Global Solvability of the Anharmonic Oscillator Model From Nonlinear Optics

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Abstract. The field equations describing the propagation of electromagnetic waves in a nonlinear dielectric medium whose polarization responds locally to the electric field as an anharmonic oscillator with potential $V(P)$ have smooth solutions global in space and time for arbitrary smooth initial data as soon as V has bounded derivatives of order less than or equal to three. This is true in spite of the fact that solutions of the nonlinear Shrödinger equation which approximate the fields in the slowly varying envelope approxiamation may blow up in finite time.

Key words. Nonlinear optics, nonlinear Maxwell equations, nonlinear Schrödinger equation, saturated susceptibility. equation.

AMS-MOS Classification. 35L60, 35Q60, 78A60.

§1. **Introduction.** A standard model, due to Lorentz [L] (see also [F, chap. I-31, and II-33]) of the linear dispersive behavior of electromagnetic waves is given by the system of partial differential equations

$$\partial_t B + curl\, E = 0 \tag{1.1}$$

$$\partial_t\, E - curl\, B = -\partial_t P \tag{1.2}$$

$$\partial_t^2 P + \partial_t P/T_1 + aP = bE \tag{1.3}$$

with positive constants a , b, and T_1. The physical origin of equation (1.3) is a model of the electron as bound to the nucleus by a Hooke's law spring force. Here E and B are the electric and magnetic fields, and, the vector field P is the polarization of the medium. A simple and natural model (see [Bl],[O]) to explain nonlinear dispersive phenomena is to replace the linear restoring force with a nonlinear law

$$\partial_t^2 P + \partial_t P/T_1 + \nabla V(P) = bE \qquad (1.4).$$

If the Taylor expansion of V at the origin is

$$V(P) = \frac{a}{2}\,|P|^2 - \beta|P|^4 + \text{ higher order terms}\,,$$

then asymptotic analysis of small amplitude solutions reveals a focussing cubic term, that is the nonlinear susceptibility, n_2, is strictly positive (see [B, pg.21ff],[DR],[NM]). In addition, the slowly varying envelope approximation (see [NM],[DR2]), leads to amplitudes which are solutions to nonlinear Schrödinger equations. In the monochromatic case, the equation has a focusing cubic nonlinearity when β is positive. In this case, there are solutions of the Schrödinger equation which explode in finite time. Our main theorem shows that the solutions of the underlying field equations, (1.1), (1.2), (1.4) do not break down. These equations have global smooth solution for arbitrary smooth data under appropriate hypotheses on the potential energy function V. An analogous result for the Maxwell- Bloch equations which come from modelling the matter as a gas of finite state quantum systems is proved in [DR1]. In both cases, the fundamental nonlinear field equations are globally solvable even when the reduced Schrödinger equation is not. These contradictory predictions are resolved by the observation that near the focal point, amplitudes grow and the assumptions underlying the slowly varying envelope approximation are no longer satisfied. Once it is known that the underlying equations have smooth solutions, it is natural to ask what is the behavior near a focal point. This appears to us to be a very difficult problem. A first step in considering large amplitude solutions is to go beyond the regime of Taylor expansions about $P = 0$. It might seem reasonable to simply take the potential

$$V(P) = \frac{a}{2}\,|P|^2 - \beta|P|^4.$$

However, for large displacements this is strongly repulsive . In fact, it is so repulsive that solutions of the classical spring equation

$$\partial_t^2 P + \partial_t P/T_1 + \nabla V(P) = 0 \tag{1.5}$$

with large initial energy diverge to infinity in finite time.

Main Hypothesis. *The potential energy function $V : \mathbf{R}^3 \to \mathbf{R}$ satisfies $V(0) = 0$ and is infinitely differentiable with second and third order partial derivatives uniformly bounded.*

This implies that $|\nabla V(P)|$ [resp. $V(P)$] grows at most linearly [resp.quadraticaly] as $P \to +\infty$. That is, there is a C such that for all $\beta \in \mathbf{N}^3$ with $|\beta| \le 3$ and $P \in \mathbf{R}^3$,

$$|\partial^\beta V(P)| \le C\ (1\ +\ |P|)^{\max(2-|\beta|,0)}. \tag{1.6}$$

An example is the potential

$$V(P) := \frac{\frac{a}{2}\,|P|^2 - \beta|P|^4}{1+\gamma|P|^2}, \quad \gamma > 0\,.$$

On one hand, this hypothesis is very strong since the nonlinear term $\nabla V(P)$ is then a globally Lipshitzean function of P. In particular, the ordinary differential equation (1.5) is globally solvable. On the other hand, the hypothesis is reasonable since, as observed above, what is needed to produce a Kerr nonlinearity is that $\partial^2 V/\partial P_i \partial P_j$ be smaller than the Hooke's law when $P \approx 0$. The hypothesis asserts roughly that this comparison is valid for all P. A second plausibility argument is that models of nonlinear susceptibility often include saturation effects. For such models one would have $|\partial^\beta V(P)| \to 0$ as $P \to \infty$ for $|\beta| \geq 2$ and so the hypothesis would be satisfied.

Assuming the main hypothesis, it is routine to prove the global solvability of a semi-linear equation of the form $(\partial_{tt} - \triangle)P + \nabla V(P) = 0$. However, there is no Laplacian in the anharmonic spring equation (1.4), and the existence proof is delicate relying on the detailed structure of the system.

Taking the divergence of (1.1) and (1.2) implies that

$$\partial_t\, div\, (E + P) = \partial_t\, div\, (B) = 0. \tag{1.7}$$

The physically relevant solutions are those which satisfy

$$div(E + P) = div(B) = 0. \tag{1.8}$$

Thanks to 1.7 this holds as soon as it holds at t=0, so (1.8) is only a constraint on the initial data.

Main Theorem. *If $s \geq 2$ and the initial data $B(0), E(0), P(0), \partial_t P(0)$ belong to $H^s(\mathbf{R}^3)$ and satisty $div\,\big(E(0) + P(0)\big) = div\,(B(0)) = 0$, then there is one and only one solution of the equations (1.1), (1.2), (1.4), (1.8) which achieves these initial data and is such that $B, E, P, \partial_t P$ are continuous functions of $t \in [0, \infty[$ with values in $H^s(\mathbf{R}^3)$. The values of the solution at $\underline{t}, \underline{x}$ depend only on the values of the initial data on the ball of radius $\underline{t}$ with center $\underline{x}$.*

It follows that if the initial data belong to $C_0^\infty(\mathbf{R}^3)$ then the solution has compact support whose diameter grows linearly with t. Using the equation to express time derivatives in terms of spatial derivatives shows that the solution is an infinitely differentiable function of t, x. Let

$$U(t,x) := \big(E(t,x)\,, B(t,x)\,, P(t,x)\,, \partial_t P(t,x)\big). \tag{1.9}$$

Equation 1.4 is written as a system for the pair (P, Q) with $Q := \partial_t P$,

$$\partial_t\, P = Q, \qquad \partial_t\, Q = bE - \frac{Q}{T_1} - \nabla V(P).$$

The equations for U then take the form of a semilinear symmetric hyperbolic system

$$
\begin{aligned}
LU &:= \partial_t U - \sum_{1\le j\le 3} A_j \partial_j U = F(U)\,, \\
F(U) &:= \big(-Q\,,\,0\,,Q\,,bE - \frac{Q}{T_1} - \nabla V(P)\big)\,.
\end{aligned}
\tag{1.10}
$$

The symmetry means that the matrices A_j are symmetric and real. In this case, the A_j are 12×12 real symmetric matrices whose last six rows vanish. The next result is classical, dating to Schauder [S]. A short modern proof uses the first theorem in [Re] in the Hilbert space of $U \in H^s(\mathbf{R}^3)$ such that $div\,(E+P) = div\,(B) = 0$. The operator $\sum_{1\le j\le 3} A_j\partial_j$ is antiselfadjoint and for $s > 3/2$ the nonlinear term $F(U)$ is locally lipshitzean.

Local Existence Theorem. *If $s > 3/2$ and $U(0,.) \in H^s(\mathbf{R}^3)$, then there is a $T_* \in]0,\infty]$ and a unique $U \in C([0,T_*[: H^s(\mathbf{R}^3))$ which satisfies (1.1), (1.2), (1.4) and attains these initial values. The solution depends continuously on the initial data in the sense that if $\varphi \in H^s(\mathbf{R}^3)$ and $T < T_*(\varphi)$, then there is an $H^s(\mathbf{R}^3)$ neighborhood $\mathcal{O}$ of φ such that $T_*(\psi) > T$ for all $\psi \in \mathcal{O}$ and the map from initial data to solution is continuous from $\mathcal{O}$ to $C([0,T]: H^s(\mathbf{R}^3))$. There is a lower bound $T_* \ge c\,(s,\,\|U(0)\|_{H^s(\mathbf{R}^3)}) > 0$ where $c(s,\lambda)$ is a nonincreasing function of λ. The values of U at $(\underline{t},\underline{x})$ depend only on the values of the initial data on the ball of radius $\underline{t}$ with center $\underline{x}$.*

Our main result asserts that the solutions do not blow up in finite time. The Main Theorem then takes the followingform

Main Theorem 1. *If $s \ge 2$ and the initial data $U(0,.) \in H^s(\mathbf{R}^3)$ satisfies $div\,(E(0) + P(0)) = div\,(B(0)) = 0$, then $T_* = \infty$.*

This result is a special case of a general result which isolates the essential features of 1.10 which guarantee global existence. Write the nonlinear term as a sum of linear and nonlinear parts by Taylor expansion at $U = 0$,

$$
\begin{aligned}
&F(U) := -BU + f(U)\,, \\
&BU := \Big(-Q\,,\ 0\,,\ Q\,,\ bE - \frac{Q}{T_1} - \left(\nabla^2 V(0)\right) P\Big)\,, \\
&f(U) := \Big(0\,, 0\,, 0\,,\ -\nabla V(P) + \left(\nabla^2 V(0)\right) P\Big)\,.
\end{aligned}
\tag{1.11}
$$

The basic system then has the form

$$
\partial_t U - \sum_{1 \le j \le 3} A_j \partial_j U - BU = f(U), \tag{1.12}
$$

We next introduce a class of $N \times N$ systems of the form 1.12 including 1.10 as a special case.

Assumption 1. *The $N \times N$ system 1.12 has hermitian symmetric constant matrix coefficients A_j and constant B. For $\xi \neq 0$, the kernel of $A(\xi) := \sum \xi_j A_j$ has dimension independent of ξ.*

For the anharmonic oscillator model this dimension is equal to 8, the kernel consisting of vectors such that E and B are parallel to ξ.

Assumption 2. *The nonlinear function $f : \mathbf{C}^N \to \mathbf{C}^N$ has range contained in $\bigcap \{KerA(\xi) \ :\ \xi \neq 0\}$. In addition, $f(0) = 0$, $\nabla f(0) = 0$, and the first derivatives $\partial f / \partial U_j$ are uniformly bounded in $\mathbf{C}^N$.*

For the anharmonic oscillator model, the main hypothesis yields the derivative bound in assumption 2. Moreover, the nonlinear term takes values in the vectors whose first 9 components vanish, and, the kernel of the A_j includes the vectors whose first 6 components vanish, so, assumption 2 is satisfied . Thus, Main Theorem 1 is a special case of the following result.

Main Theorem 2. *For semilinear symmetric hyperbolic systems satisfying Assumptions 1 and 2, $s \geq 2$, and $U(0) \in H^2(\mathbf{R}^3)$, the Cauchy problem is gloablly solvable, that is $T_* = \infty$.*

To prove Main Theorem 2, it is sufficient to prove the following *a priori* estimate. For any $T \in]0,\infty[$ and any $M > 0$ there is a constant $C(T,M)$ such that if $\underline{t} \in]0,T]$ and U is a smooth compactly supported solution of (1.12) on $[0,\underline{t}] \times \mathbf{R}^3$ such that

$$\|U(0)\|_{H^2(\mathbf{R}^3)} \leq M, \tag{1.13}$$

then

$$\|U(t,.)\|_{H^2(\mathbf{R}^3)} \leq C(T,M), \quad \text{for} \qquad 0 \leq t \leq \underline{t}. \tag{1.14}$$

To prove that (1.14) is sufficient, one must show that if $U(0) \in H^s(\mathbf{R}^3)$, $s \geq 2$ and $T > 0$ then $T_* \geq T$. Gagliardo-Nirenberg estimates imply that for a smooth solution U and $s \geq 2$, one has

$$\|U(t,.)\|_{H^s(\mathbf{R}^3)} \leq C \, \|U(0)\|_{H^s(\mathbf{R}^3)} \,, \quad \text{for} \qquad 0 \leq t \leq \underline{t} \tag{1.15}$$

where C only depends on the L^∞ norm of U on $[0,\underline{t}] \times \mathbf{R}^3$. These estimates and the the local existence theorem imply the following Corollary (see e.g. [M],[GR]). If $T_* < +\infty$, then

$$\limsup_{t \to T_*} ||U(t,.)||_{L^\infty(\mathbf{R}^3)} = +\infty. \tag{1.16}$$

Moreover, by continuity, the *a priori* estimates (1.14) extend to soluitons $U \in C^0([0,\underline{t}] : H^s(\mathbf{R}^3))$. The Sobolev imbedding $H^2(\mathbf{R}^3) \subset L^\infty(\mathbf{R}^3)$, and (1.13) show that the L^∞ norm of an H^s solution U cannot blow up before time T and therefore $T_* \geq T$.

The proof of Main Theorem 2 proceeds by a sequence of estimates leading to (1.14). These estimates are presented in sections **2-6**.

Remark 1. The divergence conditions $div\,\big(E(0) + P(0)\big) = div\,(B(0)) = 0$ are not necessary for the global existence. We included them in the statement of Main Theorem 1 since they are required for the physically relevant solutions.

Remark 2. Theorem 2 applies to the sytem (1.1) (1.2) with (1.4) replaced by the more general law

$$\partial_t^2 P = F(E, B, P, \partial_t P) \tag{1.17}$$

where $F \in C^3(\mathbf{R}^{12} : \mathbf{R}^3)$, $F(0) = 0$, and F has bounded first and second derivatives. Another generalization, which is not covered by the theorem but which can be analysed with exactly the same method, is that of a finite number of anharmonic oscillators. This corresponds more closely to what one would find from nonlinear terms in quantum perturabtion theory. The dynamics of the polarization is then given by

$$P = \sum_{j=1}^{n} P^j\,, \qquad \partial_t^2 P^j + \partial_t P^j / T_1^j + \nabla V^j(P^j) = b^j E\,.$$

Here each of the potentials V^j satisfies the main hypothesis.

§2. $L^2(\mathbf{R}^3)$ Estimates for U. Let

$$K := ||(B + B^*)/2|| + \sup_{u \in \mathbf{C}^N} \{ ||\nabla f(U)|| \} \tag{2.1}$$

which is finite thanks to Assumption 2. Then since $f(0) = 0$, one has the Lipshitz bound

$$||f(U)||_{\mathbf{C}^N} \le K ||U||_{\mathbf{C}^N}\,. \tag{2.2}$$

For a smooth compactly supported solution U satisfying (1.13), we perform the standard energy estimate. Namely take the $\mathbf{C}^N$ scalar product of the the partial differential equation (1.12) with U and integrate over $\mathbf{R}^3$. Then take the real part to find that

$$\partial_t\, ||U(t)||^2_{L^2(\mathbf{R}^3)} \le 2K\, ||U(t)||^2_{L^2(\mathbf{R}^3)}\,.$$

It follows that

$$||U(t)||_{L^2(\mathbf{R}^3)} \le e^{Kt}\, ||U(0)||_{L^2(\mathbf{R}^3)}\,. \tag{2.3}$$

§3. An $H^1(\mathbf{R}^3)$ estimate for U. Let ∂ denote one of $\partial/\partial x_j$, for $1 \le j \le 3$. Then

$$(L - B)(\partial U) = -\partial\big(f(U)\big) \;= -(\nabla f(U))\,\partial U. \tag{3.1}$$

The main hypothesis implies that ∇f is bounded and thus

$$|\,\partial\big(f(U)\big)\,(t,x)\,| \le K\,|\partial U(t,x)|\,. \tag{3.2}$$

The standard energy method, namely scalar product of (3.1) with ∂U and integrating dx over $\mathbf{R}^3$ yields

$$\partial_t \, ||\partial U(t)||^2_{L^2(\mathbf{R}^3)} \leq 2K \, ||\partial U(t)||^2_{L^2(\mathbf{R}^3)} \,, \tag{3.3}$$

Summing the resulting expressions over the three values of ∂, and applying Gronwall's inequality shows that for $0 \leq t \leq \underline{t} \leq T$

$$||\partial U(t)||_{L^2(\mathbf{R}^3)} \leq C(T, M) \,. \tag{3.4}$$

In particular, we control the $H^1(\mathbf{R}^3)$ norm of $U(t)$.

§4. An $H^2(\mathbf{R}^3)$ estimate for the propagating part of U. Denote by $\Pi_0(\xi)$ the orthogonal projector on $Ker\, A(\xi)$, defining a smooth matrix valued function homogeneous of degree zero on $\xi \neq 0$. Let $\Pi_1(\xi) := I - \Pi_0(\xi)$. The next estimates depend on the decomposition

$$U = u_0 + u_1 \,, \qquad u_\nu := \Pi_\nu(D)\, U := \mathcal{F}^{-1}\, \Pi_\nu(\xi)\, \mathcal{F}\, U \,. \tag{4.1}$$

The key intuition is that u_0 corresponds to zero speeds and the u_1 to nonzero speeds. This will show that the u_0 can only weakly influence u_1 since the corresponding waves cross transversely. Furthermore, the nonlinear term does not influence the u_1 part thanks to Assumption 2. These two facts suffice to pass from an H^1 estimate for U to an H^2 estimate for u_1. Let

$$B_{\mu\nu}(D) := \Pi_\mu(D)\, B\, \Pi_\nu(D) \,. \tag{4.2}$$

Multiply (1.12) by $\Pi_1(D)$ using the facts that $\Pi_1\, f = 0$ and $\Pi(\xi)A(\xi) = A(\xi)\Pi(\xi) = A(\xi)$ to find

$$\big(\, \partial_t - \sum A_j \partial_j \,\big) u_1 - B_{11}(D) u_1 - B_{10}(D) u_0 = 0 \,. \tag{4.3}$$

Similarly, multiplying by $\Pi_0(D)$ yields

$$\partial_t u_0 - B_{01}(D)\, u_1 = \Pi_0(D)\, f(U) \,.$$

Differentiate with respect to x to find

$$\partial_t D u_0 - B_{01}(D)\, D u_1 = \Pi_0(D) \,\big(f'(U) D U \big) \,. \tag{4.4}$$

Since f' is bounded and the $\Pi_\nu(D)$ are bounded in L^2, this yields the estimate

$$||\partial_t u_0(t)||_{H^1(\mathbf{R}^3)} \leq C\, ||U(t)||_{H^1(\mathbf{R}^3)}\,. \tag{4.5}$$

Let $<\xi>:=(1+|\xi|^2)^{1/2}$ and notice that u_1 then satisfies the modified equation

$$\big(\,\partial_t - \sum A_j\partial_j - i<D>\Pi_0(D)\,\big)u_1 - B_{11}u_1 - B_{10}u_0 = 0\,. \tag{4.6}$$

involving the antiselfadjoint elliptic operator

$$G(D) := \sum A_j\partial_j + i<D>\Pi_0(D)\,, \quad G(\xi) := -i<\xi>\Pi_0(\xi) + \sum i\,A_j\xi_j\,. \tag{4.7}$$

Then $u_1 = z + w$ where z and w are the solutions of the initial value problems

$$\big(\,\partial_t - G(D)\big)z - B_{11}z = 0\,, \quad z(0,\,.\,) = u_1(0,\,.\,)\,, \tag{4.8}$$

$$\big(\,\partial_t - G(D)\,\big)w - B_{11}w - B_{10}u_0 = 0\,, \quad w(0,\,.\,) = 0\,. \tag{4.9}$$

For z we have the elementary estimate

$$||z(t)||_{H^2(\mathbf{R}^3)} \leq e^{Kt}\,||z(0)||_{H^2(\mathbf{R}^3)} \leq e^{Kt}\,M\,. \tag{4.10}$$

Duhamel's formula yields the following formula for the Fourier transform of w

$$\hat{w}(t,\xi)) = \int_0^t e^{(G(\xi)+B_{11}(\xi))(t-s)}\;B_{10}(\xi)\,\hat{u}_0(s,\xi)\;ds\,. \tag{4.11}$$

Introduce

$$\Phi(t,s,\xi) := -\int_s^t e^{(G(\xi)+B_{11}(\xi))\,(t-\sigma)}\;d\sigma$$

so that

$$\partial_s\,\Phi(t,s,\xi) = e^{(G(\xi)+B_{11}(\xi))\,(t-s)} \quad \text{and} \quad \Phi(t,t,\xi) = 0\,.$$

Integration by parts in (4.11) yields

$$\hat{w}(t,\xi)) = B_{10}(\xi)\,\big(\Phi(t,0,\xi)\,\hat{u}_0(s,\xi) - \int_0^t \Phi(t,s,\xi)\,\partial_t\hat{u}_0(s,\xi)\;ds\,\big)\,. \tag{4.12}$$

The symmetric hyperbolicity implies that

$$||\Phi(t,s,\xi)|| \leq \int_0^{|t-s|} e^{K\sigma}\;d\sigma\,. \tag{4.13}$$

The ellipticity of G implies that for $|\xi|$ large $G+B$ is invertible and has norm $O(|\xi|^{-1})$. For such ξ,

$$\Phi(t,s,\xi) = -(G(\xi)+B_{11}(\xi))^{-1}\big(e^{(G(\xi)+B_{11}(\xi))(t-s)} - I\big)\,.$$

Together with (4.13) this implies

$$|\Phi(t,s,\xi)| \le C(T) < \xi >^{-1}\,, \quad \text{for} \quad s,t\in[0,T] \;\text{ and }\; \xi\in\mathbf{R}^3\,. \tag{4.14}$$

Then (4.12) yields

$$\begin{aligned} ||w(t)||_{H^2(\mathbf{R}^3)} &\le C(T)\big(||u_0(s)||_{H^1(\mathbf{R}^3)} + \int_0^t ||\partial_t u_0(s)||_{H^1(\mathbf{R}^3)}\ ds\,\big) \\ &\le C(T,M)\,, \end{aligned} \tag{4.15}$$

where (3.4) and (4.5) are used in the last step. Estimates (4.10) and (4.15) show that

$$||u_1(t)||_{H^2(\mathbf{R}^3)} \le C(T,M)\,. \tag{4.16}$$

§5. An $L^6(\mathbf{R}^3)$ estimate for Du_0. Consider the right hand side of (4.4). Since f' is bounded it follows that

$$||f'(U(t))DU||_{L^6(\mathbf{R}^3)} \le C||DU(t))||_{L^6(\mathbf{R}^3)}\,. \tag{5.1}$$

Since the singular integral operator $\Pi_0(D)$ is bounded from $L^6(\mathbf{R}^3)$ to itself, (4.4) and (5.1) yield

$$\begin{aligned} ||\partial_t Du_0||_{L^6(\mathbf{R}^3)} &\le C\big(||Du_1||_{L^6(\mathbf{R}^3)} + ||DU(t))||_{L^6(\mathbf{R}^3)}\big) \\ &\le 2C\,||Du_1||_{L^6(\mathbf{R}^3)} + C\,||\,Du_0||_{L^6(\mathbf{R}^3)}\ . \end{aligned} \tag{5.2}$$

The Sobolev imbedding $H^1(\mathbf{R}^3)\subset L^6(\mathbf{R}^3)$ implies that

$$||Du_1||_{L^6(\mathbf{R}^3)} \le C'||Du_1(t)||_{H^1(\mathbf{R}^3)} \le C(T,M) \tag{5.3}$$

thanks to (4.16). Thus integrating the inequality (5.2) yields

$$||Du_0||_{L^6(\mathbf{R}^3)} \le C(T,M)\,. \tag{5.4}$$

§**6. Endgame.** The second derivatives $v := D_i D_j U$ satisfy

$$Lv = f'(U)\, v + \sum_{1\le\alpha,\beta\le 3} \frac{\partial^2 f(U)}{\partial U_\alpha \partial U_\beta}\ (D_i U)\ (D_j U). \tag{6.1}$$

Thanks to the boundedness of f' and f'' the standard energy method applied to (6.1) yields

$$\begin{aligned} \partial_t ||D^2 U(t)||^2_{L^2(\mathbf{R}^3)} &\le C||D^2U(t)||^2_{L^2(\mathbf{R}^3)} + C\int_0^t ||D_iU(s)\, D_jU(s)||^2_{L^2(\mathbf{R}^3)}\, ds \\ &\le C||D^2U(t)||^2_{L^2(\mathbf{R}^3)} + C\int_0^t ||DU(s)||^2_{L^4(\mathbf{R}^3)}\, ds\,. \end{aligned} \tag{6.2}$$

Interpolating between the L^2 estimate (3.4) and the L^6 estimates (5.3)–(5.4) shows that the integrand is bounded by $C(T,M)$. Then Gronwall's method shows that

$$||D^2U(t)||_{L^2(\mathbf{R}^3)} \le C(T,M), \quad \text{for} \quad 0\le t\le T\,. \tag{6.3}$$

This together with (2.3) and (3.4), proves the desired estimate (1.14) and the proof of the Main Theorems is complete.

Acknowlegements. The authors gratefully acknowlege the support of The North Atlantic Treaty Organization, the U.S. Office of Naval Research, and the National Science Foundation under grants CRG 890904, N O14 92 J 1245, and, DMS 90 03256 respectively. They also thank the Centre de Mathématiques and the Centre de Mathématiques Appliquées of the Ecole Polytechnique in Palaiseau for the welcome offered to J. Rauch during the academic year 1992-1993. All three thank Simone, Florence, Ludovic, Geraldine, and the citizens St. Marsal for their hospitality in July 1993 when much of this research was performed.

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