CAUSTICS FOR DISSIPATIVE SEMILINEAR OSCILLATIONS

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1. Introduction

This paper describes focusing effects near general caustics for high frequency solutions of semilinear dissipative equations. Several consequences of focusing mechanisms have already been described. In [JMR 1] and [JMR 2] it was shown that nonlinear interactions can amplify the growth of amplitudes due to focusing and lead to non continuation of solutions, even as weak solutions. For globally Lipschitzean or disssipative nonlinearities this phenomena does not occur. There are unique global solutions and one can pose the problem of describing the behaviour of oscillations beyond caustics. In [JMR 2] we proved that strong dissipation can absorb radial oscillations. In [JMR 3] we showed how nonlinear oscillatory waves cross caustics when the nonlinearity is globally Lipschitzean. In the latter paper the solutions are described by oscillatory integrals which define bounded families in L^2 spaces.

The analyis in this paper relies on sharp uniform bounds for the L^p norm of such oscillatory integrals. Such estimates have already been obtained in [R 1] [R 2] [Sv]. However, the assumptions used in these papers are not adapted to our purpose. In section 3, we state and prove the uniform L^p estimates which we need. We use them to study a class of dissipative equations and consider general caustics. The analysis reveals a critical exponent p_c, associated to each point of the caustic set where rays focus. For generic caustic points, this exponent is 3. For radial focusing in $\mathbb{R}^d$, $d \geq 2$, it is $1 + 2/(d-1)$.

When the nonlinearity grows at infinity faster than $|u|^{p_c}$ the dissipative mechanism is strong and oscillations are absorbed at caustics : there are no oscillations past the focusing point. This extends [JMR 2] to general caustics. When the nonlinear term grows at infinity at a strictly slower rate than $|u|^{p_c}$, oscillations cross caustics and amplitudes can be computed after focusing. This generalizes [JMR 3] to superlinear dissipative equations.

We now sketch the results for a typical example. Consider the semilinear dissipative wave equation

$$\Box u \;+\; |\partial_t u|^{p-1}\,\partial_t u \;=\; 0\,. \tag{1.1}$$

with oscillatory Cauchy data

$$u^\varepsilon_{|t=0} \;=\; \underline{u}_0(x) + \varepsilon\,\mathbf{U}_0(x,\psi(x)/\varepsilon)\,, \qquad \partial_t u^\varepsilon_{|t=0} \;=\; \underline{u}_1(x) + \;\mathbf{U}_1(x,\psi(x)/\varepsilon)\,, \tag{1.2}$$

where ψ is a smooth function with nonvanishing differential on $\omega \subset \mathbb{R}^d$, $\mathbf{U}_0(x,\theta)$ and $\mathbf{U}_1(x,\theta)$ are smooth, 2π-periodic in θ with mean 0 and compactly supported in $\omega\times\mathbb{T}$ and $\underline{u}_0, \underline{u}_1 \in C_0^\infty(\mathbb{R}^d)$. Later these regularity assumptions are considerably weakened.

Since the Cauchy data (1.2) define bounded families in $H^1(\mathbb{R}^d)$ and $L^2(\mathbb{R}^d)$, the corresponding weak solutions u^ε are bounded in $C^0([0,+\infty[;H^1(\mathbb{R}^d))$ with $\partial_t u^\varepsilon$ bounded $C^0([0,+\infty[;L^2(\mathbb{R}^d))\cap L^{p+1}([0,+\infty[\times\mathbb{R}^d)$, see [L], [LS], [Str].

The behaviour of u^ε for small times is given by [JR]. The solution satisfies

$$u^\varepsilon \;=\; \underline{u} + \varepsilon\,\mathbf{U}_+(t,x,\varphi_+(t,x)/\varepsilon) \;+\; \varepsilon\,\mathbf{U}_-(t,x,\varphi_-(t,x)/\varepsilon) \;+\; \varepsilon^2\ldots\,, \tag{1.3}$$

where $\varphi_\pm$ are the solutions of the eikonal equation for $\Box$ with data ψ at $t=0$ and the profiles $\mathbf{U}_\pm$ satisfy transport equations, coupled to a nonlinear wave equation for $\underline{u}$.

In general, $\varphi_\pm$ develope singularities in finite time. The singular locus is called the caustic set $\mathcal{C}$. As one approaches $\mathcal{C}$, amplitudes tend to infinity and the description (1.3) breaks down. For linear equations, the substitute to (1.3) is well known. The phases are replaced by Lagrangian manifolds $\Lambda_\pm$ which are globally defined and smooth and the oscillations $\mathbf{U}_\pm(\,.\,,\varphi_\pm/\varepsilon)$ are replaced by Lagrangian distributions associated with $\Lambda_\pm$ (see e.g. [Du],[DH], [Hö 1], [Hö 2]). This representation shows that after $\mathcal{C}$ more phases are required to describe the asymptotics of u^ε. In the nonlinear case the situation can be worse since amplitudes may blow up before caustics [JMR 2]. This does not happen for dissipative equations.

For the wave equation, the Lagrangian distributions are given by integrals of the form

$$(2\pi\varepsilon)^{-d}\iint e^{i\Phi_\pm(t,x,y,\xi)/\varepsilon}\;a(t,y)\;dy\,d\xi\,, \tag{1.4}$$

with

$$\Phi_\pm(t,x,y,\xi) \;:=\; \pm\,t\,|\xi| \;+\; (x-y)\cdot\xi \;+\; \psi(y)\,. \tag{1.5}$$

In the nonlinear case one must take care of harmonics and consider symbols depending on the extra variable $\theta \in \mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$. This leads to periodic symbols with mean 0

$$(1.6) \qquad \mathcal{A}(t,y,\theta) \;=\; \sum_{n\neq 0} a_n(t,y)\; e^{i\; n\;\theta}$$

and oscillatory integrals

$$(1.7) \qquad I^{\varepsilon}_{\pm}(\mathcal{A}) \;=\; \sum_{n\neq 0} \frac{|n|^d}{(2\pi\varepsilon)^d} \iint e^{i\;n\;\Phi_\pm(t,x,y,\xi)/\varepsilon}\; a_n(t,y)\; dy\, d\xi \,.$$

Recall that in the linear case (see e.g. [Du], [Hö 2])

$$(1.8) \qquad u^\varepsilon \;\sim\; \underline{u} \;+\; \varepsilon\, I^{\varepsilon}_{+}(\mathcal{U}_{+}) \;+\; \varepsilon\, I^{\varepsilon}_{-}(\mathcal{U}_{-})\,.$$

The caustic set $\mathcal{C}$, is the set of points (t,x) where at least one of the critical point of $(y,\xi) \to \Phi_\pm(t,x,y,\xi)$ is degenerate. Thus, outside the caustic set, the classical stationnary phase expansion yields a phase-amplitude expansion of the form

$$(1.9) \qquad I^{\varepsilon}_{\pm}(\mathcal{U}) \;=\; \sum_k \mathbf{U}_{\pm,k}(t,x,\varphi_{\pm,k}(t,x)/\varepsilon) \;+\; O(\varepsilon)\,.$$

The sum (1.9) runs over the set of critical points of $(y,\xi) \to \Phi_\pm(t,x,y,\xi)$. In particular, for small times, there is a unique critical point and (1.8) is identical to (1.3). Sometimes refined stationnary phase theorems yield precise descriptions of u^ε near the caustic, [Lu], [Du], [Hö 2].

In [JMR 3], Lagrangian distributions are at the center of our analysis for the nonlinearities which are globally Lipschitzean. The goal of this paper, is to show that the description (1.8) is true for all dissipative equations (1.1), provided that the Lagrangians $\Lambda_\pm$ satisfy a non resonance condition and some technical assumptions.

The analysis in [JMR 3] uses the fact that families of integrals like (1.4) or (1.7) are bounded in L^2, even on neighborhoods of caustics. This expresses the boundedness of energy. Near the caustics, these families are not bounded in L^∞. Therefore, it is a natural question to determine the supremum of the real numbers $q \geq 2$ such that $I^{\varepsilon}_{\pm}(\mathcal{A})$ is bounded in L^q. In addition these L^q estimates are needed to control nonlinear terms.

The analysis reveals a critical exponent

$$q_c := 2 + 2/\mu$$

which does not depend on the full complexity of the caustic but, for the phases $\Phi_\pm$ in (1,5), only on the algebraic order μ to which the determinant of the Hessian matrix of $\Phi_\pm$ vanishes at degenerate critical points.

When the C^∞ symbol $\mathcal{A}$ is supported in a small neigborhood of a degenerate critical point of multiplicity μ, then $I^\varepsilon_\pm(\mathcal{A})$ is bounded in the weak L^{q_c} space, i.e the Lorentz space $L^{q_c,\infty}$. In particular, for all $q < q_c$, $I^\varepsilon_\pm(\mathcal{A})$ is bounded in L^q.

This is proved under a locally constant rank condition for degenerate critical points. We refer to §2 and §3 for a precise statement. Similar estimates can be found in [R 1], [R 2] and [Sv], where convexity assumptions are used. The critical index q_c is sharp. If $\mathcal{A} \neq 0$ at the degenerate critical points, then for $q \geq q_c$, the L^q norms of $I^\varepsilon_\pm(\mathcal{A})$ are not bounded. To guarantee their boundedness in L^q requires some vanishing of $\mathcal{A}$ on $\mathcal{S}_\pm$ the set of degenerate critical points of $\Phi_\pm$. Results in this direction can be found in [CDMM] (see also references in [Ste]). Note that the condition $q < q_c$ is also necessary for the right hand side of (1.9) to define bounded families in L^q.

These estimates are not and do not follow the from sharp L^p estimates on Fourier Integral Operators which can be found in [Ste] and [So] and references therein.

We show that

there are $\underline{u}$ and profiles $\mathcal{U}_+$ and $\mathcal{U}_-$ such that the solutions of (1.1) (1.2) *satisty* (1.8)*. $\underline{u}$ and $\mathcal{A}_\pm$ are related by a coupled system of transport equations and a wave equation. With $\mathcal{B}_\pm := \partial_\theta \mathcal{U}_\pm$, the equations are of the form :*

$$\partial_t \mathcal{B}_\pm \; + \; \mathcal{E}_\pm(\underline{u}, \mathcal{B}_+, \mathcal{B}_-) \; = \; 0 \; , \quad \Box \underline{u} \; + \; \underline{\mathcal{E}}\,(\underline{u}, \mathcal{B}_+, \mathcal{B}_-) \; = \; 0 \, . \tag{1.10}$$

(1.9) provides a phase-amplitude representation of u^ε outside $\mathcal{C}$. The propagation equation for the amplitude $\mathbf{U}_{\pm,j}$ follows from the equation for $\mathcal{U}_\pm$. The $\mathbf{U}_{\pm,j}$ satisfy transmission conditions which include the usual phase shift, as in [JMR 3]. The system (1.10) inherits dissipativity from the original one. The nonlinear interactions $\mathcal{E}_\pm$ are singular above the caustic set $\mathcal{C}$. This fact was exploited in [JMR 3] to reveal the possible occurence of nonsmooth tranfer of energy at caustics. Here, the singularity of $\mathcal{E}_\pm$ is augmented by the stronger nonlinearity in (1.1), and $\mathcal{E}_\pm(\underline{u}, \mathcal{B}_+, \mathcal{B}_-)$ is not always locally integrable near the caustic points. Thus the transport equations in (1.10) must be interpreted with care. The discussion again involves the critical index q_c attached to degenerate critical points, indicating consistency with the property that $\partial_t u^\varepsilon$ belongs to L^{p+1}. If $p + 1 \geq q_c$, then $\mathcal{E}_\pm(\underline{u}; \mathcal{B}_+, \mathcal{B}_-)$ is not necessarily integrable near the corresponding point. But there, *strong* dissipation and *absorption* occur. The transport equation (1.10) is satisfied along the rays before the singular point, $\mathcal{B}_\pm$ tends to zero at this point, and $\mathcal{B}_\pm$ vanishes after it. On the contrary, when $p + 1 < q_c$, then $\mathcal{E}_\pm(\underline{u}; \mathcal{B}_+, \mathcal{B}_-)$ is locally integrable so that the transport equation in (1.10) makes sense and is satisfied across the singular point.

Denote by $T_\pm(y)$, the smallest focusing time t along the ray $y \pm t\, d\psi(y)/|d\psi(y)|$ and such that the critical exponent q_c at (t, y) is less than or equal to $p + 1$. For the wave equation, the times of focusing along the rays from $\underline{y}$ are $\pm$ the radii of curvature of the level surface $\psi(y) = \psi(\underline{y})$. The multiplicity $\mu_\pm(t, \underline{y})$ is the multiplicity of the corresponding principal curvature. Then the precise meaning of (1.10) is the following.

$\mathcal{B}_\pm = 0$ for $t > T_\pm(y)$ and the transport equations $\partial_t \mathcal{B}_\pm = \mathcal{E}_\pm(\underline{u}, \mathcal{B}_+, \mathcal{B}_-)$ are satisfied in L^1_{loc} for $t < T_\pm(y)$.

When $p \geq 3$, then $p+1$ is always larger than $2 + 2/\mu$ since $\mu \geq 1$. Thus $T_\pm(y)$ is the first time of focusing along the ray from y.

For generic caustics, the multiplicity $\mu = 1$ and the critical index $q_c = 4$. Thus, if (almost) all the rays have only generic focusing points, absorption occurs at caustics points if and only if $p \geq 3$, which is the supercritical case. On the other hand, in the subcritical case, that is when $p < 3$, oscillations cross the caustics (Figure 1.1 below).

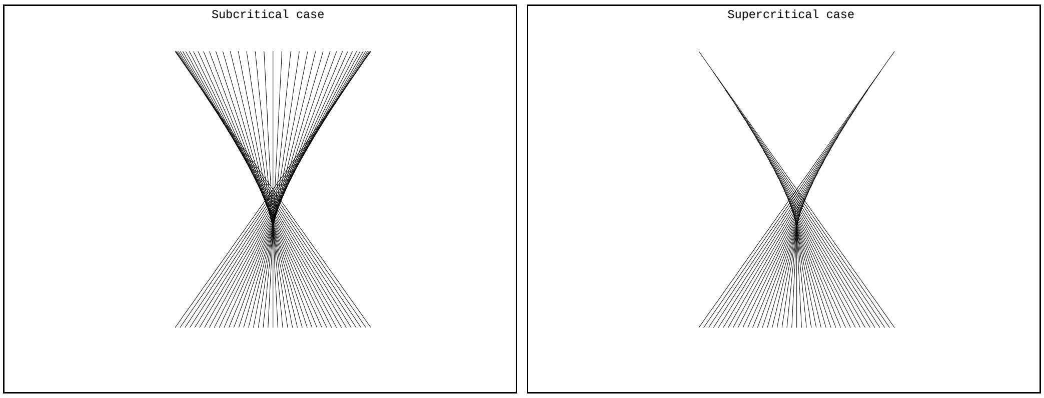


Figure 1.1

For radial phases $\psi(y) = \chi(|y|)$, the multiplicity μ is equal to $d-1$. Thus absorption occurs at the caustic set $\{x = 0\}$ if and only if $p \geq 1 + 2/(d-1)$. For such radial phases, and for $p \geq 1 + 2/(d-1)$, absorption was proved in [JMR 2]. The present paper improves the asymptotics near the caustic, shows that the index $1 + 2/(d-1)$ is sharp and that, for smaller p, oscillations continue after focusing.

When $p < 1 + 2/(d-1)$, absorption never occurs and oscillations persist after crossing the caustics, since the multiplicity μ is at most equal to $d-1$. This includes the limit case, $p = 1$ which corresponds to linear equations (1.1) and also to globally Lipschitzean nonlinearities.

2. Notations and main results

2.1. The time evolution

In $\mathbb{R}^{1+d}$, consider the Cauchy problem for a semilinear system

$$\partial_t u - \sum_{j=1}^{d} A_j \partial_j u \; + \; f(t, x, u) \; = \; 0 \,, \quad u_{|t=0} = u_0 \tag{2.1.1}$$

where $\partial_j := \partial/\partial x_j$. $L := \partial_t - \sum_{j=1}^{d} A_j\partial_j$ is assumed to be a symmetric hyperbolic first order system of constant multiplicity. For simplicity, we assume it has constant coefficients.

ASSUMPTION 2.1.1. *The A_j are constant real symmetric $N\times N$ matrices and the eigenvalues $\lambda_k(\xi)$, $1 \le k \le k_0$, of $A(\xi) := \sum_{j=1}^{d} \xi_j\, A_j$ have multiplicity independant of $\xi \in \mathbb{R}^d\backslash 0$.*

The nonlinear interaction f is assumed to be *dissipative*.

ASSUMPTION 2.1.2. *f is a C^1 function from $[0,+\infty[\times\mathbb{R}^d \times \mathbb{C}^N$ to $\mathbb{C}^N$, with $f(t,x,0) = 0$ and there are $p \in [1,+\infty[$ and constants $0 < c < C < +\infty$ such that for all $(t,x) \in \mathbb{R}^{1+d}$ and all u and v in $\mathbb{C}^N$,*

$$|\, f(t,x,u)\,| \le \; C\,|\,u\,|^p \;, \quad |\,\partial_{u,\overline{u}} f(t,x,u)\,| \le \; C\,|\,u\,|^{p-1}\,, \tag{2.1.2}$$

$$\mathrm{Re}\Big(\big(f(t,x,u) - f(t,x,v)\big)\cdot(\overline{u}-\overline{v})\Big) \;\ge\; c\,|\,u\, - v\,|^{p+1}\,, \tag{2.1.3}$$

In (2.1.2), $\partial_{u,\overline{u}} f$ denotes the set of derivatives $\partial_u f$ and $\partial_{\overline{u}} f$. Remark that $f(t,x,0) = 0$ and (2.1.3) imply that

$$c\,|\,u\,|^{p+1} \;\le\; f(t,x,u)\cdot\overline{u}\,. \tag{2.1.4}$$

EXAMPLE. $f := \partial_{\overline{u}}\Phi : \mathbb{C}^N \to \mathbb{C}^N$ satisfies $\mathrm{Re}\Big(\big(f(u) - f(v)\big)\cdot(\overline{u}-\overline{v})\Big) \ge 0$, when Φ is a convex function from $\mathbb{C}^N$ to $\mathbb{R}$. The stronger condition (2.1.3), and the other assumptions are satisfied for example for $\Phi(u) := |u|^{p+1}$.

In most applications, u is real valued and f is defined for real values of u, but the use of complex exponentials in place of sin and cos is convenient and we systematically introduce complex values for u. This is not a restriction, since when f is a C^1 function from $[0,+\infty[\times\mathbb{R}^d \times \mathbb{R}^N$ to $\mathbb{R}^N$ which satisfies the analogues of (2.1.2) (2.1.3) for u and v in $\mathbb{R}^N$, the extension to complex $u + iu' \in \mathbb{C}^N$, defined by $\widetilde{f}(u+iu') := f(u) + i\,g(u')$, satisfies (2.1.2) (2.1.3) if g has the same properties as f.

Assumption 2.1.2 can be slightly weakened. In particular, in §7, we give modifications of (2.1.2) (2.1.3) adapted to the first order reduction of the wave equation (1.1).

Solutions of the Cauchy problem (2.1.1) are constructed in [Str] (see also [LS] or [Ha]).

THEOREM 2.1.3. *For $u_0 \in L^2(\mathbb{R}^d)$, the Cauchy problem (2.1.1) has a unique solution $u \in C^0(\mathbb{R}\,; L^2(\mathbb{R}^d)) \cap L^{p+1}([0,+\infty[\times\mathbb{R}^d)$. Moreover, if u_0 and v_0 are Cauchy data in $L^2(\mathbb{R}^d)$, the solutions u and v satisfy*

$$\text{(2.1.5)}\quad \sup_{t\geq 0}\ \| u(t) - v(t) \|^2_{L^2(\mathbb{R}^d)} \ +\ 2c\, \| u - v \|^{p+1}_{L^{p+1}([0,+\infty[\times\mathbb{R}^d)} \ \leq\ \| u_0 - v_0 \|^2_{L^2(\mathbb{R}^d)}\,.$$

2.2. Lagrangians, caustics and phases

Consider Cauchy data of the form

$$\text{(2.2.1)}\qquad u_0^\varepsilon(x) \ =\ \underline{u}_0(x) \ +\ \mathcal{U}_0(x, \psi(x)/\varepsilon),$$

where $\mathcal{U}_0(x,\theta)$ is periodic in θ with vanishing mean value. We assume that ψ satisfies

ASSUMPTION 2.2.1. *ω is an open subset of $\mathbb{R}^d$ and $\partial\omega$ has d-dimensional Lebesgue measure equal to zero. The initial phase ψ is real valued and C^∞ on a neighborhood of $\overline{\omega}$ and $d\psi$ does not vanish on $\overline{\omega}$.*

Extensions to phases with isolated critical points or singular points are possible, as in [JMR 3] §6.5.

We assume that $\mathcal{U}_0(x,\theta) = 0$ when $x \notin \omega$. Then the Cauchy data (2.2.1) is defined on $\mathbb{R}^d$.

Thanks to Assumption 2.1.1, the eiconal equation $\det L(d\varphi) = 0$ splits into

$$\text{(2.2.2)}\qquad \partial_t\varphi \ -\ \lambda_k(\partial_x\varphi) = 0\,, \quad \varphi_{|t=0} = \psi\,,$$

for $k \in \{1,\dots,k_0\}$. The geometric global solutions of (2.2.2) are Lagrangian manifolds Λ_k in $T^*\mathbb{R}^{1+d}$. As in [JMR 3], we take global coordinates (t,y) on theses manifolds. Precisely, introduce

$$\text{(2.2.3)}\qquad G \ :=\ [0,+\infty[\times\omega\,,$$

and the mapping $\jmath_k\ :\ \overline{G} =\ [0,+\infty[\times\overline{\omega}\ \to\ [0,+\infty[\times\mathbb{R}^d$

$$\text{(2.2.4)}\qquad \jmath_k(t,y) = (t,x) \quad \text{with} \quad x = q_k(t,y) := y - t\lambda_k'(d\psi(y))\,.$$

Introduce the set of points where $\jmath_k$ is not a local diffeomorphism,

$$\text{(2.2.5)}\qquad \mathcal{S}_k := \{(t,y)\in G \quad | \quad \det \partial_y\, q_k(t,y) = 0\}\,.$$

The caustic set $\mathcal{C}_k$ is

$$\mathcal{C}_k := \jmath_k(\mathcal{S}_k) \subset [0,+\infty[\times\mathbb{R}^d\,. \tag{2.2.6}$$

Introduce

$$g_k(y) \;:=\; \lambda_k'(d\psi(y)) \tag{2.2.7}$$

and its Jacobian matrix $g_k'(y)$. Then $(t,y) \in \mathcal{S}_k$ if and only if $1/t$ is a positive eigenvalue of $g_k'(y)$.

The analysis of the singularities near the caustic set depends on the properties of the mappings $\jmath_k$. We make the following *finiteness* hypothesis which is automaticlly satisfied when ψ is real analytic, since λ is also real analytic.

Assumption 2.2.2. *There is an integer ℓ_*, such that for all k, and all $(t,x) \notin \mathcal{C}_k$, the number of points $y \in \omega$ such that $x = q_k(t,y)$, is less than or equal to ℓ_*.*

As in [JMR 3], introduce the following terminology

Definition 2.2.3. *A subset $\Omega \subset [0,+\infty[\times\mathbb{R}^d$ is k-regular if and only if*

i) Ω is relatively open in $[0,+\infty[\times\mathbb{R}^d$ and connected,

ii) there are smooth functions $\rho_{k,1},..,\rho_{k,\ell}$ from Ω to ω such that $\rho_{k,j} \neq \rho_{k,l}$ for $j \neq l$ and

$$\forall (t,x) \in \Omega\,, \quad \{y \in \omega \mid q_k(t,y) = x\} = \{\rho_{k,j}(t,x)\}_{1\leq j\leq \ell}\,. \tag{2.2.8}$$

A subset $\Omega \subset [0,+\infty[\times\mathbb{R}^d$ is called regular, when it is k-regular for all k.

Note that the $\rho_{k,j}(t,x)$ are the feet of the bicharacteristics passing through (t,x). In *ii)* the number ℓ is allowed to be zero. In this case, condition *ii)* means that $\{y \in \omega \mid q_k(t,y) = x\}$ is empty. A k-regular set is necessarily contained in $[0,+\infty[\times\mathbb{R}^d\backslash\mathcal{C}_k$. As noticed in [JMR 3], the converse is not true, since the number of preimages of x can change when one of them reach the boundary $\partial\omega$. However one has the following result (see §4 in [JMR 3]).

Proposition 2.2.4. *Introduce*

$$\widetilde{\mathcal{C}}_k \;:=\; \mathcal{C}_k \;\cup\; \jmath_k([0,+\infty[\times\partial\omega)\,, \quad \widetilde{\mathcal{C}} \;=\; \cup_k\widetilde{\mathcal{C}}_k\,. \tag{2.2.9}$$

Then $\widetilde{\mathcal{C}} \subset [0,+\infty[\times\mathbb{R}^d$ and, for all k, $\jmath_k^{-1}(\widetilde{\mathcal{C}}) \subset \overline{G}$ have zero Lebesgue measure. For all $(t,x) \notin \widetilde{\mathcal{C}}_k$, $\jmath_k^{-1}(\{t,x\})$ is contained in G. The number of points in $\jmath_k^{-1}(\{t,x\})$ is locally constant on $[0,+\infty[\times\mathbb{R}^d\backslash\widetilde{\mathcal{C}}_k$. In particular, every point $(t,x) \notin \widetilde{\mathcal{C}}_k$ has a k-regular neighborhood.

If Ω is a k-regular open set, $\jmath_k^{-1}(\Omega)$ is the disjoint union of open sets $G_{k,j} \subset G$ such that $\jmath_k$ is a diffeomorphism from $G_{k,j}$ onto Ω. The inverse mappings are $(t,x) \to (t, \rho_{k,j}(t,x))$ where the $\rho_{k,j}$ are as in (2.2.8). This defines a family of phases $\varphi_{k,j}$ on Ω, $(k,j) \in Z$,

$$\varphi_{k,j}(t,x) \ := \ \psi(\rho_{k,j}(t,x)) \,. \tag{2.2.10}$$

The $\varphi_{k,j}$ satisfy the eikonal equation (2.2.2). They are present in the oscillations of the solutions of the linear equation $Lu^\varepsilon = 0$ with Cauchy data (2.2.1). These phases may resonate and nonlinear interaction occur, creating new oscillations for the solutions of (2.1.1) (see e.g. [McLPT], [HMR], [JMR 4] and references therein, for several studies on the mechanism of nonlinear resonance of oscillations). In this paper, which is mainly concerned with caustics, we assume that the phases $\varphi_{k,j}$ do not resonate. The reader is refered to [JMR 3] where it is shown that this assumption is satisfied by a wide class of examples.

Assumption 2.2.5. *For all regular open Ω the set of phases $\varphi_{k,j}$, $(k,j) \in Z$, given by* (2.2.10)*, satisfy the following property :*

for all integers $\{\alpha_{k,j}\}_{(k,j)\in Z}$ at least two of which do not vanish,

$$\det L\,(\sum_{k,j} \alpha_{k,j}\, d\varphi_{k,j}) \ \neq \ 0 \quad \text{a.e. on } \Omega \,. \tag{2.2.11}$$

2.3. L^p estimates of oscillatory integrals

Our goal is to represent the solutions u^ε of equation (2.1.1) with Cauchy data (2.2.1), as a sum of Lagrangian distributions associated to Λ_k. Let $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. Suppose $\mathbb{T}$ equipped with the measure $d\theta$ with total mass equal to 1. Each term is given by an oscillatory integral with a symbol $\mathcal{U}$ defined on $G \times \mathbb{T}$ with mean 0 on $\mathbb{T}$. The variable θ in $\mathbb{T}$ takes care of harmonics.

Definition 2.3.1. *A trigonometric polynomial is a finite sum*

$$\mathcal{A}(t,y,\theta) \ := \ \sum_{n\neq 0} a_n(t,y)\; e^{in\theta} \;, \quad a_n \in C_0^\infty([0,+\infty[\times\omega) \tag{2.3.1}$$

The space of trigonometric polynomials is denoted by $\mathcal{P}$.

For $\mathcal{A} \in \mathcal{P}$, introduce the oscillatory integral

$$I_k^\varepsilon(\mathcal{A})\,(t,x) \ := \ \sum_n \, (|n|/2\pi\varepsilon)^d \int e^{i\,n\,\Phi_k(t,x,y,\xi)/\varepsilon}\; a_n(t,y)\; dy\, d\xi \,, \tag{2.3.2}$$

with

$$\Phi_k(t,x,y,\xi) \ := \ t\,\lambda_k(\xi) \ + \ (x-y)\cdot\xi \ + \ \psi(y) \,. \tag{2.3.3}$$

A crucial point in this work is to study the asymptotic behaviour of such integrals, in L^q spaces. The boundedness in L^2 is clear, as well as the blow up in L^∞ near caustics. The question is to determine the set of q such that $I_k^\varepsilon(\mathcal{A})$ is bounded in L^q. The analysis reveals a critical exponent, which surprisingly does not depend on the full complexity of the caustic but only on the multiplicity μ_k.

Definition 2.3.2. *The multiplicity of $\mathcal{S}_k$ at a point (t, y) is the algebraic multiplicity of $1/t$ as an eigenvalue of $g_k'(y)$. We denote it by $\mu_k(t, y)$.*

The geometric multiplicity, that is the dimension of $\ker(Id - t g_k'(y))$, is denoted by $\nu_k(y)$.

Assumption 2.3.3. *There is an open subset $\omega_0 \subset \omega$ such that $\omega \backslash \omega_0$ has Lebesgue measure equal to 0 and for all k and all $(t, y) \in \mathcal{S}_k$ with $y \in \omega_0$, one has*

i) μ_k and ν_k are constant on a neighborhood of (t, y) in $\mathcal{S}_k$,

ii) either $\nu_k = 1$ or $\nu_k = \mu_k$.

The assumption is satisfied for the wave equation or Maxwell equation when ψ is real analytic. It implies that on a neighborhood of $(t, y) \in \mathcal{S}_k$ with $y \in \omega_0$, $\mathcal{S}_k$ is a smooth manifold of codimension one.

Definition 2.3.4. *For $q \in [2, +\infty]$, $\mathcal{S}_k^-(q)$ denotes the set of points $(t, y) \in \mathcal{S}_k$ such that $y \in \omega_0$ and $\mu_k(t, y) < 2/(q - 2)$. The remaining set is denoted by $\mathcal{S}_k^+(q) := \mathcal{S}_k \backslash \mathcal{S}_k^-(q)$.*

Theorem 2.3.5. *Suppose that Assumptions 2.2.2 and 2.3.3 hold. Let $q \geq 2$, and $\mathcal{A} \in \mathcal{P}$. Suppose that the coefficients a_n vanish on a neighborhood of the points $(t, y) \in \mathcal{S}_k^+(q)$. Then,*

$$\sup_{\varepsilon \in]0,1]} \| I_k^\varepsilon(\mathcal{A}) \|_{L^q(\mathbb{R}^{1+d})} < +\infty . \tag{2.3.4}$$

Remark 2.3.6. The limit $\mu_k < 2/(q - 2)$ is sharp. If $y \in \omega_0$ and $\mathcal{A}$ does not vanish at a point $(t, y) \in \mathcal{S}_k^+$ where $\mu_k \geq 2/(q - 2)$, then $I_k^\varepsilon(\mathcal{A})$ is not bounded in L^q (see §3).

2.4. Oscillations and profiles

When $(t, x) \notin \mathcal{C}_k$ or when the coefficients a_n vanish on a neighborhood of $\mathcal{S}_k$, the relevant stationnary points with respect to (y, ξ) of the phase Φ_k, are nondegenerate. Therefore, the standard stationnary phase expansion implies that

$$I_k^\varepsilon(\mathcal{A}) = J_k^\varepsilon(\mathcal{A}) + O(\varepsilon) \tag{2.4.1}$$

where

$$(2.4.2)\qquad J_k^\varepsilon(\mathcal{A})\,(t,x) \;:=\; \sum_{\{y\,|\,q_k(t,y)=x\}} \frac{1}{\Delta_k(t,y)} \Big(\sum_{n>0} i^{m_k(t,y)}\; a_n(t,y)\; e^{i\,n\,\psi(y)/\varepsilon} + \sum_{n<0} i^{-m_k(t,y)}\; a_n(t,y)\; e^{i\,n\,\psi(y)/\varepsilon}\Big)$$

$$= \sum_{\{y\,|\,q_k(t,y)=x\}} \frac{1}{\Delta_k(t,y)} \,\big(\mathcal{H}^{m_k(t,y)}\mathcal{A}\big)\,(t,y,\psi(y)/\varepsilon)\,,$$

with

$$(2.4.3)\qquad \Delta_k(t,y) := |\,\det d_y\, q_k(t,y)\,|^{1/2}$$

$$(2.4.4)\qquad m_k(t,y) := \frac{1}{2}\;\mathrm{sign}\;\partial^2_{(y,\xi)}\Phi_k(t,x,y,d\psi(y))\,,\ \text{with}\ \ q_k(t,y)=x$$

and $\mathcal{H}$ is the Hilbert transform acting on Fourier series by

$$(2.4.5)\qquad \mathcal{H}\Big(\sum_{n\neq 0} a_n\; e^{in\theta}\Big) \;:=\; \sum_{n\neq 0} i^{\mathrm{sign} n}\; a_n\; e^{in\theta}\,.$$

In (2.4.4), sign A denotes the signature of the symmetric matrix A. Remark that $\Delta_k(t,y)\neq 0$ for $(t,y)\notin\mathcal{S}_k$ and that $m_k(t,y)$ is locally constant on $G\backslash\mathcal{S}_k$.

Suppose that Ω is a regular open set. The matrix $\partial^2_{(y,\xi)}\Phi_k(t,x,y,d\psi(y))$ is nondegenerate for $(t,y)\notin\mathcal{S}_k$ and $x=q_k(t,y)$. Since Ω is connected, the signature $m_k(t,y)$ is constant on each component $G_{k,j}$ of $\jmath_k^{-1}(\Omega)$. Call $m_{k,j}$ its value. Then, using the notations (2.2.8) and (2.2.9), one has on Ω

$$(2.4.6)\qquad J_k^\varepsilon(\mathcal{A})(t,x) \;=\; \sum_{j=1}^{\ell_k} \mathbf{A}_{k,j}\,(t,\;x,\;\varphi_{k,j}(t,x)/\varepsilon)$$

with

$$(2.4.7)\qquad \mathbf{A}_{k,j}(t,x,\theta) \;:=\; \Delta_k(t,\rho_{k,j}(t,x))^{-1}\,(\mathcal{H}^{m_{k,j}}\mathcal{A})\;(t,\rho_{k,j}(t,x),\theta)\,.$$

This formula links the profile $\mathcal{A}$ on the Lagrangian and the profile $\mathbf{A}$ on the base. Note that Δ_k vanishes exactly to order $\mu_k/2$ on $\mathcal{S}_k$. Using the nonresonance Assumption, the L^q norm of $J_k^\varepsilon(\mathcal{A})$ is asymptotically estimated to be the L^q norm of the profiles $\mathbf{A}_{k,j}$. Since $\jmath_k$ is a diffeomorphism from $G_{k,j}$ to Ω whose Jacobian is Δ^2, the L^p norm of $\mathbf{A}_{k,j}$ is equal to a weighted L^q norm of $\mathcal{A}$ on $G_{k,j}\times\mathbb{T}$. This suggests the following definition.

Definition 2.4.1. *For $q \in]1,+\infty[$ and $T > 0$, $\mathcal{L}^q_k([0,T] \times \omega \times \mathbb{T})$ denotes the space of L^q integrable functions on $[0,T[\times\omega \times \mathbb{T}$ with respect to the measure $\Delta_k(t,y)^{2-q}\, dt\, dy\, d\theta$.*

For $0 < s < 1$, $\mathcal{L}^{s,q}_k([0,T] \times \omega \times \mathbb{T})$ denotes the space of functions $\mathcal{U} \in \mathcal{L}^q_k([0,T] \times \omega \times \mathbb{T})$ such that

$$\int_0^1 \int_{[0,T]\times\omega\times\mathbb{T}} \Delta_k(t,y)^{2-q} \ \Big| \frac{\mathcal{U}_k(t,y,\theta+h) \,-\, \mathcal{U}_k(t,y,\theta)}{h^s} \Big|^q dt\, dy\, d\theta \, \frac{dh}{h} \ < \ +\infty$$

The definition trivially extends to $T = +\infty$. For a trigonometric polynomial, belonging to $\mathcal{L}^q_k$ requires no condition on $\mathcal{S}^-_k(q)$ but some vanishing on $\mathcal{S}^+_k(q)$. $\mathcal{L}^{s,q}_k$ is the space of L^q integrable functions on $[0,+\infty[\times\omega$ with respect to the measure $\Delta_k(t,y)^{2-q} dt\, dy$ with values in the Sobolev space $W^{s,q}(\mathbb{T})$. Recall that

$$\|w\|^q_{W^{s,q}(\mathbb{T})} \ = \ \|w\|^q_{L^q(\mathbb{T})} \ + \ \int_0^1 \int_{\mathbb{T}} \ \Big| \frac{w(\theta+h) \,-\, w(\theta)}{h^s} \Big|^q d\theta \, \frac{dh}{h} \,.$$

When $q = 2$, note that $\mathcal{L}^2_k([0,T] \times \omega \times \mathbb{T}) = L^2([0,T] \times \omega \times \mathbb{T})$.

Proposition 2.4.2. *i) For all $q \in]1,+\infty[$, there is a constant $C > 0$ such that for all trigonometric polynomials $\mathcal{A}$ which vanish on a neighborhood of $\mathcal{S}_k$ and all $T \geq 0$,*

$$\begin{aligned} \frac{1}{C} \ \| \mathcal{A} \|_{\mathcal{L}^q_k([0,T]\times\omega\times\mathbb{T})} \ &\leq \ \liminf_{\varepsilon\to 0} \ \| J^\varepsilon_k(\mathcal{A}) \|_{L^q([0,T]\times\mathbb{R}^d)} \\ \leq \limsup_{\varepsilon\to 0} \ \| J^\varepsilon_k(\mathcal{A}) \|_{L^q([0,T]\times\mathbb{R}^d)} \ &\leq \ C \ \| \mathcal{A} \|_{\mathcal{L}^q_k([0,T]\times\omega\times\mathbb{T})}. \end{aligned} \tag{2.4.8}$$

ii) For $1/q < s < 1$ and each $\varepsilon \in]0,1]$, J^ε_k extends to a bounded linear operator from $\mathcal{L}^{s,q}_k$ to $L^q([0,+\infty[\times\mathbb{R}^d)$ and

$$\sup_{\varepsilon\in]0,1]} \ \| J^\varepsilon_k(\mathcal{A}) \|_{L^q([0,T]\times\mathbb{R}^d)} \ \leq \ C \ \| \mathcal{A} \|_{\mathcal{L}^{s,q}_k([0,T]\times\omega\times\mathbb{T})} \,. \tag{2.4.9}$$

Our method is by duality and requires nonsmooth profiles. Estimate (2.4.8) implies that J^ε_k can be extended *asymptotically* as a map from $\mathcal{L}^q_k$ to $L^q([0,+\infty[\times\mathbb{R}^d)$, so that the substitution $\theta = \psi(y)$ in $\mathcal{A}(t,y,\theta)$, which has no direct meaning for $\mathcal{A} \in \mathcal{L}^q_k$, makes sense *asymptotically*. More precisely, one has the following result.

Proposition 2.4.3. *For all $T \in]0, +\infty]$ and for all $\mathcal{U} \in \mathcal{L}^q_k([0,T] \times \omega \times \mathbb{T})$ there exists a bounded family u^ε in $L^q([0,T[\times\mathbb{R}^d)$ such that for all $\delta > 0$, there are $\varepsilon_\delta > 0$ and a trigonometric polynomial $\mathcal{U}_\delta$ which vanishes near $\mathcal{S}_k$, such that*

$$\| \mathcal{U} - \mathcal{U}_\delta \|_{\mathcal{L}^q_k}([0,T] \times \omega \times \mathbb{T}) \leq \delta , \tag{2.4.10}$$

$$\forall \varepsilon \in]0, \varepsilon_\delta] , \quad \| u^\varepsilon - J^\varepsilon_k(\mathcal{U}_\delta) \|_{L^q([0,T[\times\mathbb{R}^d)} \leq \delta . \tag{2.4.11}$$

When u^ε satisfies (2.4.10) (2.4.11), we write $u^\varepsilon \sim \tilde{J}^\varepsilon_k(\mathcal{U})$ in $L^q([0,T] \times \mathbb{R}^d)$. If $u^\varepsilon \sim \tilde{J}^\varepsilon_k(\mathcal{U})$ and $v^\varepsilon \sim \tilde{J}^\varepsilon_k(\mathcal{U})$ in L^q, then $u^\varepsilon - v^\varepsilon$ converges strongly to 0 in $L^q([0,T[\times\mathbb{R}^d)$. More generally, we write $u^\varepsilon \sim v^\varepsilon$ in L^q, when u^ε and v^ε are bounded families in $L^q([0,T[\times\mathbb{R}^d)$ such that $u^\varepsilon - v^\varepsilon$ converges strongly to 0 in $L^q([0,T[\times\mathbb{R}^d)$.

The notations are consistent : when $\mathcal{U} \in \mathcal{L}^{s,q}_k([0,T] \times \omega \times \mathbb{T})$ and $s > 1/q$, $J^\varepsilon_k(\mathcal{U})$ is defined for each ε, and

$$J^\varepsilon_k(\mathcal{U}) \sim \tilde{J}^\varepsilon_k(\mathcal{U}) \quad \text{in} \quad L^q([0,T[\times\mathbb{R}^d) \tag{2.4.12}$$

in the sense defined above.

There is a similar treatment for functions on $\mathbb{R}^d$, i.e. for initial data. Given $\mathcal{U}_0 \in L^2(\omega \times \mathbb{T})$, one defines bounded families in $L^2(\mathbb{R}^d)$, denoted $u^\varepsilon_0 \sim \tilde{J}^\varepsilon_0(\mathcal{U}_0)$, and such that

$$J^\varepsilon_0(\mathcal{U}_0)\,(y) := \mathcal{U}_0(y, \psi(y)/\varepsilon) , \tag{2.4.13}$$

when $\mathcal{U}_0$ is smooth, or more generally whenever the substitution $\theta = \psi(y)$ makes sense, for example when $\mathcal{U}_0 \in L^2(\omega \times \mathbb{T})$ and $\partial_\theta \mathcal{U}_0 \in L^2(\omega \times \mathbb{T})$.

2.5. Sketch of the main results

Consider Cauchy data such that

$$u^\varepsilon_0 \sim \underline{u}_0 + \tilde{J}^\varepsilon_0(\mathcal{U}_0) \quad \text{in } L^2 , \tag{2.5.1}$$

where $\underline{u}_0 \in L^2(\mathbb{R}^d)$, $\mathcal{U}_0 \in L^2(\omega \times \mathbb{T})$ and the mean value $\int_{\mathbb{T}} \mathcal{U}_0(y, \theta)\, d\theta$ vanishes for (almost) all $y \in \omega$.

The family u^ε_0 is bounded in $L^2(\mathbb{R}^d)$, and therefore, there is a unique bounded family u^ε in $C^0([0, +\infty[; L^2(\mathbb{R}^d)) \cap L^{p+1}([0, +\infty[\times\mathbb{R}^d)$, such that u^ε satisfies (2.1.1) with the initial condition

$$u^\varepsilon_{|t=0} = u^\varepsilon_0 . \tag{2.5.2}$$

An abbreviated version of the main result is the following. We suppose that the various assumptions in §§2.1, 2.2 and 2.3 are satisfied.

THEOREM 2.5.1. *There are unique* $\mathcal{U}_k \in \mathcal{L}_k^{p+1} \cap C^0([0,+\infty[;L^2(\omega\times\mathbb{T}))$ *and* $\underline{u} \in C^0([0,+\infty[;L^2(\mathbb{R}^d)) \cap L^{p+1}([0,+\infty[\times\mathbb{R}^d)$*, such that for all* $T<+\infty$*,*

$$u^\varepsilon \;\sim \underline{u} \;+\; \sum_{k=1}^{k_0} \tilde{J}_k^\varepsilon(\mathcal{U}_k) \quad \text{in} \quad L^{p+1}([0,T]\times\mathbb{R}^d)\,. \tag{2.5.3}$$

When $\partial_\theta\mathcal{U}_0 \in L^2(\omega\times\mathbb{T})$*, the profiles* $\mathcal{U}_k$ *belong to* $\mathcal{L}_k^{s,p+1}$ *for all* $s<2/(p+1)$ *and one has the stronger result*

$$u^\varepsilon \;\sim \underline{u} \;+\; \sum_{k=1}^{k_0} J_k^\varepsilon(\mathcal{U}_k) \quad \text{in} \quad L^{p+1}([0,T[\times\mathbb{R}^d)\,. \tag{2.5.4}$$

The first step is to introduce the weak limits $\underline{u}$ in L^{p+1} and $\underline{f}$ in $L^{1+1/p}$ of u^ε and $f^\varepsilon := f(u^\varepsilon)$, extracting subsequences if necessary. Extracting further subsequences, we introduce profiles $\mathcal{U}_k$ and $\mathcal{F}_k$ such that for all $\mathcal{A}\in\mathcal{P}$ which vanish on a neighborhood of $\mathcal{S}_k$

$$\int u^\varepsilon\, \overline{J_k^\varepsilon(\mathcal{A})}\, dt\, dx \longrightarrow \int \mathcal{U}_k\, \overline{\mathcal{A}}\, dt\, dx\, d\theta\,, \tag{2.5.5}$$

$$\int f^\varepsilon\, \overline{J_k^\varepsilon(\mathcal{A})}\, dt\, dx \longrightarrow \int \mathcal{F}_k\, \overline{\mathcal{A}}\, dt\, dx\, d\theta\,. \tag{2.5.6}$$

The convergence (2.5.5) (resp. (2.5.6)) holds for $\mathcal{A}\in\mathcal{L}_k^{1+1/p}$ (resp. $\mathcal{A}\in\mathcal{L}_k^{p+1}$).

Alternatively one can replace J_k^ε by I_k^ε in (2.5.5), (2.5.6) to find

$$\int f^\varepsilon\, \overline{I_k^\varepsilon(\mathcal{A})}\, dt\, dx \longrightarrow \int \mathcal{F}_k\, \overline{\mathcal{A}}\, dt\, dx\, d\theta\,. \tag{2.5.7}$$

It is crucial to observe that (2.5.7) holds for $\mathcal{A}\in\mathcal{P}$ which vanishes on a neighborhood of supercritical points $\mathcal{S}_k^+(p+1)$. This is true because f^ε is bounded in $L^{1+1/p}$ and $I_k^\varepsilon(\mathcal{A})$ is bounded in L^{p+1} for such $\mathcal{A}$, thanks to Theorem 2.3.5.

PROPOSITION 2.5.2. *The profiles* $\mathcal{U}_k\in\mathcal{L}_k^{p+1}$ *satisfy the polarization condition*

$$P_k\,\mathcal{U}_k = \mathcal{U}_k\,. \tag{2.5.8}$$

Moreover, $\mathcal{F}_k\in\mathcal{L}_k^{1+1/p}$ *and* $\mathcal{U}_k$ *satisfies on* $G\backslash\mathcal{S}_k^+(p+1)\times\mathbb{T}$

$$\partial_t\mathcal{U}_k \;+\; P_k\,\mathcal{F}_k \;=\; 0\,, \quad \mathcal{U}_{k|t=0} \;=\; P_k\mathcal{U}_0\,. \tag{2.5.9}$$

The weak limits satisfy $\underline{u}\in L^{p+1}([0,+\infty[\times\mathbb{R}^d)$*,* $\underline{f}\in L^{1+1/p}([0,+\infty[\times\mathbb{R}^d)$ *and*

$$L\underline{u} \;+\; \underline{f} \;=\; 0 \quad \text{on } [0,+\infty[\times\mathbb{R}^d\,, \quad \underline{u}_{|t=0} \;=\; \underline{u}_0\,. \tag{2.5.10}$$

In (2.5.8) (2.5.9) $P_k(y)$ denotes the spectral projector of $A(d\psi(y))$ corresponding to the eigenvalue $\lambda_k(d\psi(y))$. Formula (2.5.9) follows from (2.5.7) and the known asymptotic behavior of $L(I_k^\varepsilon(\mathcal{A}))$ (§6). It holds outside $\mathcal{S}_k^+(p+1)$ because test profiles in (2.5.7) are required to vanish on $\mathcal{S}_k^+(p+1)$.

The proof of (2.5.3) uses the energy identity

$$\| u^\varepsilon(t) - v^\varepsilon(t) \|_{L^2}^2 \ - \ 2\,\mathrm{Re} \int_{[0,t]\times\mathbb{R}^d} L(v^\varepsilon - u^\varepsilon)\cdot(\overline{v^\varepsilon} - \overline{u^\varepsilon}) \ = \ \| u^\varepsilon(0) - v^\varepsilon(0) \|_{L^2}^2 \,. \tag{2.5.11}$$

This implies

$$\begin{aligned} &\| u^\varepsilon(t) - v^\varepsilon(t) \|_{L^2}^2 + \ c\, \| u^\varepsilon - v^\varepsilon \|_{L^{1+p}([0,t]\times\mathbb{R}^d)}^{1+p} \ \le \\ &\| u^\varepsilon(0) - v^\varepsilon(0) \|_{L^2}^2 \ + \ 2\,\mathrm{Re} \int_{[0,t]\times\mathbb{R}^d} (Lv^\varepsilon + f(v^\varepsilon))\cdot(\overline{v^\varepsilon} - \overline{u^\varepsilon})\ dt\, dx \,. \end{aligned} \tag{2.5.12}$$

We chose $v^\varepsilon \sim \underline{u} + \sum \tilde{J}_k^\varepsilon(\mathcal{U}_k)$. Note that the profiles are not yet known to be smooth and we need here the extended definition $\tilde{J}$. An important step is to show, using Proposition 2.5.2, that one can also choose v^ε so that

$$Lv^\varepsilon \sim \underline{f} + \sum \tilde{J}_k^\varepsilon(P_k\mathcal{F}_k) \quad \text{in} \quad L^{1+1/p} \tag{2.5.13}$$

The next important step is to analyse $f(v^\varepsilon)$. For $v^\varepsilon \sim \underline{u} + \sum \tilde{J}_k^\varepsilon(\mathcal{U}_k)$, one has

$$f(v^\varepsilon) \ \sim \ \underline{\mathcal{E}}(\underline{v}, \mathcal{U}_*) \ + \ \sum_k \tilde{J}_k^\varepsilon(P_k \mathcal{E}_k(\underline{v}, \mathcal{U}_*)) \ + \ h^\varepsilon \,, \tag{2.5.14}$$

where $\mathcal{U}_* := (\mathcal{U}_1, \ldots, \mathcal{U}_{k_0})$, $\underline{\mathcal{E}}$ and $\mathcal{E}_k$ are nonlinear operators acting from $L^{p+1} \times \prod \mathcal{L}_l^{p+1}$ into $L^{1+1/p}$ and $\mathcal{L}_k^{1+1/p}$ respectively. The family h^ε is bounded in $L^{1+1/p}$ and has no oscillations in the characteristic variety of L. This follows from the non resonance Assumption 2.2.5. Since $L(u^\varepsilon - v^\varepsilon)$ is bounded in $L^{1+1/p}$ by (2.5.13), Proposition 5.2.2 and Corollary 5.2.4 show that

$$\lim_{\varepsilon=0} \int h^\varepsilon (u^\varepsilon - v^\varepsilon) = 0$$

The weak limit and the profiles of $u^\varepsilon - v^\varepsilon$ vanish. This implies that

$$\int_{[0,t]\times\mathbb{R}^d} \Big(\underline{f} \ + \underline{\mathcal{E}}(\underline{u}, \mathcal{U}_*) + \sum \tilde{J}_k^\varepsilon \big(P_k(\mathcal{F}_k \ + \mathcal{E}_k(\underline{u}, \mathcal{U}_*)) \big) \Big) \cdot (\overline{v^\varepsilon} - \overline{u^\varepsilon})\ dt\, dx \longrightarrow 0 \,. \tag{2.5.15}$$

Thus the right hand side of (2.5.12) tends to zero and $u^\varepsilon - v^\varepsilon$ tends to zero strongly in L^{1+p} ending the proof of (2.5.3).

The strong convergence implies that $f(u^\varepsilon) \sim f(v^\varepsilon)$ in $L^{1+1/p}$. Therefore (2.5.13) yields

$$\underline{f} \;=\; \underline{\mathcal{E}}(\underline{u},\mathcal{U}_*)\,, \quad P_k\mathcal{F}_k \;=\; P_k\,\mathcal{E}_k(\underline{u},\mathcal{U}_*)\,. \tag{2.5.16}$$

With Proposition 2.5.2, we obtain the following equations for the profiles

$$\begin{aligned} &L\underline{u} \;+\; \underline{\mathcal{E}}(\underline{u},\mathcal{U}_*) \;=\; 0\,,\\ &P_k\,\mathcal{U}_k \;=\; \mathcal{U}_k,\\ &\partial_t\mathcal{U}_k \;+\; P_k\,\mathcal{E}_k(\underline{u},\mathcal{U}_*) \;=\; 0 \quad \text{on} \quad G\backslash\mathcal{S}_k^+(p+1)\times\mathbb{T}\,. \end{aligned} \tag{2.5.17}$$

The initial conditions are

$$\underline{u}_{|t=0} \;=\; \underline{u}_0\,, \quad \mathcal{U}_{k\,|t=0} \;=\; P_k\,\mathcal{U}_0\,. \tag{2.5.18}$$

Knowing that $\mathcal{U}_*$ satisfies the profile equations (2.5.17) (2.5.18), one shows that $\mathcal{U}_k \in \mathcal{L}_k^{s,p+1}$ for all $s < 2/(p+1)$ when $\partial_\theta\mathcal{U}_0 \in L^2$. Thus the $J_k^\varepsilon(\mathcal{U}_k)$ are defined for each ε, and (2.4.12) implies the stronger form (2.5.4) of the asymptotics.

Note that $\mathcal{E}_k(\underline{u},\mathcal{U}_*)$ belongs to $\mathcal{L}_k^{1+1/p}$. Near supercritical points, $\mathcal{L}_k^{1+1/p}$ is *not* included in L^1_{loc}, since this would imply that $L^\infty \subset \mathcal{L}_k^{1+p}$ near $\mathcal{S}_k^+(1+p)$ which is impossible from Definition 2.4.1. Thus the meaning of the third equation in (2.5.17) is not clear near $\mathcal{S}_k^+(p+1)$. However, the third equation *does* determine $\mathcal{U}_k$ because near supercritical points, *strong* dissipation intervenes, and there, the *absorption* phenomenon occurs. This suggests the following terminilogy.

Definition 2.5.3. *For $y \in \omega$, the k-time of absorption along the ray issued from y is the smallest value of $t > 0$ such that $(t, y) \in S_k^+(p+1)$ the set of $p+1$-supercritical point in $\mathcal{S}_k$, i.e. satisfying $\mu_k(t, y) \geq 2/(p-1)$. It is denoted by $T_k^{abs}(y)$.*

$\frac{1}{T_k^{abs}(y)}$ is the largest positive eigenvalue of $g_k'(y)$ of multiplicity larger than or equal to $\frac{2}{p-1}$. $T_k^{abs}(y)$ is infinite if there is no such eigenvalue.

Introduce

$$\begin{cases} G_k^- \;:=\; \{(t,y) \in]0,+\infty[\times\omega_0 \mid t < T_k^{abs}(y)\ \} \\[2ex] G_k^+ \;:=\; \{(t,y) \in]0,+\infty[\times\omega_0 \mid t > T_k^{abs}(y)\ \} \end{cases} \tag{2.5.19}$$

T_k^{abs} is C^∞ on ω_0 where the eigenvalues of $g_k'(y)$ are smooth. Thus G_k^- is open. Moreover, $G\backslash\mathcal{S}_k^+(p+1)$ is the union of G_k^-, G_k^+ and a negligible set.

Theorem 2.5.4. *With notations as in Theorem 2.5.1, the profiles $\mathcal{U}_k$ vanish on $G_k^+\times\mathbb{T}$, and the third equation in (2.5.17) holds in L^1_{loc} on the domains $G_k^-\times\mathbb{T}$.*

2.6. Remarks and Examples

1) Note that $g_k'(y) = \lambda_k''(d\psi(y))\,\psi''(y)$. Since λ_k' is homogeneous of degree 0, $\lambda_k''(\xi)$ has always a nontrivial kernel which contains ξ. Thus 0 is always an eigenvalue of $g_k'(y)$. As a consequence, the multiplicities μ_k satisfies

$$\forall k \in \{1, \dots, k_0\}\,, \forall (t,y) \in \mathcal{S}_k\,, \quad 1 \le \mu_k(t,y) \ \le \ d-1\,. \tag{2.6.1}$$

In particular, in space dimension $d = 2$, the only possibility is $\mu_k = 1$ and Assumption 2.3.3 is always satisfied.

2) The Lagrangian Λ_k associated to the eiconal equation (2.2.2) is the set of (t, x, τ, ξ) such that

$$x = q_k(t,y)\,, \quad \xi = d\psi(y)\,, \quad \tau = \lambda_k(d\psi(y))\,. \tag{2.6.2}$$

It is also associated to the defining phase function

$$\Phi_k(t,x,y,\xi) \ := \ t\,\lambda_k(\xi) \ + \ (x-y)\cdot\xi \ + \ \psi(y)\,. \tag{2.6.3}$$

$\mathcal{S}_k$ corresponds to the set of points in Λ_k where the projection $\pi \, : \, (t,x,\tau,\xi) \to (t,x)$ restricted to Λ_k, is not of maximal rank $d+1$. The multiplicity μ_k is equal to the order of vanishing of $\det d(\pi_{|\Lambda_k})$. It is larger than or equal to the dimension of the kernel of $d(\pi_{|\Lambda_k})$. For the wave equation, these two numbers are equal, see §3.5.

3) If $p \ge 3$, all points in $\mathcal{S}_k$ are supercritical, since then $2/(p-1) \le 1 \le \mu_k(t,y)$. Thus $T_k^{abs}(y)$ is the first time of focusing along the ray.

In the extreme opposite case, all points in $\mathcal{S}_k$ are subcritical when $p < 1 + \frac{2}{d-1}$, since then $\frac{2}{p-1} > d - 1 \ge \mu_k(t,y)$, see (2.6.1). In this case T_k^{abs} is infinite.

4) For radial focusing in a wave equation, $\lambda(\xi) := |\xi|$ and $\psi(y) := \chi(|y|)$. Then, g' has only one positive eigenvalue equal to $1/|y|$, of multiplicity $d-1$. Thus absorption occurs if and only if $p \ge 1 + 2/(d-1)$. The if part was proved in [JMR 2].

5) In many examples, $\mu_k(t,y) = 1$ for all points $(t,y) \in \mathcal{S}_k$. Then Assumption 2.3.3 is satisfied. This happens when all the rays meet the caustic at generic points (folds) (see Remark 3.5.2). It also occurs when $\pi|_{\Lambda_k}$ has a cusp. In these two cases The critical exponent is $p = 3$. If $p < 3$, $T_k^{abs}(y)$ is infinite. Conversely if $p \ge 3$, absorption occurs at the first crossing of caustics. Examples in space dimension 2 for the wave equation are the familiar phases

$$\psi_1(y_1,y_2) \ = \ y_2 + y_1^2\,, \quad \psi_2(y_1,y_2) \ = \ a_1 y_1^2 + a_2 y_2^2 \quad \text{with} \quad 0 < a_1 < a_2\,. \tag{2.6.4}$$

For ψ_1 the caustic set is the manifold $\mathcal{C}$ with equation $x_1^2 = (t^{2/3} - 1)^3$, which has a cusp singularity at $x_1 = 0, t = 1$, corresponding to the rays from $y_1 = 0$.

6) Again for the wave equation, $\lambda_k(\xi) = |\xi|$, in $\mathbb{R}^d$ consider

$$\psi_3(y) := y_d \ + \ (y_1^2 \ + \cdots + \ y_{d-1}^2)/2 \ =: y_d + |y'|^2/2. \tag{2.6.5}$$

For $y' \neq 0$, there are two distinct positive eigenvalues : $1/\delta$ of multiplicity $d-2$ and $1/\delta^3$ of multiplicity 1, where $\delta^2 := 1 + |y'|^2$. For $y' = 0$, these eigenvalues are equal. Therefore, $(t = 1, y' = 0)$ is a crossing point for two different eigenvalues, but it is the only point where multiplicities are not locally constant. Nevertheless Assumptions 2.2.2 and 2.3.3 are still satisfied.

There are two focusing times, δ and δ^3. Accordingly, the caustic set has two components. The first one is the plane $\mathcal{C}_1 = \{x' = 0\}$. The second is the manifold $\mathcal{C}_2$ with equation $x'^2 = (t^{2/3} - 1)^3$, which has a singular locus at $x' = 0, t = 1$, corresponding to the rays from $y' = 0$. Since $\delta < \delta^3$, the rays cross $\mathcal{C}_1$ before $\mathcal{C}_2$.

If $p \geq 1 + 2/(d-2)$, then $\mathcal{C}_1$ is supercritical. Oscillations are absorbed at $\mathcal{C}_1$ and never reach $\mathcal{C}_2$. If $p < 1+2/(d-2)$, then both $\mathcal{C}_1$ and $\mathcal{C}_2$ are subcritical. Oscillations cross $\mathcal{C}_1$ and $\mathcal{C}_2$.

3. L^p estimates for oscillatory integrals

The aim of this section is to provide uniform L^q estimates for oscillatory integrals including $I^\varepsilon(\mathcal{A})$. Because the question is interesting in its own, we consider more general classes of oscillatory integrals. At the end of this section the results are applied to the context developped in §2. The estimates are proved under assumptions which are satisfied for generic caustics, cusps, and spherical focusing.

3.1 Statement of the main estimate

Consider integrals of the form

$$u^\varepsilon(z) \ := \ \varepsilon^{-n/2} \int_{\mathbb{R}^n} e^{i\,\Phi(z,\alpha)/\varepsilon} \ a(z,\alpha,\varepsilon)\ d\alpha \ , \quad z \in \mathbb{R}^m, \tag{3.1.1}$$

where Φ and a are C^∞ functions on $\mathbb{R}^m \times \mathbb{R}^n$ and $\mathbb{R}^m \times \mathbb{R}^n \times [0,1]$ respectively. Φ is real valued and a is compactly supported. Our goal is to obtain bounds for the L^q-norm of u^ε.

Results in this direction can be found in [R 1], [R 2] and [Sv]. Typically, they concern phases of the form $\Phi(z,\alpha) = z \cdot \alpha - h(\alpha)$ with h real analytic and convex, or $n = m = 1$ and h'' vanishes at finite order at isolated points. For example, it is proved in [R 2] that, for amplitudes with small support,

$$\sup_{\varepsilon \in]0,1]} |u^\varepsilon(z))| \ \leq \ C\ |\det h''(\alpha(z)|^{-1/2} \tag{3.1.2}$$

where, $\alpha(z)$ is the unique critical point of $\alpha \mapsto \Phi(z,\alpha)$. Note that the stationary phase theorem implies that a control of u^ε by $|\det h''|^{-1/2}$ is the best estimate that one can expect. However, an estimate like (3.1.2) is certainly not true in general. In the simplest case where $n = m = 1$ and $\Phi = z\alpha - \alpha^3$ which leads to Airy integrals, there is no critical point in the shadow region $z < 0$ but u^ε is not uniformly bounded in this region. We refer to [R 1] for a substitute of (3.1.2) for phases of the form $z\alpha - \alpha^k$ which lead to regular caustics.

The convexity assumptions are not always satisfied. For example, a typical phase for which the caustic set is a cusp, is $\Phi(z_1, z_2, \alpha) = z_1\alpha + z_2\alpha^2 + \alpha^4$. For the phases (2.3.3), there is no substitute in general for the convexity assumptions. We are not able to prove estimates like (3.1.2), but we have a reasonable answer to the question of determining the supremum of the real numbers q such that the family u^ε is bounded in L^q.

Patching local estimates with a partition of unity, it is sufficient to assume that the amplitude a in (3.1.1) is supported in a small neighborhood of a point $\underline{\rho} := (\underline{z}, \underline{\alpha})$. Because $u^\varepsilon = O(\varepsilon^\infty)$ when the phase has no stationary points in a neighborhood of the support of a, and $u^\varepsilon = O(1)$ for a nondegenerate stationary point, we study the remaining case where $\underline{\rho}$ is a *degenerate stationary point*, i.e.

$$(3.1.3) \qquad d_\alpha \Phi(\underline{\rho}) = 0\,, \quad D(\underline{\rho}) \;=\; 0$$

where $D \;:=\; \det \Phi''_{\alpha\alpha}$ is the determinant of the Hessian matrix

$$(3.1.4) \qquad \Phi''_{\alpha\alpha}(z,\alpha) \;:=\; \Big(\frac{\partial^2 \Phi}{\partial \alpha_j\, \partial \alpha_k}\,(z,\alpha) \Big)_{1 \le j,\, k \le n} .$$

The behavior of u^ε depends on the structure of the set of degenerate stationary points. We consider simple cases of degeneracy, which naturally occur in the study of integrals $I^\varepsilon(\mathcal{A})$, see §3.5. Our first assumption is that the phase Φ is nondegenerate (see e.g. [Hö 1]), which means that Φ'_α is of maximal rank at $\underline{\rho}$:

$$(3.1.5) \qquad \operatorname{rank} \big(d_{(z,\alpha)}\, \Phi'_\alpha(\underline{z}, \underline{\alpha}) \big) \;=\; n\,.$$

Then, near $\underline{\rho}$, the set of stationary points

$$(3.1.6) \qquad C_\Phi := \{ (z,\alpha)\ ;\ \Phi'_\alpha(z,\alpha) = 0 \,\}$$

is a smooth manifold of dimension m. The set of degenerate stationary points is

$$(3.1.7) \qquad \mathcal{S}_\Phi := \{ (z,\alpha) \in C_\Phi\ ;\ D(z,\alpha) = 0 \,\}\,.$$

First we give a necessary condition for the family u^ε to be bounded in L^q.

Theorem 3.1.1. *There is a neighborhood ω of $\underline{\rho}$ such that if $q > 2$ and $a \in C_0^\infty(\omega \times [0,1])$ are such that the family (3.1.1) is bounded in L^q, then the restriction of $|D|^{1-q/2}(\rho)\, |a(\rho, 0)|^q$ to C_Φ is integrable.*

Next we turn to sufficient conditions. Sharper results are obtained if one uses the Lorentz spaces. Recall that a measurable function u belongs to the weak-L^p space if

$$\|u\|_{p,\infty} := (\sup_{s>0} s^p \ \mathrm{meas}\{x, |u(x)| > s\})^{1/p} < +\infty,$$

and that locally $L^q_{loc} \subset L^{p,\infty}_{loc}$ for every $q \geq p$.

Denote by D_* the restriction of D to C_Φ.

THEOREM 3.1.2. *Suppose that there exist a neighborhood $\mathcal{O}$ of $\underline{z}$ and an integer ℓ such that for almost all $z \in \mathcal{O}$ the number of critical points $(z, \alpha) \in C_\Phi$ is not larger than ℓ. Suppose in addition that the dimension of* $\ker \Phi''_{\alpha\alpha}(\underline{\rho})$ *is equal to* 1.

If $r > 0$ is such that $D_*^{-r} \in L^{1,\infty}$ [resp. L^1] on a neighborhood of $\underline{\rho}$ in C_Φ, then there exists a neighborhood ω of $\underline{\rho}$ such that for all $a \in C_0^\infty(\omega \times [0,1])$ the family u^ε is bounded in $L^{2+2r,\infty}$ [resp. L^{2+2r}].

When the phase Φ is real analytic, the first assumption in Theorem 3.1.2 is automatically satisfied. Thus when the dimension of $\ker \Phi''_{\alpha\alpha}(\underline{\rho})$ is equal to one and Φ is real analytic, Theorems 3.1.1 and 3.1.2 give a a complete answer to our question.

When the dimension of $\ker \Phi''_{\alpha\alpha}(\underline{\rho})$ is larger than one, our results are much less complete. The following theorem covers the important case of radial focusing in $\mathbb{R}^d$, $d \geq 3$.

THEOREM 3.1.3. *Suppose that $\mathcal{S}_\Phi$ is a smooth submanifold of codimension one in C_Φ, the dimension ν of* $\ker \Phi''_{\alpha\alpha}(\rho)$ *is constant and ≥ 2 for ρ in a neighborhood of $\underline{\rho}$ in $\mathcal{S}_\Phi$ and D_* vanishes exactly to order $\mu = \nu$ on $\mathcal{S}_\Phi$. Then there exists a neighborhood ω of $\underline{\rho}$ such that for all $a \in C_0^\infty(\omega \times [0,1])$ the family u^ε is bounded in $L^{2+2/\mu,\infty}$.*

REMARK 3.1.4. When $\mathcal{S}_\Phi$ is a smooth submanifod of codimension one in C_Φ and D_* vanishes exactly to order μ on $\mathcal{S}_\Phi$, then $D_*^{-2/\mu} \in L^{1,\infty}$. Thus either when the dimension of $\ker \Phi''_{\alpha\alpha}(\underline{\rho})$ is equal to 1 and the number of critical points is bounded, or when this dimension is localy constant and equal to μ, the family u^ε is bounded in $L^{2+2/\mu,\infty}$, provided that a is supported in a small neighborhood of $\underline{\rho}$. In particular u^ε is bounded in L^q for all $q < 2 + 2/\mu$. The index $q_c = 2 + 2/\mu$ is sharp, since Theorem 3.1.1 implies that whem $a(\underline{\rho}, 0) \neq 0$, the family u^ε is not bounded in L^q for $q \geq 2 + 2/\mu$.

REMARK 3.1.5. The assumptions are invariant under change to equivalent phase functions. They have the following intrinsic formulation. Consider

$$\Lambda := \iota(C_\Phi \cap \omega) \subset T^*\mathbb{R}^m \ , \quad \text{where} \quad \iota \ : \ (z,\alpha) \to (z, \Phi'_z(z,\alpha)) \tag{3.1.8}$$

an ω is a neighborhood of $\underline{\rho}$. If ω is small enough, then the equation (3.1.5) implies that Λ is a smooth Lagrangian manifold and that ι is a local diffeomorphism from $C_\Phi \cap \omega$ to Λ. Let π denote the restriction to Λ of the projection $(z,\zeta) \to z$. The kernel of π' at $(z,\zeta) = \iota(\rho)$ is exactly the image by $\iota'(\rho)$ of the space $\{0\} \times \ker \Phi''_{\alpha\alpha}$ which is tangent to C_Φ at ρ. In particular,

$$
(3.1.9) \qquad \begin{cases} \iota(\mathcal{S}_\Phi) = \mathcal{S} := \{(z,\zeta) \; ; \; \operatorname{rank} \pi'(z,\zeta) < m \}\,, \\ \dim \ker \pi'(z,\zeta) \;=\; \dim \ker \Phi''_{\alpha\alpha}(\rho)\,, \quad \text{for} \quad (z,\zeta) = \iota(\rho) \in \mathcal{S}\,. \end{cases}
$$

Therefore, the assumptions in Theorem 3.1.2 are : 1) for almost all z in a neighborhood of $\underline{z}$ the number of preimages of x by π in Λ is less than or equal to ℓ, 2) $\ker \pi'(\underline{z},\underline{\zeta})$ has dimension 1.

Similarly, the assumptions in Theorem 3.1.3 are 1) $\mathcal{S}$ is a smooth submanifold of codimension 1 in Λ, 2) $\ker \pi'(z,\zeta)$ has constant dimension ν for (z,ζ) in a neighborhood of $(\underline{z},\underline{\zeta})$ in $\mathcal{S}$, 3) $\det \pi'$ vanishes exactly to order ν on $\mathcal{S}$.

This shows that the assumptions are invariant under change of phase functions which define the same germ of Lagrangian Λ.

3.2. Proof of Theorem 3.1.1

Using the Theorem of equivalence of phase functions (see [Hö 1] [Du]), shrinking neighborhoods and multiplying u^ε by $e^{i\varphi(z)/\varepsilon}$ if necessary, we can assume that $m = n$ and on a neighborhood $\omega = \Omega \times U$ of $\underline{\rho} = (\underline{z},\underline{\zeta})$, Φ is of the form

$$
(3.2.1) \qquad \Phi(z,\alpha) \;=\; x\cdot\alpha \;-\; h(\alpha)\,.
$$

where h is C^∞ on a neighborhood of $\overline{U}$. In this case, $C_\Phi = \{(z,\alpha) \in \Omega \times U \mid z = h'(\alpha)\}$, and $D_*(\alpha) = \det h''(\alpha)$. Consider $a \in C_0^\infty(\omega \times [0,1])$.

For $f \in C_0^\infty(\Omega)$, one has

$$
(3.2.2) \qquad \int f(z)\, |u^\varepsilon(z)|^2\, dz \;=\; \int F^\varepsilon(\eta,\alpha,\alpha+\varepsilon\eta)\; e^{i\,(h(\alpha)-h(\alpha+\varepsilon\eta))/\varepsilon}\, d\xi\, d\eta\,,
$$

where

$$
(3.2.3) \qquad F^\varepsilon(\eta,\alpha,\alpha') \;:=\; \int e^{i\,z\eta} f(z)\, a(z,\alpha,\varepsilon)\, \overline{a(z,\alpha',\varepsilon)}\, dz \;=\; O((1+|\eta|)^{-\infty})\,.
$$

The estimate is uniform, and Lebesgues' dominated convergence theorem implies that

$$
(3.2.4) \qquad \int f(z)\, |u^\varepsilon(z)|^2\, dx \quad \to \quad (2\pi)^d \int f(h'(\alpha))\, |a(h'(\alpha),\alpha,0)|^2\;\, d\alpha\,,
$$

as $\varepsilon \to 0$. Therefore, if the family u^ε is bounded in L^q, there is a constant C such that for all $f \in C_0^\infty(\Omega)$,

$$
(3.2.5) \qquad \Big| \int f(h'(\xi))\, |a(h'(\xi),\xi)|^2\, d\xi \Big| \;\le\; C\, \|\, f\,\|_{L^r(\Omega)}\;, \quad r := q/(q-2)\,.
$$

Introduce the measure λ on Ω which is the image of $|a(h'(\alpha),\alpha,0)|^2\,d\alpha$ by h'. The left hand side of (3.2.5) is equal to $\big|\int f(z)\,\lambda(dz)\big|$. Thus (3.2.5) implies that

$$\lambda \;=\; \ell(x)\,dx\ , \quad \text{where} \quad \ell \in L^{r'}(\Omega)\ , \quad r' = q/2\,. \tag{3.2.6}$$

Introduce $\mathcal{S} = \{\alpha \in \overline{U} \;:\; D_*(\alpha) = 0\}$, where $D_* := \det h''$. Introduce next $\mathcal{C} = h'(\mathcal{S}_1) \cap \overline{\Omega_1}$. For all $z \in \Omega\backslash\mathcal{C}$, the set of $\alpha \in \overline{U}$ such that $h'(\alpha) = z$ is finite and there is a neighborhood G of z such that $h'^{-1}(G)$ is the union of finitely many pairwise disjoint open sets V_j, such that h' is a diffeomorphism from V_j onto G. This implies that on $\Omega\backslash\mathcal{C}$

$$\ell(z) \;=\; \sum_{\alpha\in h'^{-1}(\{z\})} \frac{|a(z,\alpha)|^2}{|D_*(\alpha)|} \;\geq\; 0\,. \tag{3.2.7}$$

Sard's theorem implies that the Lebesgue measure of $\mathcal{C}$ is equal to 0. Therefore the density ℓ is completely determined by (3.2.7). Since $\ell \in L^{r'}$, one has

$$\begin{aligned} \int \ell(z)^{r'}\,dz \;&=\; \int \ell(z)^{r'-1}\,\lambda(dz) \\ &=\; \int \ell(h'(\alpha))^{r'-1}\,|a(h'(\alpha),\alpha,0)|^2\,d\alpha \;<\; +\infty\,. \end{aligned} \tag{3.2.8}$$

For $\alpha \notin h'^{-1}(\mathcal{C})$, (3.2.7) shows that

$$\ell(h'(\alpha))^{r'-1}\,|a(h'(\alpha),\alpha,0)|^2 \;\geq\; |D_*(\alpha)|^{1-q/2}\,|a(h'(\alpha),\alpha,0)|^q\,. \tag{3.2.9}$$

In addition, (3.2.6) implies that $\lambda(\mathcal{C}) = 0$, hence $h'^{-1}(\mathcal{C})$ is a null set for the measure $|a(h'(\alpha),\alpha,0)|^2\,d\alpha$. Therefore (3.2.8) and (3.2.9) imply that

$$\int |D_*(\alpha)|^{1-q/2}\,|a(h'(\alpha),\alpha,0)|^q\,d\alpha \;<\; +\infty\,, \tag{3.2.10}$$

which finishes the proof of Theorem 3.1.1.

3.3 Proof of Theorem 3.1.2

1. By a linear change of variables, one can split the variables α into $(\alpha',\alpha'') \in \mathbb{R}\times\mathbb{R}^{n-1}$, so that $\Phi''_{\alpha',\alpha'}(\underline{\rho}) = 0$ and $\Phi''_{\alpha''\alpha''}(\underline{\rho})$ is invertible. Thus, near $\underline{\rho}$, the phase $\alpha'' \to \Phi(z,\alpha',\alpha'')$ has a unique critical point, $\alpha''(z,\alpha')$ which is nondegenerate. Therefore, if a is supported in a sufficiently small neighborhood of $\underline{\rho}$, the stationnary phase theorem implies that

$$\varepsilon^{-(n-1)/2}\int_{\mathbb{R}^{n-1}} e^{i\,\Phi(z,\alpha',\alpha'')/\varepsilon}\;a(z,\alpha',\alpha'',\varepsilon)\;d\alpha'' \;=\; b(z,\alpha',\varepsilon)\;e^{i\,\Psi(x,\alpha')/\varepsilon}\,, \tag{3.3.1}$$

where b is C^∞ on $U'\times[0,1]$ and $\Psi(z,\alpha') := \Phi(z,\alpha',\alpha''(z,\alpha'))$. Thus

$$u^\varepsilon(z) \;=\; \varepsilon^{-\nu/2}\int_{\mathbb{R}^\nu} e^{i\,\Psi(z,\alpha')/\varepsilon}\;b(z,\alpha',\varepsilon)\;d\alpha'\,. \tag{3.3.2}$$

Since Ψ and Φ define the same germ of Lagrangian manifold, Ψ satisfies the same assumptions as Φ. Therefore, it is sufficient to prove Theorem 3.1.2 when $n = 1$ which we assume from now on.

2. The phase Φ is nondegenerate and $\underline{\rho} \in \mathcal{S}_\Phi$. Therefore, there are coordinates $z = (t, y) \in \mathbb{R}^{m-1} \times \mathbb{R}$ such that $\partial^2_{y\alpha}\Phi(\underline{\rho}) \neq 0$. Using the Theorem of equivalence of phase functions (see [Hö 1] [Hö 2] [Du]) with t as parameters reduces to the case where

$$\Phi(z, \alpha) \ := \ y\,\alpha \ - \ h(t, \alpha)\ , \quad z = (t, y) \in \mathbb{R}^{d-1} \times \mathbb{R}\,. \tag{3.3.3}$$

From now on, we assume that Φ is given by (3.3.3) on $\omega = \Omega \times \mathcal{I}$ where $\Omega = \Omega_{\underline{t}} \times \Omega_{\underline{y}}$ is a relatively compact neighborhood of $\underline{z} = (\underline{t}, \underline{y})$, $\mathcal{I}$ is an open interval which contains $\underline{\alpha}$, h is C^∞ on $\overline{\Omega_{\underline{t}}} \times \overline{\mathcal{I}}$ and $h'_\alpha(\Omega_{\underline{t}} \times \mathcal{I}) \subset \Omega_{\underline{y}}$. In this case, C_Φ is parametrized by $y = h'_\alpha(t, \alpha)$ with (t, α) as parameters. Taking (t, α) as coordinates on C_Φ, one has $D_*(t, \alpha) = h''_{\alpha,\alpha}(t, \alpha)$. $\mathcal{S}_\varphi$ is the set of (t, α) such that $D_*(t, \alpha) = 0$.

Lemma 3.3.1. *There is a negligible set $\mathcal{N} \subset \Omega$ such that for all $x \in \Omega \backslash \mathcal{N}$, the equation $\Phi'_\alpha(z, \alpha) = \lambda$ has at most ℓ solutions α in $\mathcal{I}$ for almost all $\lambda \in \mathbb{R}$.*

Proof. Shrinking neighborhoods if necessary, the assumptions in Theorem 3.1.2 imply that there is a negligible set $\mathcal{N}' \subset \mathcal{O}$ such that for all $z \in \mathcal{O} \backslash \mathcal{N}'$, the equation $\Phi'_\alpha(z, \alpha) = y - h'_\alpha(t, \alpha) = 0$ has at most ℓ roots in $\mathcal{I}$. Therefore, there is a negligible set $\mathcal{N}_{\underline{t}} \subset \Omega_{\underline{t}}$ such that for all $t \in \Omega_{\underline{t}} \backslash \mathcal{N}_{\underline{t}}$, the set $\{y : (t, y) \in \mathcal{N}'\}$ is negligible, implying that for almost all $y \in \mathcal{O}_{\underline{y}}$, the equation $h'_\alpha(t, \alpha) = y$ has at most ℓ roots in $\mathcal{I}$. Since $h'_\alpha(\mathcal{O}_{\underline{t}} \times \mathcal{I}) \subset \Omega_{\underline{y}}$, the lemma follows with $\mathcal{N} = \mathcal{N}_{\underline{t}} \times \Omega_{\underline{y}}$.

¿From now on, we fix $a \in C_0^\infty(\mathcal{O} \times \mathcal{I} \times [0, 1])$.

3. Consider $\delta > 0$ and $z \in \Omega \backslash \mathcal{N}$. Introduce

$$\mathcal{I}(z, \delta) := \{\alpha \in \mathcal{I} \ : \ |\Phi'_\alpha(z, \alpha)| > \delta\,\}\ , \quad \mathcal{J}(z, \delta) := \{\alpha \in \mathcal{I} \ : \ |\Phi'_\alpha(z, \alpha)| < 2\delta\,\}\,.$$

Fix $\chi \in C^\infty(\mathbb{R})$ such that $\chi(\lambda) = 0$ for $|\lambda| \leq 1$ and $\chi(\lambda) = 1$ for $|\lambda| \geq 2$. In the integral (3.1.1), use the partition of unity $1 = \chi(\Phi'_\alpha/\delta) + (1 - \chi(\Phi'_\alpha/\delta))$. This splits u^ε into into two pieces. The second integral is carried by $\mathcal{J}(z, \delta)$ and is estimated by

$$\|\, a\,\|_{L^\infty} \ \mathrm{meas}\mathcal{J}(z, \delta)/\sqrt{\varepsilon}\,. \tag{3.3.4}$$

In the first integral, one integrates by parts to find

$$\begin{aligned} & i\,\sqrt{\varepsilon} \int e^{i\,\Phi/\varepsilon}\ \partial_\alpha(a\,\chi(\Phi'_\alpha/\delta)/\Phi'_\alpha)\ d\alpha \\ & \quad = \ i\,\sqrt{\varepsilon} \int e^{i\,\Phi/\varepsilon}\ a\ \partial_\alpha(\chi(\Phi'_\alpha/\delta)/\Phi'_\alpha)\ d\alpha \ + O(\sqrt{\varepsilon}\,\|\,\partial_\alpha\, a\,\|_{L^\infty}/\delta)\,. \end{aligned}$$

Note that $\partial_\alpha\big(\chi(\Phi'_\alpha/\delta)/\Phi'_\alpha\big) = O(|\Phi''_{\alpha,\alpha}|/(\Phi'_\alpha)^2)$ is supported in $\mathcal{I}(z, \delta)$. Therefore, the first term is estimated by

$$C\ \sqrt{\varepsilon} \int_{\mathcal{I}(z,\delta)} \frac{|\Phi''_{\alpha,\alpha}|}{|\Phi'_\alpha|^2}\ d\alpha \ + \ C\sqrt{\varepsilon}/\delta \tag{3.3.5}$$

Next, we apply to the function $1/\Phi'_\alpha$ the following lemma.

Lemma 3.3.2. *Suppose that f is a bounded and C^1 function from and open set $O \subset \mathbb{R}$ to $\mathbb{R}$. Suppose for almost all $\lambda \in \mathbb{R}$ the equation $f(\alpha) = \lambda$ has at most ℓ solutions. Then*

$$\int_O |\, f'(\alpha)\, |\, d\alpha \ \leq\ 2\ \ell\ \|\, f\, \|_{L^\infty(O)}\,. \tag{3.3.6}$$

In fact, (3.3.6) follows from the more precise following result. Consider a C^1 function f from the open set $O \subset \mathbb{R}^d$ into $\mathbb{R}^d$. Suppose that, for almost all $\lambda \in \mathbb{R}^d$, the number $\ell(\lambda)$ of points $\alpha \in O$ such that $f(\alpha) = \lambda$ is finite. Then

$$\int_O |\ \det f'(\alpha)\, |\, d\alpha \ =\ \int_{f(O)} \ell(\lambda)\, d\lambda\,, \tag{3.3.7}$$

This proof is by a combination of Sard's Theorem for eliminating the critical values and the theorem of change of variables near regular points. (See e.g. [E G]).

Thanks to Lemma 3.3.1, for $z \in \Omega\backslash\mathcal{N}$ one can apply Lemma 3.3.2 to the function $\alpha \mapsto 1/\Phi'_\alpha$ to find the estimate

$$\int_{\mathcal{I}(z,\delta)} \frac{|\Phi''_{\alpha,\alpha}|}{|\Phi'_\alpha|^2}\, d\alpha \ \leq\ C/\delta\,.$$

With (3.3.4) et (3.3.5), we have proved that there is a constant C such that for all $\varepsilon \in]0,1]$, all $\delta > 0$ and all $z \in \Omega\backslash\mathcal{N}$

$$|\, u^\varepsilon(z)\, | \ \leq\ C\, \Big(\frac{\sqrt{\varepsilon}}{\delta} + \frac{\mathrm{meas}\mathcal{J}(z,\delta)}{\sqrt{\varepsilon}} \Big)\,. \tag{3.3.8}$$

4. Next we apply Lemma 3.3.2 to the function $\alpha \mapsto \Phi'_\alpha(z,\alpha)$ when $z \notin \mathcal{N}$. This shows that

$$\int_{\mathcal{J}(z,\delta)} |\Phi''_{\alpha,\alpha}(z,\alpha)|\, d\alpha \ \leq\ 2\ \ell\, \delta\,.$$

Therefore

$$\begin{aligned} &\mathrm{meas}\big(\{\alpha \in \mathcal{J}(z,\delta) \ :\ |\Phi''_{\alpha,\alpha}(z,\alpha)| \geq \delta^2/\varepsilon\}\big) \\ &\qquad \leq\ \varepsilon/\delta^2 \int_{\mathcal{J}(z,\delta)} |\Phi''_{\alpha,\alpha}(z,\alpha)|\, d\alpha \ \leq\ 2\ell\varepsilon/\delta\,. \end{aligned} \tag{3.3.9}$$

Introduce

$$\mathcal{K}(z,\delta,\varepsilon) \ :=\ \ \{\alpha \in \mathcal{I} \ :\ |\Phi'_\alpha(z,\alpha)| \leq 2\delta \ \text{ and } \ |\Phi''_{\alpha,\alpha}(z,\alpha)| \leq \delta^2/\varepsilon\ \}\,. \tag{3.3.10}$$

Then, (3.3.8) and (3.3.9) imply that there is a constant C_1 such that for all $\varepsilon \in]0,1]$, all $\delta > 0$ and all $z \in \Omega \backslash \mathcal{N}$

$$(3.3.11) \qquad |\, u^\varepsilon(x)\,| \ \le\ C_1 \,\Big(\frac{\sqrt{\varepsilon}}{\delta} + \frac{\operatorname{meas} \mathcal{K}(x,\delta,\varepsilon)}{\sqrt{\varepsilon}} \Big) .$$

5. For $s > 0$ and $\varepsilon \in]0,1]$, introduce

$$(3.3.12) \qquad \mathcal{A}(s,\varepsilon) \ := \ \{ z \in \mathcal{O} \ : \ |\, u^\varepsilon(x)\,| \ge 2C_1 s \}$$

Our goal is to estimate the measure of $\mathcal{A}(s,\varepsilon)$. We apply (3.3.11) with $\delta = \delta_s := \sqrt{\varepsilon}/s$. For $z \in \Omega \backslash \mathcal{N}$,

$$|\, u^\varepsilon(z)\,| \ \le\ C_1 \ s \ + \ C_1 \ \frac{\operatorname{meas} \mathcal{K}(z, \sqrt{\varepsilon}/s, \varepsilon)}{\sqrt{\varepsilon}} .$$

Therefore, for $z \in \mathcal{A}(s,\varepsilon)\backslash \mathcal{N}$, one has

$$(3.3.13) \qquad \operatorname{meas} \mathcal{K}(x, \sqrt{\varepsilon}/s, \varepsilon) \ \ge\ \sqrt{\varepsilon}\ s \ :=\ \rho_s .$$

Introduce

$$(3.3.14) \qquad \begin{aligned} &\mathcal{B}(s{,}\varepsilon) \ := \\ &\qquad \{ (z,\alpha) \in \Omega \times \mathcal{I} \ : \ |\Phi'_\alpha(z,\alpha)| \le 2\sqrt{\varepsilon}/s \ \text{ and } \ |\Phi''_{\alpha\alpha}(z,\alpha)| \le 1/s^2 \} . \end{aligned}$$

Then $\mathcal{B}(z,s,\varepsilon) := \{ \alpha \in \mathcal{I} \,:\, (z,\alpha) \in \mathcal{B}(s,\varepsilon) \} \ = \ \mathcal{K}(z, \sqrt{\varepsilon}/s, \varepsilon)$. Since $\operatorname{meas} \mathcal{N} = 0$, (3.3.13) implies that

$$(3.3.15) \qquad \begin{aligned} \operatorname{meas} \mathcal{B}(s,\varepsilon) \ &= \ \int \operatorname{mes} \mathcal{B}(z,s,\varepsilon)\, dz \\ &\ge \ \int_{\mathcal{A}(s,\varepsilon)} \operatorname{meas} \mathcal{B}(z,s,\varepsilon)\, dz \ \ge\ \rho_s \ \operatorname{meas} \mathcal{A}(s,\varepsilon) . \end{aligned}$$

Since $\Phi'_\alpha(z,\alpha) = y - h'_\alpha(t,y)$ and $\Phi''_{\alpha\alpha}(z,\alpha) = -h''_{\alpha\alpha}(t,\alpha) = -D_*(t,\alpha)$, one has

$$(3.3.16) \qquad \operatorname{meas} \mathcal{B}(s,\varepsilon) \ \le\ 4\ \delta_s \ \operatorname{meas} \mathrm{A}(s^2)$$

where $\mathrm{A}(\sigma) := \{ (t,\alpha) \in \Omega_{\underline{t}} \times \mathcal{I} \ : \ |D_*(t,\alpha)|^{-1} \ge \sigma \}$. Therefore, (3.3.15) implies

$$(3.3.17) \qquad \operatorname{mes} \mathcal{A}(s,\varepsilon) \ \le\ \frac{4\,\delta_s}{\rho_s} \ \operatorname{mes} \mathrm{A}(s^2) \ =\ \frac{4}{s^2} \ \operatorname{mes} \mathrm{A}(s^2) .$$

6. If $D_*^{-1} \in L^{r,\infty}$, then $\operatorname{meas} \mathrm{A}(\sigma) \ \le\ \Gamma\, \sigma^{-r}$. Thus $\operatorname{meas} \mathcal{A}(s,\varepsilon) \ \le\ \Gamma\, s^{-(2+2r)}$ and therefore

$$(3.3.18) \qquad \sup_{\varepsilon \in]0,\varepsilon_0[} \ \|\, u^\varepsilon \,\|_{L^{2+2r,\infty}} \ <\ +\infty .$$

Similarly, (3.3.17) implies that

$$\begin{aligned} \|\, u^\varepsilon \,\|_{L^{2+2r}} \ &\le\ C \ \int_0^{+\infty} s^{1+2r} \ \operatorname{mes} \mathcal{A}(s,\varepsilon)\ ds \\ &\le 4\ C \ \int_0^{+\infty} s^{2r-1} \ \operatorname{mes} \mathrm{A}(s^2)\ ds \le\ C' \|\, D_*^{-r} \,\|_{L^1(\mathcal{O}_{\underline{t}} \times \mathcal{I})} , \end{aligned}$$

which completes the proof of Theorem 3.1.2.

Examples and Remarks 3.3.3. The assumptions of Theorem 3.1.2 are satisfied when

$$\Phi(t,y,\alpha) \;=\; y\,\alpha \;-\; \psi(t,\alpha)\,, \quad (t,y)\in\mathbb{R}^{d-1}\times\mathbb{R}\,, \quad \alpha\in\mathbb{R}\,, \tag{3.3.19}$$

and ψ is a polynomial in α of degree $k\geq 2$, whose leading coefficient does not vanish for almost all t. In this case, $C_\Phi = \{y=\varphi'_\alpha(t,\alpha)\}$ and $\mathcal{S}_\varphi = C_\Phi\cap\{\psi''_{\alpha,\alpha}(t,\alpha)=0\}$. No smoothness assumption is imposed on $\mathcal{S}_\Phi$.

The examples $\psi(t,\alpha)=\alpha^3$ and $\psi(t,\alpha)=t_1\,\alpha^2+\alpha^4$, yield caustics which are folds and cusps. In these two examples the critical index q_c is equal to 4. In particular, this shows that the critical index q_c is not related to the order of the caustic as introduced in [Du].

3.4. Proof of Theorem 3.1.3

We consider oscillatory integral (3.1.1), assuming that $\mathcal{S}_\Phi$ is a smooth submanifold of C_Φ of codimension one, that the dimension ν of $\ker\Phi''_{\alpha\alpha}(\rho)$ is constant for $\rho\in\mathcal{S}_\Phi$ and that D_*, the restriction of $\det\Phi''_{\alpha\alpha}$ to C_Φ vanishes to order $\mu=\nu$ on $\mathcal{S}_\varphi$. In addition, $\nu\geq 2$.

By a linear change of variables, one can split the variables α into $(\alpha',\alpha'')\in\mathbb{R}^\nu\times\mathbb{R}^{n-\nu}$, so that $\Phi''_{\alpha',\alpha'}(\underline{\rho})=0$ and $\Phi''_{\alpha''\alpha''}(\underline{\rho})$ is invertible. Therfore, using the stationary phase theorem in the α'' variables reduces the proof to the case $n=\nu$.

¿From now on, we assume that $\nu=n$, which means that

$$\forall\rho\in\mathcal{S}_\Phi\;\;:\quad \Phi''_{\alpha\alpha}(\rho)\;=\;0\,. \tag{3.4.1}$$

Since the phase is non degenerate

$$\operatorname{rank}\Phi''_{z\alpha}(\underline{\rho})\;=\;\nu. \tag{3.4.2}$$

In particular, $m\geq\nu$. After a linear change of coordinates, one can split the variables z into $z:=(t,x)\in\mathbb{R}^{m-\nu}\times\mathbb{R}^\nu$, so that

$$\det\,\Phi''_{x\alpha}\;\neq\;0\,. \tag{3.4.3}$$

Considering t as a parameter, (3.4.3) implies that

$$C_\Phi(t):=\{(x,\alpha)\;;\Phi'_\alpha(t,x,\alpha)=0\}$$

is a smooth manifold of dimension ν. The associated Lagrangian manifold is

$$\Lambda_\Phi(t):=\{(x,d_x\Phi(t,x,\alpha))\;|\;(x,\alpha)\in C_\Phi(t)\,\}\,.$$

Moreover, the mapping $(x,\xi)\to\xi$ is a local diffeomorphism from $\Lambda_\Phi(t)$ to $\mathbb{R}^\nu$. Therefore $\Lambda_\Phi(t)$ can be parametrized by ξ as the set $\{(d_\xi h(t,\xi),\xi)\}$, with a smooth function h defined near the point $(\underline{t}, \underline{\xi})$ corresponding to $\underline{\rho}$. Consider the phase function

$$\Psi(t,x,\xi) \ := \ x \cdot \xi \ - \ h(t,\xi)\,. \tag{3.4.4}$$

The function h is uniquely determined if one requires that $\Phi(\underline{\rho}) = \Psi(\underline{t}, \underline{x}, \underline{\xi})$.

The Theorem of equivalence of phase functions (see [Hö 1] [Du]), with t as parameters, implies that there exists a mapping $(z,\alpha) \to \xi(z,\alpha)$ which is, for all fixed z, a local diffeomorphism $\alpha \to \xi$ such that

$$\Phi(z,\alpha) \ = \ \Psi(z, \xi(z,\alpha))\,. \tag{3.4.5}$$

Changing variables implies that, for a supported in a sufficiently small neighborhood of $\underline{\rho}$, one gets

$$u^\varepsilon(t,x) \ = \ \varepsilon^{-\nu/2} \int_{\mathbb{R}^\nu} e^{i\,\Psi(t,x,\xi)/\varepsilon}\ b(t,x,\xi,\varepsilon)\ d\xi\,, \tag{3.4.6}$$

where b is a smooth function on $U \times [0,1]$, with U a neighborhood of $(\underline{t}, \underline{x}, \underline{\xi})$.

The new phase Ψ defines the same Lagrangian manifold Λ as Φ. Thus Ψ satisfies the same assumptions as Φ. The manifold C_Ψ is given by the relation $x = h'_\xi(t,\xi)$ and is parametrized by (t,ξ). In this parametrization the manifold $\mathcal{S}_\Psi$ is locally given by an equation

$$f(t,\xi) = 0 \quad \text{with} \quad df(\underline{t}, \underline{\xi}) \neq 0\,. \tag{3.4.7}$$

Then (3.4.1) means that

$$h''_{\xi\xi}(t,\xi) = 0 \quad \text{when} \quad f(t,\xi) = 0\,. \tag{3.4.8}$$

In addition, the condition that D_* vanishes exactly to order μ on $\mathcal{S}_\Phi$ means that

$$\det h'' = f^\mu\, e \quad \text{ with } \quad e(\underline{t}, \underline{x}, \underline{\xi}) \neq 0\,. \tag{3.4.9}$$

Lemma 3.4.1. *There is a neighborhood ω of $(\underline{t}, \underline{x}, \underline{\xi})$ and a C^∞ function $g(t)$ near $\underline{t}$, such that $g(\underline{t}) = 0$, $dg(\underline{t}) \neq 0$ and*

$$h''_{\xi\xi}(t,\xi) \ = \ g(t)^r\, E(t,\xi)\,, \quad \text{with} \quad \det E(t,\xi) \ \neq \ 0\,. \tag{3.4.10}$$

Proof. (3.4.8) implies that there is a smooth symmetric matrix $E(t,x)$ with

$$h''_{\xi\xi}(t,\xi) \;=\; f(t,\xi)\,E(t,\xi)\,. \tag{3.4.11}$$

Since $\mu = \nu$, (3.4.9) implies that $\det E(t,\xi) \neq 0$. We show that (3.2.13) implies that $d_\xi f = 0$ when $f = 0$. This implies that $\mathcal{S}_\Psi := \{f = 0\}$ is also given by an equation $g(t) = 0$ and the proposition then follows.

Differentiating (3.4.11) with respect to ξ shows that

$$\begin{aligned} &h'''_{\xi\xi\xi}(t,\xi)(\delta\xi_1,\delta\xi_2,\delta\xi_3) \;= \\ &\qquad f'_\xi(t,\xi)\,(\delta\xi_3)\,E(t,\xi)\,(\delta\xi_1,\delta\xi_2) + f(t,\xi)\,E'_\xi(t,\xi)\,(\delta\xi_3,\delta\xi_1,\delta\xi_2)\,. \end{aligned}$$

The left hand side is symmetric with respect to $(\delta\xi_1,\delta\xi_2,\delta\xi_3)$. Therefore, where $f(t,\xi) = 0$,

$$f'_\xi(t,\xi)\,(\delta\xi_3)\,E(t,\xi)\,(\delta\xi_1,\delta\xi_2) \;=\; f'_\xi(t,\xi)\,(\delta\xi_1)\,E(t,\xi)\,(\delta\xi_2,\delta\xi_3)\,. \tag{3.4.12}$$

Since $\nu \geq 2$, there is $\delta\xi_1 \neq 0$ such that $f'_\xi(t,\xi)(\delta\xi_1) = 0$. Since $E(t,\xi)$ is symmetric and nondegenerate, there is $\delta\xi_2$ such that $E(t,\xi)\,(\delta\xi_1,\delta\xi_2), \neq 0$. Therefore (3.4.12) implies that $f'_\xi(t,\xi)\,(\delta\xi_3) = 0$ for all $\varphi\xi_3$. This shows that $d_\alpha f(t,\xi) = 0$ when $f(t,\xi) = 0$ and the lemma is proved.

Changing Ψ into $\Psi(t,x,\eta) + \psi(t,x)$ does not change the modulus $|u^\varepsilon|$ of the oscillatory integrals (3.4.6). Thus, we can also assume that $\underline{\eta} = 0$ and that $h(t,0) = 0$. Therefore, for η in a neighborhood of 0, one has

$$h(t,\eta) \;=\; h'_\eta(t,0)(\eta) \;+\; \int_0^1 (1-s)\; h''_{\eta\eta}(t,s\eta)\,(\eta\,,\,\eta)\; ds\,.$$

The change of variables $(t,x) \to (t, x - h'_\eta(t,0))$ reduces the proof of Theorem 3.1.3 to the case where $h'_\eta(t,0) = 0$. In this case, (3.4.10) implies that

$$h(t,\eta) \;=\; g(t)\,\psi(t,\xi)\,, \quad \text{with} \quad \det \psi''_{\xi\xi}(t,0) \neq 0\,. \tag{3.4.13}$$

In addition, $dg(\underline{t}) \neq 0$. All the reductions above being performed, Theorem 3.1.3 is a consequence of the following estimate.

Proposition 3.4.2. *Consider the phase function* (3.4.4) *with h satisfying* (3.4.13). *There is a neighborhood ω of $\underline{\rho} = (\underline{t},\underline{x},\underline{\xi})$, such that for all $b \in C_0^\infty(\omega \times [0,1])$, there is a constant C such that the oscillatory integrals* (3.4.6) *satisfy*

$$|\, u^\varepsilon(t,x)\,| \;\leq\; C\;(|g(t)| \,+\, |x|)^{-\nu/2} \;+\infty\,. \tag{3.4.14}$$

Proof. Introduce the oscillatory integrals

$$v^\delta(t,y,x) \;:=\delta^{-\nu/2}\int e^{i\,(y\cdot\xi\,-\,\psi(t,\xi))/\delta}\; b(t,x,\xi)\,d\xi\,. \tag{3.4.15}$$

There are $R>0$ and $c>0$ such that for $|y|\ge R$, $|d_\xi(y\cdot\xi-\psi(t,\xi)|\ge c|y|$ on the support of b. Integrations by parts show that for all N there is C_N such that

$$\forall |y|\ge R\,,\ \forall\delta\le|y|\,,\quad |\,v^\delta(t,y,x)\,|\ \le\ C_N(|\delta|^{N-\nu/2}\,|y|^{-N})\,. \tag{3.4.16}$$

Since $\psi''_{\eta\eta}$ is nondegenerate at the origin, if the support of b is sufficiently small, then v^δ is also uniformly bounded for bounded y and bounded δ : there is C such that

$$\forall |y|\le R\,,\ \forall|\delta|\le 1\,,\quad |\,v^\delta(t,y,x)\,|\ \le\ C\,. \tag{3.4.17}$$

Moreover v^δ is compactly supported in (t,x).

For $g(t)\neq 0$, one has

$$u^\varepsilon(t,x)\ =\ g(t)^{-\nu/2}\;v^\delta(t,\,x/g(t),\,x)\,,\qquad\text{with}\quad \delta=\varepsilon/g(t)\,. \tag{3.4.18}$$

1. For $|x|\le R\,|g(t)|$ and $\varepsilon\le|g(t)|$, (3.4.17) shows that $u^\varepsilon=O(|g(t)|^{-\nu/2})=O((|g(t)|+|x|)^{-\nu/2})$.

2. For $|x|\ge R\,|g(t)|$ and $\varepsilon\le|x|$, (3.4.16) implies that $u^\varepsilon=O(\varepsilon^{N-\nu/2}|x|^{-N})$. Taking $N\ge\nu/2$, this implies that $u^\varepsilon=O(|x|^{-\nu/2})=O((|g(t)|+|x|)^{-\nu/2})$. This estimate extends to the case $g(t)=0$, since then the phase has no critical point.

3. In the other cases, one has $|g(t)|+|x|\le 2\varepsilon$, and $u^\varepsilon=O(\varepsilon^{-\nu/2})=O((|g(t)|+|x|)^{-\nu/2})$.

Summarizing, we have proved that when the support of b is contained in a sufficiently small neighborhood of $(\underline{t},\underline{x})$, the estimate (3.4.14) holds.

Example 3.4.3. Typical phases are $\Phi(t,x,\xi)\ =\ x\cdot\xi\,-\,t\,|\xi|^2$, for $(t,x,\xi)\in\mathbb{R}\times\mathbb{R}^\nu\times\mathbb{R}^\nu$. Spherical focusing leads to such phases.

3.5. Subcritical L^p estimates for $I^\varepsilon(\mathcal{A})$

This section is devoted to the proof of Theorem 2.3.5. For the convenience of the reader we recall severall notations and assumptions from §2. Consider oscillatory integrals of the form

$$I^\varepsilon(\mathcal{A})\,(t,x)\ :=\ \sum_n\ (|n|/2\pi\varepsilon)^d\int e^{i\,n\,\Phi(t,x,y,\xi)/\varepsilon}\;a_n(t,y)\;dy\,d\xi\,, \tag{3.5.1}$$

with

$$\Phi(t,x,y,\xi):=t\,\lambda(\xi)\ +\ (x-y)\cdot\xi\ +\ \psi(y)\,, \tag{3.5.2}$$

where ψ satisfies Assumptions 2.2.1 on $\omega \subset \mathbb{R}^d$, $\lambda(\xi)$ is a C^∞ function on $\mathbb{R}^d \backslash \{0\}$ homogeneous of degree one and

$$(3.5.3) \qquad \mathcal{A}(t,y,\theta) := \sum_{n\neq 0} a_n(t,y)\, e^{in\theta}\,, \quad a_n \in C_0^\infty([0,+\infty[\times\omega)$$

is a trigonometric polynomial. This is a particular case of integrals (3.1.1) with space variables $z = (t,x)$ and phase variables $\alpha = (y,\xi)$ except that the domain of integration is not compact in ξ.

Φ is nondegenerate and the set C_Φ is parametrized by $\mathbb{R} \times \omega$ as follows

$$(3.5.4) \quad \iota \;:\; (t,y) \;\mapsto\; \big(t, y - t\,g(y), \lambda(d\psi(y)), d\psi(y)\big)\,, \quad \text{with} \quad g(y) := \lambda'(d\psi(y))\,.$$

One has

$$(3.5.5) \qquad \det \Phi''_{\alpha\alpha}(\iota(t,y)) \;=\; \det(I - t\,g'(y))\,.$$

Therefore the singular set $\mathcal{S}_\Phi$ is equal to $\iota(\mathcal{S})$ where

$$(3.5.6) \qquad \mathcal{S} \;:=\; \{(t,y) \;:\; \det(I - t\,g'(y)) = 0\}\,.$$

The Caustic set is $\mathcal{C} := \jmath(\mathcal{S})$ where $\jmath(t,y) := (t,x)$ with $x = q(t,y) := y - tg(y)$. (3.5.6) shows that $(t,y) \in \mathcal{S}$ if and only if $1/t$ is an eigenvalue of $g'(y)$. Following Definition 2.3.2 we call $\mu(t,y)$ the algebraic multiplicity of the corresponding eigenvalue and $\nu(t,y)$ the dimension of the eigenspace. Saying that the multiplicity μ is constant on a neighborhood of (t,y) in $\mathcal{S}$ is equivalent to saying that the eigenvalue has constant multiplicity on a neighborhood of y. Therefore,

Lemma 3.5.1. *Consider $(\underline{t},\underline{y}) \in \mathcal{S}$. Suppose that μ and ν are constant on a neighborhood of $(\underline{t},\underline{y})$ in $\mathcal{S}$. Then, on a neighborhood of $(\underline{t},\underline{y})$, $\mathcal{S}$ is a smooth manifold of codimension one and the determinant $\det(I - t\,g'(y))$ vanishes exactly to order μ on $\mathcal{S}$.*

Remarks and Examples 3.5.2. a) One has $1 \le \nu \le \mu \le d-1$, see (2.6.1). In particular, in space dimension $d = 2$, the only possibility is $\nu = \mu = 1$.

b) The condition $\mu = \nu$ means that the corresponding eigenvalue is semi-simple. If either λ''_k or ψ'' is positive (or negative) semidefinite, then the nonzero eigenvalues of $g'_k(y) = \lambda''(d\psi(y))\,\psi''(y)$ are semi-simple.

This applies in particular to the wave equation, where $\lambda_k(\xi) = \pm|\xi|$ is convex (concave). In this case, the eigenvalues of $g'_k(\underline{y})$ are 0 and $\pm$ the principal curvatures of the manifold $\psi(y) = \psi(\underline{y})$.

The same remark applies to the Maxwell equations, the linearized Euler equations, and linear elasticity for each of which all eigenvalues are convex.

c) Generic caustics occur when $\nu = \mu = 1$, and $Vc \neq 0$ where $V(y)$ is a smooth nonvanishing eigenvectorfield $V(y) \in \ker(c(y)I - g'(y))$ and $c(y)$ is the eigenvalue of $g'(y)$ under consideration (see [Hö 2][Du][JMR 3]). In this case, $\mathcal{C}$ is smooth and $\pi_{|\Lambda}$ is a fold.

d) When $\nu = \mu = 1$ and $Vc(\underline{y}) = 0$, $V^2c(\underline{y}) \neq 0$, the caustic $\mathcal{C}$ has a cusp. This applies for example to the wave equation in $\mathbb{R}^{1+2}$ and the phases (2.6.4), where only folds and cusps are present.

e) For the wave equation and radial phases $\psi(y) = \chi(|y|)$, there is one positive eigenvalue, $c(y) = |y|$, semisimple and of multiplicity $\mu = \nu = d - 1$.

The Assumptions 2.2.2 and 2.3.3 read

Assumption 3.5.3. *i) There is an integer ℓ_*, such that for and all $(t,x) \notin \mathcal{C}$, the number of points $y \in \omega$ such that $x = q(t,y)$, is less than or equal to ℓ_*.*

ii) There is an open subset $\omega_0 \subset \omega$ such that $\omega \backslash \omega_0$ has Lebesgue measure equal to 0 and for all k and all $(t,y) \in \mathcal{S}k$ with $y \in \omega_0$, one has

a) μ and ν are constant on a neighborhood of (t,y) in $\mathcal{S}$,

b) either $\nu = 1$ or $\nu = \mu$.

Remark 3.5.4. a) When λ and ψ are real analytic, the phase Φ is real analytic and the Assumption *i)* is satisfied.

b) The eigenvalues of $g'(y)$ are locally of constant algebraic and geometric multiplicities multiplicity on a dense open subset of ω. Part a) of Assumption *ii)* says that this set has full measure in ω. In particular, this is true when λ and ψ are real analytic.

c) Combined with Remarks 3.5.2 a) and b) above, this implies that Assumption 3.5.3 is satisfied in the following three cases :

1) real analytic λ and real analytic convex phases ψ

2) real analytic phases for the wave equations,

3) real analytic λ and real analytic phases in space dimension $d = 2$.

d) Assumption 3.5.3 is also satisfied in the example 6 of §2.6, where $\lambda_k(\xi) = |\xi|$ and

$$\psi(y) := y_d \ + \ (y_1^2 \ + \cdots \ + \ y_{d-1}^2)/2\,, \tag{3.5.7}$$

with $\omega_0 := \{y' \neq 0\}$.

For $q \in [2, +\infty]$, the set of supercritical points $\mathcal{S}^+(q)$ is the set of $(t,y) \in \mathcal{S}$ such that either $y \in \omega \backslash \omega_0$ or $\mu(t,y) \geq 2/(q-2)$. Theorem 2.3.5 is an immediate corollary of the following result.

THEOREM 3.5.5. *Suppose that Assumption* 3.5.3 *is satisfied. Let* $q \geq 2$. *Assume that the coefficients* a_n *of* $\mathcal{A}$ *are compactly supported in* $[0, +\infty[\times \omega$ *and vanish on a neighborhood of points* $(t, y) \in \mathcal{S}^+(q)$. *Then,*

$$\sup_{\varepsilon \in]0,1]} \| I^\varepsilon(\mathcal{A}) \|_{L^q(\mathbb{R}^{1+d})} \; < \; +\infty\,. \tag{3.5.8}$$

Proof. **a)** It is sufficient to consider the case where $\mathcal{A}$ is a monomial. Changing ε to $\varepsilon/|n|$ and taking complex conjugate it is sufficient to consider integrals of the form

$$u^\varepsilon(t,x) \; := \; \varepsilon^{-d} \int a(t,y)\, e^{i\,\Phi(t,x,y,\xi)/\varepsilon}\, dy\, d\xi\,, \tag{3.5.9}$$

b) Choose $\chi \in C^\infty(\mathbb{R}^d)$ such that $1 - \chi$ has compact support disjoint from the origin and $\chi = 0$ on a neighborhood of the support of $d\psi$. Let

$$u^\varepsilon_\chi(t,x) \; := \; \varepsilon^{-d} \int a(t,y)\, \chi(\xi)\, e^{i\,\Phi(t,x,y,\xi)/\varepsilon}\, dy\, d\xi\,. \tag{3.5.10}$$

Since $|\partial_y \Phi| \geq c(1 + |\xi|)$ on the support of $a(t,y)\,\chi(\xi)$, integration by parts in y yields $u^\varepsilon_\chi = O(\varepsilon^\infty)$ uniformly in (t,x).

To estimate the L^2 norm, write

$$u^\varepsilon_\chi(t\,.\,) \;=\; e^{i\,t\,\lambda(D)}\; \chi(\varepsilon\, D)\; \big(e^{i\,\psi/\varepsilon}\; a(t,\,.\,)\big)\,.$$

The $L^2(\mathbb{R}^d)$ norm is the same as that without the factor $e^{i\,t\,\lambda(D)}$. The expression (3.5.10) without λ has phase $\widetilde{\Phi} = (x-y)\cdot\xi + \psi(y)$ which satisfies $|\partial_y \widetilde{\Phi}| \geq c(1+|\xi|)$ on the support of χ and $|\partial_\xi \widetilde{\Phi}| \geq c|x|$ on the support of a if $|x|$ is large. So, integrations by parts in (y,ξ) yield

$$\big|\,(e^{-i\,t\,\lambda(D)}\, u^\varepsilon_\chi)\,(t,x)\,\big| \;\leq\; O(\,\varepsilon^N/(1+|x|)^N\,)\,,$$

for all N. Therefore $u^\varepsilon_\chi(t,\,.\,) = O(\varepsilon^\infty)$ in L^2. This proves that for all $p \geq 2$,

$$\| \, u^\varepsilon_\chi(t,\,\cdot\,)\|_{L^p(\mathbb{R}^d)} \;=\; O(\varepsilon^\infty)\,. \tag{3.5.11}$$

c) Consider

$$u^\varepsilon_{1-\chi}(t,x) \; := \; \varepsilon^{-d} \int a(t,y)\,(1-\chi(\xi))\, e^{i\,\Phi(t,x,y,\xi)/\varepsilon}\, dy\, d\xi\,, \tag{3.5.12}$$

Since a is compactly supported in y, $|d_\xi \Phi| \geq c(|x| + 1)$ when $|x|$ is larger than R. Therefore, $u^\varepsilon_{1-\chi}$ is $O((\varepsilon/|x|)^\infty)$ for $|x| \geq R$. Therefore

$$\| \, u^\varepsilon_{1-\chi}(t,\,\cdot\,)\|_{L^p(|x|\geq R)} \;=\; O(\varepsilon^\infty)\,. \tag{3.5.13}$$

d) Thus, it is sufficient to bound the L^q norms of $u^\varepsilon_{1-\chi}$ on compact sets in $[0,+\infty[\times\mathbb{R}^d$. The nonstationary points contribute $O(\varepsilon^\infty)$, and the nondegenerate stationaty points contibutes $O(1)$. Therefore, using a partition of unity, it is sufficient to prove that the L^q norm of $u^\varepsilon_{1-\chi}$ is bounded independently of ε when a is supported in a small neighborhood of $(\underline{t},\underline{y})\in\mathcal{S}$ and $(1-\chi)$ is supported in a small neighborhood of $\underline{\xi}:=d\psi(\underline{y})$. In addition, since the original a is supported away from $\mathcal{S}^+(q)$ it is sufficient to consider points $(\underline{t},\underline{y})\in\mathcal{S}$ such that $\underline{y}\in\omega_0$ and $\mu(\underline{t},\underline{y})<2/(q-2)$. In this case, Assumptions 3.5.3 implies that either Theorem 3.1.2 or Theorem 3.1.3 applies (see Remark 3.1.4) and Theorem 3.5.5 follows.

Remark 3.5.6. If the hypothesis of Theorem 3.5.5 is satisfied for $q\geq 4$ then the conclusion 3.5.8 can be strengthened. In fact, 3.5.8 then holds for all $q\geq 2$. To see this note that $2/(q-2)\leq 1$ for $q\geq 4$, which implies that all points in $\mathcal{S}$ are supercritical. Thus the hypothesis implies that $\mathcal{A}$ vanishes for (t,y) in a neighborhood of $\mathcal{S}$. Therefore the critical points of the phase Φ_k located in the support of the coefficients are nondegenerate, so the family $I^\varepsilon k(\mathcal{A})$ is bounded in L^∞. With L^2 estimates provided by the Plancherel identity, this implies that (3.5.8) holds for all $q\geq 2$.

On the other hand, when $q<2+2/(d-1)$, (3.5.8) holds for all trigonometric polynomials with coefficients in $C_0^\infty([0,+\infty[\times\omega_0)$, since the multiplicity μ is at most $d-1<2/(q-2)$. Thus the index $2+2/(d-1)$ which occurs for spherical focusing is the worst case.

4. Oscillations and profiles

In the first part of this section, we introduce families $I^\varepsilon(\mathcal{A})$ and $J^\varepsilon(\mathcal{A})$. They are defined in (2.3.2) and (2.4.6) when $\mathcal{A}$ is a smooth trigonometric polynomial. The definition of J^ε is extended to $\mathcal{L}^{s,q}_k$ for $s>1/q$ and also, in an asymptotic sense, to $\mathcal{L}^q_k$. In the second part of the section, we introduce profiles associated to families u^ε. This is a modification of the multiscale analysis introduced in [N], [A], [E], [ES], [JMR 2] with phases replaced by Lagrangians.

In this section we suppose that Assumptions 2.1.1, 2.2.1, 2.2.2, 2.2.5 and 2.3.3 are satisfied.

4.1. Estimates for $J^\varepsilon(\mathcal{A})$ and $I^\varepsilon(\mathcal{A})$

Definition 4.1.1. *$\mathcal{P}_k\subset\mathcal{P}$ denotes the space of trignometric polynomials whose coefficients vanish in a neighborhood of $\mathcal{S}_k$.*

$\mathcal{P}_k(q)$ is the space of trigonometric polynomials which vanish on a neighborhood of the set of supercritical point $\mathcal{S}^+_k(q)$ introduced in Definition 2.3.4.

For $\mathcal{A}\in\mathcal{P}_k$ the family $J^\varepsilon_k(\mathcal{A})$ is defined by (2.4.2).

PROPOSITION 4.1.2. *For $\mathcal{A} \in \mathcal{P}_k$, and $\varepsilon > 0$, $J_k^\varepsilon(\mathcal{A}) \in C_0^\infty([0,+\infty[\times\mathbb{R}^d)$. For each $q \in]1,+\infty[$, there is a constant C such that for all $\mathcal{A} \in \mathcal{P}_k$,*

$$\limsup_{\varepsilon\to 0} \; || \, J_k^\varepsilon(\mathcal{A}) \, ||_{L^q([0,\infty[\times\mathbb{R}^d)} \;\leq\; C \;\; || \, \mathcal{A} \, ||_{\mathcal{L}_k^q} . \tag{4.1.1}$$

For $1/q < s < 1$, there is a constant C such that for all $\mathcal{A} \in \mathcal{P}_k$,

$$\sup_{0<\varepsilon\leq 1} \; || \, J_k^\varepsilon(\mathcal{A}) \, ||_{L^q([0,\infty[\times\mathbb{R}^d)} \;\leq\; C \;\; || \, \mathcal{A} \, ||_{\mathcal{L}_k^{s,q}} . \tag{4.1.2}$$

Proof. **a)** The support of $\mathcal{A} \in \mathcal{P}_k$ is contained in $K \times \mathbb{T}$ where K is a compact subset of G which does nor intersect $\mathcal{S}_k$. Therefore, there is a covering of K by relatively open sets $G_\alpha \subset G$, such that $\jmath_k$ is a diffeomorphism from G_α onto $\Omega_\alpha := \jmath_k(G_\alpha)$. Denote by σ_α the inverse mapping from Ω_α onto G_α. Introduce a finite partition of unity, $\chi_\alpha \in C_0^\infty(G_\alpha)$, such that $\chi_\alpha \geq 0$ and $\sum \chi_\alpha = 1$ on K. Thus $\mathcal{A} = \sum \chi_\alpha \, \mathcal{A}$, $J_k^\varepsilon(\mathcal{A}) = \sum J_k^\varepsilon(\chi_\alpha \, \mathcal{A}_\alpha)$ and

$$J_k^\varepsilon(\chi_\alpha \, \mathcal{A}) \, (t,x) \;=\; \frac{\chi_\alpha(\sigma_\alpha(t,x))}{\Delta_k(\sigma_\alpha(t,x))} \; (\mathcal{H}^{m_{k,\alpha}} \mathcal{A}) \, (\sigma_\alpha(t,x), \varphi_\alpha(t,x)/\varepsilon) \, , \tag{4.1.3}$$

where $\varphi_\alpha(t,x) := \psi(\sigma_\alpha(t,x))$ and $m_{k,\alpha}$ is the value of $m_k(t,y)$ on G_α. This proves that $J_k^\varepsilon(\mathcal{A})$ is C^∞ and has compact support contained in $\jmath_k(K)$.

b) The finiteness Assumption 2.2.2 implies that

$$| \, J_k^\varepsilon(\mathcal{A}) \, (t,x) \, |^q \;\leq\; \ell_*^{q-1} \sum_{\{y \,|\, q_k(t,y) = x\}} \frac{1}{|\Delta_k(t,y)|^q} \; \big| \, \big(\mathcal{H}^{m_k(t,y)} \mathcal{A}\big) \, (t,y,\psi(y)/\varepsilon) \, \big|^q .$$

Introduce $\mathcal{B} := \mathcal{H}^{m_k} \mathcal{A}$. $\mathcal{B}$ is smooth in t, y since m_k is locally constant on the support of $\mathcal{A}$ and therefore $\mathcal{B} \in \mathcal{P}_k$. The identity $|\mathcal{B}|^q = \sum \chi_\alpha |\mathcal{B}|^q$ where χ_α is the partition of unity introduced in a), implies that

$$| \, J_k^\varepsilon(\mathcal{A}) \, (t,x) \, |^q \;\leq\; \ell_*^{q-1} \sum_\alpha \frac{\chi_\alpha(\sigma_\alpha(t,x))}{|\Delta_k(\sigma_\alpha(t,x))|^q} \; \big| \, \mathcal{B} \, (\sigma_\alpha(t,x), \varphi_\alpha(t,x)/\varepsilon) \, \big|^q .$$

Since Δ_k^2 is the Jacobian of q_k, one has

$$|| \, J_k^\varepsilon(\mathcal{A}) \, ||_{L^q}^q \;\leq\; \ell_*^{q-1} \sum_\alpha \int_{G_\alpha} \frac{\chi_\alpha(t,y)}{|\Delta_k(t,y)|^{q-2}} \; \big| \, \mathcal{B} \, (t,y,\psi(y)/\varepsilon) \, \big|^q \; dtdy \, . \tag{4.1.4}$$

c) $|\mathcal{B}(t,y,\theta)|^q$ is a compactly supported and continuous function. Expanding in Fourier series, we see that

$$\int_{G_\alpha} \frac{\chi_\alpha(t,y)}{|\Delta_k(t,y)|^{q-2}} \; \big| \, \mathcal{B} \, (t,y,\psi(y)/\varepsilon) \, \big|^q \; dtdy \quad \to \quad \int_{G_\alpha\times\mathbb{T}} \frac{\chi_\alpha(t,y)}{|\Delta_k(t,y)|^{q-2}} \; \big| \, \mathcal{B} \, (t,y,\theta) \, \big|^q \; dtdyd\theta$$

as $\varepsilon \to 0$. Therefore

$$\limsup_{\varepsilon\to 0} \| \, J_k^\varepsilon(\mathcal{A}) \, \|_{L^q}^q \ \leq \ \ell_*^{q-1} \int |\Delta_k(t,y)|^{2-q} \, \big| \, \mathcal{B}\,(t,y,\theta) \, \big|^q \; dtdyd\theta \,. \tag{4.1.5}$$

For $q \in]1,+\infty[$, the Hilbert transform is bounded in $L^q(\mathbb{T})$. Thus

$$\int \big| \, \mathcal{B}\,(t,y,\theta) \, \big|^q \; d\theta \ \leq \ C_q \int \big| \, \mathcal{A}\,(t,y,\theta) \, \big|^q \; d\theta \,.$$

With (4.1.5) this implies (4.1.1).

d) For $s > 1/q$, we use the Sobolev inequality

$$\sup_{\theta\in\mathbb{T}} \ | \, \mathcal{B}(t,y,\theta) \, | \quad \leq \ C \, \| \, \mathcal{B}(t,y,\,.\,) \, \|_{W^{s,q}(\mathbb{T})} \,.$$

Then, by definition of $\mathcal{L}_k^{s,q}$, there is a constant C such that

$$\sup_{\theta\in\mathbb{T}} \ | \, \mathcal{B}(t,y,\theta) \, |^q \quad \leq \ C b(t,y) \tag{4.1.6}$$

with

$$\int_G \ \Delta_k^{2-q} \ b(t,y) \, dt \, dy \ \leq \ \| \, \mathcal{B} \, \|_{\mathcal{L}_k^{s,q}}^q \ \leq \ C \, \| \, \mathcal{A} \, \|_{\mathcal{L}_k^{s,q}}^q \,, \tag{4.1.7}$$

where the second inequality uses the fact that the Hilbert transform is bounded in $W^{s,q}(\mathbb{T})$. Plugging (4.1.6) in (4.1.4), and using (4.1.7) yields (4.1.2) and Proposition 4.1.2 is proved.

Proposition 4.1.3. *i) For all trigonometric polynomial $\mathcal{A}$,*

$$\limsup_{\varepsilon\to 0} \ \| \, I_k^\varepsilon(\mathcal{A}) \, (t, \, \cdot \,) \, \|_{L^2(\mathbb{R}^d)} \quad = \quad \| \, \mathcal{A}(t, \, \cdot \,) \, \|_{L^2(\omega\times\mathbb{T})} \,. \tag{4.1.8}$$

The limit is uniform on bounded intervals of time. In addition, for all $T > 0$

$$\| \, I_k^\varepsilon(\mathcal{A}) \ - \ J_k^\varepsilon(\mathcal{A}) \, \|_{L^2([0,T]\times\mathbb{R}^d)} \quad \to \quad 0 \,, \quad \text{as } \varepsilon \to 0 \,. \tag{4.1.9}$$

ii) For all $q \in [2,+\infty[$, there is a constant C such that for all trigonometric polynomial $\mathcal{A} \in \mathcal{P}_k(q)$,

$$\limsup_{\varepsilon\to 0} \ \| \, I_k^\varepsilon(\mathcal{A}) \, \|_{L^q([0,T]\times\mathbb{R}^d)} \quad \leq \quad C \ \ \| \, \mathcal{A} \, \|_{\mathcal{L}_k^q} \,, \tag{4.1.10}$$

and

$$\| \, I_k^\varepsilon(\mathcal{A}) \ - \ J_k^\varepsilon(\mathcal{A}) \, \|_{L^q([0,\infty[\times\mathbb{R}^d)} \quad \to \quad 0 \,, \quad \text{as } \ \varepsilon \to 0 \,. \tag{4.1.11}$$

Proof. **a)** Part *i)* is a consequence of the standard L^2 estimates for oscillatory integrals (2.3.2). For a detailed proof, see [JMR 3].

b) To prove part *ii)*, consider $\mathcal{A} \in \mathcal{P}_k(q)$. Estimates (3.5.11) (3.5.13) imply that the L^q norm of $I_k^\varepsilon(\mathcal{A})$ outside a compact subset K in $[0,+\infty[\times\mathbb{R}^d$ is $O(\varepsilon^\infty)$. Therefore, it is sufficient to prove that

$$\limsup_{\varepsilon\to 0} \; ||\, I_k^\varepsilon(\mathcal{A})\,||_{L^q(K)} \quad \leq \quad C \;\; ||\,\mathcal{A}\,||_{\mathcal{L}_k^q}\,. \tag{4.1.12}$$

c) $\mathcal{A}$ vanishes near points in $\mathcal{S}_k^+(q)$. Thus, points (t,y) in $\mathcal{S}_k \cap \operatorname{supp}\mathcal{A}$ satisfy $\mu_k(t,y) < 2/(q-2)$. By compactness, one can find $r > q$ such that $\mu_k(t,y) < 2/(r-2)$ and hence $\mathcal{A} \in \mathcal{P}_k(r)$. Therefore, Theorem 2.3.5 implies that $I_k^\varepsilon(\mathcal{A})$ is bounded in $L^r([0,+\infty[\times\mathbb{R}^d)$. Thus, there exists M such that for all bounded domains D, one has

$$||\, I_k^\varepsilon(\mathcal{A})\,||_{L^q(D)} \quad \leq \quad M \;\; (\text{meas} D)^{1/q-1/r}\,. \tag{4.1.13}$$

d) Since the caustic set $\mathcal{C}_k$ has Legesgue measure equal to zero, (4.1.13) implies that in order to prove (4.1.12), it is sufficient to show that for all bounded open sets Ω such that $\overline{\Omega}\cap\mathcal{C}_k = \emptyset$

$$\limsup_{\varepsilon\to 0} \; ||\, I_k^\varepsilon(\mathcal{A})\,||_{L^q(\Omega)} \quad \leq \quad C \;\; ||\,\mathcal{A}\,||_{\mathcal{L}_k^q}\,. \tag{4.1.14}$$

On such domains, the stationary phase theorem applies, and $I_k^\varepsilon(\mathcal{A}) - J_k^\varepsilon(\mathcal{A}) = O(\varepsilon)$ in L^∞. Therefore (4.1.10) follows from Proposition 4.1.2.

e) Similarly, Proposition 4.1.2 implies that $J_k^\varepsilon(\mathcal{A})$ satisfies estimates analogous to (4.1.13). Since $J_k^\varepsilon(\mathcal{A})$ is compactly supported, to prove (4.1.11), it is sufficient to show that

$$||\, I_k^\varepsilon(\mathcal{A}) \; - \; J_k^\varepsilon(\mathcal{A})\,||_{L^q(\Omega)} \quad \to \quad 0\,, \quad \text{as } \varepsilon \to 0\,. \tag{4.1.15}$$

for all bounded open sets Ω such that $\overline{\Omega}\cap\mathcal{C}_k = \emptyset$. But there, $I_k^\varepsilon(\mathcal{A}) - J_k^\varepsilon(\mathcal{A}) = O(\varepsilon)$ in L^∞, implying (4.1.15). The proof of Proposition 4.2.3 is complete.

PROPOSITION 4.1.4. *If $\mathcal{A}$ and $\mathcal{B}$ belong to $\mathcal{P}_k$ then, as $\varepsilon \to 0$,*

$$\int J_k^\varepsilon(\mathcal{A})\,\overline{J_k^\varepsilon(\mathcal{B})}\; dt\,dx \quad \to \quad \int \mathcal{A}\,\overline{\mathcal{B}}\; dt\,dy\,d\theta\,. \tag{4.1.16}$$

Proof. It is sufficient to prove (4.1.16) when $\mathcal{A} = a(t,y)\, e^{i\, n_\alpha\, \theta}$ and $\mathcal{B} = b(t,y)\, e^{i\, n_\beta\, \theta}$ where a and b have small supports. Thus one can assume that they are supported in G_α and G_β respectively, where $\jmath_k$ is a diffeomorphism from both G_α and G_β onto a regular open set Ω. In this case, with notations as in (4.1.3),

$$J_k^\varepsilon(\mathcal{A})\,(t,x) \;=\; \frac{i^{m_{k,\alpha}}}{\Delta_k(\sigma_\alpha(t,x))}\; a\,(\sigma_\alpha(t,x))\; e^{i\, n_\alpha\, \varphi_\alpha(t,x)/\varepsilon}\,, \tag{4.1.17}$$

and a similar formula for $J_k^\varepsilon(\mathcal{B})$. There are two cases.

1) $G_\alpha = G_\beta$, $\sigma_\alpha = \sigma_\beta$ and $\varphi_\alpha = \varphi_\beta$. If $n_\alpha = n_\beta$ the two sides of (4.1.16) are independent of ε and are equal since $\Delta_k^2(t,y)$ is the Jacobian of $\jmath_k$. If $n_\alpha \neq n_\beta$, the right hand side vanishes and the left hand side tends to 0, since $d\varphi_\alpha$ does not vanish.

2) $G_\alpha \cap G_\beta = \emptyset$. Then the integrand in the right hand side vanishes. The left hand side tends to 0, since $n_\alpha d\varphi_\alpha - n_\beta d\varphi_\beta \neq 0$ almost everywhere, thanks to the nonresonance Assumption 2.2.5.

4.2. Extension to nonsmooth profiles

Proposition 4.2.1. *i) For all* $q \in [1,+\infty[$ *and* $s \in]0,1[$, $\mathcal{P}_k$ *is dense in* $\mathcal{L}_k^q$ *and in* $\mathcal{L}_k^{s,q}$.

ii) For $1/q < s < 1$ *and each* $\varepsilon \in]0,1]$, J_k^ε *extends to a bounded linear operator from* $\mathcal{L}_k^{s,q}$ *to* $L^q([0,+\infty[\times\mathbb{R}^d)$ *and inequality* (4.1.2) *extends to* $\mathcal{L}_k^{s,q}$.

Proof. $\mathcal{S}_k$ is the set of (t,y) such that $1/t$ is a positive eigenvalue of $g_k'(y)$. Thus its Lebesgue measure is zero, and the space of compactly supported functions which vanish on a neighborhood of $\mathcal{S}_k$ is dense in $\mathcal{L}_k^q$ and in $\mathcal{L}_k^{s,q}$ for all $q \in [1,+\infty[$ and $s \in]0,1[$. Using mollifiers, this implies that the space of compactly supported C^∞ functions on $[0,+\infty[\times\omega\times\mathbb{T}$, which vanish on a neighborhood of $\mathcal{S}_k$, are dense. Truncating the Fourier series proves *i)*. Part *ii)* follows.

For general profiles, the definition of J^ε must be understood in an asymptotic sense, i.e. they are bounded families in a Banach space defined up to families which converge strongly to zero in the same space. A convenient setting is the following.

For a Banach space E, BE denotes the Banach space of bounded families $\{u^\varepsilon\}_{\varepsilon\in]0,1]}$ normed by $\sup \|u^\varepsilon\|_E$. NE is the closed subspace of families with $u^\varepsilon \to 0$ as $\varepsilon \to 0$. AE is the quotient space BE/NE. The norm in AE is equal to

$$||\,[u^\varepsilon]\,||_{AE} \;=\; \limsup_{\varepsilon}\; ||\,u^\varepsilon\,||_E\,. \tag{4.2.1}$$

As usual the aspect quotient space in the following discussion is systematically suppressed, and we denote $u^\varepsilon \sim 0$ in E, whenever $u^\varepsilon \in NE$ and similarly $u^\varepsilon \sim v^\varepsilon$ in E when $u^\varepsilon - v^\varepsilon \in NE$.

PROPOSITION 4.2.2. *For all $q \in]1,+\infty[$, the family of mappings J_k^ε defines a bounded mapping $\mathcal{J}_k$, from $\mathcal{L}_k^q$ to $AL^q([0,+\infty[\times\mathbb{R}^d)$. It satisfies*

$$(4.2.2)\qquad \frac{1}{C}\,||\,\mathcal{A}\,||_{\mathcal{L}_k^q}\ \le\ ||\,\mathcal{J}_k(\mathcal{A})\,||_{L^q([0,\infty[\times\mathbb{R}^d)}\ \le\ C\ \ ||\,\mathcal{A}\,||_{\mathcal{L}_k^q}\,.$$

Suppressing as always the quotients, the notation $\mathcal{J}_k(\mathcal{A})$ is replaced by $\tilde{J}_k^\varepsilon(\mathcal{A})$. Then

$$(4.2.3)\qquad u^\varepsilon\ \sim\ \tilde{J}_k^\varepsilon(\mathcal{A})\quad \text{in } L^q\,,$$

means that the class of $[u^\varepsilon]$ of u^ε in $AL^q([0,+\infty[\times(\mathbb{R}^d))$ is $\mathcal{J}_k(\mathcal{A})$. This is equivalent to saying that u^ε is a bounded family in $L^q([0,+\infty[\times\mathbb{R}^d)$ such that for all $\delta > 0$ there is an $\mathcal{A} \in \mathcal{P}_k$ and $\varepsilon_\delta > 0$ such that

$$(4.2.4)\qquad \begin{cases} ||\,\mathcal{A}\,-\,\mathcal{A}_\delta\,||_{\mathcal{L}_k^q}\ \le\ \delta\,, \qquad \text{and} \\ \forall\varepsilon \in]0,\varepsilon_\delta]\,,\quad ||\,u^\varepsilon\,-\,J_k^\varepsilon(\mathcal{A}_\delta)\,||_{L^q([0,+\infty[\times\mathbb{R}^d)}\ \le\ \delta\,. \end{cases}$$

Thus, Proposition 4.2.2 is a restatement of Propositions 2.4.2 and 2.4.3.

Proof. The extension and the second inequality follow from the density Lemma 4.2.1 and Proposition 4.1.2.

Propositions 4.1.2 and 4.1.4 imply that for trigonometric polynomials $\mathcal{A}$ and $\mathcal{B}$ in $\mathcal{P}_k$, one has

$$(4.2.5)\qquad \Big|\int \mathcal{A}\,\overline{\mathcal{B}}\;dt\,dy\,d\theta\,\Big|\ \le C\ \liminf\,||\,J_k^\varepsilon(\mathcal{A})||_{L^q}\ ||\,\mathcal{B}\,||_{\mathcal{L}_k^{q'}}$$

with $1/q + 1/q' = 1$. The density of $\mathcal{P}_k$ in $\mathcal{L}_k^{q'}$ and the converse Hölder inequality imply that for $\mathcal{A} \in \mathcal{P}_k$,

$$\frac{1}{C}\,||\,\mathcal{A}\,||_{\mathcal{L}_k^q}\ \le\ \liminf_{\varepsilon\to 0}\ ||\,J_k^\varepsilon(\mathcal{A})\,||_{L^q([0,\infty[\times\mathbb{R}^d)}\,.$$

By density, this extends to $\mathcal{A} \in \mathcal{L}_k^q$ and the proposition is proved. This estimate also finishes the proof of Proposition 2.4.2.

There is a similar treatment for the Cauchy data, and the subtitution

$$\mathcal{U}_0(y,\psi(y)/\varepsilon)$$

which is defined for $\mathcal{U}_0 \in C^0(\omega\times\mathbb{T})$ with compact support and extends asymptotically from $L^2((\mathbb{R}^d\times\mathbb{T})$ to $AL^2(\mathbb{R}^d)$. The analogous notation to (4.2.3) is

$$u_0^\varepsilon\ \sim\ \tilde{J}_0^\varepsilon(\mathcal{U}_0)\quad \text{in } L^2(\mathbb{R}^d)\,.$$

REMARK 4.2.3. The operators J_k^ε are defined for fixed ε when $\mathcal{A} \in \mathcal{P}_k$ or more generally for $\mathcal{A} \in \mathcal{L}_k^{s,q}$, $s > 1/q$. In these cases, $J_k^\varepsilon(\mathcal{A})$ is a representative of $\mathcal{J}_k(\mathcal{A})$, that is,

$$J_k^\varepsilon(\mathcal{A}) \ \sim\ \tilde{J}_k^\varepsilon(\mathcal{A})\,.$$

It would be tempting to use (4.1.10) to extend I^ε asymptotically. By (4.1.11) this would yield $\tilde{I}^\varepsilon \sim \tilde{J}^\varepsilon$ and therefore nothing new. Note that the problems of extending I^ε and J^ε are different. For J^ε the difficulty is the substitution $\theta = \psi(y)/\varepsilon$ in (2.4.2). On the other hand, when $\mathcal{A}$ is smooth in θ, for example when $\mathcal{A}(t,y,\theta) := a(t,y)\,e^{i\,\theta}$, $I^\varepsilon(\mathcal{A})$ is defined for each ε, at least in the sense of distributions, but need not be bounded in L^q when $a \in L^q$. This is not due to the caustic but to the fact that Fourier Integral Operators of degree 0 are not bounded in L^q for $q \neq 2$. In particular, the limsup in (4.1.10) cannot be replaced by sup.

In the sequel, the operators I_k^ε act only on smooth profiles $\mathcal{A} \in \mathcal{P}_k(q)$. The $I_k^\varepsilon(\mathcal{A})$ serve as test functions.

4.3. Weak profiles

The analysis of weak convergence has been refined to a multiscale analysis, associating profiles to sequences of functions (see [N], [A], [E], [ES], and [JMR 2]). Given a phase φ, the idea is to test u^ε against functions of the form $\mathbf{B}(t,x,\varphi(t,x)/\varepsilon)$. Near a caustic there are several phases involved and the objects are better described by tests against functions of the form $J^\varepsilon(\mathcal{A})$.

DEFINITION 4.3.1. *If $1 < q < \infty$ and $\{u^\varepsilon\}_{0<\varepsilon<1}$ is a bounded family in $L^q(\mathbb{R}^{1+d})$ then $\mathcal{U} \in \mathcal{L}_k^q(G \times \mathbb{T})$ is called a Λ_k-profile if there is a subsequence $u^{\varepsilon'}$ so that for all $\mathcal{A} \in \mathcal{L}_k^{q'}(G \times \mathbb{T})$ and all $v^\varepsilon \sim \tilde{J}_k^\varepsilon(\mathcal{A})$ in $L^{q'}$.*

$$\lim_{\varepsilon' \to 0} \int \overline{v^{\varepsilon'}}\ u^{\varepsilon'}\ dt\,dx \quad = \quad \int \overline{\mathcal{A}}\,\mathcal{U}_k\ dt\,dy\,d\theta\,. \tag{4.3.1}$$

We say that a sequence u^ε is k-pure when it converges weakly and the convergence (4.3.1) holds for the full family u^ε. It is pure, when it is k-pure for all k.

The estimate (4.1.1) and Lemma 4.2.1 show that it suffices to verify (4.3.1) for $\mathcal{A} \in \mathcal{P}_k$. The next proposition implies that any bounded family in L^q has pure subsequences.

PROPOSITION 4.3.2. *If $1 < q < +\infty$ and u^ε is bounded in $L^q(\mathbb{R}^{1+d})$, then the set of Λ_k-profiles of u^ε is nonempty.*

There is a constant C independent of the family u^ε such that every Λ_k-profile satisfies

$$||\,\mathcal{U}_k\,||_{\mathcal{L}_k^q} \ \leq\ C\,\limsup_{\varepsilon \to 0}\,||\,u^\varepsilon\,||_{L^q}\,. \tag{4.3.2}$$

In addition, if $\mathcal{A} \in \mathcal{L}_k^{q'}(G \times \mathbb{T})$ *and*

$$v^\varepsilon \sim \tilde{J}_k^\varepsilon(\mathcal{A}) \qquad \text{in} \qquad L^{q'}(\mathbb{R}^{1+d})\,, \tag{4.3.3}$$

then

$$\int v^{\varepsilon'}\ \overline{u^{\varepsilon'}}\ dt\,dx \quad \to \quad \int \mathcal{A}\,\overline{\mathcal{U}_k}\ dt\,dy\,d\theta\,. \tag{4.3.4}$$

Proof. **a)** Choose G' such that $\jmath_k$ is a diffeomorphism from $G' \subset G$ onto $\Omega' \subset [0,+\infty[\times\mathbb{R}^d$. Denote by σ the inverse mapping from Ω' onto G'. Then, for $\mathcal{A} \in \mathcal{P}$ supported in $G' \times \mathbb{T}$,

$$J_k^\varepsilon(\mathcal{A})\,(t,x) \;=\; \frac{1}{\Delta_k(\sigma(t,x))}\ (\mathcal{H}^m\mathcal{A})\,(\sigma(t,x), \varphi(t,x)/\varepsilon)\,, \tag{4.3.5}$$

where $\varphi(t,x) = \psi(\sigma(t,x))$ and m is the value of $m_k(t,y)$ on G'. Assumption 2.2.1 implies that $d\varphi(t,x) \neq 0$ for all $(t,x) \in \Omega'$. Therefore, there exists a subsequence ε' and $\mathbf{U} \in L^q(\Omega' \times \mathbb{T})$ such that for all $\mathbf{B} \in C_0^\infty(\Omega' \times \mathbb{T})$,

$$\begin{aligned} &\int_{\Omega'} \mathbf{B}(t,x,\varphi(t,x)/\varepsilon')\ \overline{u^{\varepsilon'}(t,x)}\ dt\,dx \quad \to \\ &\qquad\qquad \int \mathbf{B}(t,x,\theta)\ \overline{\mathbf{U}}(t,x,\theta)\ dt\,dx\,d\theta\,. \end{aligned} \tag{4.3.6}$$

(see [N], [A], [E], [ES], [JMR 2]). In particular, for all $\mathcal{A} \in \mathcal{P}$ supported in G', the integrals

$$\int J_k^{\varepsilon'}(\mathcal{A})\ \overline{u^{\varepsilon'}}\ dt\,dx \tag{4.3.7}$$

converge as $\varepsilon' \to 0$.

b) Since $G\backslash\mathcal{S}_k$ is the countable union of open sets U_α which satisfy the conditions in a), we see that there is a subsequence ε' such that the integrals (4.3.7) converge, for all $\mathcal{A} \in \mathcal{P}_k$. The limit defines a linear form ℓ on $\mathcal{P}_k$. Proposition 4.1.2 applied with the conjugate exponent $q' \in]1+\infty[$, implies that

$$|\,\ell(\mathcal{A})\,| \;\leq\; C\,||\,\mathcal{A}\,||_{\mathcal{L}_k^{q'}}\ \limsup_{\varepsilon\to 0}\,||\,u^\varepsilon\,||_{L^q}\,. \tag{4.3.8}$$

Since $\mathcal{P}_k$ is dense in $\mathcal{L}_k^{q'}$ this implies that ℓ extends uniquely as a continuous linear form on $\mathcal{L}_k^{q'}$. Hence, there is $\mathcal{U}_k \in \mathcal{L}_k^q$ such that

$$\ell(\mathcal{A}) \;=\; \int \mathcal{A}\,\overline{\mathcal{U}_k}\ dtdyd\theta\,. \tag{4.3.9}$$

This proves (4.3.1), and (4.3.2) follows from (4.3.8).

c) If (4.3.3) holds and $\delta > 0$, introduce the approximations (4.2.4) and use the convergence (4.3.1) for $\mathcal{A}_\delta$. The convergence (4.3.4) follows for the subsequence ε', completing the proof of the proposition.

Proposition 4.3.3. *If $1 < q < +\infty, \mathcal{U} \in \mathcal{L}^q_\ell$, and $u^\varepsilon \sim \tilde{J}^\varepsilon_\ell(\mathcal{U})$ in $L^q(\mathbb{R}^{1+d})$ then, the family u^ε is pure and converges weakly to 0. If $k = \ell$, the Λ_k-profile is equal to $\mathcal{U}$ while if $k \neq \ell$ the Λ_k-profile is equal to 0.*

Proof. **a)** Using the approximations (4.2.4), it suffices to show that for all trigonometric polynomials $\mathcal{A} \in \mathcal{P}_k$ and $\mathcal{B} \in \mathcal{P}_\ell$, one has

$$\int J^\varepsilon_k(\mathcal{A})\ \overline{J^\varepsilon_\ell(\mathcal{B})}\ dtdx \quad \to \quad \delta_{k,\ell} \int \mathcal{A}\ \overline{\mathcal{B}}\ dtdyd\theta \tag{4.3.10}$$

where $\delta_{k,\ell}$ denotes Kronecker's symbol. One can reduce further the problem to the case where $\mathcal{A}$ and $\mathcal{B}$ are supported in $G' \times \mathbb{T}$ and $G'' \times \mathbb{T}$ respectively, such that $\jmath_k$ and $\jmath_\ell$ are diffeomorphisms from G' to Ω' and from G'' to Ω'' respectively. Denote by σ' and σ'' the inverse mappings. Then

$$J^\varepsilon_k(\mathcal{A})\,(t,x) \ = \ \mathbf{A}(t,x,\varphi_k(t,x)/\varepsilon)\,, \quad J^\varepsilon_k(\mathcal{B})\,(t,x) \ = \ \mathbf{B}(t,x,\varphi_\ell(t,x)/\varepsilon)\,, \tag{4.3.11}$$

where

$$\mathbf{A}(t,x,\theta) \ := \ \frac{1}{\Delta_k(\sigma'(t,x))}\ (\mathcal{H}^m\mathcal{A})\,(\sigma'(t,x),\theta)\,, \tag{4.3.12}$$

and the definition of $\mathbf{B}$ is similar. In (4.3.11) $\varphi_k := \psi \circ \sigma'$ and $\varphi_\ell := \psi \circ \sigma''$.

b) When $k \neq \ell$, (4.3.10) follows from the convergence

$$\int_{\Omega'\cap\Omega''} \mathbf{A}(t,x,\varphi_k(t,x)/\varepsilon)\ \overline{\mathbf{B}(t,x,\varphi_\ell(t,x)/\varepsilon)}\ dt\,dx \quad \to \quad 0\,. \tag{4.3.13}$$

To prove this convergence, expand $\mathbf{A}$ and $\mathbf{B}$ in Fourier series. Since the mean values of both vanish, it is sufficient to show that for all $\alpha \in \mathbb{Z}\backslash\{0\}$, $\beta \in \mathbb{Z}\backslash\{0\}$ and $a \in C^\infty_0(\Omega' \cap \Omega'')$,

$$\int_{\Omega'\cap\Omega''} a(t,x)\ e^{i\ (\ \alpha\,\varphi_k(t,x) + \beta\,\varphi_\ell(t,x)\,)/\varepsilon)}\ dt\,dx \quad \to \quad 0\,. \tag{4.3.14}$$

This convergence follows from the nonresonance Assumption 2.2.5, which implies that $\alpha d\varphi_k + \beta d\varphi_\ell \neq 0$ almost everywhere on $\Omega' \cap \Omega''$.

c) When $k = \ell$, one can assume that $\Omega' = \Omega''$ and $G' = G''$. Expanding $\mathbf{A}$ $\mathbf{B}$ in Fourier series, and using the property that $d\varphi_k \neq 0$ on Ω', one obtains that

$$\begin{aligned}\int_{\Omega'} \mathbf{A}(t,x,\varphi_k(t,x)/\varepsilon)\ \overline{\mathbf{B}(t,x,\varphi_k(t,x)/\varepsilon)}\ dt\,dx \quad &\to \\ \int_{\Omega'\times\mathbb{T}} \mathbf{A}(t,x,\theta)\ \overline{\mathbf{B}(t,x,\theta)}\ dt\,dx\,d\theta\,.&\end{aligned} \tag{4.3.15}$$

Using (4.3.12) and the fact that Δ_k^2 is the Jacobian of $\jmath_k$, we see that the right hand side is equal to

$$\int_{G'\times\mathbb{T}} (\mathcal{H}^m\mathcal{A})\,(t,y,\theta)\ \overline{(\mathcal{H}^m\mathcal{B})\,(t,y,\theta)}\ dt\,dy\,d\theta\,. \tag{4.3.15}$$

Since $\mathcal{H}$ is unitary, this integral is equal to the right hand side of (4.3.10) and the proof of Proposition 4.3.2 is complete.

Proposition 4.3.4. *Suppose that u^ε is is a pure bounded family in L^q, $1 < q \le 2$, with profiles $\mathcal{U}_k$. Then, for all $\mathcal{A} \in \mathcal{P}_k(q')$,*

$$(4.3.16) \qquad \int I^\varepsilon_k(\mathcal{A})\, \overline{u^\varepsilon}\; dt\, dx \quad \to \quad \int \mathcal{A}\, \overline{\mathcal{U}_k}\; dt\, dy\, d\theta\,.$$

This is a consequence of Proposition 4.3.1 and the equivalence $I^\varepsilon(\mathcal{A}) \sim J^\varepsilon(\mathcal{A})$ in $L^{q'}$ given by Proposition 4.1.3, part *ii)*.

5. The nonlinear interaction operators

In this section we define and study the nonlinear operators $\underline{\mathcal{E}}$ and $\mathcal{E}_k$ which appear in the profile equations (2.5.17). They involve the nonlinear term $f(t,x,u)$ which satisfies Assumption 2.1.2.

5.1. The interaction operators for profiles

Consider $\underline{u} \in L^{p+1}([0,+\infty[\times\mathbb{R}^d)$ and for each eigenvalue λ_k, a profile $\mathcal{U}_k(t,y,\theta)$ on $G \times \mathbb{T}$ belonging to the weighted space $\mathcal{L}^{p+1}_k$. Suppose that

$$(5.1.1) \qquad u^\varepsilon \;\sim\; \underline{u} \;+\; \sum_k \tilde{J}^\varepsilon_k(\mathcal{U}_k) \quad \text{in } L^{p+1}\,.$$

Our goal is to analyse $f(t,x,u^\varepsilon)$.

Suppose that Ω is a regular open set in the sense of Definition 2.2.3. Then $\jmath_k$ is a diffeomorphism from open sets $G_{k,j}$ onto Ω. We denote by $\sigma_{k,j}$ the inverse mappings, and by $m_{k,j}$ the value of $m_k(t,y)$ on $G_{k,j}$. Let Z denote the relevant set of indices (k,j). The formula (2.4.7) introduces profiles $\mathbf{U}_{k,j}$ on $\Omega \times \mathbb{T}$ and, formally, one has on Ω,

$$(5.1.2) \qquad u^\varepsilon(t,x) \;\sim\; \underline{u}(t,x) \;+\; \sum_{(k,j)\in Z} \mathbf{U}_{k,j}(t,x,\varphi_{k,j}/\varepsilon)\,,$$

with $\varphi_{k,j} := \psi \circ \sigma_{k,j}$. Introduce on $\Omega \times (\mathbb{T})^Z$, the function

$$(5.1.3) \qquad \mathbf{I}(\underline{u},\mathcal{U}_*)\,(t,x,\theta_*) \;=\; \underline{u}(t,x) \;+\; \sum_{(k,j)\in Z} \mathbf{U}_{k,j}(t,x,\theta_{k,j})\,,$$

where $\theta_* := \{\theta_{k,j}\}$ and $\mathcal{U}_* := \{\mathcal{U}_k\}$. Then, formally

$$(5.1.4) \qquad f(t,x,u^\varepsilon(t,x)) \;\sim\; f\big(t,x,\mathbf{I}(\underline{u},\mathcal{U}_*)(t,x,\varphi_*(t,x)/\varepsilon)\big)\,,$$

As in [JMR 3], we analyze the oscillations of $f(u^\varepsilon)$ by expanding $f(\mathbf{I}(\underline{u},\mathcal{U}_*))$ in Fourier series. Introduce the average over all the angular variables.

$$(5.1.5) \qquad \underline{\mathcal{E}}(\underline{u},\mathcal{U}_*)\,(t,x) \;:=\; \int_{(\mathbb{T})^Z} f\big(t,x,\mathbf{I}(\underline{u},\mathcal{U}_*)\,(t,x,\theta_*)\big)\; d\theta_*\,,$$

Recall that the total mass of $d\theta$ is equal to 1. The function $\underline{\mathcal{E}}(\underline{u},\mathcal{A}_*)$ is defined on any regular Ω and thus outside $\widetilde{\mathcal{C}} = \bigcup \widetilde{\mathcal{C}}_k$. In particular, it is defined almost everywhere on $[0,+\infty[\times\mathbb{R}^d$ as noted in Proposition 2.2.4.

Consider next one index k, and a component $G_{k,j}$ of $\jmath_k^{-1}(\Omega)$. Introduce Z' the complement of $\{k,j\}$ in Z. For $(t,y)\in G_{k,j}$ introduce

$$\mathcal{F}^{(1)}_{k,j}(\underline{u},\mathcal{U}_*)\,(t,y,\theta) \;:=\; \int_{(\mathbb{T})^{Z'}} f\big(\jmath_k(t,y),\mathbf{I}(\underline{u},\mathcal{U}_*)\,(\jmath_k(t,y),\theta_*)\big)d\theta'_*\,. \tag{5.1.6}$$

where θ_* is identified with (θ,θ'_*), with θ in the (k,j)-th position. The average in θ of $\mathcal{F}^{(1)}_{k,j}(\underline{u},\mathcal{U}_*)(t,y,\theta)$ is $\underline{\mathcal{E}}(\underline{u},\mathcal{U}_*)(\jmath_k(t,y))$ and thus its oscillating part is

$$\mathcal{F}_{k,j}(\underline{u},\mathcal{U}_*)\,(t,y,\theta) \;:= \mathcal{F}^{(1)}_{k,j}(\underline{u},\mathcal{U}_*)\,(t,y,\theta) \;-\; \underline{\mathcal{E}}(\underline{u},\mathcal{U}_*)\,(\jmath_k(t,y))\,. \tag{5.1.7}$$

Finally, lift $\mathcal{F}_{k,j}(\underline{u},\mathcal{U}_*)$ to $G_{k,j}$, introducing

$$\mathcal{E}_k(\underline{u},\mathcal{U}_*)\,(t,y,\theta) \;:=\; \Delta_k(t,y)\;(\mathcal{H}^{-m_{k,j}}\mathcal{F}_{k,j})\,(t,y,\theta)\,. \tag{5.1.8}$$

This defines $\mathcal{E}_k(\underline{u},\mathcal{U}_*)$ on $G_{k,j}$, thus on the disjoint union $\bigcup_j G_{k,j} = \jmath_k^{-1}(\Omega)$ and therefore outside $\jmath_k^{-1}(\widetilde{\mathcal{C}}_k)$, in particular almost everywhere on $G\times\mathbb{T}$. In the same way , define $\mathcal{F}_k$ and $\mathcal{F}^{(1)}_k$ from their restrictions $\mathcal{F}_{k,j}$ and $\mathcal{F}^{(1)}_{k,j}$

Proposition 5.1.1. *i) If $\underline{u}\in C^\infty_0([0,+\infty[\times\mathbb{R}^d)$ and $\mathcal{U}_k\in\mathcal{P}_k$, then $\underline{\mathcal{E}}(\underline{u},\mathcal{U}_*)\in C^1_0([0,+\infty[\times\mathbb{R}^d)$ and $\mathcal{E}_k(\underline{u},\mathcal{U}_*)\in C^1_0(G\times\mathbb{T})$.*

ii) The mappings $\underline{\mathcal{E}}$ and $\mathcal{E}_* := \{\mathcal{E}_k\}$ are bounded from $L^{p+1}([0,+\infty[\times\mathbb{R}^d)\;\times\;\mathcal{L}^{p+1}_*$ to $L^{1+1/p}([0,+\infty[\times\mathbb{R}^d)$ and $\mathcal{L}^{1+1/p}_*$ respectively, where $\mathcal{L}^q_*$ denotes the product space $\prod_k\mathcal{L}^q_k$. They are Lipschitzean on bounded sets of $L^{p+1}([0,+\infty[\times\mathbb{R}^d)\;\times\;\mathcal{L}^{p+1}_*$

iii) For each k, the support of $\mathcal{E}_k(\underline{u},\mathcal{U}_k)$ is contained in the support of $\mathcal{U}_k$.

Proof. It is analogous to the proof of Proposition 4.1.2.

a) Suppose that $\underline{u}\in C^\infty_0([0,+\infty[\times\mathbb{R}^d)$, and $\mathcal{U}_k\in\mathcal{P}_k$. Since $\mathcal{U}_k$ vanishes on a neighborhood of $\mathcal{S}_k$, all $(\underline{t},\underline{x})$ has a neighborhood Ω' such that $\operatorname{supp}\mathcal{U}_k\cap\jmath_k^{-1}(\Omega')$ is the disjoint union of $G'_{k,j}\subset G$ such that $\jmath_k$ is a diffeomorphism from $G'_{k,j}$ onto Ω'. Thus, only the corresponding variables intervene in the definition of $\mathbf{I}(\underline{u},\mathcal{U}_*)$ implying that $\underline{\mathcal{E}}(\underline{u},\mathcal{U}_*)$ is C^1 with compact support.

A similar argument shows that $\mathcal{E}_k(\underline{u},\mathcal{U}_*)$ belongs to C^1 and vanishes outside a compact, which achieves the proof of $i)$.

b) For $a\in C^\infty_0(G)$ which vanishes on a neighborhood of $\mathcal{S}_k$, introduce

$$(S_k a)\,(t,x) \;:=\; \sum_{\{y\,;\,q_k(t,y)=x\}} \frac{1}{\Delta_k(t,y)^2}\;a(t,y)\,. \tag{5.1.9}$$

Then, by linearity,

$$\int (S_k a)\,(t,x)\ dt\,dx \ = \ \int a(t,y)\ dt\,dy\,, \tag{5.1.10}$$

since this is true when a is supported in $G' \subset G$ which is diffeomorphic to $\jmath_k(G')$. By density, this extends to $a \in L^1(G)$.

Conversely, consider a nonnegative function $g \in C_0^\infty([0,+\infty[\times\mathbb{R}^d)$. The finiteness Assumption 2.2.2 implies that

$$S_k\big(\Delta_k^2\ g\circ \jmath_k\big) \ \le \ \ m_*\,g\,. \tag{5.1.11}$$

Thus

$$\int\ \Delta_k(t,y)^2\ g(\jmath_k(t,y))\ dt\,dy \ \le\ m_*\ \int g(t,x)\ dt\,dx\,. \tag{5.1.12}$$

By density, this extends to $g \in L^1([0,+\infty[\times\mathbb{R}^d)$.

c) Consider $\underline{u} \in L^{p+1}$ and $\mathcal{A}_* \in \mathcal{L}_*^{p+1}$. Introduce

$$a_k(t,y) \ := \ |\Delta_k(t,y)|^{1-p}\ \int_{\mathbb{T}}\ |\mathcal{U}_k(t,y,\theta)|^{p+1}\ d\theta\,. \tag{5.1.13}$$

Thus

$$\int_G a_k(t,y)\ dt\,dy \ := \ ||\,\mathcal{U}_k\,||^{p+1}_{\mathcal{L}_k^{p+1}}\,. \tag{5.1.14}$$

Consider $(t,x) \notin \mathcal{C}$ and Ω a regular neighborhood of (t,x). Introduce $\mathbf{I}(\underline{u},\mathcal{A}_*)$ as in (5.1.3). Assumption 2.1.2 implies that

$$|\, f(t,x,\mathbf{I}(\underline{u},\mathcal{U}_*)(t,x,\theta_*))\,|^{1+1/p} \ \le\ C\ |\,\mathbf{I}(\underline{u},\mathcal{U}_*)(t,x,\theta_*))\,|^{p+1}\,. \tag{5.1.15}$$

Integrating in θ, and using the finiteness Assumption 2.2.2, shows that there is a constant C' such that

$$|\,\underline{\mathcal{E}}(\underline{u},\mathcal{U}_*)(t,x)\,|^{1+1/p} \ \le\ C'\ \Big(\ |\,\underline{u}(t,x)\,|^{p+1}\ +\ \sum_Z \int |\,\mathbf{U}_{k,j}(t,x,\theta_{k,j})\,|^{p+1}\ d\theta_{k,j}\ \Big)\,. \tag{5.1.16}$$

Using the boundeness of the Hilbert transform in $L^{p+1}(\mathbb{T})$ and the notations (5.1.9) (5.1.13), (5.1.16), yields

$$|\,\underline{\mathcal{E}}(\underline{u},\mathcal{U}_*)(t,x)\,|^{1+1/p} \ \le\ C'\ \big(|\underline{u}(t,x)|^{p+1}\ +\ \sum_k (S_k a_k)(t,x)\big) \ =:\ C' g(t,x)\,. \tag{5.1.17}$$

This estimate holds for $(t,x) \notin \mathcal{C}_k$, and therefore almost everywhere. Integrating and using (5.1.10) (5.1.12), shows that

$$\|\,\underline{\mathcal{E}}(\underline{u},\mathcal{U}_*)\,\|^{1+1/p}_{L^{1+1/p}} \;\le\; C' \,\big(\, \|\,\underline{u}\,\|^{p+1}_{L^{p+1}} \;+\; \sum_k \;\|\,\mathcal{U}_k\,\|^{p+1}_{\mathcal{L}^{p+1}_k}\,\big)\,. \tag{5.1.18}$$

d) Similarly, consider $(t,y) \in G$ such that $(t,x) := \jmath_k(t,y) \notin \mathcal{C}_k$. Consider Ω a regular neighborhood of (t,x) and introduce $\mathcal{F}_k$ by (5.1.7). Using (5.1.15), one obtains

$$\begin{aligned}\int |\,\mathcal{F}_k(t,y,\theta)\,|^{1+1/p}\;d\theta \;&\le\; 2\,C \;\int |\,\mathbf{I}(\underline{u},\mathcal{U}_*)\,(\jmath_k(t,y),\theta_*)\,|^{p+1}\;d\theta_* \\ &\le\; C''g(\jmath_k(t,y))\,.\end{aligned} \tag{5.1.19}$$

Since $\mathcal{H}$ is bounded in $L^{1+1/p}(\mathbb{T})$, the definition (5.1.8) of $\mathcal{E}_k(\underline{u},\mathcal{U}_*)$ shows that

$$\begin{aligned}b_k(t,y) \;:=\; |\,\Delta_k(t,y)\,|^{1-1/p}\int |\,\mathcal{E}_k(\underline{u},\mathcal{U}_*)(t,y,\theta)\,|^{1+1/p}\;d\theta \\ \le C''\;\Delta_k(t,y)^2 g(\jmath_k(t,y))\,.\end{aligned} \tag{5.1.20}$$

This is true for all $(t,y) \notin \jmath_k^{-1}(\mathcal{C}_k)$ hence almost everywhere on G. Integrating over G and using (5.1.12), this proves that

$$\|\,\mathcal{E}_k(\underline{u},\mathcal{U}_*)\,\|^{1+1/p}_{\mathcal{L}^{1+1/p}_k} \;\le\; C' \,\big(\, \|\,\underline{u}\,\|^{p+1}_{L^{p+1}} \;+\; \sum_k \;\|\,\mathcal{U}_k\,\|^{p+1}_{\mathcal{L}^{p+1}_k}\,\big)\,. \tag{5.1.21}$$

e) Similarly, using the estimate on the derivatives of f, one shows that

$$\begin{aligned}\|\,\underline{\mathcal{E}}(\underline{u},\mathcal{U}_*) \;-\; \underline{\mathcal{E}}(\underline{v},\mathcal{V}_*)\,\|_{L^{p+1}} \;+\; \sum_k \|\,\mathcal{E}_k(\underline{u},\mathcal{U}_*) \;-\; \mathcal{E}_k(\underline{v},\mathcal{V}_*)\,\|_{\mathcal{L}^{1+1/p}_k} \\ \le\; C\;K\,\big(\,\|\,\underline{u} \;-\; \underline{v}\,\|_{L^{p+1}} \;+\; \sum_k \;\|\,\mathcal{U}_k \;-\; \mathcal{V}_k\,\|_{\mathcal{L}^{p+1}_k}\,\big)\end{aligned} \tag{5.1.22}$$

where

$$K \;:= \|\,\underline{u}\,\|^{p-1}_{L^{p+1}} \;+\; \|\,\underline{v}\,\|^{p-1}_{L^{p+1}} \;+\; \sum_k \;\|\,\mathcal{U}_k\,\|^{p-1}_{\mathcal{L}^{p+1}_k} \;+\; \|\,\mathcal{V}_k\,\|^{p-1}_{\mathcal{L}^{p+1}_k} \tag{5.1.23}$$

f) To prove *iii)* we show that if $\mathcal{U}_k$ vanishes on $G' \subset G$, then $\mathcal{E}_k(\underline{u},\mathcal{E}_*) = 0$ a.e. on G'. Using Proposition 2.2.4, it is sufficient to show that for all regular open set Ω, $\mathcal{E}_k(\underline{u},\mathcal{E}_*) = 0$ a.e. on $G'_{k,j} := G' \cap \jmath_k^{-1}(\Omega)$. This latter set is contained in one component $G_{k,j}$. Since $\mathcal{U}_k = 0$ on G', $\mathbf{I}(\underline{u},\mathcal{U}_*)$ and $f(t,x,\mathbf{I}(t,x,\theta_*)$ do not depend on the variable $\theta_{k,j}$ for $(t,x) \in \jmath_k(G'_{k,j})$. Formulas (5.1.7) (5.1.8) imply that $\mathcal{E}_k(\underline{u},\mathcal{U}_*)$ vanishes on $G'_{k,j}$. The proof of Proposition 5.1.1 is complete.

5.2. Profiles of nonlinear functions of oscillations

DEFINITION 5.2.1 *A bounded family h^ε in $L^{1+1/p}([0,+\infty[\times\mathbb{R}^d)$ has no propagated oscillations for L if, for all bounded families w^ε in $C^0([0,+\infty;L^2(\mathbb{R}^d))\cap L^{p+1}([0,+\infty[\times\mathbb{R}^d)$ such that Lw^ε is bounded in $L^{1+1/p}([0,+\infty[\times\mathbb{R}^d)$,*

$$\int h^\varepsilon\cdot\overline{w^\varepsilon}\,dtdx\quad\to\quad 0\,,\quad \text{as}\quad \varepsilon\to 0\,. \tag{5.2.1}$$

The property above is stronger than the fact that $L^{-1}(h^\varepsilon)$ converges to zero in the sense of distributions. Note that if $h^\varepsilon\sim g^\varepsilon$ in $L^{1+1/p}$ and h^ε has no propagated oscillations, then neither does g^ε. Thus the definition extends to asymptotic families, that is to elements of $AL^{1+1/p}$ (with notations of §4.2).

PROPOSITION 5.2.2. *Suppose that $\underline{u}\in L^{p+1}([0,+\infty[\times\mathbb{R}^d)$, $\mathcal{U}_k\in\mathcal{L}_k^{p+1}$ and*

$$u^\varepsilon\ \sim\ \underline{u}\ +\ \sum_k\tilde{J}_k^\varepsilon(\mathcal{U}_k)\quad in\quad L^{p+1}([0,+\infty[\times\mathbb{R}^d)\,. \tag{5.2.2}$$

Then $f(u^\varepsilon)$ is a bounded family in $L^{1+1/p}([0,+\infty[\times\mathbb{R}^d)$ which is pure in the sense of Definition 4.3.1. Its weak limit is $\underline{\mathcal{E}}(\underline{u},\mathcal{U}_*)$ and its Λ_k-profiles are $\mathcal{E}_k(\underline{u},\mathcal{U}_*)$).

Moreover,

$$g^\varepsilon\ \sim\ f(u^\varepsilon)\ -\ \underline{\mathcal{E}}(\underline{u},\mathcal{U}_*)\ -\ \sum_k\tilde{J}_k^\varepsilon(\mathcal{E}_k(\underline{u},\mathcal{U}_*))\quad in\quad L^{1+1/p}([0,+\infty[\times\mathbb{R}^d) \tag{5.2.3}$$

has no propagated oscillations for L.

Proof. **a)** Assumption 2.1.2 implies that $f(u^\varepsilon)$ is bounded in $L^{1+1/p}$. It also implies that

$$||\,f(u)-f(v)\,||_{L^{1+1/p}}\ \le C\ (\,||\,u\,||_{L^{1+p}}\ +\ ||\,v\,||_{L^{1+p}}\,)^{p-1}\ ||\,u-v\,||_{L^{1+p}}\,.$$

Therefore, changing u^ε by a small term in L^{1+p} induces a small change of $f(u^\varepsilon)$ in $L^{1+1/p}$. By Propositions 5.1.1 and 4.2.2, (5.2.3) defines g^ε as an equivalence class of bounded families in $L^{1+1/p}$. Thus it is sufficient to consider the case where $\underline{u}\in C_0^\infty([0,+\infty[\times\mathbb{R}^d)$ and $\mathcal{U}_k\in\mathcal{P}_k$ and

$$u^\varepsilon=\underline{u}+\sum_k J_k^\varepsilon(\mathcal{U}_k)\,,\quad g^\varepsilon=f(u^\varepsilon)-\underline{\mathcal{E}}(\underline{u},\mathcal{U}_*)-\sum_k J_k^\varepsilon(\mathcal{E}_k(\underline{u},\mathcal{U}_*))\,. \tag{5.2.4}$$

Note that Proposition 5.1.1 implies that $\mathcal{E}_k(\underline{u},\mathcal{U}_*)$ belongs to C^1 and vanishes on a neighborhood of $\mathcal{S}_k$. Therefore the right hand side of (5.2.4) is well defined for each ε. One can further assume that the $\mathcal{U}_k$ are supported in $\jmath_k^{-1}(\Omega)$ where Ω is a regular open set Ω. In this case

$$(5.2.5)\qquad u^\varepsilon(t,x) \ = \ \underline{u}(t,x) \ + \ \sum_Z \ \mathbf{U}_{k,j}(t,x,\varphi_{k,j}/\varepsilon))\,,$$

where the $\mathbf{U}_{k,j} \in C_0^\infty(\Omega\times\mathbb{T})$ are defined in (2.4.7) and $\varphi_{k,j}$ are the local phases on Ω, introduced in (2.2.9). Then

$$(5.2.6)\qquad f(u^\varepsilon) \ = \ \mathbf{F}(t,x,\varphi_*/\varepsilon)\,,$$

where

$$(5.2.7)\qquad \mathbf{F}(t,x,\theta_*) \ := \ f\big(\,\underline{u}(t,x) + \sum_Z \mathbf{U}_{k,j}(t,x,\theta_{k,j})\,\big) \ \in C_0^\infty(\Omega\times\mathbb{T}^Z)\,,$$

Consider the rapidly convergent Fourier expansion

$$(5.2.8)\qquad \mathbf{F}(t,x,\theta_*) \ = \ \sum_{\alpha_*\in(\mathbb{Z})^Z} \ c_{\alpha_*}(t,x)\ e^{i\,\alpha_*\cdot\theta_*}$$

Then, the definition of $\underline{\mathcal{E}}$ and $\mathcal{E}_k$ is made so that

$$(5.2.9)\qquad g^\varepsilon \ = \sum_{\alpha_*\in R} \ c_{\alpha_*}(t,x)\ e^{i\,\alpha_*\cdot\theta_*} \quad ,$$

with R the set of those $\alpha_*\in\mathbb{Z}^Z$ such that at least two components $\alpha_{k,j}$ do not vanish.

b) To compute the profiles, it suffices to show that, for all $\alpha_*\in R$ and $c\in C_0^\infty(\Omega)$, the sequence

$$(5.2.10)\qquad \ell^\varepsilon(t,x) \ := \ c(t,x)\ e^{i\ \alpha_*\,\varphi_*(t,x)/\varepsilon}$$

converges weakly to 0, and that all its profiles vanish.

Assumption 2.2.5 implies that the phase $\varphi := \alpha_*\varphi_*$ satisfies $d\varphi\neq 0$ almost everywhere. Therefore ℓ^ε converges weakly to 0.

Consider $\mathcal{A}\in\mathcal{P}_k$. On Ω, $J_k^\varepsilon(\mathcal{A})$ is of the form

$$(5.2.11)\qquad J_k^\varepsilon(\mathcal{A})(t,x) = \sum_j \mathbf{A}_j(t,x,\varphi_{k,j}(t,x)/\varepsilon) \ = \sum_{j,n} a_{j,n}(t,x)\, e^{i\ n\ \varphi_{k,j}(t,x)/\varepsilon}\,,$$

where n runs over a finite subset of $\mathbb{Z}$. The phase $\alpha_*\varphi_* + n\varphi_{k,j}$ is either of the form $\gamma\varphi_{k',j'}$ with $\gamma\neq 0$, or of the form $\alpha'_*\varphi_*$ with $\alpha'_*\in R$. In the first case, the differential of the phase does not vanish by Assumption 2.2.1. In the second case, it does not vanish almost everywhere by Assumption 2.2.5. This implies that

$$\int \ell^\varepsilon(t,x)\ J_k^\varepsilon(\mathcal{A})(t,x)\ dtdx\ \to\ 0\,, \tag{5.2.12}$$

showing that ℓ^ε is k-pure with Λ_k-profile equal to 0.

c) It remains to show that ℓ^ε defined in (5.2.10) has no propagated oscillations for L. Assumption 2.2.5 implies that $\det L(d\varphi) \neq 0$ almost everywhere on Ω. Thus the Lebesgue measure of the set of points (t,x) where $|\det L(d\varphi(t,x))| \le \delta\}$ tends to 0 as δ tends to 0, implying that the $L^{1+1/p}$ norm of ℓ^ε on this set converges to 0 as $\delta \to 0$, uniformly with respect to ε. Therefore, it is sufficient to prove the convergence (5.2.1) for $\ell^\varepsilon = c\, e^{i\varphi/\varepsilon}$, when c is compactly supported in the open set where $\det L(d\varphi) \neq 0$.

In this case,

$$b\ :=\ L(i\, d\varphi)^{-1} c\ \in\ C_0^\infty(\Omega) \tag{5.2.13}$$

and

$$c\, e^{i\varphi/\varepsilon}\ =\ \varepsilon\ L(b\, e^{i\,\varphi/\varepsilon})\ -\ \varepsilon\ (Lb)\ e^{i\varphi/\varepsilon}\,. \tag{5.2.14}$$

Suppose that w^ε is a bounded family in L^{1+p} such that Lw^ε is bounded in $L^{1+1/p}$. Then using (5.2.13) and (5.2.14), integration by parts show that

$$\int c\ e^{i\varphi/\varepsilon} \cdot \overline{w^\varepsilon}\ dtdx\ =\ O(\varepsilon)\,. \tag{5.2.15}$$

This finishes the proof for ℓ^ε.

Recall that the spectral projector P_k is defined in Proposition 2.5.2.

Proposition 5.2.3. *If $\mathcal{A} \in \mathcal{L}_k^{1+1/p}$ and $P_k\mathcal{A} = 0$, then $\tilde J_k^\varepsilon(\mathcal{A})$ has no propagated oscillations for L.*

Proof. Using the remark following Definition 5.2.1 and Proposition 4.2.2, it suffices to prove the result when $\mathcal{A} \in \mathcal{P}_k$ and is a monomial : $\mathcal{A}(t,y,\theta) = a(t,y)\ e^{i\,n\,\theta}$. Then $J_k^\varepsilon(\mathcal{A})$ is a sum of terms of the form

$$c_{k,j}(t,x)\ e^{i\,n\,\varphi_{k,j}/\varepsilon}\,, \tag{5.2.16}$$

where $c_{k,j}(t,x) := \Delta_k^{-1}(t,\rho_{k,j}(t,x))\, a((t,\rho_{k,j}(t,x))$. Let $P_{k,j}(t,x) := P_k(\rho_{k,j}(t,x))$ denote the orthogonal projector on the kernel of $L(d\varphi_{k,j})$. It remains to show that (5.2.1) is satisfied when $h^\varepsilon = c\, e^{i\,\varphi/\varepsilon}$ with $\varphi := \varphi_{k,j}$ and $c \in C_0^\infty(\Omega)$ satisfies

$P_{k,j}c = 0$. In such case, there exists $b \in C_0^\infty(\Omega)$ such that $L(id\varphi)b = c$. Then, (5.2.14) holds, implying (5.2.15), and the proof of Proposition 5.2.3 is complete.

COROLLARY 5.2.4. *For all $\mathcal{F}_k \in \mathcal{L}_k^{1+1/p}$, $\tilde{J}_k^\varepsilon((I - P_k)\mathcal{F}_k)$ has no propagated oscillations for L.*

6. PROOF OF ASYMPTOTICS

In this section we prove Theorems 2.5.1 and 2.5.4 and show that the profile equations (2.5.16) are satisfied. The strategy is the following. We extract from both families u^ε and $f^\varepsilon := f(u^\varepsilon)$ pure subsequences, in the sense of §4. Equations are derived for the weak limits $\underline{u}$ and $\underline{f}$ and the profiles $\mathcal{U}_k$ and $\mathcal{F}_k$. Next, we introduce an appropriate $v^\varepsilon \sim \underline{u} + \sum \tilde{J}_k^\varepsilon(\mathcal{U}_k)$, such that $Lv^\varepsilon \sim \underline{f} + \sum \tilde{J}_k^\varepsilon(P_k\mathcal{F}_k)$. Using the dissipativity of the equation, we prove that the subsequence $u^\varepsilon - v^\varepsilon$ converges strongly to 0. This property is used to identify the weak limits $\underline{f}$ and the profiles $\mathcal{F}_k$ as functions of $\underline{u}$ and $\mathcal{U}_k$ and to deduce that $\underline{u}$ and $\mathcal{U}_k$ satisfy the profile equations. Finally, we show that the solution of the profile equation is unique, implying that no extraction of subsequences was necessary and that $u^\varepsilon \sim \underline{u} + \sum \tilde{J}_k^\varepsilon(\mathcal{U}_k)$ as claimed .

6.1. Weak convergence

We consider the family of solutions u^ε of (2.1.1) with Cauchy data (2.5.1). Then, u^ε is bounded in $C^0([0, +\infty[\,; L^2(\mathbb{R}^d))$ and in $L^{p+1}([0, +\infty[\times\mathbb{R}^d)$. This implies that $f^\varepsilon := f(t, x, u^\varepsilon)$ is bounded in $L^{1+1/p}([0, +\infty[\times\mathbb{R}^d)$ By Proposition 4.3.2 we can extract pure subsequences which, for simplicity, we still note u^ε and f^ε.

Recall that $P_k(y)$ denotes the orthogonal projetor on the kernel of $\lambda_k(\xi)Id - \sum \xi_j A_j$ for $\xi = d\psi(y)$ and $\mathcal{S}_k^+(p + 1)$ is the set of supercritical points introduced in Definition 2.3.4.

PROPOSITION 6.1.1. *The Λ_k-profiles $\mathcal{U}_k$ of $\{u^\varepsilon\}$, $\mathcal{F}_k$ of $\{f^\varepsilon\}$ satisfy $\mathcal{U}_k \in \mathcal{L}_k^{p+1}$, $\mathcal{F}_k \in \mathcal{L}_k^{1+1/p}$ and the polarization condition*

$$P_k\,\mathcal{U}_k = \mathcal{U}_k\,. \tag{6.1.1}$$

Moreover, they satisfy on $(G\backslash\mathcal{S}_k^+(p + 1)) \times \mathbb{T}$

$$\partial_t\mathcal{U}_k \;+\; P_k\,\mathcal{F}_k \;=\; 0\,, \quad \mathcal{U}_{k|t=0} \;=\; P_k\,\mathcal{U}_0\,. \tag{6.1.2}$$

The weak limits satisfy $\underline{u} \in L^{p+1}([0, +\infty[\times\mathbb{R}^d)$, $\underline{f} \in L^{1+1/p}([0, +\infty[\times\mathbb{R}^d)$ and

$$L\underline{u} \;+\; \underline{f} \;=\; 0 \quad \text{on } [0, +\infty[\times\mathbb{R}^d\,, \quad \underline{u}_{|t=0} \;=\; \underline{u}_0\,. \tag{6.1.3}$$

Proof. **a)** Let $\mathcal{A}$ be a trigonometric polynomial. For ε fixed, $I_k^\varepsilon(\mathcal{A})$ is rapidly decreasing as $|x| \to \infty$ (see the parts b) and c) in the proof of Theorem 3.5.5) and is compactly supported in t. Let $a_0^\varepsilon := \mathcal{A}(0, y, \psi(y)/\varepsilon)$. Then, since L is skew adjoint, one has

$$(6.1.4) \qquad (Lu^\varepsilon, I_k^\varepsilon(\mathcal{A})) \;=\; -\,(u^\varepsilon, LI_k^\varepsilon(\mathcal{A})) \;-\; (u_0^\varepsilon, a_0^\varepsilon)\,,$$

where (,) denotes scalar product in L^2.

b) One has (see e.g. [Hö 2], Theorem 7.7.7)

$$(6.1.5) \quad L(I_k^\varepsilon(\mathcal{A})) \;\sim\; \frac{1}{\varepsilon}\, I_k^\varepsilon(\mathcal{B}_{-1}) \;+\; I_k^\varepsilon(\mathcal{B}_0) \;+\; O(\varepsilon) \quad \text{in } L^2 \cap L^\infty([0, +\infty[\times \mathbb{R}^d)\,,$$

with

$$(6.1.6) \qquad \mathcal{B}_{-1} \;:=\; i\,\big(\lambda_k(d\psi)\, Id \;-\; A(d\psi)\big)\mathcal{A}\,, \quad \mathcal{B}_0 \;:=\; \mathcal{D}_k\, \mathcal{A}\,,$$

$$(6.1.7) \qquad \mathcal{D}_k \;:=\; \frac{\partial}{\partial t} \;+\; \sum_j \;\Big(\frac{\partial \lambda_k}{\partial \xi_j}(d\psi)\, Id - A_j\Big)\, \frac{\partial}{\partial y_j} \;+\; \gamma\,.$$

$$(6.1.8) \qquad \gamma \;:=\; \frac{1}{2}\, \sum_l \sum_j \; \frac{\partial^2 \lambda_k}{\partial \xi_j\, \partial \xi_l}(d\psi)\; \frac{\partial^2 \psi}{\partial y_j\, \partial y_l}$$

Differentiating twice the identity $(\lambda_k(\xi) Id - A(\xi))\Pi_k(\xi) = 0$, where $\Pi_k(\xi)$ is the spectral projector of $A(\xi)$, and multiplying the result on the left by $\Pi_k(\xi)$, one proves the fundamental identity

$$(6.1.9) \qquad P_k\, \mathcal{D}_k\, (P_k\, \mathcal{A}) \;:=\; \partial_t\, P_k \mathcal{A}\,.$$

c) Suppose that $\mathcal{B} \in \mathcal{P}_k$ satisfies $P_k \mathcal{B} = 0$. Then there is a unique $\mathcal{A} \in \mathcal{P}_k$ such that

$$(6.1.10) \qquad \mathcal{B} \;:=\; i\,\big(\lambda_k(d\psi)\, Id \;-\; A(d\psi)\big)\mathcal{A}\,, \quad P_k \mathcal{A} = 0\,.$$

Therefore, (6.1.5) implies that $\varepsilon L I_k^\varepsilon(\mathcal{A})) \sim I_k^\varepsilon(\mathcal{B})$ in L^2. Since $\mathcal{A}$ vanishes near $\mathcal{S}_k$, $I_k^\varepsilon(\mathcal{A})$ is bounded in L^q for all q. Thus (6.1.4) implies that for all such $\mathcal{B}$,

$$(6.1.11) \qquad \int \mathcal{U}_k \cdot \overline{\mathcal{B}}\; dt dy d\theta \;=\; 0\,,$$

which proves (6.1.1).

d) For $\mathcal{A} \in \mathcal{P}_k(p+1)$ (see Definition 4.1.1) such that $P_k\mathcal{A} = \mathcal{A}$, (6.1.5) and (6.1.9) imply that $LI_k^\varepsilon(\mathcal{A}) \sim I^\varepsilon(\partial_t\mathcal{A})$ in L^2. Since u^ε is pure in L^2, this implies that

$$(u^\varepsilon,\, LI_k^\varepsilon(\mathcal{A})) \;\to\; \int \mathcal{U}_k \cdot \overline{\partial_t\mathcal{A}}\; dtdyd\theta\,. \tag{6.1.12}$$

Since the subsequence f^ε is pure in $L^{1+1/p}$, Proposition 4.3.4 implies that

$$(f^\varepsilon,\, I_k^\varepsilon(\mathcal{A})) \;\to\; \int \mathcal{F}_k \cdot \overline{\mathcal{A}}\; dtdyd\theta\,. \tag{6.1.13}$$

Finally, the Cauchy data (2.5.1) u_0^ε is pure in L^2 and we can pass to the limit in (6.1.4) to find

$$\int \mathcal{F}_k \cdot \overline{\mathcal{A}}\; dtdyd\theta \;=\; \int \mathcal{U}_k \cdot \overline{\partial_t\mathcal{A}}\; dtdyd\theta \;+\; \int \mathcal{U}_0 \cdot \overline{\mathcal{A}}_{|t=0}\; dyd\theta\,. \tag{6.1.14}$$

For $\mathcal{B} \in \mathcal{P}_k(p+1)$, one can apply (6.1.14) to $\mathcal{A} = P_k\mathcal{B}$. Using (6.1.1), we see that

$$\int P_k\,\mathcal{F}_k \cdot \overline{\mathcal{B}}\; dtdyd\theta \;=\; \int \mathcal{U}_k \cdot \overline{\partial_t\mathcal{B}}\; dtdyd\theta \;+\; \int P_k\,\mathcal{U}_0 \cdot \overline{\mathcal{B}}_{|t=0}\; dyd\theta\,. \tag{6.1.15}$$

This proves (6.1.2).

e) The equation (6.1.3) for the weak limits is immediate.

6.2. Construction of approximate solutions

The next result is fundamental in the understanding of equations (6.1.2).

PROPOSITION 6.2.1. *Suppose that $\mathcal{U} \in \mathcal{L}_k^{p+1}$, $\mathcal{F} \in \mathcal{L}_k^{1+1/p}$ satisfy $\partial_t\mathcal{U} = \mathcal{F}$ on $(G\backslash\mathcal{S}_k^+(p+1)) \times \mathbb{T}$ and $\mathcal{U}_{|t=0} \in L^2(\omega \times \mathbb{T})$. Then there exist trigonometric polynomials $\mathcal{U}_n \in \mathcal{P}_k(p+1)$ and $\mathcal{F}_n \in \mathcal{P}_k$ such that*

i) $\partial_t\mathcal{U}_n = \mathcal{F}_n$ on $G \times \mathbb{T}$

ii) $\mathcal{U}_n \to \mathcal{U}$ strongly in $\mathcal{L}_k^{p+1}$,

iii) $\mathcal{F}_n \to \mathcal{F}$ strongly in $\mathcal{L}_k^{1+1/p}$,

iv) $\mathcal{U}_{n|t=0} \to \mathcal{U}_{|t=0}$ strongly in $L^2(\omega \times (\mathbb{R}/2\pi\mathbb{Z}))$.

When $\mathcal{U}$ satisfies $P_k\mathcal{U} = \mathcal{U}$, one can choose the approximations $\mathcal{U}_n$ such that $P_k\mathcal{U}_n = \mathcal{U}_n$.

Proof. **a)** Multiplying $\mathcal{U}$ by a function $\chi(y)$ does not affect the assumptions nor the result. Since $\omega\backslash\omega_0$ has measure zero thanks to Assumption 2.3.3, a cutoff of this type yields an approximation with support in $[0, +\infty[\times\omega_0$. In addition the proof of Proposition 6.2.1 is trivial when $\mathcal{U}$ is supported in a tube $[T_1, T_2] \times \omega'$ which does not intersect $\mathcal{S}_k$. Therefore one can assume that $\mathcal{U}(t, y)$ and $\mathcal{F}(t, y)$

are supported in a tube $]a,b[\times\omega'$ where $\omega' \subset \omega_0$ is a small open ball such that $g_k'(y)$ has a unique positive eigenvalue, $1/r(y)$, of constant multiplicity μ, such that $a < a' \le r(y) \le b' < b$ for all $y \in \omega'$. Then

$$\Delta_k(t,y) \;\approx\; |t - r(y)|^{\mu/2}\,. \tag{6.2.1}$$

There are two cases. First $\mathcal{S}_k$ intersects $]a,b[\times\omega'$ at supercritical points, which corresponds to $\alpha := (p-1)\mu/2 \ge 1$, and the equation $\partial_t\mathcal{U} = \mathcal{F}$ holds oustide $\mathcal{S}_k$. Second the intersection is subcritical, that is $\alpha < 1$ and the equation holds throughout $]a,b[\times\omega'$.

Changing coordinates $s := t - r(y)$, and setting $z := (y,\theta)$ we are reduced to the situation where $u(s,z)$ and $f(s,z)$ are compactly supported functions on $V :=]-1,+1[\times V'$, such that

$$\int_V |s|^{-\alpha}\,|u|^{p+1}\,dsdz \;<+\infty\,, \qquad \int_V |s|^{\alpha/p}\,|f|^{1+1/p}\,dsdz \;<+\infty\;\;, \tag{6.2.2}$$

and

$$\partial_s u = f \quad \text{on } V\,, \quad \text{if } \alpha < 1\,, \tag{6.2.3}$$

$$\partial_s u = f \quad \text{on } V\cap\{s\neq 0\}\,, \quad \text{if } \alpha \ge 1\,. \tag{6.2.4}$$

Mollifyings in z, we can approximate u and f by trigonometric polynomials with coefficients infinitely smooth in y. It remains to smooth the functions with respect to the variable s.

b) When $\alpha < 1$, f is integrable since Hölder's inequality yields

$$\int_V |f| \;\le \Big(\int_V |s|^{\alpha/p}|f|^{1+1/p}\Big)^{p/(p+1)}\;\Big(\int_V |s|^{-\alpha}\Big)^{1/(p+1)} \;<\; +\infty\,. \tag{6.2.5}$$

Thus, u is continuous. Mollifying in s yields approximations which converge uniformly and therefore in the weighted spaces since $|s|^{-\alpha}$ is integrable.

c) Suppose $\alpha \ge 1$. Let $\chi(s)$ be equal to 1 for $|s| \ge 1$ and equal to 0 for $|s| \le 1/2$. For $\delta > 0$, let $\chi_\delta(s) := \chi(s/\delta)$, $u_\delta := \chi_\delta u$ and $f_\delta := \chi_\delta f$. Then

$$\partial_s u_\delta = f_\delta \;+ g_\delta\,, \quad \text{with } g_\delta := \frac{1}{\delta}\,\chi'(\frac{s}{\delta})\,u\,. \tag{6.2.6}$$

Thus, with $\beta := \alpha/p$,

$$\begin{aligned}\int_V |s|^\beta\,|g_\delta|^{1+1/p} \;&\le\; C\;\delta^{\beta-1-1/p}\int_{V_\delta}|u|^{1+1/p} \;\le\; C\;\delta^{\beta-1+1-2/p}\Big(\int_{V_\delta}|u|^{p+1}\Big)^{1/p}\\ &\le\; C\;\delta^{2(\alpha-1)/p}\Big(\int_{V_\delta}|s|^{-\alpha}|u|^{p+1}\Big)^{1/p}\;\;,\end{aligned} \tag{6.2.7}$$

where V_δ is the set of (s,z) in V such that $\delta/2 \le |s|\;\delta$. Since $\alpha \ge 1$, the right hand side of (6.2.7) tends to zero as δ tends to zero. This constructs approximations which vanish on a neighborhood of $s = 0$. Smoothing such functions away from $s = 0$ is easy.

d) If $\mathcal{U}$ satisfies the polarization condition $P_k\mathcal{U} = \mathcal{U}$, approximate $\mathcal{U}$ by $\mathcal{U}_n$ as above, and then replace $\mathcal{U}_n$ by $P_k\mathcal{U}_n$.

COROLLARY 6.2.2. *Suppose that $\mathcal{U}$ and $\mathcal{F}$ satisfy the assumptions of Proposition* 6.2.1. *Then $\mathcal{U} \in C^0([0,+\infty[;L^2(\omega \times \mathbb{T}))$ and*

$$(6.2.8) \qquad ||\,\mathcal{U}(T)\,||^2_{L^2(\omega\times\mathbb{T})} \;=\; ||\,\mathcal{U}(0)\,||^2_{L^2(\omega\times\mathbb{T})} \;+\; 2\,\mathrm{Re}\Big(\int_{[0,T]\times\omega\times\mathbb{T}} \mathcal{F}\,\overline{\mathcal{U}}\;dt\,dyd\theta\Big)\,.$$

Moreover, the sequence $\mathcal{U}_n$ given by Proposition 6.1.1 *converges strongly to $\mathcal{U}$ in $C^0([0,+\infty[;L^2(\omega \times \mathbb{T}))$.*

Proof. The estimate (6.2.8) is clear for smooth $\mathcal{U}$ and $\mathcal{F}$. Applying it for $\mathcal{U}_n - \mathcal{U}_{n'}$, implies that $\mathcal{U}_n$ is a Cauchy sequence in $C^0([0,+\infty[;L^2(\omega \times \mathbb{T}))$. Therefore, $\mathcal{U}$ belongs to this space and the convergence holds in $C^0([0,+\infty[;L^2(\omega\times\mathbb{T}))$. Passing to the limit this implies that $\mathcal{U}$ also satisfies (6.2.8).

PROPOSITION 6.2.3. *Suppose that*

$$u \in L^{p+1}([0,+\infty[\times\mathbb{R}^d)\ , \quad Lu \in L^{1+1/p}([0,+\infty[\times\mathbb{R}^d)\ , \quad u_{|t=0} \in L^2(\mathbb{R}^d)\,.$$

Then $u \in C^0([0,+\infty[;L^2(\mathbb{R}^d))$ and there are C^∞ functions with compact support u_n such that

i) $u_n \to u$ strongly in $L^{p+1}([0,+\infty[\times\mathbb{R}^d)$ and in $C^0([0,+\infty[;L^2(\mathbb{R}^d))$,

ii) $Lu_n \to Lu$ strongly in $L^{1+1/p}([0,+\infty[\times\mathbb{R}^d)$.

iii) $u_{n|t=0} \to u_{|t=0}$ strongly in $L^2(\mathbb{R}^d)$.

Moreover

$$(6.2.9) \qquad 2\,\mathrm{Re}\int_{[0,T]\times\mathbb{R}^d} Lu\cdot\overline{u}\;dt\,dx \;=\; ||u(T)||^2_{L^2} \;-\; ||u(0)||^2_{L^2}\,.$$

Proof. The approximations are constructed by standard mollification. The usual energy identity (6.2.9) is satisfied for smooth u_n. Taking limits, it implies that $u \in C^0([0,T];L^2(\mathbb{R}^d))$, and satisfies (6.2.9).

Our goal is to prove that $u^\varepsilon \sim \underline{u} + \sum \tilde{J}^\varepsilon_k(\mathcal{U}_k)$. Our strategy is to introduce

$$(6.2.10) \qquad v^\varepsilon \;\sim\; \underline{u} \;+\; \sum \tilde{J}^\varepsilon_k(\mathcal{U}_k) \quad \textit{in} \;\; L^{p+1}([0,T]\times\mathbb{R}^d)\,,$$

and to prove that $\|u^\varepsilon - v^\varepsilon\|_{L^{p+1}} \to 0$. To do so, we use the energy estimates and the monotonicity of the equation. In particular we need use $Lv^\varepsilon \in L^{1+1/p}$. A difficulty is that (6.2.10) defines v^ε up to a sequence which converges strongly to zero in L^{p+1}, and Lv^ε may not belong to $L^{1+1/p}$. The next proposition shows that, knowing (6.1.2) (6.1.3), one can choose a representative v^ε such that Lv^ε is bounded in $L^{1+1/p}$.

Proposition 6.2.4. *Suppose that $\underline{u}$, $\underline{f}$, $\mathcal{U}_k$ and $\mathcal{F}_k$ are as in Proposition* 6.1.1. *Then, for all $T > 0$, there exist a bounded family v^ε in $C^0([0,T];L^2(\mathbb{R}^d)) \cap L^{p+1}([0,T]\times\mathbb{R}^d)$, such that Lv^ε is bounded in $L^{1+1/p}([0,T]\times\mathbb{R}^d)$ and satisfies* (6.2.10) *and*

$$Lv^\varepsilon \sim \underline{f} + \sum \tilde{J}_k^\varepsilon(P_k\mathcal{F}_k) \quad in \quad L^{1+1/p}([0,T]\times\mathbb{R}^d)\,, \tag{6.2.11}$$

$$v^\varepsilon_{|t=0} \sim \underline{u}_0 + \mathcal{U}_0(y,\psi(y)/\varepsilon) \quad in \quad L^2(\mathbb{R}^d)\,. \tag{6.2.12}$$

Proof. **a)** Choose

$$\begin{aligned} &w_k^\varepsilon \sim \tilde{J}_k^\varepsilon(\mathcal{U}_k) \quad \text{in} \quad L^{p+1}([0,T]\times\mathbb{R}^d)\,,\\ &h_k^\varepsilon \sim \tilde{J}_k^\varepsilon(P_k\mathcal{F}_k) \quad \text{in} \quad L^{1+1/p}([0,T]\times\mathbb{R}^d)\,,\\ &w_0^\varepsilon \sim \tilde{J}_0^\varepsilon(\mathcal{U}_0) \quad \text{in} \quad L^2(\mathbb{R}^d)\,, \end{aligned}$$

We look for v^ε, as

$$v^\varepsilon := \underline{u} + \sum v_k^\varepsilon\,. \tag{6.2.13}$$

Since $\underline{u} \in L^{p+1}$, $\underline{f} \in L^{1+1/p}$ and $L\underline{u} = \underline{f}$, Proposition 6.2.3 shows that $u \in C^0([0,+\infty[;L^2(\mathbb{R}^d))$. The same proposition shows that it is sufficient to construct v_k^ε such that

$$v_k^\varepsilon \sim \tilde{J}_k^\varepsilon(\mathcal{U}_k) \quad \text{in } L^{p+1}([0,T]\times\mathbb{R}^d)\,, \tag{6.2.14}$$

$$Lv^\varepsilon \sim \tilde{J}_k^\varepsilon(P_k\mathcal{F}_k) \quad \text{in } L^{1+1/p}([0,T]\times\mathbb{R}^d)\,, \tag{6.2.15}$$

and

$$v^\varepsilon_{k|t=0} \sim P_k\mathcal{U}_0(y,\psi(y)/\varepsilon) \quad \text{in } L^2(\mathbb{R}^d)\,. \tag{6.2.16}$$

b) Fix $\delta > 0$ and $T > 0$. Propositions 6.1.1 and 6.2.1 imply that there are $\mathcal{V}_k \in \mathcal{P}_k(p+1)$ and $\mathcal{G}_k \in \mathcal{P}_k$, such that

$$||\,\mathcal{V}_k - \mathcal{U}_k\,||_{L^{1+p}_{w,k}(G\times\mathbb{T})} \le \delta\,, \quad ||\,\mathcal{V}_k|_{t=0} - P_k\mathcal{U}_0\,||_{L^2(\omega\times\mathbb{T})} \le \delta\,, \tag{6.2.17}$$

$$||\,\mathcal{G}_k - P_k\mathcal{F}_k\,||_{\mathcal{L}_k^{1+1/p}(G\times\mathbb{T})} \le \delta\,, \tag{6.2.18}$$

and

$$(6.2.19)\qquad P_k\mathcal{V}_k \;=\; \mathcal{V}_k\,, \quad \partial_t\mathcal{V}_k \;=\; \mathcal{G}_k\,.$$

The identities (6.1.5) (6.1.6) and (6.1.9) imply that

$$(6.2.20)\qquad LI_k^\varepsilon(\mathcal{V}_k) \;-\; I_k^\varepsilon(\mathcal{G}_k) \;=\; O(\varepsilon) \quad \text{in } L^2\cap L^\infty([0,T]\times\mathbb{R}^d)\,.$$

Introduce

$$(6.2.21)\qquad v_k^{\varepsilon,\delta} \;:=\; I_k^\varepsilon(\mathcal{V}_k)\,.$$

Since $\mathcal{V}_k \in \mathcal{P}_k(p+1)$, part *ii)* of Proposition 4.1.3 implies that $I_k^\varepsilon(\mathcal{V}_k) \sim J_k^\varepsilon(\mathcal{V}_k)$ in L^{1+p}. Moreover, (6.2.17) and Proposition 4.2.2 imply that $J_k^\varepsilon(\mathcal{V}_k) - \tilde J_k^\varepsilon(\mathcal{U}_k)$ is $O(\delta)$ in AL^{p+1}. Therefore, there is $\varepsilon(\delta) > 0$, such that for all $\varepsilon < \varepsilon(\delta)$,

$$(6.2.22)\qquad ||\, v_k^{\varepsilon,\delta} - w_k^\varepsilon\,||_{L^{1+p}} \;\le\; C\,\delta\,.$$

Similarly, since $\mathcal{G}_k \in \mathcal{P}_k$, $\mathcal{G}_k$ vanishes near $\mathcal{S}_k$ and $I_k^\varepsilon(\mathcal{G}_k) \sim J_k^\varepsilon(\mathcal{G}_k)$ in L^q for all $q \in [1,+\infty]$. With (6.2.20), (6.2.18) and Propostition 4.2.2, this implies that one can choose $\varepsilon(\delta,T) > 0$, such that for all $\varepsilon < \varepsilon(\delta,T)$,

$$(6.2.23)\qquad ||\, Lv_k^{\varepsilon,\delta} - h_k^\varepsilon\,||_{L^{1+1/p}} \;\le\; \delta\,.$$

Finally, (6.2.17) together with the initial condition for $\mathcal{U}_k$ in (6.1.2) show that, for $\varepsilon < \varepsilon(\delta,T)$

$$(6.2.24)\qquad ||\, {v_k^{\varepsilon,\delta}}_{|t=0} - P_k\, w_0^\varepsilon)\,||_{L^2(\mathbb{R}^d)} \;\le\; \delta\,.$$

One can assume that $\varepsilon(\delta,T)$ decreases as δ decreases.

c) There is a positive function $\delta(\varepsilon)$ such that $\varepsilon < \varepsilon(\delta(\varepsilon))$ and $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$. Therefore, (6.2.22) (6.2.23) (6.2.24) imply that $v_k^\varepsilon := v_k^{\varepsilon,\delta(\varepsilon)}$ satisfies (6.2.14) (6.2.15) and (6.2.16), and Proposition 6.2.4 is proved.

6.3. Strong convergence.

Proposition 6.3.1. *If u^ε, $\underline{u}$, $\mathcal{U}_k$, $\underline{f}$, $\mathcal{F}_k$, T and v^ε are as in Proposition 6.2.4, then $u^\varepsilon - v^\varepsilon$ converges strongly to zero in $C^0([0,T];L^2(\mathbb{R}^d))$ and in $L^{1+p}([0,T]\times\mathbb{R}^d)$. In particular,*

$$(6.3.1)\qquad u^\varepsilon \;\sim\; \underline{u} \;+\; \sum \tilde J_k^\varepsilon(\mathcal{U}_k) \quad \text{in } L^{p+1}([0,T]\times\mathbb{R}^d)\,,$$

Proof. **a)** u^ε and v^ε are bounded in L^{p+1} and Lu^ε and Lv^ε are bounded in $L^{1+1/p}$. Therefore, Proposition 6.2.3 implies that

$$\begin{aligned}\|\, u^\varepsilon(t) - v^\varepsilon(t)\, \|^2_{L^2} \;-\; 2\,\mathrm{Re}\int_{[0,t]\times\mathbb{R}^d} L(u^\varepsilon - v^\varepsilon)\cdot(\overline{v^\varepsilon} - \overline{u^\varepsilon}) \;=\; \\ \|\, u^\varepsilon(0) - v^\varepsilon(0)\, \|^2_{L^2}\,.\end{aligned} \tag{6.3.2}$$

Introduce

$$h^\varepsilon \;:=\; L(v^\varepsilon - u^\varepsilon) \;+\; f(v^\varepsilon) - f(u^\varepsilon) \;=\; Lv^\varepsilon \;+\; f(v^\varepsilon)\,. \tag{6.3.3}$$

Substituting $L(u^\varepsilon - v^\varepsilon) = -\,(f(u^\varepsilon) - f(v^\varepsilon)) + h^\varepsilon$ in (6.3.2) and using Assumption 2.1.2, we obtain that

$$\begin{aligned}\|\, u^\varepsilon(t) - v^\varepsilon(t)\, \|^2_{L^2} \;+\; c\,\|\, u^\varepsilon - v^\varepsilon\, \|^{1+p}_{L^{1+p}([0,t]\times\mathbb{R}^d)} \;\le \\ \|\, u^\varepsilon(0) - v^\varepsilon(0)\, \|^2_{L^2} \;+\; 2\,\mathrm{Re}\int_{[0,t]\times\mathbb{R}^d} h^\varepsilon\cdot(\overline{v^\varepsilon} - \overline{u^\varepsilon})\,.\end{aligned} \tag{6.3.4}$$

b) Set

$$\underline{w} \;:=\; \underline{f} \;+\; \underline{\mathcal{E}}(\underline{u},\mathcal{U}_*)\,, \qquad \mathcal{W}_k \;:=\; P_k\big(\mathcal{F}_k \;+\; \mathcal{E}_k(\underline{u},\mathcal{U}_*)\big)\,, \tag{6.3.5}$$

$$g^\varepsilon \;:=\; h^\varepsilon \;-\; \underline{w} \;-\; \sum_k J^\varepsilon_k(\mathcal{W}_k)\,. \tag{6.3.6}$$

Then, Propositions 5.2.2, Corollary 5.2.4 and Proposition 6.2.4 imply that g^ε is a bounded family in $L^{1+1/p}$ which has no propagated oscillations for L.

Proposition 4.3.3 implies that the weak limit of v^ε is $\underline{u}$, v^ε is pure and the profiles are $\mathcal{U}_k$. In particular, $u^\varepsilon - v^\varepsilon$ is bounded in L^{1+p}, converges weakly to 0, is pure, and all its profiles vanish. Thus, since $\underline{w}\in L^{1+1/p}$,

$$\int_{[0,t]\times\mathbb{R}^d} \underline{w}\cdot(\overline{v^\varepsilon} \;-\; \overline{u^\varepsilon}) \;\to\; 0\,. \tag{6.3.7}$$

Next, since $\mathcal{W}_k \in \mathcal{L}^{1+1/p}_k$ and the profiles of $u^\varepsilon - v^\varepsilon$ vanish,

$$\int_{[0,t]\times\mathbb{R}^d} J^\varepsilon_k(\mathcal{W}_k)\cdot(\overline{v^\varepsilon} \;-\; \overline{u^\varepsilon}) \;\to\; 0\,. \tag{6.3.8}$$

Finally, since u^ε and v^ε are bounded families in L^{1+p} such that Lu^ε and Lv^ε are bounded in $L^{1+1/p}$, and since g^ε has no propagated oscillations for L, one has

$$\int_{[0,t]\times\mathbb{R}^d} g^\varepsilon\cdot(\overline{v^\varepsilon} \;-\; \overline{u^\varepsilon}) \;\to\; 0\,. \tag{6.3.9}$$

Therefore,

$$(6.3.11) \qquad \int_{[0,t]\times\mathbb{R}^d} h^\varepsilon \cdot (\overline{v^\varepsilon} - \overline{u^\varepsilon}) \;\to\; 0\,.$$

c) The Cauchy data for u^ε and v^ε satisfy (2.5.1) and (6.2.12) respectively. Therefore,

$$(6.3.12) \qquad ||\, u^\varepsilon(0) - v^\varepsilon(0)\, ||_{L^2}^2 \;\to\; 0 \quad \text{as } \varepsilon \to 0\,.$$

This shows that the right hand side of (6.3.4) tends to 0 as ε tends to 0, implying the Proposition.

Corollary 6.3.2. *The weak limit $\underline{u}$ and the profiles $\mathcal{U}_k$ satisfy the equations*

$$(6.3.13) \qquad L\underline{u} \;+\; \underline{\mathcal{E}}(\underline{u},\mathcal{U}_*) \;=\; 0\,, \qquad \text{on } \;[0,+\infty[\times\mathbb{R}^d\,,$$

$$(6.3.14) \quad P_k\mathcal{U}_k \;=\; \mathcal{U}_k \quad \text{and} \quad \partial_t\mathcal{U}_k \;+\; P_k\,\mathcal{E}_k(\underline{u},\mathcal{U}_*) \;=\; 0\,, \quad \text{on } (G\backslash\mathcal{S}_k^+(p+1))\times\mathbb{T}\,.$$

$$(6.3.15) \qquad \underline{u}_{|t=0} \;=\; \underline{u}_0 \quad \text{on } \mathbb{R}^d\,, \quad \mathcal{U}_{k\,|t=0} \;=\; P_k\mathcal{U}_0 \quad \text{on } \omega\,.$$

Proof. Propositions 6.3.1 and 5.2.2 imply that

$$(6.3.16) \qquad f^\varepsilon \;\sim\; f(u^\varepsilon) \;=\; \underline{\mathcal{E}}(\underline{u},\mathcal{U}_*) \;+\; \sum_k \tilde{J}_k^\varepsilon(\mathcal{E}_k(\underline{u},\mathcal{U}_*)) \;+\; g^\varepsilon$$

where g^ε is bounded in $L^{1+1/p}$, converges weakly to 0 and is pure with null profiles. Therefore Proposition 4.3.3 implies that the weak limit $\underline{f}$ of f^ε is $\underline{\mathcal{E}}(\underline{u};\mathcal{U}_*)$ and the profiles $\mathcal{F}_k$ are equal to $\mathcal{E}_k(\underline{u},\mathcal{U}_*)$. Thus equations (6.3.13) (6.3.14) follow from Propositions 6.1.1.

6.4. Uniqueness for the profile equations.

Theorem 6.4.1. *Equations* (6.3.13) (6.3.14) (6.3.15) *have a unique solution* $(\underline{u},\mathcal{U}_*)$ *in* $L^{p+1}([0,+\infty[\times\mathbb{R}^d)\times\mathcal{L}_*^{p+1}$.

The existence of a solution follows from Corollary 6.3.2. Our goal is to prove uniqueness. It is a consequence of the dissipativity of system (6.3.13) (6.3.12), which is inherited from the dissipativity of the original equation. Remark that this dissipativity could be used to prove directly the existence of weak solutions, using the methods of [LS] [S] [Ha] for monotone operators.

LEMMA 6.4.2. *Suppose that Ω is a regular domain. Suppose that $\underline{u}$ and $\underline{v}$ belong to $C_0^\infty(\Omega)$, that $\mathcal{U}_k$ and $\mathcal{V}_k$ are trigonometric polynomials supported in $\jmath_k^{-1}(\Omega)$ and that $P_k\mathcal{V}_k = \mathcal{V}_k$. Then*

$$\begin{aligned} \int \underline{\mathcal{E}}(\underline{u},\mathcal{U}_*) \cdot \overline{\underline{v}}\; dt\, dx \;+\; \sum_k \int P_k\, \mathcal{E}_k(\underline{u},\mathcal{U}_*) \cdot \overline{\mathcal{V}_k}\; dt\, dy\, d\theta \quad = \\ \int_{\Omega\times(\mathbb{T})^Z} f\Big(t,x,\mathbf{I}(\underline{u},\mathcal{U}_*)\Big) \cdot \overline{\mathbf{I}(\underline{v},\mathcal{V}_*)}\; dt\, dx\, d\theta_*\,, \end{aligned} \tag{6.4.1}$$

where $\mathbf{I}$ *is defined in* (5.1.3).

Proof. Introduce

$$A_k(t,y) := \int_{\mathbb{T}} P_k\, \mathcal{E}_k(u,\mathcal{U}_*) \cdot \overline{\mathcal{V}_k}\; d\theta\,. \tag{6.4.2}$$

For simplicity of notation, the (t,y) variables are omitted in the right hand side. With notation as in (5.1.8), we see that

$$A_k(t,y) := \Delta_k \int_{\mathbb{T}} \mathcal{F}_k(\theta) \cdot \overline{(\mathcal{H}^{m_k}\mathcal{V}_k)\,(\theta)}\; d\theta\,, \tag{6.4.3}$$

where we have used that P_k is an orthogonal projector, $P_k\mathcal{V}_k = \mathcal{V}_k$ and that $\mathcal{H}$ is unitary in $L^2(\mathbb{T})$. According to (5.1.7), $\mathcal{F}_k$ is equal to $\mathcal{F}_k^{(1)}$ plus something independent of θ. Since the mean value of $\mathcal{V}_k$ is zero, this implies that

$$A_k(t,y) := \Delta_k(t,y) \int_{\mathbb{T}} \mathcal{F}_k^{(1)}(t,y,\theta) \cdot \overline{(\mathcal{H}^{m_k}\mathcal{V}_k)\,(t,y,\theta)}\; d\theta\,. \tag{6.4.4}$$

Introduce next the components $G_{k,j}$ of $\jmath_k^{-1}(\Omega)$ and

$$a_k(t,x) \;:=\; \sum_j \Delta_k(\sigma_{k,j}(t,x))^{-2}\; A(\sigma_{k,j}(t,x))\,, \tag{6.4.5}$$

where $\sigma_{k,j}$, an inverse of $\jmath_k$, maps Ω to $G_{k,j}$. Then, with (5.1.6), we see that

$$\begin{aligned} a_k(t,x) \;:=\; \sum_j\; & \Delta_k(\sigma_{k,j}(t,x))^{-1} \;\times \\ & \int_{(\mathbb{T})^Z} f\big(t,x,\mathbf{I}(\underline{u},\mathcal{U}_*)\,(t,x,\theta_*)\big) \cdot\; \overline{(\mathcal{H}^{m_{k,j}}\mathcal{V}_k)\,(\sigma_{k,j}(t,x),\theta_{k,j})}\; d\theta_*\,, \end{aligned} \tag{6.4.6}$$

Introduce

$$a_0(t,x) \;:=\; \underline{\mathcal{E}}(\underline{u},\mathcal{U})\,(t,x) \cdot \overline{\underline{v}(t,x)}\,. \tag{6.4.7}$$

Then, (6.4.6) and the definition (5.1.3) of $\mathbf{I}(\underline{v},\mathcal{V}_*)$ imply that

$$a_0(t,x) \;+\; \sum_k a_k(t,x) \;=\; \int_{(\mathbb{T})^Z} f\Big(t,x,\mathbf{I}(\underline{u},\mathcal{U}_*)\Big) \cdot \overline{\mathbf{I}(\underline{v},\mathcal{V}_*)}\; d\theta_*\,. \tag{6.4.8}$$

On the other hand, since Δ_k^2 is the Jacobian of q_k, one has

$$\int_\Omega a_k(t,x)\; dt\, dx \;=\; \sum_j \int_{G_{k,j}} A_k(t,y)\; dt\, dy \;=\; \int_{G_k} A_k(t,y)\; dt\, dy\,. \tag{6.4.9}$$

With (6.4.8), this implies Lemma 6.4.2.

PROPOSITION 6.4.3 *There is a constant $c > 0$ such that for all $(\underline{u},\mathcal{U}_*)$ and $(\underline{v},\mathcal{V}_*)$ in $L^{p+1}([0,+\infty[\times\mathbb{R}^d)\times\mathcal{L}_*^{p+1}$ which satisfy $P_k\mathcal{U}_k=\mathcal{U}_k$ and $P_k\mathcal{V}_k=\mathcal{V}_k$, one has*

$$\begin{aligned}&\mathrm{Re}\int\Big(\underline{\mathcal{E}}(\underline{u},\mathcal{U}_*)-\underline{\mathcal{E}}(\underline{v},\mathcal{V}_*)\Big)\cdot(\overline{\underline{u}}-\overline{\underline{v}})\;dt\,dx\;+\\ &\mathrm{Re}\sum_k\int\Big(P_k\,\mathcal{E}_k(u,\mathcal{U}_*)-P_k\mathcal{E}_k(\underline{v},\mathcal{V}_*)\Big)\cdot(\overline{\mathcal{U}_k}-\overline{\mathcal{V}_k})\;dt\,dy\,d\theta\;\;\geq\\ &\qquad c\;\Big(\;||\,\underline{u}-\underline{v}\,||^{p+1}_{L^{p+1}}\;+\;\sum_k||\,\mathcal{U}_k-\mathcal{V}_k\,||^{p+1}_{\mathcal{L}_k^{p+1}}\;\Big)\,.\end{aligned}\tag{6.4.10}$$

Proof. By density, one can assume that $\underline{u}$ and $\underline{v}$ belong to $C_0^\infty([0,+\infty[\times\mathbb{R}^d)$ and are supported in a regular Ω. One can also suppose that the $\mathcal{U}_k$ and $\mathcal{V}_k$ are trigonometric polynomials in $\mathcal{P}_k$, supported in $G_k:=\jmath_k^{-1}(\Omega)$. In this case, Lemma 6.4.2 implies that the left hand side of (6.4.10) is equal to the integral over $\Omega\times(\mathbb{T})^Z$ of

$$\mathrm{Re}\,\Big(f(t,x,\mathbf{I}(\underline{u},\mathcal{U}_*))\;-\;f(t,x,\mathbf{I}(\underline{v},\mathcal{V}_*))\Big)\cdot\Big(\;\overline{\mathbf{I}(\underline{u},\mathcal{U}_*)}\;-\;\overline{\mathbf{I}(\underline{v},\mathcal{V}_*)}\;\Big)\,.\tag{6.4.11}$$

Assumption 2.1.2 implies that this is larger or equal to cM where

$$M\;:=\;\int_{\Omega\times(\mathbb{T})^Z}|\,\mathbf{I}(\underline{u},\mathcal{U}_*)\;-\;\mathbf{I}(\underline{v},\mathcal{V}_*)\,|^{1+p}\;dtdxd\theta_*\,.\tag{6.4.12}$$

By definition,

$$\mathbf{I}(\underline{u},\mathcal{U}_*)\,(t,x,\theta_*)\;=\;\underline{u}(t,x)\;+\sum_{(k,j)\in Z}\mathbf{U}_{k,j}(t,x,\theta_{k,j})\tag{6.4.13}$$

where

$$\mathbf{U}_{k,j}(t,x,\theta)\;:=\;\frac{1}{\Delta_k(\sigma_{k,j}(t,x))}\;\big(\mathcal{H}^{m_{k,j}}\,\mathcal{U}_k\big)(\sigma_{k,j}(t,x),\theta)\tag{6.4.14}$$

Similar expressions hold for $\mathbf{I}(\underline{v},\mathcal{V})$. Since

$$\underline{u}(t,x)-\underline{v}(t,x)\;=\;\int_{(\mathbb{T})^Z}\big(\;\mathbf{I}(\underline{u},\mathcal{U}_*)\;-\;\mathbf{I}(\underline{v},\mathcal{V}_*)\big)\;d\theta_*\tag{6.4.15}$$

one has

$$\int_\Omega|\,\underline{u}(t,x)-\underline{v}(t,x)\,|^{1+p}\,dt\,dx\;\leq\;M\,.\tag{6.4.16}$$

Similarly, since $\mathbf{U}_{k,j}(t,x,\theta_{k,j})$ is the average of $\mathbf{I}(\underline{u},\mathcal{U}_*)(t,x,\theta_*)$ with respect to all the variables $\theta_{k',j'}$ except $\theta_{k,j}$, one has

$$\int_{\Omega\times\mathbb{T}} |\,\mathbf{U}_{k,j}\ -\ \mathbf{V}_{k,j}\,|^{1+p}\ dt\,dx\,d\theta\ \le\ M\,. \tag{6.4.17}$$

Lifting the integration to $G_{k,j}$, using the boundedness of $\mathcal{H}$ in L^{1+p} and summing on j, this implies that

$$\int_{G_k\times\mathbb{T}} \Delta_k(t,y)^{1-p}\ |\,\mathcal{U}_k\ -\ \mathcal{V}_k\,|^{1+p}\ dt\,dy\,d\theta\ \le\ C\ M\,. \tag{6.4.18}$$

This proves (6.4.10) for smooth functions supported over Ω, and thus implies Proposition 6.4.3.

Proof of Theorem 6.4.1. Suppose that $(\underline{u},\mathcal{U}_*)$ and $(\underline{v},\mathcal{V}_*)$ are two solutions of (6.3.13) (6.3.14). Corollary 6.2.2 implies that

$$\begin{aligned}\|\,\mathcal{U}_k(t)-\mathcal{V}_k(t)\,\|^2_{L^2}\ &= \|\,\mathcal{U}_k(0)-\mathcal{V}_k(0)\,\|^2_{L^2}\ - \\ &2\ \mathrm{Re}\int_{[0,t]\times\omega\times\mathbb{T}}\Big(P_k\,\mathcal{E}_k(\underline{u},\mathcal{U}_*)-P_k\mathcal{E}_k(\underline{v},\mathcal{V}_*)\Big)\cdot(\overline{\mathcal{U}_k}-\overline{\mathcal{V}_k})\ dt\,dy\,d\theta\,.\end{aligned} \tag{6.4.19}$$

Proposition 6.2.3 implies that

$$\begin{aligned}\|\,\underline{u}(t)-\underline{v}(t)\,\|^2_{L^2}\ &=\ \|\,\underline{u}(0)-\underline{v}(0)\,\|^2_{L^2}\ - \\ &2\ \mathrm{Re}\int_{[0,t]\times\omega}\Big(\underline{\mathcal{E}}(\underline{u},\mathcal{U}_*)-\underline{\mathcal{E}}(\underline{v},\mathcal{V}_*)\Big)\cdot(\overline{\underline{u}}-\overline{\underline{v}})\ dt\,dy\,.\end{aligned} \tag{6.4.20}$$

Adding up, and using Proposition 6.4.3 applied to functions cut off after the time t, yields

$$\begin{aligned}&\|\,\underline{u}(t)-\underline{v}(t)\,\|^2_{L^2}\ +\ \sum_k\|\,\mathcal{U}_k(t)-\mathcal{V}_k(t)\,\|^2_{L^2}\ + \\ &\qquad c\ \Big(\ ||\,\underline{u}-\underline{v}\,||^{p+1}_{L^{p+1}}\ +\ \sum_k||\,\mathcal{U}_k-\mathcal{V}_k\,||^{p+1}_{\mathcal{L}^{p+1}_k}\ \Big) \\ &\qquad\qquad \le \|\,\underline{u}(0)-\underline{v}(0)\,\|^2_{L^2}\ +\ \sum_k\|\,\mathcal{U}_k(0)-\mathcal{V}_k(0)\,\|^2_{L^2}\,.\end{aligned} \tag{6.4.21}$$

This implies uniqueness and Theorem 6.4.1 is proved.

6.5. The main result.

Theorem 2.5.1 is included in the more precise statement. We consider an equation (2.1.1) which satisfies the hyperbolicity Assumption 2.1.1 and the dissipativity Assumption 2.1.2. We consider an initial phase ψ which satisfies Assumption 2.2.1

on $\omega \subset \mathbb{R}^d$. We also suppose that the regularity Assumption 2.3.3, the finiteness Assumption 2.2.2 and the nonresonance Assumption 2.2.5 are satisfied.

THEOREM 6.5.1. *Suppose that* $\underline{u}_0 \in L^2(\mathbb{R}^d)$, $\mathcal{U}_0 \in L^2(\omega \times \mathbb{T})$ *and* $\{u_0^\varepsilon\}$ *is a bounded family in* $L^2(\mathbb{R}^d))$ *such that*

$$u_0^\varepsilon \ \sim \ \underline{u}_0 \ + \ \tilde{J}_0^\varepsilon(\mathcal{U}_0) \qquad in \quad L^2(\mathbb{R}^d)\,. \tag{6.5.1}$$

Denote by $(\underline{u}, \mathcal{U}_k)$ *the unique solution of the profile equations* (6.3.13-14-15) *with*

$$\begin{aligned} &\underline{u} \in C^0([0,+\infty[; L^2(\mathbb{R}^d)) \cap L^{1+p}([0,+\infty[\times\mathbb{R}^d) \\ &\mathcal{U}_k \in C^0([0,+\infty[; L^2(\omega \times \mathbb{T}) \cap \mathcal{L}_k^{1+p}([0,+\infty[\times\omega \times \mathbb{T})\,, \end{aligned} \tag{6.5.2}$$

and $u^\varepsilon \in C^0([0,+\infty[; L^2(\mathbb{R}^d)) \cap L^{1+p}([0,+\infty[\times\mathbb{R}^d)$ *the unique solution of* (2.1.1) *with initial data* u_0^ε.

Then, the family u^ε *is bounded in* $\in C^0([0,+\infty[; L^2(\mathbb{R}^d)) \cap L^{1+p}([0,+\infty[\times\mathbb{R}^d)$ *and for all* $T > 0$,

$$u^\varepsilon \ \sim \ \underline{u} \ + \ \sum_k \ \tilde{J}_k^\varepsilon(\mathcal{U}_k) \qquad in \quad L^{1+p}([0,+T[\times\mathbb{R}^d)\,. \tag{6.5.3}$$

Proof. Existence, uniqueness and boundedness of u^ε is a classical result for dissipative equations (2.1.1) (see [S], [LS], [Ha]). Existence and uniqueness of $(\underline{u}, \mathcal{U}_*)$, solutions to the profile equations, are given by Theorem 6.4.1. Consider

$$v^\varepsilon \ \sim \ \underline{u} \ + \ \sum_k \ \tilde{J}_k^\varepsilon(\mathcal{U}_k) \qquad \text{in} \quad L^{1+p}([0,+\infty[\times\mathbb{R}^d)\,. \tag{6.5.4}$$

For example, one can choose the family constructed in Proposition 6.2.4. Then, Proposition 6.3.1 implies that any pure subsequence u^{ε_n} satisfies

$$u^{\varepsilon_n} \ \sim \ v^{\varepsilon_n} \qquad \text{in} \quad L^{1+p}([0,+T[\times\mathbb{R}^d)\,. \tag{6.5.5}$$

Since any boundeed sequence in L^{1+p} has at least one pure subsequence by Proposition 4.3.1, this implies that the complete family u^ε satisfies

$$u^\varepsilon \ \sim \ v^\varepsilon \qquad \text{in} \quad L^{1+p}([0,+T[\times\mathbb{R}^d)\,, \tag{6.5.6}$$

proving Theorem 6.5.1.

6.6. Absorption at supercritical points

In this paragraph, we prove Theorem 2.5.4, which show that oscillations are absorbed at supercritical points. Recall the Definition 2.5.3 of $T_k^{abs}(y)$, the first time where the k-ray from y hits a supercritical point.

THEOREM 6.6.1. *If* $(\underline{u}, \mathcal{U}_*)$ *is the solution of the profile equations given by Theorem* 6.4.1, *then* $\mathcal{U}_k = 0$ *a.e. for* $t > T_k^{abs}(y)$.

LEMMA 6.6.2. *There exists $c > 0$ such that for all $(\underline{v}, \mathcal{V}_*) \in L^{1+p} \times L^{1+p}_w$ satisfying $P_k \mathcal{V}_k = \mathcal{V}_k$, one has for all k and almost all (t, y) :*

$$\text{(6.6.1)} \qquad \mathrm{Re} \int_{\mathbb{T}} P_k\, \mathcal{E}_k(\underline{v}, \mathcal{V}_*) \cdot \overline{\mathcal{V}_k}\, d\theta \;\geq\; c\, \Delta_k^{1-p} \int_{\mathbb{T}} |\, \mathcal{V}_k\,|^{1+p}\, d\theta\,.$$

Proof. By density, one can assume that $\underline{v} \in C_0^\infty([0, +\infty[\times\mathbb{R}^d)$ and that the profiles $\mathcal{V}_k \in \mathcal{P}_k$. We can also assume that $(t, x) := \jmath_k(t, y)$ belongs to a regular open set Ω. Introduce

$$\text{(6.6.2)} \qquad A_k(t, y) \;:=\; \mathrm{Re} \int_{\mathbb{T}} P_k\, \mathcal{E}_k(\underline{v}, \mathcal{V}_*) \cdot \overline{\mathcal{V}_k}\, d\theta\,.$$

For simplicity, the (t, y) dependence is omitted in the right hand side. Estimate (6.6.1) is a consequence of the strict monotonicity in Assumption 2.1.2. We now find a suitable form for A_k, in order to apply inequality (2.1.3). With notations as in (5.1.7) (5.1.8), using that $P_k \mathcal{V}_k = \mathcal{V}_k$ and that $\mathcal{H}$ is unitary, we see that

$$\text{(6.6.3)} \qquad A_k(t, y) = \Delta_k \int_{(\mathbb{T})^2} \mathrm{Re}\, \big(\mathcal{F}_k^{(1)}(\theta) - \mathcal{F}_k^{(1)}(\theta_1)\,\big) \cdot \overline{\mathcal{W}\,(\theta)}\; d\theta\, d\theta_1\,,$$

with $\mathcal{W} := \mathcal{H}^{m_k} \mathcal{V}_k$. Thus

$$\text{(6.6.4)} \qquad A_k(t, y) = \frac{\Delta_k}{2} \int_{\mathbb{T}^2} \mathrm{Re}\, \big(\mathcal{F}_k^{(1)}(\theta) - \mathcal{F}_k^{(1)}(\theta_1)\big) \cdot \big(\overline{\mathcal{W}(\theta)} - \overline{\mathcal{W}(\theta_1)}\big)\; d\theta\, d\theta_1\,.$$

By (5.1.6), the function $\mathcal{F}_k^{(1)}$ is of the form

$$\text{(6.6.5)} \qquad \mathcal{F}_k^{(1)}(t, y, \theta) := \int_{\mathbb{T}^{Z'}} f\big(t, x, \Delta_k^{-1}(t, y) \mathcal{W}(t, y, \theta) \;+\; \mathbf{W}(t, x, \theta')\big)\; d\theta'\,,$$

where, in the right hand side of (6.6.5), $(t, x) = \jmath_k(t, y)$ and $\mathbf{W}$ is a periodic function of $\theta' \in \mathbb{T}^{Z'}$. Plugging this expression in (6.6.4) transforms the right hand side into an integral over $\mathbb{T}^2 \times \mathbb{T}^{Z'}$. Assumption 2.1.2 implies that the integrated term is larger than or equal to

$$\text{(6.6.6)} \qquad c\, \Delta_k^2\, |\, \Delta_k^{-1} \mathcal{W}(\theta) \;-\; \Delta_k^{-1} \mathcal{W}(\theta_1)\,|^{1+p} \;=\; c\, \Delta_k^{1-p}\, |\, \mathcal{W}(\theta) \;-\; \mathcal{W}(\theta_1)\,|^{1+p}\,.$$

Thus

$$\text{(6.6.7)} \qquad A_k(t, y) \;\geq\; c\, \Delta_k^{1-p} \int_{\mathbb{T}^2} |\, \mathcal{W}(\theta) \;-\; \mathcal{W}(\theta_1)\,|^{1+p}\; d\theta d\theta_1\,.$$

Since the average of $\mathcal{W}$ is zero, $\mathcal{W}(\theta) = \int (\mathcal{W}(\theta) - \mathcal{W}(\theta_1))\, d\theta_1$. Thus recalling $d\theta_1$ is normalized,

$$\text{(6.6.8)} \qquad \int_{\mathbb{T}} |\, \mathcal{W}(\theta)\,|^{p+1}\; d\theta \;\leq\; \int_{\mathbb{T}^2} |\, \mathcal{W}(\theta) - \mathcal{W}(\theta_1)\,|^{p+1}\; d\theta\, d\theta_1\,.$$

Since $\mathcal{H}$ is bounded in $L^{p+1}(\mathbb{T})$, (6.6.7) and (6.6.8) imply (6.6.1) and the lemma is proved.

Proposition 6.6.3. *Consider the solution* $(\underline{u}, \mathcal{U}_*)$ *of the profile equations given by Theorem* 6.4.1. *Introduce*

$$\sigma_k(t,y) \;:=\; \int_{\mathbb{T}} |\,\mathcal{U}_k(t,y,\theta)\,|^2 \; d\theta \tag{6.6.9}$$

Then, $\sigma_k \in C^0([0,+\infty[;L^1(\mathbb{R}^d))$, $\partial_t\sigma_k$ *and* $\Delta(t,y)^{1-p}\,\sigma(t,y)^{(1+p)/2}$ *belong to* $L^1([0,+\infty[\times\mathbb{R}^d)$ *and there is a constant* $c>0$ *such that*

$$\partial_t\sigma_k(t,y) \;+\; c\;\Delta_k(t,y)^{1-p}\;\sigma_k(t,y)^{(1+p)/2} \;\le\; 0\,. \tag{6.6.10}$$

Note that inequality (6.6.10) holds everywhere, including neighborhoods of supercritical points. This is in contrast to the profile equation (6.3.14) which holds only on the complement of the supercritical points.

Proof. We know that $\mathcal{U}_k \in \mathcal{L}_k^{1+p}$, $P_k\mathcal{U}_k = \mathcal{U}_k$, $\mathcal{F}_k := P_k\mathcal{E}_k(\underline{u},\mathcal{U}_*) \in \mathcal{L}_k^{1+1/p}$ and that $\partial_t\mathcal{U}_k + \mathcal{F}_k = 0$ on G_k. From Corollary 6.2.2, $\mathcal{U}_k$ is continuous in t with values in L^2, implying that

$$\sigma_k \in C^0([0,+\infty[\,;L^1(\mathbb{R}^d))\,. \tag{6.6.11}$$

Next we show that

$$\partial_t\sigma_k \;+\; 2\;\mathrm{Re}\int_{\mathbb{T}} \mathcal{F}_k\cdot\overline{\mathcal{U}_k}\;d\theta \;=\; 0\,, \tag{6.6.12}$$

in the sense of distributions on G. To prove this, we approximate $\mathcal{U}_k$ and $\mathcal{F}_k$ by trigonometric polynomials $\mathcal{U}_k^n$ and $\mathcal{F}_k^n$ according to Proposition 6.2.1. The identity (6.6.12) for the approximations are trivial consequences of the equation $\partial_t\mathcal{U}_k^n + \mathcal{F}_k^n = 0$. To pass to the limit, we use Corollary 6.2.2 which implies that σ_k^n converges to σ_k in $C^0([0,+\infty[;L^1(\omega))$, and Proposition 6.2.1 which implies that $\mathcal{F}_k^n\cdot\overline{\mathcal{U}_k^n}$ converges to $\mathcal{F}_k\cdot\overline{\mathcal{U}_k}$ in $L^1(G)$. In particular, this proves that $\partial_t\sigma_k \in L^1$.

Lemma 6.6.2 asserts that

$$\mathrm{Re}\int_{\mathbb{T}} \mathcal{F}_k\cdot\overline{\mathcal{U}_k}\;d\theta \;\ge c\;\Delta(t,y)^{1-p}\int_{\mathbb{T}} \;|\,\mathcal{U}_k(t,y,\theta)\,|^{p+1}\;d\theta\,. \tag{6.6.13}$$

Finally, we note that

$$\sigma_k^{(1+p)/2}(t,y) \;\le\; \int_{\mathbb{T}} |\mathcal{U}_k(t,y,\theta)|^{1+p}\;d\theta\,. \tag{6.6.14}$$

Since $\mathcal{U}_k$ belongs to $\mathcal{L}_k^{1+p}$ this implies that $\Delta_k^{1+p}\,\sigma_k^{(1+p)/2} \in L^1$. Thus, (6.6.10) and the proposition follow from (6.6.11), (6.6.13), (6.6.14).

Proof of Theorem 6.6.1. We have to show that $\mathcal{U}_k = 0$ for almost all (t, y) such that $t > T_k^{abs}$. Thus it is sufficient to consider y in a small neighborhood ω' of $\underline{y}$, contained in ω_0. In this case, $r(y) := T_k^{abs}(y)$ is a smooth function of y and Assumption 2.3.3 implies that

$$\Delta_k(t,y) \;\approx\; |t - r(y)|^{\mu/2} \tag{6.6.15}$$

for $y \in \omega'$ and $t \in I$ a neighborhood of $\underline{t} = r(\underline{y})$. In addition, $(\underline{t}, \underline{y})$ is supercritical, which means that

$$\alpha \;:=\; \mu\,(p-1)/2 \;\geq\; 1\,. \tag{6.6.16}$$

Proposition 6.6.3 implies that on $I \times \omega'$, σ_k satisfies

$$\begin{aligned}
&\sigma_k \in C^0(I; L^1(\omega'))\,, \quad \sigma_k \geq 0\,,\\
&\partial_t \sigma_k \in L^1(I \times \omega')\,, \quad |t - r(y)|^{-\alpha}\ \sigma_k^{(1+p)/2} \in L^1(I \times \omega')\,,\\
&\partial_t \sigma_k \ + \ c\ |t - r(y)|^{-\alpha}\ \sigma_k^{(1+p)/2} \ \leq 0\,.
\end{aligned} \tag{6.6.17}$$

To complete the proof of Theorem 6.6.1, it remains to show that (6.6.17) implies that $\sigma_k = 0$ for $t > r(y)$.

To prove this, one can change the time variable t into $t - r(y)$, and thus reduce the problem to the case where $r \equiv 0$. Introduce

$$g(t) \;:=\; \int_{\omega'} \sigma_k(t,y)\ dy. \tag{6.6.18}$$

Then

$$g(t)^{(1+p)/2} \;\leq\; (\text{meas } \omega')^{(p-1)/2} \int_{\omega'} \sigma_k(t,y)^{(1+p)/2}\ dy. \tag{6.6.19}$$

Thus (6.6.17) implies that there exist $c > 0$ such that on $I := [-a, +a]$, g satisfies

$$\begin{aligned}
&g \in C^0(I)\,, \quad g \geq 0\,, \quad g' \in L^1(I)\,, \quad |t|^{-\alpha}\ g^{(1+p)/2} \in L^1(I)\\
&g' \ + \ c\ |t|^{-\alpha}\ g^{(1+p)/2} \ \leq 0\,.
\end{aligned} \tag{6.6.20}$$

This implies that $g' \leq 0$, hence that g is nonincreasing in particular $g(t) \geq g(0)$ for $t < 0$. Since $|s|^{-\alpha}\ g(s)^{(1+p)/2}$ is integrable over $[-a, 0[$ and $\alpha \geq 1$, this implies that $g(0) = 0$. Therefore $g(t) = 0$ for $t > 0$, so $\sigma_k(t,y) = 0$ a.e. for $t > 0$ and Theorem 6.6.1 is proved.

Proposition 6.6.3 has another corollary. It shows that no oscillations can be created in the k-mode by nonlinear interaction, if there are no oscillations in this mode at time $t = 0$.

THEOREM 6.6.4. *Let $(\underline{u}, \mathcal{U}_*)$ be the solution of the profile equations given by Theorem 6.4.1. Suppose that $P_k(y)\,\mathcal{U}_0(y,\theta)$, the initial condition for $\mathcal{U}_k$, vanishes on $\omega' \times \mathbb{T}$. Then $\mathcal{U}_k = 0$ a.e. on $[0,+\infty[\times\omega' \times \mathbb{T}$.*

Proof. Proposition 6.6.3 implies that the function

$$g(t) \ := \ \int_{\omega'} \sigma_k(t,y)\ dy \tag{6.6.21}$$

is continuous and that its derivative is integrable and nonpositive. Thus g is nonincreasing and $g(t)$ vanishes identically if $g(0) = 0$.

6.7. Angular smoothness for profiles

One of the difficulties in our analysis is that, for profiles $\mathcal{U}_k$ which are not smooth, the expression $\mathcal{U}_k(t, y, \psi(y)/\varepsilon)$ does not make sense for fixed ε. However, if $\mathcal{U}$ is smooth in θ, the substitution is meaningful. One can prove regularity in θ by using the contractivity estimate (6.4.21) together with the fact that that the profile equations are invariant under translations in θ. The angular smoothness is described using the spaces $\mathcal{L}_k^{s,q}$ introduced in Definition 2.4.1.

THEOREM 6.7.1. *In addition to the assumptions of Theorem 6.5.1, suppose that $\partial_\theta \mathcal{U}_0 \in L^2(\omega \times \mathbb{T})$. Then the profiles $\mathcal{U}_k$ belong to $\mathcal{L}_k^{s,p+1}$ for all $s < 2/(p+1)$ and $\partial_\theta \mathcal{U}_k$ belong to $L^\infty([0,+\infty[: L^2(\omega \times \mathbb{T}))$. In particular, the solutions u^ε satisfy*

$$u^\varepsilon \ \sim \ \underline{u} \ + \ \sum_k \ J_k^\varepsilon(\mathcal{U}_k) \qquad in \quad L^{p+1}([0,T]\times\mathbb{R}^d)\,. \tag{6.7.1}$$

Proof. The profile equations are invariant by translations in the θ variable. Thus $(\underline{v}, \mathcal{U}^h) \ := \ (\,\underline{u}, \mathcal{U}(t,y,\theta+h)\,)$ is another solution. Therefore inequality (6.4.21) implies that

$$\begin{aligned} \sup_{0\le t\le T} \sum_k \,\|\,\mathcal{U}_k(t) - \mathcal{U}_k^h(t)\,\|_{L^2}^2 \ + \ c\ \sum_k ||\,\mathcal{U}_k - \mathcal{U}_k^h\,||_{\mathcal{L}_k^{p+1}}^{p+1} \\ \le\ \sum_k \|\,\mathcal{U}_k(0) - \mathcal{U}_k^h(0)\,\|_{L^2}^2\,. \end{aligned} \tag{6.7.2}$$

Since $\partial_\theta \mathcal{U}_0 \in L^2(\omega \times \mathbb{T})$, the right hand side of (6.7.2) is $O(h^2)$ and one concludes that $\partial_\theta \mathcal{U}_k \in L^\infty([0,+\infty[\,; L^2(\omega \times \mathbb{T}))$ and

$$\sup_{0\le t\le T}\ \Big\|\frac{\partial \mathcal{U}_k(t)}{\partial\theta}\Big\|_{L^2(\omega\times\mathbb{T})} \ \le \ \Big\|\frac{\partial \mathcal{U}_k(0)}{\partial\theta}\Big\|_{L^2(\omega\times\mathbb{T})}\,. \tag{6.7.3}$$

(6.7.2) also implies that

$$\sup_{0<h\le 1}\int_{[0,T]\times\omega\times\mathbb{T}} \Delta_k(t,y)^{1-p}\,\Big|\,\frac{\mathcal{U}_k(t,y,\theta+h)\;-\;\mathcal{U}_k(t,y,\theta)}{h^{2/(p+1)}}\,\Big|^{p+1} dt\,dy\,d\theta$$
$$\le\; C\,\Big\|\frac{\partial\mathcal{U}_k(0)}{\partial\theta}\Big\|_{L^2(\omega\times\mathbb{T})}\,.$$

Thus, for all $s<2/(p+1)$,

$$\begin{aligned}\int_0^1\int_{[0,T]\times\omega\times\mathbb{T}} \Delta_k(t,y)^{1-p}\,\Big|\,\frac{\mathcal{U}_k(t,y,\theta+h)\;-\;\mathcal{U}_k(t,y,\theta)}{h^s}\,\Big|^{p+1} dt\,dy\,d\theta\,\frac{dh}{h}\\ \le\; C\,\Big\|\frac{\partial\mathcal{U}_k(0)}{\partial\theta}\Big\|_{L^2(\omega\times\mathbb{T})}\,.\end{aligned} \tag{6.7.4}$$

which implies that $\mathcal{U}_k\in\mathcal{L}^{s,p+1}_k$. Proposition 4.2.1 then implies that $J^\varepsilon_k(\mathcal{U}_k)$ is defined for each ε and that and is a representative of $\tilde J^\varepsilon_k(\mathcal{U}_k)$ and (6.7.1) is equivalent to (6.5.3).

7. Additional results and remarks

7.1. The wave equation

With minor changes which we now explain, our results apply to the semilinear wave equation

$$\Box v\;+\;g(t,x,\partial_t v)\;=\;0\,. \tag{7.1.1}$$

Assumption 7.1.1. *$g\in C^1([0,+\infty[\times\mathbb{R}^d\times\mathbb{C}\,;\,\mathbb{C})$, $g(t,x,0)=0$ and there are constants $p\in[1,+\infty[$ and $0<c<C<+\infty$ such that for all $(t,x)\in\mathbb{R}^{1+d}$ and all v and v' in $\mathbb{C}$,*

$$|\,g(t,x,v)\,|\le\; C\,|\,v\,|^p\,,\qquad |\,\partial_{v,\overline{v}}g(t,x,v)\,|\le\; C\,|\,v\,|^{p-1}\,,$$

$$\mathrm{Re}\,\big(g(t,x,v)-g(t,x,v')\big)\,(\overline{v}-\overline{v}')\;\ge\; c\,|\,v\,-\,v'\,|^{p+1}\,,$$

Consider oscillatory Cauchy data

$$v^\varepsilon_{|t=0}\;=\;\underline{h}_0(x)\;+\;\varepsilon\,\mathbf{H}_0(x,\psi(x)/\varepsilon)\,,\quad \partial_t v^\varepsilon_{|t=0}\;=\;\underline{h}_1(x)\;+\;\mathbf{H}_1(x,\psi(x)/\varepsilon)\,, \tag{7.1.2}$$

where ψ is a smooth function with nonvanishing differential on $\omega\subset\mathbb{R}^d$, $\mathbf{H}_0(x,\theta)$ and $\mathbf{H}_1(x,\theta)$ are smooth, 2π-periodic in θ with mean 0 and compactly supported in $\omega\times\mathbb{T}$, and $\underline{h}_0,\underline{h}_1\in C^\infty_0(\mathbb{R}^d)$.

Introduce $u := (\partial_t v, \partial_x v) := (u^0, u^1, \ldots, u^d)$. Then (7.1.1) (7.1.2) take the form

$$Lu \ := \ \partial_t u \ - \ A(\partial_x)\, u \ + \ f(t,x,u) \ = \ 0\,, \tag{7.1.3}$$

$$u^\varepsilon_{|t=0} \ = \ \underline{u}_0(x) \ + \ \mathbf{U}_0(x, \psi(x)/\varepsilon) \ + \ O(\varepsilon)\,, \tag{7.1.4}$$

with

$$A(\xi) \ := \ \begin{pmatrix} 0 & {}^t\xi \\ \xi & 0 \end{pmatrix}, \qquad f(t,x,u) \ := \ \begin{pmatrix} g(t,x,u_0,) \\ 0 \end{pmatrix}. \tag{7.1.5}$$

$$\underline{u}_0 := (\underline{h}_1, \partial_x \underline{h}_0)\,, \qquad \mathbf{U}_0 := (\mathbf{H}_1, \partial_x\psi\, \partial_\theta \mathbf{H}_0) \tag{7.1.6}$$

Equation (7.1.3) is of the form (2.1.1). The Assumption 2.1.1 of hyperbolicity and constant multiplicity is satisifed, but the dissipativity Assumption 2.1.2 is not satisfied. However, f depends only on u^0 and the energy methods for (7.1.1) involve scalar products $f(t,x,u)\cdot\overline{u} = g(t,x,u^0)\cdot\overline{u^0}$, suggesting that Assumption 7.1.1 is sufficient. More generally, our methods apply to systems (2.1.1) when the nonlinearity f satisfies

Assumption 7.1.2. *There is a subspace $E \subset \mathbb{C}^N$, and a projector p_E from $\mathbb{C}^N$ to E, and a mapping $g(t,x,v)$ from $[0,+\infty[\times\mathbb{R}^d \times E$ to E, which satisfies Assumption 2.1.2 on $[0,+\infty[\times\mathbb{R}^d \times E$, and such that*

$$f(t,x,u) := g(t,x,p_E u)\,. \tag{7.1.7}$$

In the reduction of the wave equation above, E is the space of $u = (u^0, u') \in \mathbb{C}\times\mathbb{C}^d$ such that $u' = 0$. This assumption guarantees the global existence and uniqueness of solutions to (2.1.1), $u \in C^0([0,+\infty[\,; L^2(\mathbb{R}^d))$, with $p_E u \in L^{p+1}([0,+\infty[\times\mathbb{R}^d)$. $\blacksquare$

Next, note that the reduction of $\Box$ to a first order system introduces the artificial eigenvalue $\tau = 0$ and thus the possibility of artificial resonances. To circumvent this difficulty, we note, as in [JMR 3], that the system (7.1.3) satisfies

Assumption 7.1.3. *L satisfies Assumption 2.1.1 and there is a subset $K \subset \{1,\ldots,k_0\}$ so that the subvariety*

$$\Gamma \ := \ \bigcup_{k\in K} \{\tau = \lambda_k(\xi)\} \subset \mathrm{char}\, L$$

satisfies the following condition. There are linear mappings $R(\tau,\xi)$ from E to $\mathbb{C}^N$, which depend smoothly on $(\tau,\xi) \in \mathbb{R}^{1+d}\backslash(\Gamma\cup\{0\})$, such that

$$\forall (\tau,\xi) \notin \Gamma\cup\{0\}\,, \quad \forall u \in E\,, \quad L(\tau,\xi)\, R(\tau,\xi)\, u \ = \ u\,. \tag{7.1.8}$$

For the system (7.1.3) this assumption is satisfied with $\Gamma := \{\tau^2 = |\xi|^2\}$ and

$$R(\tau,\xi) \;:\; \begin{pmatrix} b \\ 0 \end{pmatrix} \;\to\; \frac{b}{\tau^2 - |\xi|^2} \begin{pmatrix} \tau \\ \xi \end{pmatrix}. \tag{7.1.9}$$

Assumption 7.1.3 implies that if $\varphi \in C^\infty(\Omega)$ and $b \in C_0^\infty(\Omega)$ satisfy

$$d\varphi(t,x) \notin \Gamma \quad \text{and} \quad b(t,x) \in E \quad \text{a.e.} \tag{7.1.10}$$

then $b\, e^{i\varphi/\varepsilon}$ has no propagated oscillations for L. To prove this modification of Propositions 5.2.2 and 5.2.3, it is sufficient to replace $L^{-1}(d\varphi)$ by $R(d\varphi)$ in (5.2.13). This shows that only the resonances with $d\varphi \in \Gamma$ need be considered. In this spirit, introduce $\jmath_k$, $\mathcal{S}_k$, J_k^ε etc, only for $k \in K$. In particular the nonresonance Assumption 2.2.5 is weakened to

Assumption 7.1.4 *For all regular open Ω, the set of phases $\varphi_{k,j}$, $(k,j) \in Z \subset K \times \mathbb{N}$, given by* (2.2.14)*, satisfy for all $\alpha \in \mathbb{Z}^Z$ with at least two nonvanishing components,*

$$\sum \alpha_{k,j} d\varphi_{k,j} \;\notin \Gamma \cup \{0\} \quad \text{a.e. on } \Omega\,. \tag{7.1.11}$$

Theorem 7.1.5. *Consider Cauchy data of the form* (2.5.1)*, with*

$$\forall k \notin K\;, \qquad P_k \mathcal{U}_0 \;=\; 0\,. \tag{7.1.12}$$

Then, there are unique $\underline{u} \in C^0([0,+\infty[\,;\, L^2(\mathbb{R}^d)) \cap L^{p+1}([0,+\infty[\times\mathbb{R}^d)$ and $\mathcal{U}_k \in \mathcal{L}_k^{p+1} \cap C^0([0,+\infty[\,;\, L^2(\omega\times\mathbb{T}))$, $k \in K$, such that

$$u^\varepsilon \;\sim \underline{u} \;+\; \sum_{k\in K} \tilde{J}_k^\varepsilon(\mathcal{U}_k) \quad \text{in} \quad L^{p+1}([0,T[\times\mathbb{R}^d)\,. \tag{7.1.13}$$

The $\underline{u}$ and $\mathcal{U}_k$ are solutions of equations, analogous to (6.3.13-14-15)*, where the operators $\underline{\mathcal{E}}$ and $\mathcal{E}_k$ only involve the Lagrangians Λ_l for $l \in K$.*

For the system (7.1.3), Γ is the union of $\tau = |\xi|$ and $\tau = -|\xi|$ and K has cardinality two. The phases $\varphi_{k,j}$, all satisfy the eikonal equation for $\Box$

$$(\partial_t\varphi)^2 \;=\; |\partial_x\varphi|^2\,. \tag{7.1.14}$$

Assumption 7.1.11, means that over regular open sets Ω, one has

$$(7.1.15)\qquad \big(\sum \alpha_{k,j}\partial_t\varphi_{k,j}\big)^2 \ - \ \big|\sum \alpha_{k,j}\partial_x\varphi_{k,j}\big|^2 \ \neq \ 0 \quad \text{a.e. on } \Omega\,.$$

Finally, note that the condition (7.1.12) is satisfied for Cauchy data (7.1.4). For the solutions v^ε of (7.1.1) (7.1.2), one has

$$(7.1.16)\qquad \begin{cases} \partial_t v^\varepsilon \ \sim \ \partial_t \underline{v} \ + \ \tilde J^\varepsilon_+(\mathcal{B}_+) \ + \ J^\varepsilon_-(\mathcal{B}_-) \\ \partial_{x_j} v^\varepsilon \ \sim \ \partial_{x_j}\underline{v} \ + \ J^\varepsilon_+(\mathcal{B}_{+,j}) \ + \ J^\varepsilon_-(\mathcal{B}_{-,j}) \end{cases} \quad \text{in} \quad L^{p+1}([0,T[\times\mathbb{R}^d)\,,$$

with

$$(7.1.17)\qquad \mathcal{B}_{\pm,j} \ = \ \pm\,\frac{\partial_{y_j}\psi}{|d\psi|}\,\mathcal{B}_\pm\,.$$

The profile equations are

$$(7.1.18)\qquad \begin{cases} \Box\underline{v} & + \quad \underline{\mathcal{E}}(\partial_t\underline{u},\mathcal{B}_+,\mathcal{B}_-) \ = \ 0\,,\\ 2\,\partial_t\mathcal{B}_+ & + \quad \mathcal{E}_+(\partial_t\underline{u},\mathcal{B}_+,\mathcal{B}_-) \ = \ 0\,,\\ 2\,\partial_t\mathcal{B}_- & + \quad \mathcal{E}_-(\partial_t\underline{u},\mathcal{B}_+,\mathcal{B}_-) \ = \ 0\,. \end{cases}$$

where the operators $\underline{\mathcal{E}}$ and $\mathcal{E}_\pm$ are those associated to the nonlinearity $g(t,x,\partial_t v)$. Note that the profiles of $u^\varepsilon := (\partial_t v^\varepsilon, \partial_x v^\varepsilon)$ are $\mathcal{B}_\pm\,(1,\pm d\psi/|d\psi|)$ and the orthogonal projector on the line generated by $\ell := (1,\pm d\psi/|d\psi|)$ is $\ell\otimes\ell/2$. This explains the factor 2 in front of $\partial_t\mathcal{B}_\pm$ in (7.1.18).

The initial conditions are

$$(7.1.19)\qquad \begin{cases} \underline{v}_{|t=0}(x) \ = \ \underline{h}_0(x)\,, \qquad \partial_t\underline{v}_{|t=0}(x) \ = \ \underline{h}_1(x) \\ \mathcal{B}_{\pm|t=0}(x,\theta) \ = \ \frac{1}{2}\,\big(\mathbf{H}_1(x,\theta) \ \pm \ |d\psi(x)|\ \partial_\theta\mathbf{H}_0(x,\theta)\big)\,. \end{cases}$$

7.2. Lipshitzean nonlinearities and nonstrictly monotone nonlinearities

The analysis and results extend immediately to nonlinear terms

$$f(t,x,u) + h(t,x,u)$$

where f is as in §2 and the perturbation h vanishes at $u=0$ and is globally lipshitzean in u. In this way the results include both those of [JMR 2] and [JMR 3]. More generally, one can treat nonlinear terms satisfying

$$(7.2.1)\qquad c_1\,|u|^{p+1} - c_2\,|u|^2 \le \mathrm{Re}\ f(t,x,u)u\,, \qquad |f(t,x,u)| \le C_1|u|^p + C_2|u|^2\,,$$

and

$$(7.2.2)\qquad \mathrm{Re}\,\Big(f(t,x,u) - f(t,x,v)\,,\ u-v\Big) \ge c\,|u-v|^{p+1} - c'\,|u-v|^2$$

with positive constants c_1 and c. In this case the solutions may grow exponentially in time, but the analysis on bounded time intervals is unchanged.

The inequalities (2.1.3) or more generally (7.2.2) with $c > 0$ serve to prove that the asymptotics (6.5.3) is valid in L^{p+1}. If the monotonicity is relaxed to $c \geq 0$, the analysis still goes through but yields (6.5.3) in the weaker sense of $L^2([0,T] \times \mathbb{R}^d)$.

7.3. Nonlinear interaction and transfer of singularities above the caustic

In this section we give an example of a beam Γ_+ which focuses and is absorbed at a fold. Next, we study the nonlinear interaction between Γ_+ and another beam Γ_-. Let $(\underline{t}, \underline{x})$ be a point in the caustic of Γ_+. The focusing of Γ_+ implies that the time derivative of the profile of Γ_- which has a finite limit as $t \nearrow \underline{t}$, tends to infinity as $t \searrow \underline{t}$.

In space dimension $d = 2$, consider the semilinear wave equation

$$\Box v \;+\; (\partial_t v)^3 \;=\; 0 \tag{7.3.1}$$

with oscillatory Cauchy data

$$v^\varepsilon_{|t=0} \;=\; \varepsilon\, \mathbf{H}_0(y, \psi(y)/\varepsilon)\,, \qquad \partial_t v^\varepsilon_{|t=0} \;=\; \mathbf{H}_1(y, \psi(y)/\varepsilon)\,, \tag{7.3.2}$$

where

$$\psi(y) \;:=\; y_2 \;+\; y_1^2/2\,, \tag{7.3.3}$$

$\mathbf{H}_0(y,\theta)$ and $\mathbf{H}_1(y,\theta)$ are smooth and compactly supported and are 2π-periodic with mean 0 with respect to θ.

The eigenvalues are $\lambda_\pm(\xi) \;:=\; \pm|\xi|$. The mappings (2.2.4) $\jmath_\pm$ are

$$\jmath_\pm(t,y) = (t,x) \quad \text{with} \quad x_1 = y_1 \mp t y_1/\sqrt{1+y_1^2}\,, \quad x_2 = y_2 \mp t/\sqrt{1+y_1^2}\,. \tag{7.3.4}$$

Only $\jmath_+$ leads to a caustic for positive times. The singular set (2.2.5) is

$$\mathcal{S}_+ \;:=\; \{(t,y) \in [0,+\infty[\times\mathbb{R}^2 \;;\; t = s(y_1)\,\}\,. \tag{7.3.5}$$

where $s(y_1) := (1+y_1^2)^{3/2}$. The caustic set $\mathcal{C}_+$ is parametrized by $y \in \mathbb{R}^2$ as

$$t = (1+y_1^2)^{3/2}\,, \quad x_1 = -y_1^3\,, \quad x_2 = y_2 - (1+y_1^2)\,. \tag{7.3.6}$$

and

$$\mathcal{C}_+ \;:=\; \{(t,x) \in [0,+\infty[\times\mathbb{R}^2 \;;\; x_1^2 = (t^{2/3}-1)^3\,\}\,. \tag{7.3.7}$$

Introduce next $\mathcal{T}_+$ so that $\jmath_+^{-1}(\mathcal{C}_+) = \mathcal{S}_+ \cup \mathcal{T}_+$. $\mathcal{T}_+$ is a smooth surface parametrized by $(a,b) \in \mathbb{R}^2$ as

$$\mathcal{T}_+ := \{(t,y) \in [0,+\infty[\times\mathbb{R}^2; t = (1+a^2)^{3/2}, y_1 = -a^3 - a - a(1+a^2+a^4)^{1/2}, y_2 = b\}$$

and there exists a smooth even convex function $\tau:\mathbb{R}\to\mathbb{R}$ such that $\tau(y_1)<s(y_1)$ for $y_1\neq 0$, $\tau(0)=s(0)$ and

$$\mathcal{T}_+ \ := \ \{(t,y)\in[0,+\infty[\times\mathbb{R}^2\ ;\ t=\tau(y_1)\}\,. \tag{7.3.8}$$

$G\backslash\{\mathcal{S}_+\cup\mathcal{T}_+\}$ has four connected components

$$\begin{aligned} G_1 \ &:= \ \{(t,y)\in[0,+\infty[\times\mathbb{R}^2\ ,\ t<\tau(y_1)\,\}\,,\\ G_2^+ \ &:= \ \{(t,y)\in[0,+\infty[\times\mathbb{R}^2\ ,\ \tau(y_1)<t<s(y_1)\,,\ y_1>0\}\,,\\ G_2^- \ &:= \ \{(t,y)\in[0,+\infty[\times\mathbb{R}^2\ ,\ \tau(y_1)<t<s(y_1)\,,\ y_1<0\}\,,\\ G_3 \ &:= \ \{(t,y)\in[0,+\infty[\times\mathbb{R}^2\ ,\ t>s(y_1)\,\}\,, \end{aligned} \tag{7.3.9}$$

Define

$$G_2 := G_2^+\cup G_2^-\,. \tag{7.3.10}$$

LEMMA 7.3.1. *i) If $(t,y)\in G_1$, then (t,y) is the unique preimage of $\jmath_+(t,y)$.*

ii) If $(t,y)\in G_2$, then $\jmath_+^{-1}(\jmath_+(t,y))$ has three elements, (t,y), $(t,y')\in G_2$ and $(t,y'')\in G_3$. This defines a mapping $\rho:(t,y)\to(t,y')$ from G_2 into itself. Then $\rho\circ\rho=Id$, ρ is C^∞ on G_2, $\rho(G_2^+)=G_2^-$ and $\rho(G_2^-)=G_2^+$. Moreover, ρ extends continuously to $\overline{G_2}$ so that $\rho(\mathcal{T}_+)=\mathcal{S}_+$, $\rho(\mathcal{S}_+)=\mathcal{T}_+$.

Suppose $(t,y)\in G_2^-$. We have $(t,y')=\rho(t,y)$ if y' is the unique solution of

$$y'(1-t(1+y'^2)^{-1/2}) = y(1-t(1+y^2)^{-1/2}) \tag{7.3.11}$$

such that $(t,y')\in G_2^+$.

With the notations and results of §7.1, the asymptotics of $\partial_t v^\varepsilon$ are described in (7.1.16) and (7.1.18). Since the nonlinearity $\partial_t u^3$ is an odd function of $\partial_t u$, it follows that, if $\mathbf{H}_0$ is odd in θ and $\mathbf{H}_1$ is even in θ, then the average $\underline{v}$ vanishes. One has

$$\partial_t v^\varepsilon \ \sim\ J_+^\varepsilon(\mathcal{B}_+)\ +\ J_-^\varepsilon(\mathcal{B}_-) \tag{7.3.12}$$

and the profiles are determined by the transport equations

$$\left\{\begin{array}{rcl} 2\,\partial_t\mathcal{B}_+ \ + \ \mathcal{E}_+(\mathcal{B}_+,\mathcal{B}_-) &=& 0\,,\\ 2\,\partial_t\mathcal{B}_- \ + \ \mathcal{E}_-(\mathcal{B}_+,\mathcal{B}_-) &=& 0\,, \end{array}\right. \tag{7.3.13}$$

with initial conditions

$$\left\{\begin{array}{rcl} \mathcal{B}_{+|t=0}(y,\theta) &=& \frac12\Big(\mathbf{H}_1(y,\theta)\ +\ |d\psi(y)|\ \partial_\theta\mathbf{H}_0(y,\theta)\Big)\,,\\ \mathcal{B}_{-|t=0}(y,\theta) &=& \frac12\Big(\mathbf{H}_1(y,\theta)\ -\ |d\psi(y)|\ \partial_\theta\mathbf{H}_0(y,\theta)\Big)\,. \end{array}\right. \tag{7.3.14}$$

If, in addition, $\mathbf{H}_1 = |d\psi|\,\partial_\theta \mathbf{H}_0$, then Theorem 6.6.4 implies that $\mathcal{B}_-$ vanishes identically. Thus

$$\partial_t v^\varepsilon \;\sim\; J_+^\varepsilon(\mathcal{B}_+)\,, \quad \text{with} \quad 2\,\partial_t \mathcal{B}_+ \;+\; \mathcal{E}_+(\mathcal{B}_+) \;=\; 0\,. \tag{7.3.15}$$

As mentionned in section 2.6, the exponent $p = 3$ is always supercritical, and therefore Theorem 6.6.1 implies that $\mathcal{B}_+ = 0$ on G_3.

We now describe the interaction term $\mathcal{E}_+(\mathcal{B}_+)$. On $G_1 \times \mathbb{T}$, Lemma 7.3.1 *i)* implies that

$$\mathcal{E}_+(\mathcal{B}_+)\,(t,y,\theta) \;=\; \frac{1}{2}\,\frac{1}{\Delta_+(t,y)^2}\,\mathcal{B}_+(t,y,\theta)^3\,, \tag{7.3.16}$$

where $\Delta_+(t,y)^2 := \big|\,1 - t/s(y_1)\,\big|$.

When $(t,y) \in G_2$, the general description of $\mathcal{E}_+$, as given in (5.1.8), involves the three preimages of $\jmath_+(t,y)$. Since $\mathcal{B}_+ = 0$ on G_3, only the preimages in G_2 have to be considered, that is (t,y) and $(t,y') = \rho(t,y)$. To (t,y'), the construction (5.1.3) assigns a variable $\theta' \in \mathbb{T}$, and (5.1.8) shows that

$$\mathcal{E}_+(\mathcal{B}_+)\,(t,y,\theta) \;=\; \frac{1}{2}\,\Delta(t,y)\,\int_{\mathbb{T}} \left(\frac{\mathcal{B}_+(t,y,\theta)}{\Delta_+(t,y)} \;+\; \frac{\mathcal{B}_+(\rho(t,y),\theta')}{\Delta_+(\rho(t,y))} \right)^3 d\theta'\,. \tag{7.3.17}$$

We have used that the index $m(t,y)$ is equal to 0 on G_2, so that no Hilbert transform is needed. Since $\mathcal{B}_+$ is even in θ, one obtains

$$\begin{aligned} \mathcal{E}_+(\mathcal{B}_+)\,(t,y,\theta) \;&=\; \frac{1}{2}\,\frac{\mathcal{B}_+(t,y,\theta)^3}{\Delta_+(t,y)^2} \;+\; \frac{1}{2}\,a(t,y)\,\mathcal{B}_+(t,y,\theta) \\ a(t,y) &:= \frac{1}{\Delta_+(\rho(t,y))^2}\,\int_{\mathbb{T}} \mathcal{B}_+(\rho(t,y),\theta')^2\,d\theta' \end{aligned} \tag{7.3.18}$$

Example 7.3.2 Suppose that the initial oscillations are supported in $0 < \alpha \le y_1 \le \beta < +\infty$. Then, Theorem 6.6.4 implies that $\mathcal{B}_+$ is supported in the strip $\mathcal{D} := \{\,(t,y) \in [0,\infty[\times\mathbb{R}^2 \;:\; \alpha \le y_1 \le \beta\}$. If $(t,y) \in \mathcal{D} \cap G_2 \subset G_2^+$, then $\rho(t,y) \in G_2^-$ and $\mathcal{B}_+(\rho(t,y),\,.\,)$ vanishes identically on $\mathbb{T}$. Therefore, $a(t,y) = 0$. Combining (7.3.16) and (7.3.18) and the absorption result on G_3, we conclude that $\mathcal{B}_+$ is supported in $K \times \mathbb{T}$, $K := \{\,(t,y) \in [0,\infty[\times\mathbb{R}^2 \,:\, \alpha \le y_1 \le \beta\,,\ 0 \le t \le s(y_1)\}$ and $\mathcal{B}$ satisfies in the interior of $K \times \mathbb{T}$

$$2\,\partial_t\,\mathcal{B}_+(t,y,\theta) \;+\; \frac{\mathcal{B}_+(t,y,\theta)^3}{\Delta_+(t,y)^2} \;=\; 0\,. \tag{7.3.19}$$

Thus, for $t < s(y_1)$,

$$\mathcal{B}_+(t,y,\theta) \;=\; \frac{\mathcal{B}_+(0,y,\theta)}{\big(1 \;+\; \ell(t,y)\,\mathcal{B}_+(0,y,\theta)^2\big)^{1/2}}\,, \tag{7.3.20}$$

with

$$(7.3.21)\qquad \ell(t,y) \ := \ \int_0^t \frac{s(y_1)}{s(y_1)\ -\ s}\ ds\,.$$

$\mathcal{B}_+$ is smooth if the data are smooth and tends to 0 when $t \to s(y_1)$, in accordance with the absorption theorem. Formula (7.3.20) provides a precise rate of decay which is further analysed in (7.3.38).

Note that $\jmath_+$ is a bijection from K onto $\jmath_+(K)$. In the (t,x) space, one has

$$(7.3.22)\qquad \partial_t v^\varepsilon(t,x) \ \sim \ \mathbf{B}(t,x,\varphi(t,x)/\varepsilon)$$

with $\mathbf{B}$ supported in $\jmath_+(K)\times\mathbb{T}$, given by

$$(7.3.23)\qquad \mathbf{B}(t,x,\theta) := \frac{1}{\Delta_+(t,y)}\ \mathcal{B}_+(t,y,\theta)\,, \qquad \varphi(t,x)=\psi(y)$$

where y is the unique solution in K of $\jmath(t,y)=(t,x)$. In contrast to the behavor of the Lagrangian profile $\mathcal{B}$, the amplitude of $\mathbf{B}$ tends to infinity as (t,y) approaches $\mathcal{S}_+$. However, the rate of blow up is strictly smaller then in the linear case.

Example 7.3.3. Suppose that the data are supported in $\{-\infty < c \le y_1 \le d < 0\} \cup \{0 < a \le y_1 \le b < +\infty\}$. Then, Theorem 6.6.4 implies that $\mathcal{B}_+$ is supported in the union of the strips $\mathcal{D}^+ := \{\,(t,y)\in[0,\infty[\times\mathbb{R}^2\ ;\ a\le y_1\le b\,\}$ and $\mathcal{D}^- := \{\,(t,y)\in[0,\infty[\times\mathbb{R}^2\ ;\ c\le y_1\le d\,\}$.

Introduce $K^+ := \{\,(t,y)\in[0,\infty[\times\mathbb{R}^2\ ;\ a\le y_1\le b\,,\ 0\le t\le s(y_1)\}$ and $K^- := \{\,(t,y)\in[0,\infty[\times\mathbb{R}^2\ ;\ c\le y_1\le d\,,\ 0\le t\le s(y_1)\}$. Then $\jmath_+$ is a bijection from $K^\pm$ onto $\jmath_+(K^\pm)$. One has

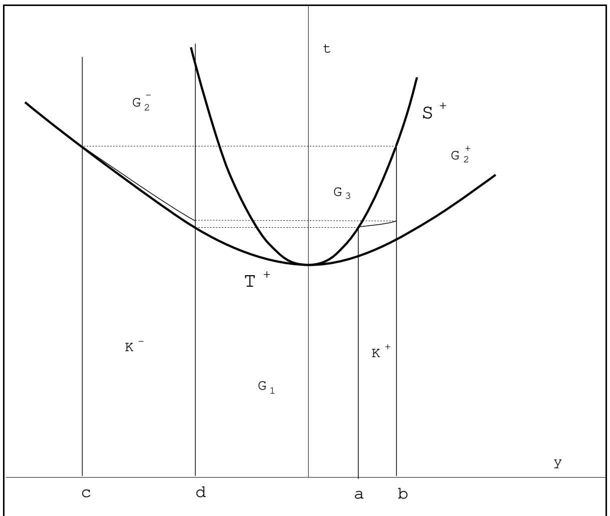
$$(7.3.24)\qquad \partial_t v^\varepsilon(t,x) \ \sim \ \mathbf{B}^+(t,x,\varphi^+(t,x)/\varepsilon) \ + \ \mathbf{B}^-(t,x,\varphi^-(t,x)/\varepsilon)$$

with $\mathbf{B}^\pm$ supported in $\jmath_+(K^\pm)\times\mathbb{T}$, given by

$$(7.3.25)\qquad \mathbf{B}^\pm(t,x,\theta) := \frac{1}{\Delta_+(t,y)}\ \mathcal{B}_+(t,y,\theta)\,, \quad \varphi^\pm(t,x)=\psi(y)$$

where y is the unique solution in $K^\pm$ of $\jmath_+(t,y)=(t,x)$. In this case, $\partial_t v^\varepsilon$ is the superposition of two wave trains, which interact on $\jmath_+(K^+)\cap\jmath_+(K^-)$. Each wave train focuses, $\mathbf{B}^+$ at $\jmath_+(\mathcal{S}_+\cap K^+)$ and $\mathbf{B}^-$ at $\jmath_+(\mathcal{S}_+\cap K^-)$.

When one follows a ray starting at $(0,y)\in K^-$, for $t<\tau(y_1)$, the point (t,y) remains in G_1, and $\jmath(t,y)\notin\jmath(K^+)$. There is no interaction, and $\mathcal{E}_+(\mathcal{B}_+)$ is given by (7.3.16).



Exemple 7.3.3 in Lagrangian coordinates

Figure 7.1

When the ray reaches $\mathcal{T}_+$, that is when $(t, y) \in G_2^-$, the interaction can start since $\jmath(t, y) = \jmath(\rho(t, y))$. See Figure 7.1, where, for simplicity, it is assumed that $\jmath_+$ maps $(\mathcal{D}^+ \cap G_2)$ onto $(\mathcal{D}^- \cap G_2)$ so that, on the base, the intersection of the two beams is exactly the interaction area we want to discuss, wide enough to contain all the focal points of the beam corresponding to $\mathcal{D}^+$ but no more. Note that $\rho(t, y) \in \mathcal{S}_+$ when $(t, y) \in \mathcal{T}_+$, and therefore a singularity is expected in the coefficient a. Equation (7.3.18) suggests that this singularity may jump to the solution $\mathcal{B}_+$. In other words, when one follows a ray for $\mathbf{B}^-$, the interaction with $\mathbf{B}^+$ starts when one reaches the caustic of $\mathbf{B}^+$, and the singulatity of $\mathbf{B}^+$ at the caustic may induce a singularity in $\mathbf{B}_-$. We now analyse this possibility in detail.

Proposition 7.3.4. *i) There is a constant C such that for all $(t, y) \in K^+ \cup K^-$ and $(t, y') := \rho(t, y)$, one has*

$$\frac{1}{C}\ |t - \tau(y_1)|^{1/2} \ \leq\ |t - s(y_1')|\ \leq\ C\ \ |t - \tau(y_1)|^{1/2}\,. \tag{7.3.26}$$

ii) With a defined in (7.3.18), *there is a constant C such that for all $(t, y) \in (K^+ \cup K^-) \cap G_2$ one has*

$$|a(t, y)|\ \leq\ C\ \big|t - \tau(y_1)\big|^{-1/2}\,. \tag{7.3.27}$$

Proof. For $(t, y) \in \overline{G_2} \cap K_\pm$ the function $t - \tau(y_1)$ vanishes only when $\rho(t, y) \in \mathcal{S}_+$, that is when $(t, y) \in \mathcal{T}_+$. Since $K^\pm$ are compact it is sufficient to prove (7.3.26) near such points. At these points, $\jmath_+$ is a local diffeomorphism near $\mathcal{T}_+$ and has a fold along $\mathcal{S}_+$. Since $\jmath_+(\mathcal{T}_+) = \jmath_+(\mathcal{S}_+)$, the estimate follows.

Proposition 6.6.3 implies that

$$\partial_t \int_{\mathbb{T}} |\mathcal{B}_+(t,y,\theta)|^2 \, d\theta \ \leq 0\,. \tag{7.3.28}$$

Since the data are assumed to be smooth, this implies that the integral is uniformly bounded on $[0,+\infty[\times\mathbb{R}^2$.

Since $\Delta(\rho(t,y)) = |t-s(y_1')|^{1/2}$, (7.3.26) and the definition (7.3.18) of a, imply (7.3.27).

We now solve the profile equations. The solution is supported in $K\times\mathbb{T}$, with $K := K^+\cup K^-$. On $(K\cap G_1)\times\mathbb{T}$, the equation is

$$2\,\partial_t\mathcal{B}_+ \ + \ \frac{1}{\Delta_+(t,y)^2}\,\mathcal{B}_+(t,y,\theta)^3 \ = \ 0\,. \tag{7.3.29}$$

where $\Delta_+(t,y)^2 := \big|\,1 - t/s(y_1)\,\big|$. The explicit solution is given by (7.3.20). We note that $\mathcal{B}_+$ is smooth on $K\cap G_1$, up to the boundary $K\cap\mathcal{T}_+$. Let

$$\mathcal{V}_0(y,\theta) \ := \ \mathcal{B}_+(\tau(y_1),y,\theta) \tag{7.3.30}$$

denote the trace of $\mathcal{B}_+$ on $\mathcal{T}_+$

On $(K\cap G_2)\times\mathbb{T}$, the equation is

$$2\,\partial_t\mathcal{B}_+(t,y,\theta) \ + \ \frac{\mathcal{B}_+(t,y,\theta)^3}{\Delta_+(t,y)^2} \ + \ a(t,y)\,\mathcal{B}_+(t,y,\theta) \ = \ 0\,, \tag{7.3.31}$$

where a is defined by (7.3.18). Introduce

$$A(t,y) \ := \ \frac{1}{2}\int_{\tau(y_1)}^{t} a(s,y)\,ds\,. \tag{7.3.32}$$

By Proposition 7.3.4, A is bounded on $G_2\cap K$ and vanishes on $\mathcal{T}_+$. Then

$$\mathcal{V}(t,y,\theta) \ := \ e^{A(t,y)}\ \mathcal{B}_+(t,y,\theta) \tag{7.3.33}$$

satisfies

$$2\,\partial_t\mathcal{V}(t,y,\theta) \ + \ \frac{e^{-2A(t,y)}}{\Delta(t,y)^2}\,\mathcal{V}(t,y,\theta)^3 = 0\ , \quad \mathcal{V}(\tau(y_1),y,\theta) = \mathcal{V}_0(y,\theta). \tag{7.3.34}$$

Since $\mathcal{B}_+$ is continuous across $\mathcal{T}_+$, one has

$$\mathcal{V}(t,y,\theta) \ = \ \frac{\mathcal{V}_0(y,\theta)}{\big(1 \ + \ h(t,y)\,\mathcal{V}_0(y,\theta)^2\big)^{1/2}}\,, \tag{7.3.35}$$

with

$$h(t,y) \;:=\; \int_{\tau(y_1)}^{t} \frac{e^{-2A(s,y)}}{\Delta(s,y)^2}\, ds\,. \tag{7.3.36}$$

Since A is bounded on $K \cap G_2$ and $\Delta(t,y)^2 = 1 - t/s(y_1)$, we see that

$$h(t,y) \;\approx\; -\;\ln(s(y_1)-t) \quad \text{as} \quad (t,y) \to \mathcal{S}_+ \cap K\,. \tag{7.3.37}$$

This implies that if $\mathcal{V}_0(y,\theta) \neq 0$ a.e. on $\mathbb{T}$, then

$$\mathcal{V}(t,y,\theta) \;\approx\; \Big|\ln(s(y_1)-t)\Big|^{-1/2} \quad \text{as} \quad t \to s(y_1), \quad (t,y) \in K^+. \tag{7.3.38}$$

In particular $\mathcal{V} \to 0$. But (7.3.38) is more precise. It implies that

$$\int_{\mathbb{T}} |\mathcal{V}(t,y,\theta)|^2\; d\theta \approx \Big|\ln(s(y_1)-t)\Big|^{-1} \quad \text{as} \quad t \to s(y_1), \quad (t,y) \in K^+ \tag{7.3.39}$$

if $\mathcal{V}_0(y,\,.\,) \neq 0$ a.e. on $\mathbb{T}$. Using again that A is bounded, we see that

$$\int_{\mathbb{T}} |\mathcal{B}_+(t,y,\theta)|^2\; d\theta \approx \Big|\ln(s(y_1)-t)\Big|^{-1} \quad \text{as} \quad t \to s(y_1), \quad (t,y) \in K^+. \tag{7.3.40}$$

Proposition 7.3.4 then implies that

$$a(t,y) \approx \Big|\tau(y_1)-t\Big|^{-1/2}\Big|\ln(\tau(y_1)-t)\Big|^{-1} \quad \text{as} \quad t \to \tau(y_1), \quad (t,y) \in K^-, \tag{7.3.41}$$

for $y \in \mathcal{D}^-$ such that $\mathcal{B}_+(0,y',\cdot) \neq 0 \in L^2(\mathbb{T})$ where $y' \in \mathcal{D}^+$ is defined by $\rho(\tau(y_1),y) = (s(y_1'),y')$. Plugging this estimate in (7.3.31) yields the following result.

Theorem 7.3.5. *i)*

$$\lim_{t \nearrow \tau(y_1)} \;\partial_t\, \mathcal{B}_+(t,y,\theta) \;=\; -\;\frac{1}{2}\;\frac{\mathcal{B}_+(\tau(y_1),y,\theta)^3}{\Delta_+(\tau(y_1),y)^2}\,. \tag{7.3.42}$$

ii) If $\mathcal{B}_0(y,\theta) \neq 0$,

$$\lim_{t \searrow \tau(y_1)} \;\partial_t\, \mathcal{B}_+(t,y,\theta) \;=\; -\;\mathrm{sgn}(\mathcal{B}_+(0,y,\theta))\;\infty\,. \tag{7.3.43}$$

Moreover, the quotient of $|\,\partial_t\, \mathcal{B}_+(t,y,\theta)\,|$ *by*

$$\big|\tau(y_1)-t\big|^{-1/2}\big|\ln(\tau(y_1)-t)\big|^{-1}$$

is bounded from above and from below by positive numbers, as $t \searrow \tau(y_1)$.

For globally Lipschitzean nonlinearities, we observed jumps for $\partial_t \mathcal{B}_+$ on $\mathcal{T}_+$. In particular, $\mathcal{B}_+$ was Lipschitzean in t. The singularity (7.3.43) is much stronger, since $\mathcal{B}_+$ belongs to no C^α, with $\alpha > 1/2$. Note that $\partial_t \, |\mathcal{B}_+(t,y,\theta)\,|^2 \to -\infty$ as $t \to \tau(y_1)$ from avove, which means that the rate of dissipation increases drastically when one crosses $\mathcal{T}_+$.

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