([JR 1]) :

$$u^\varepsilon(t,x) \ = \ \underline{u}(t,x) \ + \ \varepsilon\, U_{_+}(t,x,(|x|+t)/\varepsilon) \ + \ \varepsilon\, U_{_-}(t,x,(|x|-t)/\varepsilon) \ + \ \varepsilon^2 \dots\,, \tag{5.3}$$

The ground state $\underline{u}$ and the oscillations $V_\pm := \partial_\theta U_\pm$ of the periodic profiles satisfy a coupled system of integro-differential equations. When $\underline{u}_0 = 0$ and $U_1 = \partial_\theta U_0$, with U_0 even in θ, the equations simplify and

$$u^\varepsilon(t,x) \ = \ \varepsilon\, U_{_+}(t,x,(|x|+t)/\varepsilon) \ + \ \varepsilon^2 \dots\,. \tag{5.4}$$

Moreover, $V := \partial_\theta U_{_+}$ satisfies

$$\partial_t V \ - \partial_{|x|} V \ - \ \frac{1}{|x|}\, V \ - \ \frac{1}{2}\, |V|^2\, V \ = \ 0 \quad , \quad V(0,x,\theta) \ = \ \partial_\theta U_0(x,\theta)\,. \tag{5.5}$$

Then $W(t,x,\theta) := |x|^2\, |V(t,x,\theta)|^2$ satisfies

$$W(t,x,\theta) \ = \ \frac{W(0,y,\theta)}{1 - W(0,y,\theta)\, \dfrac{t}{|x|\,|y|}}\,, \quad y := x + t\, \frac{x}{|x|}\,. \tag{5.6}$$

Therefore, if $V(0,y,\theta) \neq 0$, even if it is very small, W and V blow up on the ray $x = y - t\,y/|y|$ starting at y, *before the caustic set* $x = 0$. Remark that the blow up occurs not only in L^∞ but also in L^1.

The blow up is a consequence of two mechanisms. The first, is due to the nonlinearity $-|V|^2V$, as in the ordinary differential equation $V' - |V|^2 V = 0$. But for this equation, the blow up time depends on the size of the data. It is very large when the data is small. The second mechanism is the amplification caused by focusing, as in the linear transport equation $\partial_t V - \partial_{|x|} V - \frac{1}{|x|} V = 0$. Even if the data is small, it forces V to be very large before one reaches the caustic. Then the first mechanism makes the solution blow up very quickly. This shows that focusing is an essential part of the mechanism which produces blow up in a finite time bounded independently of the smallness of the data.

When the profile V blows up, the family of exact solutions u^ε cannot remain bounded. Moreover, since the blow up of V occurs in L^1, this implies that u^ε is not bounded in any reasonable space in which weak solutions can be defined. For example, consider Cauchy data of the form (5.2). The first focusing time is the smallest value of $|y|$ such that $V(0, y, \,.\,)$ is not identicaly zero.

Theorem ([JMR 5]). *There are Cauchy data of the form* (5.2) *and* $T_* > 0$, *strictly smaller than the first focusing time , such that*

i) for all $T < T_*$, *there is* $\varepsilon(T) > 0$ *such that for all* $\varepsilon < \varepsilon(T)$, *the solutions* u^ε *exist and are smooth on* $[0, T] \times \mathbb{R}^3$.

ii) If the family $\partial_t u^\varepsilon$ *is bounded in* $L^3(]0, T[\times \mathbb{R}^3)$, *then* $T \leq T_*$.

Example 2. Consider in dimension $d = 5$, the equation

$$\Box u^\varepsilon = (\partial_t u^\varepsilon)^2 - |\nabla_x u^\varepsilon|^2 \tag{5.7}$$

with initial data

$$u^\varepsilon(0, x) = 0 \quad , \quad \partial_t u^\varepsilon(0, x) = U_1(x, |x|/\varepsilon)\,, \tag{5.8}$$

supported in the annulus $\{1 < |x| < 2\}$. Nirenberg's linearization, $v^\varepsilon := 1 - \exp(-\,u^\varepsilon)$, transform (5.7) into the linear initial value problem

$$\Box v^\varepsilon = 0 \quad , \quad v^\varepsilon(0, x) = 0 \quad , \quad \partial_t v^\varepsilon(0, x) = U_1(x, |x|/\varepsilon)\,. \tag{5.9}$$

By finite speed of propagation, $u^\varepsilon = v^\varepsilon = 0$ in the cone $\mathcal{C} := \{|x| \leq 1 - t\}$. Let Ω_ε denote the connected open subset of $\{v^\varepsilon < 1\}$, which contains $\mathcal{C}$. Then the solutions u^ε are defined and smooth on Ω_ε. Moreover, $\partial u^\varepsilon = \partial v^\varepsilon/(1 - v^\varepsilon)$. If $dv^\varepsilon \neq 0$ near a point $(t, x) \in \partial\Omega_\varepsilon$, then $\partial\Omega_\varepsilon$ is a smooth manifold of codimension one near (t, x), and du^ε is not integrable up to the boundary in Ω_ε, near (t, x). Explicit computations, using the fundamental solution of $\Box$ in dimension $d = 5$, show that there are sequences $t_n \to 1$ and $\varepsilon_n \to 0$ such that $v^{\varepsilon_n}(t_n, 0) = 1$.

Theorem ([JMR 5]). *There are Cauchy data of the form* (5.8) *such that for all* $\delta > 0$, *there is* ε_0 *such that the ball* B_δ, *of radius* δ *and centered at* $(t = 1, x = 0)$ *is not contained in* Ω_ε *for* $\varepsilon < \varepsilon_0$. *Moreover,* $(\partial_t u^\varepsilon)^2 - |\nabla_x u^\varepsilon|^2$ *is not integrable (up to the boundary) on* $\Omega_\varepsilon \cap B_\delta$.

In particular, u^ε *cannot* be extended to B_δ, even as a *weak* solution. Thus the first focusing time is the largest common lifetime of the (weak) solutions u^ε.

6. Absorbtion of oscillations.

When combined with strongly dissipative mechanisms, focusing can lead to a complete absorbtion of oscillations, in finite time. The oscillations disappear when they reach the caustic set. The following example of such a behaviour is taken from [JMR 5].

Consider in dimension $d = 3$, the dissipative wave equation

$$\Box u \ + \ |\partial_t u|^2 \partial_t u \ = \ 0\,, \tag{6.1}$$

$$u^\varepsilon(0,x) \ = \ \underline{u}_0(x) \ + \ \varepsilon\, U_0(x,|x|/\varepsilon) \quad , \quad \partial_t u^\varepsilon(0,x) \ = \ U_1(x,|x|/\varepsilon)\,, \tag{6.2}$$

There are global weak solutions $u^\varepsilon \in C^0([0,\infty[; H^1(\mathbb{R}^3)) \cap L^4([0,\infty[\times \mathbb{R}^4)$ (see [Li], [LS]).

For small times, [JR 1] shows that u^ε has an asymptotic expansion of the form (5.3). When $\underline{u}_0 = 0$, U_0 is even in θ and $U_1 = \partial_\theta U_0$, one has the simpler form

$$u^\varepsilon(t,x) \ = \ \varepsilon\, U_{_+}(t,x,(|x|+t)/\varepsilon) \ + \ \varepsilon^2 \ldots\,. \tag{6.3}$$

and $V := \partial_\theta U_{_+}$ satisfies

$$\partial_t V \ - \partial_{|x|} V \ - \ \frac{1}{|x|}\, V \ + \ \frac{1}{2}\, |V|^2\, V \ = \ 0\,. \tag{6.4}$$

The density of energy $W(t,x,\theta) := |x|^2\, |V(t,x,\theta)|^2$ satisfies

$$W(t,x,\theta) \ = \ \frac{W(0,y,\theta)}{1 + W(0,y,\theta)\, \dfrac{t}{|x|\,|y|}}\,, \qquad y := x + t\, \frac{x}{|x|}\,. \tag{6.5}$$

If $V(0,y,\theta) \neq 0$, then along the ray $x = y - t\, \dfrac{y}{|y|}$,

$$|\,V\,| \to +\infty\,, \quad \text{but} \quad W \to +0 \quad \text{as} \quad |x| \to 0\,. \tag{6.6}$$

This means that the amplitude V is infinite at the caustic, but the energy transported by the oscillations tends to zero, that is, is entirely dissipated. This suggests that the oscillations are absorbed when they reach the caustic set. This can be rigourously proved.

Theorem ([JMR 5]). (See figure 1) *If the Cauchy data* (6.2) *are supported in the annulus A, then the family of solutions u^ε is compact in H^1 on all compact subdomain of D.*

127mm by 86mm (fig1 scaled 750)

Figure 1

7. Oscillations past caustics.

To understand an intermediate case between blow up and absorbtion, consider the semilinear wave equation equation

$$\Box u^\varepsilon \ + \ f(\partial_t u^\varepsilon) \ = \ 0\,, \tag{7.1}$$

with oscillatory Cauchy data

$$u^\varepsilon(0,x) \ = \ \underline{u}_0(x) \ + \ \varepsilon\, U_0(x,\psi(x)/\varepsilon) \quad , \quad \partial_t u^\varepsilon(0,x) \ = \ U_1(x,\psi(x)/\varepsilon)\,. \tag{7.2}$$

The main assumption is that f is globally Lipschitzean and satisfies

(7.3) $$f(0) = 0\,, \quad \big|\, f(v) - f(w) \big| \;\le\; K \,\big|\, v - w \,\big|\,,$$

with K independent of v and w. This immediately implies that there are global weak solutions $u^\varepsilon \in C^0([0,\infty[\,; H^1(\mathbb{R}^3))$. For small times, the solutions satisfy ([JR 1]) :

(7.4) $$u^\varepsilon(t,x) \;=\; \underline{u}(t,x) \;+\; \varepsilon\, U_{_+}(t,x,\varphi_{_+}(t,x)/\varepsilon) \;+\; \varepsilon\, U_{_-}(t,x,\varphi_{_-}(t,x)/\varepsilon) \;+\; \varepsilon^2 \ldots\,,$$

where $\varphi_\pm$ are the two solutions of the eikonal equation for $\square$ with Cauchy data ψ :

(7.5) $$\partial_t \varphi_\pm \;=\; \pm\,|\partial_x \varphi_\pm|\,, \quad \varphi_{\pm|t=0} \;=\; \psi\,.$$

The ground state $\underline{u}$ and the oscillations $V_\pm := \partial_\theta U_\pm$ of the periodic profiles, are uniquely determined as solutions of a coupled system of integro-differential equations. Moreover, [JR 1] shows that asymptotically, the solutions satisfy (7.4) *as long as the the eikonal equations* (7.5) *have smooth solutions.*

The question is now to describe the solutions u^ε for arbitrary large times. In the linear theory, one idea to study the behaviour of oscillations near caustics is to replace phase-amplitude expansions like (7.4) by *Lagrangian integrals* (see [Du], [Hö 1], [Hö 2]). In the context of wave equations, they are of the form

(7.6) $$(2\pi\varepsilon)^{-d} \iint e^{i\Phi_\pm(t,x,y,\xi)/\varepsilon}\; a(t,y)\; dy\, d\xi\,,$$

with

(7.7) $$\Phi_\pm(t,x,y,\xi) \;:=\; \pm\, t\,|\xi| \;+\; (x-y)\cdot\xi \;+\; \psi(y)\,.$$

These integrals are associated to Lagrangian manifolds, $\Lambda_\pm \subset T^*\mathbb{R}^{1+d}$. $\Lambda_\pm$ is the set of points (t,x,τ,ξ) such that there exists y such that

(7.8) $$x \;=\; y \;\mp\; t\,\frac{\xi}{|\xi|}\,, \quad \xi \;=\; d\psi(y)\,, \quad \tau = \pm|\xi|\,.$$

Remark that (7.8) provides a global parametrization of the Lagrangians with (t,y) as parameters. In particular, we can consider $a(t,y)$ as the *symbol* of the integral (7.6).

When harmonics are present, the definition (7.6) has a clear extension. For

(7.9) $$A(t,y,\theta) \;=\; \sum_{n\neq 0} a_n(t,y)\; e^{i\;n\;\theta}$$

introduce

(7.10) $$I^\varepsilon_\pm(A) \;=\; \sum_{n\neq 0} \frac{|n|^d}{(2\,\pi\,\varepsilon)^d} \iint e^{i\Phi_\pm(t,x,y,\xi)/\varepsilon}\; a_n(t,y)\; dy\, d\xi\,,$$

The local behaviour of such integrals is given by stationary phase expansions. Consider the hessian of $\Phi_\pm$ with respect to the variables (y,ξ), evaluated on $\Lambda_\pm$, that is for x and ξ as in

(7.8). Introduce the square root $\Delta_\pm(t,y)$ of the absolute value of its determinant and $2\,m_\pm(t,y)$ its signature. The caustic set is $\mathcal{C} := \mathcal{C}_+ \cup \mathcal{C}_-$, where

$$\mathcal{C}_\pm \;:=\; \{\ (t,x)\ \mid\ \exists\ y\ :\ x = y \mp t\,g(y)\ \ \textit{and}\ \ \Delta_\pm(t,y) \;=\; 0\ \}\,, \tag{7.11}$$

with $g(y) := d\psi(y)/|d\psi(y)|$. *The classical stationnary phase theorem* implies that for $(t,x) \notin \mathcal{C}$

$$I^\varepsilon_\pm(A)\,(t,x) \;\sim\; J^\varepsilon(A)\,(t,x) \;:=\; \sum_{y\,|\,y\mp tg(y)=x} \frac{1}{\Delta_\pm(t,y)}\ \big(\mathcal{H}^{m(t,y)}\,A\big)(t,y,\psi(y)/\varepsilon)\,, \tag{7.12}$$

where $\mathcal{H}$ is the Hilbert transform on Fourier series

$$\mathcal{H}\Big(\ \sum_{n\neq 0} a_n\ e^{i\ n\ \theta}\Big) \;:=\; \sum_{n\neq 0} i^{\,\mathrm{sign}\,n}\ a_n\ e^{i\ n\ \theta}\,. \tag{7.13}$$

Note that the phase shift $m(t,y)\,\pi/2$ changes to $-m(t,y)\,\pi/2$ when passing from positive to negative harmonics. This explains the presence of the Hilbert transform.

Locally, there is a regular labelling $y_{\pm,k}$ of points y such that $y \mp tg(y) = x$. Then

$$\varphi_{\pm,k}(t,x) \;:=\; \psi(y_{\pm,k}) \tag{7.14}$$

are solutions of the eikonal equation, and $J^\varepsilon_\pm(A)$ is of the form

$$J^\varepsilon_\pm(A) \;=\; \sum_k\ U_{\pm,k}(t,x,\varphi_{\pm,k}(t,x)/\varepsilon)\,, \tag{7.15}$$

with

$$U_{\pm,k}(t,x,\theta) \;:=\; \frac{1}{\Delta_\pm(t,y_{\pm,k})}\ \big(\mathcal{H}^{m(t,y_{\pm,k})}\,A\big)(t,y_{\pm,k},\theta)\,, \tag{7.16}$$

The linear theory (see e.g. [Du], [H 2]), shows that

$$u^\varepsilon \;\sim\; \underline{u} \;+\; \varepsilon\,I^\varepsilon_+(A_+) \;+\; \varepsilon\,I^\varepsilon_-(A_-) \;\sim\; \underline{u} \;+\; \varepsilon\,J^\varepsilon_+(A_+) \;+\; \varepsilon\,J^\varepsilon_-(A_-)\,. \tag{7.17}$$

The phase-amplitude representation of u^ε outside the caustic then follows from (7.15). In fact, for linear equations, the results are much more precise and refined stationary phase expansions yield precise descriptions of u^ε near $\mathcal{C}$. ([Lu], [Du], [Ho2]).

For the nonlinear wave equation (7.1) with globally Lipschitzian nonlinearity f, one can show, with suitable assumptions, that the solution u^ε still have an asymptotic description of the form (7.17) (see [JMR 7]).

Assumptions on the initial phase ψ.

1. $\psi \in C^\infty(\overline{\omega})$, ω open subset of $\mathbb{R}^d$, and the measure of $\partial\omega$ in $\mathbb{R}^d$ is 0.
2. $d\psi(y) \neq 0$ for all $y \in \overline{\omega}$.
3. The number of solutions y of $y \pm tg(y) = x$ is bounded independently of (t,x).
4. **Nonresonance**. For all domains Ω which do not intersect $\mathcal{C}$, the local phases $\varphi_{\pm,k}$ (7.14) do not resonate, which means that for all families $\alpha_{\pm,k} \in \mathbb{Z}$ with at least two nonvanishing components, $\sum \alpha_{\pm,k}\ \varphi_{\pm,k}$ is noncharacteristic a.e. on Ω.

Examples and Remarks.

1. Examples of phases which satisfy these assumptions are

a) $\psi(y) \;=\; |y|$, away from the origin.

b) $\psi(y) \;=\; a_1\, y_1^2 \;+\; a_2\, y_2^2$, on $\mathbb{R}^2\backslash 0$.

c) $\psi(y) \;=\; y_d \;+\; y_1^2 + \ldots + y_{d-1}^2$ in $\mathbb{R}^d$.

2. The second assumption can be relaxed, to cover the case where ψ has isolated critical points as in example b), or isolated singular points as in the first exemple. The third condition, requires that the mapping $y \to y \pm tg(y)$ is not too complicated.

3. Much more important is the nonresonance condition 4. This assumption is *much weaker than coherence*. Coherence would require that the phases $\sum \alpha_{\pm,k}\ \varphi_{\pm,k}$ are *nowhere* characteristic. Moreover, the coherence assumption *is not satisfied* in the examples b) or c) above, while the nonresonance condition 4, is satisfied in these examples.

We refer the reader to [JMR 7] for further examples and discussions.

Assumptions on the Cauchy data.

Suppose that $\underline{u}_0 \in H^1(\mathbb{R}^d)$, that $U_0(x,\theta)$ and $U_1(x,\theta)$ are smooth enough, periodic in θ and vanish for $x \notin \omega$. Error terms r_0^ε and r_1^ε can be added to the Cauchy data, provided that they converge to zero in $H^1(\mathbb{R}^d)$ and $L^2(\mathbb{R}^d)$ respectively.

The idea is that Lagrangian integrals can be used to study the asymptotic behaviour of the solutions u^ε of the Cauchy problem (7.1) (7.2). We sketch the main steps, referring for details to [JMR 7].

A) *L^2 estimates of oscillatory integrals*. For A smooth enough, $I_\pm^\varepsilon(A)$ and $J_\pm^\varepsilon(A)$ are bounded families in $L^\infty([0,T], L^2(\mathbb{R}^d)$ and

$$\limsup_{\varepsilon\to 0}\ ||\, I_\pm^\varepsilon(A)\,(t)\,||_{L^2(\mathbb{R}^d)} \;\le\; ||\, A(t)\,||_{L^2(\omega\times\mathbb{R}/\mathbb{Z})}\,,$$

$$\limsup_{\varepsilon\to 0}\ ||\, J_\pm^\varepsilon(A)\,(t)\,||_{L^2(\mathbb{R}^d)} \;\le\; C\ ||\, A(t)\,||_{L^2(\omega\times\mathbb{R}/\mathbb{Z})}\,.$$

The limsup's are uniform on bounded intervals of time. Moreover, for all T,

$$||\, I_\pm^\varepsilon(A) \;-\; J_\pm^\varepsilon(A)\,||_{L^2([0,T]\times\mathbb{R}^d)} \;\to\; 0 \qquad \text{as}\;\; \varepsilon \to 0\,.$$

A stronger convergence in $L^\infty([0,T]; L^2(\mathbb{R}^d))$ holds, when additional transversality conditions are satisfied (see [JMR 7]). These conditions are satisfied in the three examples of phases given above.

B) *Nonlinear functions of Lagrangian integrals*. Suppose that Ω does not intersect $\mathcal{C}$. Let $\varphi_{\pm,k}$ denote the local phases on Ω defined in (7.14). Suppose that

$$v^\varepsilon \;\sim\; \underline{v} \;+\; J_+^\varepsilon(B_+) \;+\; J_-^\varepsilon(B_-)\,. \tag{7.18}$$

Then, by (7.15), v^ε is of the form $v^\varepsilon \;\sim\; \underline{v} \;+\; \sum\; V_{\pm,k}\,(\;\cdot\;, \varphi_{\pm,k}/\varepsilon)\,$. Thus,

$$f(v^\varepsilon(t,x)) \;\sim\; \mathcal{F}(t,x,\varphi_*(t,x)/\varepsilon)\,, \tag{7.19}$$

where

$$\mathcal{F}(t,x,\theta_*) \;:=\; f\Big(\underline{v}(t,x) + \sum V_{\pm,k}(t,x,\theta_{\pm,k})\Big)\,. \tag{7.20}$$

We have used the notation, $\theta_* := \{\theta_{\pm,k}\}$, $\varphi_* := \{\varphi_{\pm,k}\}$.

The idea is now to extract from (7.19) the characteristic oscillations. Here we use the nonresonance assumption. Introduce first the total average

$$\mathcal{E}_0(\underline{v}, B_+, B_-)\,(t,x) \;:=\; \int \mathcal{F}(t,x,\theta_*)\; d\theta_*\,. \tag{7.21}$$

Next, for (t,y) such that $x := y - tg(y) \notin \mathcal{C}_+$, label the $y_{\pm,k}$ so that $y = y_{+,1}$. Introduce the average of $\mathcal{F}$ with respect to the variables θ_* except the first one $\theta_{+,1}$:

$$\mathcal{F}_{+,1}(\underline{v}, B_+, B_-)\,(t,y,\theta_{\pm,1}) \;:=\; \int \mathcal{F}(t,x,\theta_*)\; d\theta_{+,2}\ldots d\theta_{-,1}\ldots\,. \tag{7.22}$$

Its oscillatory part is

$$\mathcal{F}^{osc}_{+,1}(t,y,\theta) \;:=\; \mathcal{F}_{+,1}(\underline{v}, B_+, B_-)\,(t,y,\theta) \;-\; \mathcal{E}_0(\underline{v}, B_+, B_-)(t,x)\,. \tag{7.23}$$

To lift it to the Lagrangian, introduce

$$\mathcal{E}_+(\underline{v}, B_+, B_-)\,(t,y,\theta) \;:= \Delta_+(t,y)\;\big(\mathcal{H}^{-m(t,y)}\;\mathcal{F}^{osc}_{+,1}\big)\,(t,y,\theta)\,. \tag{7.24}$$

The definition of $\mathcal{E}_-(\underline{v}, B_+, B_-)$ is similar.

The Lipschitz property (7.3) for f and the L^2 estimates for J^ε, imply that the nonlinear operators $\mathcal{E}_0$ and $\mathcal{E}_\pm$ act nicely in L^2. Moreover, Expanding $\mathcal{F}(t,x,\eta_*)$ into its Fourier series and *using the non resonance assumption*, one proves that

$$f(v^\varepsilon) \;\sim\; \mathcal{E}_0 \;+ J^\varepsilon_+(\mathcal{E}_+) \;+\; J^\varepsilon_-(\mathcal{E}_-) \;+\; g^\varepsilon\,, \tag{7.25}$$

where g^ε has *no characteristic oscillations*.

C) *The profile equations.*

Now take $v^\varepsilon = \partial_t u^\varepsilon$ and consider $f(v^\varepsilon)$ as a source term for $\Box$. Only the characteristic oscillations are propagated. Thus the terms g^ε in (7.25) are expected to contribute small errors in H^1. On the other hand, standard computations for oscillatory integrals, show that

$$\Box\big(\varepsilon\, I^\varepsilon_\pm(A_\pm)\big) \;\sim\; I^\varepsilon_\pm(2\,\partial_t\, B_\pm) \quad \text{and} \quad \partial_t\,\big(\varepsilon\, I^\varepsilon_\pm(A_\pm)\big) \;\sim\; I^\varepsilon_\pm(B_\pm)\,,$$

with $B_\pm := \pm\,|d\psi|\;\partial_\theta\, A_\pm$. Assuming that u^ε is asymptotically given by (7.17), these considerations lead, for $\underline{u}$ and $B_\pm := \pm\,|d\psi|\;\partial_\theta\, A_\pm$, to the following set of equations :

$$\left\{\begin{array}{lcl} \Box\underline{u} & + & \mathcal{E}_0(\partial_t\underline{u}, B_+, B_-) \;=\; 0\,, \\ 2\,\partial_t B_+ & + & \mathcal{E}_+(\partial_t\underline{u}, B_+, B_-) \;=\; 0\,, \\ 2\,\partial_t B_- & + & \mathcal{E}_-(\partial_t\underline{u}, B_+, B_-) \;=\; 0\,. \end{array}\right. \tag{7.26}$$

The natural initial conditions are

$$\left\{\begin{array}{l} \underline{u}_{|t=0}(x) \;=\; \underline{u}_0(x)\,, \qquad \partial_t\underline{u}_{|t=0}(x) \;=\; \underline{u}_1(x) \;:=\; \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} U_1(x,\theta)\; d\theta\,, \\[2ex] B_{\pm|t=0}(x,\theta) \;=\; \frac{1}{2}\,\Big(U_1(x,\theta) \;-\; \underline{u}_1(x) \;\;\pm\; |d\psi(x)|\;\partial_\theta U_0(x,\theta)\Big)\,. \end{array}\right. \tag{7.27}$$

Remark that the data of $B_\pm$ uniquely determines $A_\pm$, since both are periodic in θ and their mean values are zero. $B_\pm$ corresponds to the time derivatives of $\varepsilon\, I^\varepsilon_\pm(A_\pm)$. For the spatial derivatives, one has

$$\partial_{x_j}\big(\varepsilon\, I^\varepsilon_\pm(A_\pm)\big)\ \sim\ I^\varepsilon_\pm(B_{\pm,j})\quad\text{with}\quad B_{\pm,j} := \partial_{y_j}\psi\ \partial_\theta\, A_\pm\ =\ \pm\,\frac{\partial_{y_j}\psi}{|d\psi|}\ B_\pm\,. \tag{7.28}$$

D) *The main result.*

THEOREM. ([JMR 7]) *i) The profile equations* (7.26) *with Cauchy data* (7.27) *have a unique solution.*

ii) For all $T>0$*, the solutions* u^ε *of* (7.1) *with Cauchy data* (7.2)*, satisfy*

$$\left\{\begin{array}{l} u^\varepsilon - \underline{u}\ \to\ 0\\ \partial_t u^\varepsilon\ -\ \big(\partial_t\underline{u}\ +\ I^\varepsilon_+(B_+)\ +\ I^\varepsilon_-(B_-)\big)\ \to\ 0\\ \partial_{x_j} u^\varepsilon\ -\ \big(\partial_{x_j}\underline{u}\ +\ I^\varepsilon_+(B_{+,j})\ +\ I^\varepsilon_-(B_{-,j})\big)\ \to\ 0\end{array}\right. \tag{7.29}$$

strongly in $C^0([0,T],L^2(\mathbb{R}^d))$.

iii) In addition, for all $T>0$,

$$\left\{\begin{array}{l} \partial_t u^\varepsilon\ -\ \big(\partial_t\underline{u}\ +\ J^\varepsilon_+(B_+)\ +\ J^\varepsilon_-(B_-)\big)\ \to\ 0\\ \partial_{x_j} u^\varepsilon\ -\ \big(\partial_{x_j}\underline{u}\ +\ J^\varepsilon_+(B_{+,j})\ +\ J^\varepsilon_-(B_{-,j})\big)\ \to\ 0\end{array}\right. \tag{7.30}$$

strongly in $L^2([0,T]\times\mathbb{R}^d)$.

The relations (7.29) and (7.30), are the precise statements corresponding to the equivalence

$$u^\varepsilon\ \sim\ \underline{u}\ +\ \varepsilon\, I^\varepsilon_+(A_+)\ +\ \varepsilon\, I^\varepsilon_-(A_-)\ \sim\ \underline{u}\ +\ \varepsilon\, J^\varepsilon_+(A_+)\ +\ \varepsilon\, J^\varepsilon_-(A_-)\,. \tag{7.31}$$

The form (7.15) of $J^\varepsilon_\pm(A_\pm)$ provides a phase amplitude expansion for u^ε, outside the caustic set.

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