A ZERO-ONE LAW FOR INVARIANT MEASURES AND A LOCAL LIMIT THEOREM FOR COEFFICIENTS OF RANDOM WALKS ON THE GENERAL LINEAR GROUP

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Abstract. We prove a zero-one law for the stationary measure for algebraic sets generalizing the results of Furstenberg [13] and Guivarc'h and Le Page [21]. As an application, we establish a local limit theorem for the coefficients of random walks on the general linear group.

1. Introduction

1.1. **Motivation and objectives.** Let $d \geqslant 2$ be an integer. We denote by V a d-dimensional vector space over $\mathbb{R}$ and by V^* the dual space of V. We choose norms on V and V^* which will be both denoted by $|\cdot|$. For any $v \in V$ and $f \in V^*$ the corresponding duality bracket is denoted by $\langle f, v\rangle = f(v)$; sometimes instead of $f(v)$ we shall use the reverse notation $v(f) = \langle v, f\rangle$. Denote by $\mathbb{G} = \mathrm{GL}(V)$ the group of linear automorphisms of V. The projective space $\mathbb{P}(V)$ of V is the set of elements $x = \mathbb{R}v$, where $v \in V \setminus \{0\}$. The projective space of V^* is denoted by $\mathbb{P}(V^*)$. For any $x = \mathbb{R}v \in \mathbb{P}(V)$ and $y = \mathbb{R}f \in \mathbb{P}(V^*)$ we define $\delta(y, x) = \frac{|\langle f,v\rangle|}{|f||v|}$. For any $g \in \mathbb{G}$ and $x = \mathbb{R}v \in \mathbb{P}(V)$ with $v \in V \setminus \{0\}$, let $gx = \mathbb{R}gv \in \mathbb{P}(V)$, where $gv \in V$ is the image of the automorphism $v \mapsto gv$ on V. Set $\mathbb{N} = \{0, 1, 2, \ldots\}$ and $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$.

Let μ be a probability measure on $\mathbb{G}$. Consider the probability space $(\Omega, \mathscr{F}, \mathbb{P})$, where $\Omega = \mathbb{G}^{\mathbb{N}^*}$, $\mathscr{F}$ is the Borel σ-algebra on Ω and $\mathbb{P} = \mu^{\otimes \mathbb{N}^*}$. If we denote by g_i the coordinate mapping on Ω, then $g_1, g_2, \ldots$ is a sequence of independent and identically distributed random elements in $\mathbb{G}$ defined on $(\Omega, \mathscr{F}, \mathbb{P})$ with the same law μ. Set $G_n = g_n \ldots g_1$, for $n \geqslant 1$.

A measure ν is called μ-invariant (or equivalently μ-stationary) if $\mu * \nu = \nu$, where $*$ stands for the convolution of probability measures. Furstenberg [13] showed that under some mild conditions there exists a unique μ-invariant probability measure ν on $\mathbb{P}(V)$ which is not supported by any proper projective hyperplane: for any projective hyperplane $Y \subsetneq \mathbb{P}(V)$,

$$\nu(Y) = 0. \tag{1.1}$$

The Furstenberg zero-law (1.1) turns out to be one of the key properties in the study of random walks on the group $\mathbb{G}$. It is used in the proof of many limit theorems for the norm cocycle $\log |G_n v|$, where $v \in V \setminus \{0\}$ is a starting vector. We refer to Furstenberg and Kesten [14], Furstenberg [12], Le Page [25], Bougerol and Lacroix

Date: October 15, 2021.

2010 *Mathematics Subject Classification.* Primary 60B15, 15B52, 37A30; Secondary 60B20.

Key words and phrases. General linear group; zero-one law; stationary measure; random matrices; regularity; algebraic set.

[9], Goldsheid and Guivarc'h [16], Guivarc'h [19], Benoist and Quint [3, 4, 5], Xiao, Grama and Liu [28], who established the law of large numbers, the central limit theorem, the local limit theorem and large deviation asymptotics. Some of these results have been extended to the coefficients $\langle f, G_n v \rangle$, where $f \in V^*$ and $v \in V$, however much less is known in this respect. Guivarc'h and Raugi [22] have proved the law of large numbers and the central limit theorem for the coefficients. In the setting of reductive groups, Benoist and Quint [5] have established the law of iterated logarithm and the large deviation bounds. The approach developed in [22], [9] and [18] for the proof of the law of large numbers and of the central limit theorem for $\log |\langle f, G_n v \rangle|$ is based on the use of the quantitative version of the property (1.1) called Hölder regularity of the stationary measure ν which we state below: there exist positive constants α and C such that for any $y = \mathbb{R}f \in \mathbb{P}(V^*)$ and $\varepsilon > 0$,

$$\nu \left(\{ x \in \mathbb{P}(V) : \delta(y, x) \leqslant \varepsilon \} \right) \leqslant C \varepsilon^{\alpha}. \tag{1.2}$$

The next elementary identity relates the coefficient $\langle f, G_n v \rangle$ to the norm $|G_n v|$: for any $x = \mathbb{R}v \in \mathbb{P}(V)$ and $y = \mathbb{R}f \in \mathbb{P}(V^*)$ with $|f| = 1$,

$$\log |\langle f, G_n v \rangle| = \log |G_n v| + \log \delta(y, G_n x), \tag{1.3}$$

where $\delta(y, G_n x) = \frac{|\langle f, G_n v \rangle|}{|f||G_n v|}$. From (1.2) one can deduce that for any $\beta > 0$,

$$\lim_{n \to \infty} n^{-\beta} \log \delta(y, G_n x) = 0, \quad \mathbb{P}\text{-a.s.} \tag{1.4}$$

Now using (1.3) and (1.4) we can infer the limit behaviour of $\log |\langle f, G_n v \rangle|$ from that of $\log |G_n v|$. This allows for instance to prove the law of large numbers and the central limit theorem: for any $x = \mathbb{R}v \in \mathbb{P}(V)$ and $y = \mathbb{R}f \in \mathbb{P}(V^*)$,

$$\lim_{n \to \infty} \frac{1}{n} \log |\langle f, G_n v \rangle| = \lambda, \quad \mathbb{P}\text{-a.s.}$$

and for any $t \in \mathbb{R}$,

$$\lim_{n \to \infty} \mathbb{P} \left(\frac{\log |\langle f, G_n v \rangle| - n\lambda}{\sigma \sqrt{n}} \leqslant t \right) = \Phi(t),$$

where $\lambda \in \mathbb{R}$ is a constant called the top Lyapunov exponent of μ, σ is a positive number given by $\sigma^2 = \lim_{n \to \infty} \frac{1}{n} \mathbb{E}[(\log |G_n v| - n\lambda)^2]$, and Φ is the standard normal distribution function. For the law of iterated logarithm and large deviation bounds, Benoist and Quint [5, Lemma 14.11], following the approach of Bourgain, Furman, Lindenstrauss and Mozes [10], have developed another strategy based on the following inequality: for any $a > 0$ there exist positive constants c, C and n_0 such that for any $n_0 \leqslant l \leqslant n$, $x \in \mathbb{P}(V)$ and $y \in \mathbb{P}(V^*)$,

$$\mathbb{P} \left(\delta(y, G_n x) \leqslant e^{-al} \right) \leqslant C e^{-cl}. \tag{1.5}$$

The bound (1.5) implies (1.2), and therefore contains more information than (1.2). When studying the logarithm of the coefficients $\log |\langle f, G_n v \rangle|$, (1.5) gives an alternative way to prove the law of large numbers and the central limit theorem, but also allows to establish new results like the law of iterated logarithm and the large deviation bounds. However, many important properties such as the Berry-Esseen bounds, local limit theorems, large deviation principles and exact asymptotics of large deviations for the coefficients $\langle f, G_n v \rangle$ cannot be obtained by this approach.

For these latter statements the exact contribution of the term $\delta(y, G_n x)$ should be accounted, which means that we need to establish more general theorems for the couple $(G_n x, \log |G_n v|)$.

We find out that in order to transfer many asymptotic properties from the couple $(G_n x, \log |G_n v|)$ to that of the coefficients $\log |\langle f, G_n v\rangle|$, it is necessary to establish the identity (1.1) for subsets of $\mathbb{P}(V)$ which are not projective hyperplanes, in particular, for the hypersurfaces $\{x \in \mathbb{P}(V) : \delta(y, x) = t\}$, where $t \neq 0$. The main goal of the paper is to extend the result of Furstenberg (1.1) from the special case of projective hyperplanes to arbitrary algebraic subsets Y of the projective space $\mathbb{P}(V)$. There is, however, an essential difference with the Furstenberg's result, which confers the mass 0 to a projective hyperplane. We show that for an arbitrary algebraic set Y of $\mathbb{P}(V)$ it holds that $\nu(Y)$ is 0 or 1. Contrary to the Furstenberg zero-law, it is possible, as we show in Example 2.3, that the invariant measure ν is concentrated on an algebraic subset of dimension $d-2$ on the projective space $\mathbb{P}(V)$. It is also interesting to note that for projective hyperplanes the Furstenberg zero-law can be strengthened to the regularity property (1.2), while for algebraic sets the quantitative analog of (1.2) has not yet been established.

Using the zero-one law of the stationary measure ν we will prove the following local limit theorem for the coefficients $\langle f, G_n v\rangle$: under appropriate conditions, for any real numbers $a_1 < a_2$, any $f \in V^* \setminus \{0\}$ and $v \in V \setminus \{0\}$, as $n \to \infty$,

$$\mathbb{P}\Big(\log |\langle f, G_n v\rangle| - n\lambda \in [a_1, a_2] \Big) = \frac{a_2 - a_1}{\sigma\sqrt{2\pi n}}(1 + o(1)).$$

Based on a zero-one law for the invariant measure under the change of measure which is proved in Theorem 2.6, it is possible to establish a local limit theorem with large deviations for the coefficients of G_n, however this will be done elsewhere.

1.2. **Idea of the proof of the local limit thorem.** For illustration we show how to apply our zero-one law to establish the local limit theorem for the coefficients $\langle f, G_n v\rangle$, where $f \in V^* \setminus \{0\}$ and $v \in V \setminus \{0\}$. Letting $I = [a_1, a_2]$ be an interval of the real line, we have to handle the probability

$$\mathbb{P}(\log |\langle f, G_n v\rangle| - n\lambda \in I). \tag{1.6}$$

Using (1.3) and discretizing the values of $\delta(y, G_n x)$, the probability (1.6) is bounded from above by

$$\sum_{k=1}^{\infty} \mathbb{P}\Big(\log |G_n v| - n\lambda \in I_\eta + \eta k, G_n x \in Y_k^\eta \Big), \tag{1.7}$$

where $\eta > 0$ is sufficiently small and will be chosen later, $I_\eta = [a_1 - \eta, a_2 + \eta]$ and $Y_k^\eta = \{x \in \mathbb{P}(V) : -\log \delta(y, x) \in \eta[k-1, k)\}$. By the local limit theorem for the couple $(G_n x, \log |G_n v|)$ (actually in the paper we circumvent it by using the spectral gap theory and some smoothing technique), each probability in the sum (1.7) is asymptotically bounded by $\frac{\mathrm{length}(I_\eta)}{\sigma\sqrt{2\pi n}}\nu(\overline{Y}_k^\eta)$, with $\overline{Y}_k^\eta = \{x \in \mathbb{P}(V) : -\log \delta(y, x) \in \eta[k-1-\varepsilon, k+\varepsilon)\}$. Hence, the sum (1.7) asymptotically does not exceed

$$\frac{\mathrm{length}(I_\eta)}{\sigma\sqrt{2\pi n}} \sum_{k=1}^{\infty} \nu(\overline{Y}_k^\eta). \tag{1.8}$$

An important issue is to show that the sum in (1.8) converges to 1 as $\varepsilon \to 0$ and $\eta \to 0$. This turns out to be a difficult problem. By some easy calculations it reduces to showing that for any $k \geqslant 0$,

$$\nu\left(\{x \in \mathbb{P}(V) : \log \delta(y,x) = -\eta k\}\right) = 0. \tag{1.9}$$

The set $Y_0 = \{x \in \mathbb{P}(V) : \delta(y,x) = 0\}$ is a projective hyperplane in $\mathbb{P}(V)$ of dimension $d-2$. By (1.1), it holds that $\nu(Y_0) = 0$, under the strong irreducibility condition on the measure μ. For $k \geqslant 1$, the equality (1.9) may not be true for an arbitrary η, as we show in the paper. In fact we establish a zero-one law for the invariant measure ν (see Theorem 2.2), from which it follows that for any $t < 0$,

$$\nu\left(\{x \in \mathbb{P}(V) : \log \delta(y,x) = t\}\right) = 0 \text{ or } 1.$$

This statement implies that (1.9) holds true for any $k \geqslant 1$ if we choose an appropriate constant η. Indeed, if there exists $t < 0$ such that $\nu(\{x \in \mathbb{P}(V) : \log \delta(y,x) = t\}) = 1$, then we can choose η such that $-\eta k \neq t$ for any k, so that (1.9) holds true for any k. Otherwise η can be chosen arbitrarily. This proves that the sum in (1.8) converges to 1 as $\varepsilon \to 0$ and $\eta \to 0$ and so we obtain that $\limsup_{n\to\infty}$ of the probability (1.6) is bounded from above by $\frac{\mathrm{length}(I_\eta)}{\sigma\sqrt{2\pi n}}$. By some similar reasoning $\liminf_{n\to\infty}$ of (1.6) is bounded from below by the same quantity.

2. Main results

The inverse of $g \in \mathbb{G} = \mathrm{GL}(V)$ is denoted by g^{-1}. The adjoint operator g^* of $g \in \mathbb{G}$ is the automorphism g^* of V^* defined by $(g^*f)(v) = f(gv)$, where $v \in V$ and $f \in V^*$. Let $\mathcal{C}(\mathbb{P}(V))$ be the space of complex valued continuous functions on $\mathbb{P}(V)$.

All over the paper μ is a probability measure on $\mathbb{G}$. Denote by $\operatorname{supp}\mu$ the support of μ and by $\Gamma_\mu := [\operatorname{supp}\mu]$ the smallest closed subsemigroup of $\mathbb{G}$ generated by $\operatorname{supp}\mu$.

A matrix $g \in \mathbb{G}$ is called *proximal* if it has an algebraically simple dominant eigenvalue, namely, g has an eigenvalue λ_g satisfying $|\lambda_g| > |\lambda_g'|$ for all other eigenvalues λ_g' of g. It is easy to check that $\lambda_g \in \mathbb{R}$. We choose v_g^+ an eigenvector with unit norm $|v_g| = 1$, corresponding to the eigenvalue λ_g, which will be called dominant eigenvector of g. The unique element $x_g^+ = \mathbb{R}v_g^+ \in \mathbb{P}(V)$ will be called *attractor* of g. Note that it does not depend on the choice of v_g^+.

We need the following strong irreducibility and proximality conditions:

A1 (Strong irreducibility)**.** *No non-trivial finite union of proper subspaces of V is Γ_μ-invariant, that is, no non-trivial finite union of proper subspaces $V_1, \ldots, V_k$ in V satisfies $g(V_1 \cup \ldots \cup V_k) \subset V_1 \cup \ldots \cup V_k$ for any $g \in \Gamma_\mu$.*

A2 (Proximality)**.** Γ_μ *contains at least one proximal matrix.*

Assume conditions **A1** and **A2**. By a well known result of Furstenberg [13], on the projective space $\mathbb{P}(V)$ there exists a unique μ-stationary probability measure ν such that for any $\varphi \in \mathcal{C}(\mathbb{P}(V))$,

$$\int_{\mathbb{P}(V)} \int_{\mathbb{G}} \varphi(gx)\mu(dg)\nu(dx) = \int_{\mathbb{P}(V)} \varphi(x)\nu(dx). \tag{2.1}$$

Moreover, Furstenberg [13] (see also Bougerol and Lacroix [9, Chapter III, Proposition 2.3]) showed that under appropriate assumptions any proper projective subspace $Y \subset \mathbb{P}(V)$ has ν-measure 0.

Theorem 2.1 (Furstenberg)**.** *Assume condition* ***A1****. Then, for any μ-stationary measure ν and any proper projective subspace $Y \subset \mathbb{P}(V)$, it holds that $\nu(Y) = 0$.*

Our first result extends Theorem 2.1 to algebraic subsets of $\mathbb{P}(V)$. We recall that a subset X in V is algebraic if there exist polynomial functions $p_1, \dots, p_k$ on V such that $X = \{v \in V : p_1(v) = \dots = p_k(v) = 0\}$. We say that X is homogeneous if for every $t \in \mathbb{R}$ and $v \in X$ it holds that $tv \in X$. A subset Y of $\mathbb{P}(V)$ is algebraic if there exists an algebraic homogeneous subset X in V whose projective image on $\mathbb{P}(V)$ is Y.

Theorem 2.2. *Assume conditions* ***A1*** *and* ***A2****. Then, for any algebraic subset Y of $\mathbb{P}(V)$, it holds that either $\nu(Y) = 0$ or $\nu(Y) = 1$.*

The statement of Theorem 2.2 can be interpreted as a statement about the behaviour of the trajectory $(G_n x)_{n\geqslant 0}$: either for some $x \in \mathbb{P}(V)$ the trajectory $(G_n x)_{n\geqslant 0}$ stays in Y with probability 1, or for every $x \in \mathbb{P}(V)$ this trajectory mostly avoids Y. The case where the Zariski closure of Γ_μ acts transitively on $\mathbb{P}(V)$ has been considered in Proposition 10.1 (b) of Benoist and Quint [5], however, Theorem 2.2 covers also the case where this action is not transitive. For instance, in the example below this Zariski closure is contained in the group of isometries $O(q)$ which does not act transitively on $\mathbb{P}(V)$. Another point of view of the way the random walk $(G_n x)_{n\geqslant 0}$ avoids an algebraic subset is developed in the paper by Aoun [1].

The following example shows that there exist proper algebraic subsets of $\mathbb{P}(V)$ of ν-invariant measure 0 or 1.

Example 2.3. Let $d \geqslant 3$. Fix an integer p such that $1 \leqslant p \leqslant d-1$ and equip V with the quadratic form $q(v) = v_1^2 + v_2^2 + \dots + v_p^2 - v_{p+1}^2 - \dots - v_d^2$. Let $O(q)$ be the group of isometries of q, that is the group of elements $g \in \mathrm{GL}(V)$ such that $q(gv) = q(v)$ for all $v \in V$. We choose μ to be any probability measure on $O(q)$ such that Γ_μ is proximal and strongly irreducible. For instance we can take any probability measure μ with the full support $O(q)$; then it will be proximal and strongly irreducible, since the group $O(q)$ is proximal and strongly irreducible on V for $d \geqslant 3$. Denote by ν the unique μ-stationary probability measure on $\mathbb{P}(V)$. Let Y be the subset of $\mathbb{P}(V)$ defined as the set of straight lines in V which are spanned by vectors $v \in V$ with $q(v) = 0$. Then the μ-invariant measure ν is such that $\nu(Y) = 1$. In particular this implies that the support of the measure ν is contained in Y.

Consider the case when $p = 1$. Let f be the linear functional $f : u \in V \mapsto u_1 \in \mathbb{R}$ so that $|f| = 1$ and let $y = \mathbb{R}f \in \mathbb{P}(V^*)$. Define the algebraic subset $Y = \{x \in \mathbb{P}(V) : \delta(x, y) = 1/\sqrt{2}\}$. We will show that Y contains the support of the measure ν. Indeed, if v is such that $q(v) = 0$, then $v_1^2 = v_2^2 + \dots + v_d^2$ and hence, with $x = \mathbb{R}v \in \operatorname{supp} \nu$,

$$\delta(x, y) = \frac{|\langle f, v\rangle|}{|f||v|} = \frac{1}{\sqrt{2}}.$$

Therefore $\nu(Y) = 1$. Moreover, if we define $Y' = \{x \in \mathbb{P}(V) : \delta(x, y) = t\}$ with $t \neq 1/\sqrt{2}$, then $\nu(Y') = 0$.

Corollary 2.4. *Assume conditions* ***A1*** *and* ***A2****. For any* $t \in (-\infty, 0)$*, define the hypersurface* $Y_0 = \{x \in \mathbb{P}(V) : \delta(x,y) = t\}$*. Then* $\nu(Y_0)$ *is* 0 *or* 1*.*

Proof. Let $f \in V^*$ be such that $|f| = 1$ and $y = \mathbb{R}f \in \mathbb{P}(V^*)$. Define the homogenous set X_0 as the collection of all vectors $v \in V$ satisfying $\langle f, v\rangle^2 = t^2|v|^2$. Then Y_0 is the projective image of $X_0 \setminus \{0\}$ on $\mathbb{P}(V)$. As the function $\phi : v \in V \mapsto f(v)^2 - t^2|v|^2 = (f_1v_1 + \ldots + f_dv_d)^2 - t^2(v_1^2 + \ldots + v_d^2)$ is a polynomial on V, the set X_0 is algebraic. Since it is also homogeneous, by definition the set Y_0 is algebraic in $\mathbb{P}(V)$, and the conclusion follows from Theorem 2.2. □

In order to state our second result we need to introduce the transfer operators and related notions. These operators play an important role in many problems related to random walks on the group $\mathbb{G}$, see for instance Le Page [25] or Guivarc'h and Le Page [21]. We are going to show that the stationary measures related to these operators do not charge the algebraic subsets.

For any $g \in \mathbb{G}$, $x = \mathbb{R}v \in \mathbb{P}(V)$ and $y = \mathbb{R}f \in \mathbb{P}(V^*)$, set

$$\sigma(g, x) = \log \frac{|gv|}{|v|}, \quad \sigma(g^*, y) = \log \frac{|g^*f|}{|f|}. \tag{2.2}$$

Denote $N(g) = \max\{\|g\|, \|g^{-1}\|\}$. Consider the sets

$$I_\mu^+ = \left\{ s \geqslant 0 : \int_{\mathbb{G}} \|g\|^s \mu(dg) < +\infty \right\}, \quad I_\mu^- = \left\{ s \leqslant 0 : \int_{\mathbb{G}} N(g)^{-s} \mu(dg) < +\infty \right\}$$

and note that both I_μ^+ and I_μ^- contain at least the element 0. Let $s \in I_\mu^+ \cup I_\mu^-$. Define the transfer operators P_s and P_s^* as follows: for any $\varphi \in \mathcal{C}(\mathbb{P}(V))$ and $x \in \mathbb{P}(V)$,

$$P_s\varphi(x) = \int_{\mathbb{G}} e^{s\sigma(g,x)} \varphi(gx)\mu(dg) \tag{2.3}$$

and for any $\varphi \in \mathcal{C}(\mathbb{P}(V^*))$ and $y \in \mathbb{P}(V^*)$,

$$P_s^*\varphi(y) = \int_{\mathbb{G}} e^{s\sigma(g^*,y)} \varphi(g^*y)\mu(dg). \tag{2.4}$$

The next assumption will be necessary to state the results for $s < 0$.

A3 (Two-sided exponential moment)**.** *There exists* $\alpha \in (0,1)$ *such that*

$$\int_{\mathbb{G}} N(g)^\alpha \mu(dg) < +\infty.$$

Let $s \in I_\mu^+$. By [21, Theorem 2.6], under conditions **A1** and **A2**, there exist a unique $\kappa(s) > 0$ and a unique probability measure ν_s on $\mathbb{P}(V)$ such that ν_s is an eigenmeasure for the transfer operator P_s corresponding to the eigenvalue $\kappa(s)$:

$$P_s\nu_s = \kappa(s)\nu_s. \tag{2.5}$$

Similarly, the conjugate transfer operator P_s^* has a unique probability eigenmeasure ν_s^* on $\mathbb{P}(V^*)$ corresponding to the same eigenvalue $\kappa(s)$:

$$\nu_s^* P_s^* = \kappa(s)\nu_s^*.$$

For detailed account of the mentioned properties for $s > 0$ we refer the reader to [20, 21], where it is proved that the mapping $s \mapsto \kappa(s)$ is analytic on a complex neighborhood of the interval I_μ^+. Guivarc'h and Le Page [21] have also proved that the measure ν_s does not charge any proper projective subspace Y in $\mathbb{P}(V)$.

Theorem 2.5 (Guivarc'h and Le Page)**.** *Assume conditions* ***A1*** *and* ***A2****. Then, for any $s \in I_\mu^+$ and any proper projective subspace Y of $\mathbb{P}(V)$, it holds that $\nu_s(Y) = 0$.*

Our second result extends Theorem 2.5 to proper algebraic subsets of $\mathbb{P}(V)$.

Theorem 2.6. *Assume conditions* ***A1*** *and* ***A2****. Then, for any $s \in I_\mu^+$ and any proper algebraic subset Y of $\mathbb{P}(V)$, it holds that either $\nu_s(Y) = 0$ or $\nu_s(Y) = 1$.*

We are able to prove a similar assertion for small negative s. First we show in Section 3 the existence and the uniqueness of the eigenmeasure ν_s for small negative values s.

Proposition 2.7. *Assume conditions* ***A1****,* ***A2*** *and* ***A3****. Then, there exists $s_0 > 0$ with the following property: for any $s \in [-s_0, 0)$, there exist a unique $\kappa(s) > 0$ and a unique probability measure ν_s on $\mathbb{P}(V)$ such that ν_s is an eigenmeasure for the transfer operator P_s corresponding to the eigenvalue $\kappa(s)$. Similarly, the transfer operator P_s^* has a unique probability eigenmeasure ν_s^* on $\mathbb{P}(V^*)$ corresponding to the same eigenvalue $\kappa(s)$.*

For negative values of s we can prove the following analogue of Theorem 2.6.

Theorem 2.8. *Assume conditions* ***A1****,* ***A2*** *and* ***A3****. Then, there exists $s_0 > 0$ such that for any $s \in [-s_0, 0)$ and any proper algebraic subset Y of $\mathbb{P}(V)$, it holds that either $\nu_s(Y) = 0$ or $\nu_s(Y) = 1$.*

It is easy to see that all the conclusions of the Example 2.3 apply also to the measure ν_s.

As an application of the stated results we use Theorem 2.2 to establish a local limit theorem for the coefficients of random walks on the general linear group $\mathrm{GL}(V)$.

Theorem 2.9. *Assume conditions* ***A1****,* ***A2*** *and* ***A3****. Let $-\infty < a_1 < a_2 < \infty$ be real numbers. Then, as $n \to \infty$, uniformly in $f \in V^*$ and $v \in V$ with $|f| = 1$ and $|v| = 1$,*

$$\mathbb{P}\Big(\log |\langle f, G_n v \rangle| - n\lambda \in [a_1, a_2] \Big) = \frac{a_2 - a_1}{\sigma\sqrt{2\pi n}}(1 + o(1)).$$

To the best of our knowledge, a local limit theorem for the coefficients of random walks on the general linear group $\mathrm{GL}(V)$ has not been established in the literature so far. Local limit theorem for sums of independent random variables have been studied by many authors: we refer the reader to Gnedenko [15], Stone [27], Borovkov and Borovkov [8], Breuillard [11]. For the norm cocycle of random walks on $\mathrm{GL}(V)$, local limit theorems have been proved by Le Page [25], Guivarc'h [19], Benoist and Quint [5]. The local limit theorems established in [5] play an important role for studying stationary measures on finite volume homogeneous spaces, see [2] for details.

Other potential applications of Theorem 2.2 are the Berry-Esseen bound, the Edgeworth expansion and the Cramér type moderate deviation. A paper in progress is [31], where a local limit theorem with moderate deviations has been established. In its turn Theorems 2.6 and 2.8 can be used to establish various limit theorems like the large deviation principle and local limit theorem with large deviations for the coefficients [30].

3. Properties of the stationary measure

3.1. **The existence and the uniqueness of the probability eigenmeasure for negative** s**.** In this section we prove the existence and the uniqueness of the probability eigenmeasure of the transfer operator P_s for $s < 0$. Actually we shall establish it for s in a sufficiently small neighborhood of 0, i.e. for $|s| < s_0$, for some small $s_0 > 0$. The approach is inspired by that of Guivarc'h and Le Page [20, 21]. The existence of such a measure is deduced from a very general fixed point theorem due to Brouwer-Schauder-Tychonoff. To establish the uniqueness one can make use of the general results on the perturbation theory of linear operators, which simplifies slightly the proofs of Guivarc'h and Le Page [20, 21].

We proceed to state a fixed point theorem for measures in an abstract context. Let X be a compact topological space and $\mathcal{C}(X)$ be the space of complex valued continuous functions on X equipped with the uniform norm. Denote by $\mathcal{C}(X)'$ the topological dual space of $\mathcal{C}(X)$ equipped with the weak-$*$ topology. Recall that the weak-$*$ topology is the weakest topology on $\mathcal{C}(X)'$ for which the mapping $\nu \in \mathcal{C}(X)' \mapsto \nu(\varphi) \in \mathbb{C}$ is continuous for any $\varphi \in \mathcal{C}(X)$ and that by Riesz representation theorem, the space $\mathcal{C}(X)'$ coincides with the space of complex valued Borel measures on X.

Lemma 3.1. *Let $T : \mathcal{C}(X) \to \mathcal{C}(X)$ be a bounded linear operator such that $T(f) > 0$ for any $f > 0$. Then, there exist a constant $\alpha > 0$ and a Borel probability measure ν_0 on X such that $T'\nu_0 = \alpha\nu_0$, where $T' : \mathcal{C}(X)' \to \mathcal{C}(X)'$ is the adjoint operator of T.*

Proof. Recall the Brouwer-Schauder-Tychonoff theorem (see [6, Appendix]): Let $\mathcal{P}$ be a convex and compact subset inside a topological vector space V and let $A : \mathcal{P} \to \mathcal{P}$ be a continuous mapping. Then there exists $\nu_0 \in \mathcal{P}$ such that $A\nu_0 = \nu_0$.

We shall apply Brouwer-Schauder-Tychonoff theorem with $V = \mathcal{C}(X)'$. By Riesz representation theorem, the space $\mathcal{C}(X)'$ coincides with the space of complex valued Borel measures on X. Let $\mathcal{P}$ be the subspace of $\mathcal{C}(X)'$ formed by probability measures. Then the set $\mathcal{P}$ is convex and by the Banach-Alaoglu theorem it is compact in the weak-$*$ topology. Since $T1 > 0$ everywhere, the mapping $\nu \in \mathcal{P} \mapsto \nu(T(1)) \in \mathbb{R}$ does not vanish on $\mathcal{P}$. Note that for any $\varphi \in \mathcal{C}(X)$ with $\varphi \geqslant 0$, it holds $T'\nu(\varphi) = \nu(T\varphi) \geqslant 0$ since $T\varphi \geqslant 0$. This allows to define the mapping $A : \mathcal{P} \to \mathcal{P}$ by setting, for any $\nu \in \mathcal{P}$,

$$A\nu = \frac{T'\nu}{\nu(T1)}.$$

As T is a bounded linear operator on $\mathcal{C}(X)$, the adjoint operator T' is continuous for the weak-$*$ topology. Since $T1$ is a continuous function on X, the mapping $\nu \mapsto \nu(T1)$ is also continuous for the weak-$*$ topology. Therefore, the mapping A is continuous for the weak-$*$ topology. By the Brouwer-Schauder-Tychonoff theorem, the mapping A has a fixed point ν_0: $A\nu_0 = \nu_0$. The assertion follows with $\alpha = \nu_0(T(1))$. □

Let $\gamma \in (0,1)$. Consider the Banach space $\mathscr{B}_\gamma$ of γ-Hölder continuous functions on $\mathbb{P}(V)$ endowed with the norm

$$\|\varphi\|_{\mathscr{B}_\gamma} = \sup_{x\in\mathbb{P}(V)} |\varphi(x)| + \sup_{x,x'\in\mathbb{P}(V):\, x\neq x'} \frac{|\varphi(x)-\varphi(x')|}{\mathbf{d}(x,x')^\gamma},$$

where $\mathbf{d}(x,x')$ is the sin of the angle between the vector lines $x=\mathbb{R}v$ and $x'=\mathbb{R}v'$ in $\mathbb{P}(V)$: $\mathbf{d}(x,x') = \sqrt{1-\left(\frac{\langle v,v'\rangle}{|v||v'|}\right)^2}$. The topological dual of $\mathscr{B}_\gamma$ endowed with the induced norm is denoted by $\mathscr{B}_\gamma'$. Denote by $\mathscr{B}_\gamma^*$ the Banach space of γ-Hölder continuous functions on $\mathbb{P}(V^*)$ endowed with the norm

$$\|\varphi\|_{\mathscr{B}_\gamma^*} = \sup_{y\in\mathbb{P}(V^*)} |\varphi(x)| + \sup_{y,y'\in\mathbb{P}(V^*):\, y\neq y'} \frac{|\varphi(y)-\varphi(y')|}{\mathbf{d}(y,y')^\gamma},$$

where $\mathbf{d}(y,y')$ is the sin of the angle between the vector lines $y=\mathbb{R}f$ and $y'=\mathbb{R}f'$ in $\mathbb{P}(V^*)$: $\mathbf{d}(y,y') = \sqrt{1-\left(\frac{\langle f,f'\rangle}{|f||f'|}\right)^2}$.

Recall that $\mathcal{C}(\mathbb{P}(V))$ is the space of the continuous complex valued functions on $\mathbb{P}(V)$ and $\mathcal{C}(\mathbb{P}(V))'$ is the space of the complex valued Borel measures on $\mathbb{P}(V)$.

Lemma 3.2. *Assume conditions **A1**, **A2** and **A3**. Then, there exists a positive constant s_0 such that for any $|s| < s_0$ the operator P_s has a unique probability eigenmeasure ν_s associated with the unique eigenvalue $\kappa(s)$.*

Proof. Note that for s real and close to 0, $P_s : \mathcal{C}(\mathbb{P}(V)) \to \mathcal{C}(\mathbb{P}(V))$ is a bounded linear operator. Moreover $P_s\varphi > 0$ for any $\varphi > 0$ on $\mathbb{P}(V)$. Denote by $P_s' : \mathcal{C}(\mathbb{P}(V))' \to \mathcal{C}(\mathbb{P}(V))'$ the adjoint operator of P_s. Using Lemma 3.1 with $X=\mathbb{P}(V)$, we get that there exists a probability eigenmeasure ν_s such that $P_s'\nu_s = \alpha(s)\nu_s$. By the duality, for any $\varphi \in \mathcal{C}(\mathbb{P}(V))$, we have $\nu_s(P_s\varphi) = P_s'\nu_s(\varphi)$, so that

$$\nu_s(P_s\varphi) = \alpha(s)\nu_s(\varphi). \tag{3.1}$$

The measure ν_s and the eigenvalue $\alpha(s)$ might a priori not be unique. We shall prove the uniqueness using the perturbation theory of linear operators.

From the results of Le Page [25] it follows that the operator P_0 has a spectral gap property on the Banach space $\mathscr{B}_\gamma$, for some $\gamma > 0$. Since the dependence of the operator P_s in s is analytic (cf. [9, Chapter V, Lemma 3.2] or [5, Lemma 11.17]), we can apply the perturbation theory of linear operators [24, Theorem III.8]. Thus there exist constants $s_0 > 0$ and holomorphic mappings $s \mapsto \theta_s \in \mathscr{B}_\gamma'$, $s \mapsto r_s \in \mathscr{B}_\gamma$, $s \mapsto \kappa_s \in \mathbb{C}$ on $(-s_0, s_0)$ such that $\theta_0 = \nu$, $r_0 = 1$, $\nu(r_s) = 1$, $\theta_s(r_s) = 1$ and

$$\theta_s(P_s\varphi) = \kappa(s)\theta_s(\varphi), \quad P_s r_s = \kappa(s) r_s, \tag{3.2}$$

for any $\varphi \in \mathcal{C}(\mathbb{P}(V))$. Moreover, one can choose s_0 small enough so that there exist constants $C > 0$ and $\rho_0 \in (0,1)$ such that for any complex $|s| < s_0$, we have $|\kappa(s)| > \rho_0$ and

$$\|P_s^n\varphi - \kappa(s)^n\theta_s(\varphi)r_s\|_{\mathscr{B}_\gamma} \leqslant C\rho_0^n \|\varphi\|_{\mathscr{B}_\gamma}. \tag{3.3}$$

In particular, (3.3) implies that for any complex $|s| < s_0$, the complex number $\kappa(s)$ is the unique eigenvalue of P_s in $\mathscr{B}_\gamma$ with modulus strictly larger than ρ_0 and the associated eigenspace is $\mathbb{C}r_s$. Indeed, let $\varphi \in \mathscr{B}_\gamma$ with $\varphi \neq 0$ and $\lambda \in \mathbb{C}$ be such that $P_s(\varphi) = \lambda\varphi$ and $\lambda \neq \kappa(s)$. Then

$$\lambda\theta_s(\varphi) = \theta_s(P_s\varphi) = \kappa(s)\theta_s(\varphi),$$

hence $\theta_s(\varphi) = 0$. Therefore, (3.3) gives that

$$|\lambda|^n \|\varphi\|_{\mathscr{B}_\gamma} = \|P_s^n\varphi\|_{\mathscr{B}_\gamma} \leqslant C\rho_0^n \|\varphi\|_{\mathscr{B}_\gamma}.$$

This implies that $|\lambda| \leqslant \rho_0$, which means that $\kappa(s)$ is the unique eigenvalue with modulus strictly larger than ρ_0.

Let us now show that the eigenspace associated to $\kappa(s)$ is spanned by the function r_s. Indeed, if φ is in this eigenspace then $\varphi \in \mathscr{B}_\gamma$ and $P_s\varphi = \kappa(s)\varphi$. Again by (3.3), we have

$$|\kappa(s)|^n \|\varphi - \theta_s(\varphi)r_s\|_{\mathscr{B}_\gamma} = \|P_s^n\varphi - \kappa(s)^n\theta_s(\varphi)r_s\|_{\mathscr{B}_\gamma} \leqslant C\rho_0^n\|\varphi\|_{\mathscr{B}_\gamma}.$$

Since $|\kappa(s)| > \rho_0$, we get $\varphi = \theta_s(\varphi)r_s$.

We shall use the uniqueness property of $\kappa(s)$ to show that $\kappa(s)$ is real and that r_s takes real values for real $s \in (-s_0, s_0)$. Indeed, as s is real, for any $\varphi \in \mathscr{B}_\gamma$ we have $P_s\overline{\varphi} = \overline{P_s\varphi}$, which gives $P_s\overline{r}_s = \overline{P_s r_s} = \overline{\kappa}_s\overline{r}_s$. Since $\kappa(s)$ is the unique eigenvalue of P_s with modulus strictly larger that ρ_0, this proves $\overline{\kappa}_s = \kappa_s$. Besides, from the equation $P_s\overline{r}_s = \kappa_s\overline{r}_s$ it follows that $\overline{r}_s$ belongs to the eigenspace $\mathbb{C}r_s$ associated to $\kappa(s)$, so there exists $z \in \mathbb{C}$ such that $\overline{r}_s = zr_s$. Since $\nu(r_s) = 1 = \nu(\overline{r}_s)$, we get $z = 1$ and hence $\overline{r}_s = r_s$ as required.

Since $r_0 = 1$, we can assume that s_0 is small enough such that r_s is strictly positive for real $s \in (-s_0, s_0)$. We now prove that $\alpha(s) = \kappa(s)$ for real $s \in (-s_0, s_0)$. We put $\varphi = r_s$ in (3.1) and use the second identity in (3.2) to obtain

$$\alpha(s)\nu_s(r_s) = \nu_s(P_s r_s) = \kappa(s)\nu_s(r_s). \tag{3.4}$$

Since $r_s > 0$ we have $\nu_s(r_s) > 0$, which implies that $\alpha(s) = \kappa(s)$ for real valued s.

Iterating (3.1) and using the fact that $\alpha(s) = \kappa(s)$, we have that for any $\varphi \in \mathscr{B}_\gamma$,

$$\nu_s\left(P_s^n\varphi\right) = \kappa(s)^n\nu_s(\varphi). \tag{3.5}$$

From (3.5) and (3.3), taking the limit as $n \to \infty$ we obtain that

$$\nu_s(\varphi) = \nu_s\left(\theta_s(\varphi)r_s\right) = \theta_s(\varphi)\nu_s(r_s),$$

from which it follows that for any $\varphi \in \mathscr{B}_\gamma$,

$$\frac{\nu_s(\varphi)}{\nu_s(r_s)} = \theta_s(\varphi).$$

This proves that the linear functional θ_s is indeed a non-negative Borel measure, and that any non-negative Borel measure which is an eigenmeasure of P_s is proportional to θ_s. □

To show the uniqueness of the eigenfunction of the operator P_s we need more notation. For any real $|s| < s_0$, let $r_s > 0$ be the function introduced in the proof of Lemma 3.2. Introduce the operator Q_s by setting, for any $\varphi \in \mathcal{C}(\mathbb{P}(V))$,

$$Q_s\varphi = \frac{P_s(r_s\varphi)}{\kappa(s)r_s}. \tag{3.6}$$

Then Q_s is a Markov operator, namely, $Q_s\varphi \geqslant 0$ for any $\varphi \geqslant 0$, and $Q_s(1) = 1$.

Lemma 3.3. *For any $\varphi \in \mathcal{C}(\mathbb{P}(V))$, one has*

$$\lim_{n\to\infty} \sup_{x\in\mathbb{P}(V)} \left| Q_s^n(\varphi)(x) - \frac{\nu_s(\varphi r_s)}{\nu_s(r_s)} \right| = 0 \tag{3.7}$$

and

$$\lim_{n\to\infty} \sup_{x\in\mathbb{P}(V)} \left| \frac{1}{\kappa(s)^n} P_s^n(\varphi)(x) - \frac{\nu_s(\varphi)}{\nu_s(r_s)} r_s(x) \right| = 0. \tag{3.8}$$

Proof. First note that from (3.3) we have that (3.8) holds for any $\varphi \in \mathscr{B}_\gamma$. This implies that (3.7) also holds for any $\varphi \in \mathscr{B}_\gamma$. As Q_s is a Markov operator, it has norm 1 in $\mathcal{C}(\mathbb{P}(V))$, so Q_s^n is uniformly bounded in the space of bounded operators of $\mathcal{C}(\mathbb{P}(V))$. Since $\mathscr{B}_\gamma$ is dense in $\mathcal{C}(\mathbb{P}(V))$, the convergence (3.7) holds for any $\varphi \in \mathcal{C}(\mathbb{P}(V))$. This in turn implies that (3.8) also holds for $\varphi \in \mathcal{C}(\mathbb{P}(V))$. □

Lemma 3.4. *Assume conditions* ***A1***, ***A2*** *and* ***A3***. *Then, for any $|s| < s_0$, the function r_s is the unique (up to a scaling constant) non-negative continuous eigenfunction of the operator P_s.*

Proof. If we take φ to be a non-negative non-zero continuous eigenfunction of P_s associated to some eigenvalue λ, then by (3.8) we get

$$\lim_{n\to\infty} \sup_{x\in\mathbb{P}(V)} \left| \frac{1}{\kappa(s)^n} \lambda^n \varphi(x) - \frac{\nu_s(\varphi)}{\nu_s(r_s)} r_s(x) \right| = 0. \tag{3.9}$$

It follows that $\lambda = \kappa(s)$ and $\varphi = \frac{\nu_s(\varphi)}{\nu_s(r_s)} r_s$, which means that the function φ coincides (up to a scaling constant) with the eigenfunction r_s. □

Applying the previous theory to the operator P_s^* and to the adjoint projective space $\mathbb{P}(V^*)$ we obtain the following:

Lemma 3.5. *Assume conditions* ***A1***, ***A2*** *and* ***A3***. *Then, there exists a positive constant s_0 such that for any real $|s| < s_0$ the operator P_s^* has a unique probability eigenmeasure ν_s^* and a unique (up to a scaling constant) positive continuous eigenfunction r_s^* associated with the same unique eigenvalue $\kappa^*(s)$.*

We will show in Lemma 3.10 that $\kappa^*(s) = \kappa(s)$.

3.2. **The Hölder regularity of the stationary measure.** In this section we establish the Hölder regularity of the stationary measure ν_s defined in Lemma 3.2.

Note that ν_0 coincides with the stationary measure ν defined by (2.1). The Hölder regularity of the stationary measure ν has been established in [22] (see also [9, 18, 5]): under conditions **A1**, **A2** and **A3**, there exists a constant $\alpha > 0$ such that

$$\sup_{y\in\mathbb{P}(V^*)} \int_{\mathbb{P}(V)} \frac{1}{\delta(y,x)^\alpha} \nu(dx) < +\infty. \tag{3.10}$$

By the Frostman lemma (see [26]), the assertion (3.10) implies that the Hausdorff dimension of the stationary measure ν is at least α. As mentioned before, (3.10) plays a crucial role for establishing limit theorems such as the law of large numbers and the central limit theorem for the coefficients $\langle f, G_n v \rangle$ (see [22, 9, 18, 5]). In the following we establish the Hölder regularity of the stationary measure ν_s when s is in a small neighborhood of 0. The proof is based on (3.10) and the spectral gap properties of the transfer operator P_s established in subsection 3.1.

Proposition 3.6. *Assume conditions* ***A1***, ***A2*** *and* ***A3***. *Then, there exist constants $s_0 > 0$ and $\alpha > 0$ such that*

$$\sup_{s\in(-s_0,s_0)} \sup_{y\in\mathbb{P}(V^*)} \int_{\mathbb{P}(V)} \frac{1}{\delta(y,x)^\alpha} \nu_s(dx) < +\infty. \tag{3.11}$$

In particular, there exist constants $\alpha, C > 0$ such that for any $0 < t < 1$,

$$\sup_{s\in(-s_0,s_0)} \sup_{y\in\mathbb{P}(V^*)} \nu_s\left(\{x \in \mathbb{P}(V) : \delta(y,x) \leqslant t\}\right) \leqslant Ct^\alpha. \tag{3.12}$$

One implication of the Proposition 3.6 is that the eigenmeasure ν_s does not charge the projective hyperplanes. Precise formulation follows:

Corollary 3.7. *Assume conditions* ***A1***, ***A2*** *and* ***A3***. *Then there exists a constant $s_0 > 0$ such that for any $s \in (-s_0, s_0)$ and any projective hyperplane Y of $\mathbb{P}(V)$ it holds $\nu_s(Y) = 0$.*

Before proceeding to proving Proposition 3.6, let us first recall a change of measure formula which will be used in the proof of Proposition 3.6. For any $s \in (-s_0, s_0)$, the family of functions $q_n^s(x,g) = \frac{e^{s\sigma(g,x)}}{\kappa(s)^n}\frac{r_s(gx)}{r_s(x)}$, $n \geqslant 1$, satisfies the equation

$$q^s_{k+m}(x, g_2g_1) = q^s_k(g_1x, g_2)q^s_m(x, g_1)$$

for $x \in \mathbb{P}(V)$ and $g_1, g_2 \in \mathbb{G}$. This, together with the fact that $P_sr_s = \kappa(s)r_s$, implies that the probability measures

$$\mathbb{Q}^x_{s,n}(dg_1,\dots,dg_n) = q^s_n(x, g_n\dots g_1)\mu(dg_1)\dots\mu(dg_n) \tag{3.13}$$

form a projective system on $\mathbb{G}^{\mathbb{N}}$. By the Kolmogorov extension theorem, there exists a unique probability measure $\mathbb{Q}^x_s$ on $\mathbb{G}^{\mathbb{N}}$ with marginals $\mathbb{Q}^x_{s,n}$. We denote by $\mathbb{E}_{\mathbb{Q}^x_s}$ the corresponding expectation. For any measurable function φ on $(\mathbb{P}(V)\times\mathbb{R})^n$, it holds that

$$\begin{aligned}\frac{1}{\kappa(s)^n r_s(x)}&\mathbb{E}\Big[r_s(G_nx)e^{s\sigma(G_n,x)}\varphi\Big(G_1x,\sigma(G_1,x),\dots,G_nx,\sigma(G_n,x)\Big)\Big]\\ &= \mathbb{E}_{\mathbb{Q}^x_s}\Big[\varphi\Big(G_1x,\sigma(G_1,x),\dots,G_nx,\sigma(G_n,x)\Big)\Big]. \end{aligned}\tag{3.14}$$

Under the changed measure $\mathbb{Q}^x_s$, the Markov chain $(G_nx)_{n\geqslant 0}$ has a unique stationary measure π_s defined by $\pi_s(\varphi) = \frac{\nu_s(\varphi r_s)}{\nu_s(r_s)}$, for any $\varphi \in \mathcal{C}(\mathbb{P}(V))$.

We shall use the following result which has been established in [5, Lemma 14.11].

Lemma 3.8. *Assume conditions* ***A1***, ***A2***, ***A3***. *Then, for any $\varepsilon > 0$, there exist constants $c_0 > 0$ and $n_0 \geqslant 1$ such that for all $n \geqslant k \geqslant n_0$, $x \in \mathbb{P}(V)$ and $y \in \mathbb{P}(V^*)$,*

$$\mathbb{P}\Big(\delta(y, G_nx) \leqslant e^{-\varepsilon k}\Big) \leqslant e^{-c_0k}.$$

Proof of Proposition 3.6. Step 1. We choose a small enough constant $s_0 > 0$ and show that for any $\varepsilon > 0$, there exist constants $c_1 > 0$ and $n_0 \geqslant 1$ such that for any $n \geqslant n_0$,

$$\sup_{s\in(-s_0,s_0)} \sup_{y\in\mathbb{P}(V^*)} \sup_{x\in\mathbb{P}(V)} \mathbb{Q}^x_s\Big(\delta(y, G_nx) \leqslant e^{-\varepsilon n}\Big) \leqslant e^{-c_1n}. \tag{3.15}$$

To prove this, using (3.14) and the fact that the eigenfunction $x \mapsto r_s(x)$ is strictly positive and bounded on $\mathbb{P}(V)$ uniformly in $s \in (-s_0, s_0)$, we get

$$\begin{aligned}\mathbb{Q}_s^x\Big(\delta(y, G_n x) \leqslant e^{-\varepsilon n}\Big) &= \frac{1}{\kappa(s)^n r_s(x)}\mathbb{E}\left[e^{s\sigma(G_n,x)} r_s(G_n x)\mathbb{1}_{\{\delta(y,G_n x)\leqslant e^{-\varepsilon n}\}}\right] \\ &\leqslant \frac{c}{\kappa(s)^n}\mathbb{E}\left[e^{s\sigma(G_n,x)}\mathbb{1}_{\{\delta(y,G_n x)\leqslant e^{-\varepsilon n}\}}\right].\end{aligned}$$

By Hölder's inequality, it follows that

$$\mathbb{Q}_s^x\Big(\delta(y, G_n x) \leqslant e^{-\varepsilon n}\Big) \leqslant \frac{c}{\kappa(s)^n}\left[\mathbb{E}e^{2s\sigma(G_n,x)}\right]^{1/2}\left[\mathbb{P}\big(\delta(y, G_n x) \leqslant e^{-\varepsilon n}\big)\right]^{1/2}. \quad (3.16)$$

It is easy to see that $\mathbb{E}e^{2s\sigma(G_n,x)} \leqslant \Big\{\mathbb{E}\big[N(g_1)^{2|s|}\big]\Big\}^n$. Since $\kappa(0) = 1$ and the function κ is continuous in a small neighborhood of 0, we can choose a sufficiently small constant $s_0 > 0$ such that

$$\sup_{s\in(-s_0,s_0)}\sup_{x\in\mathbb{P}(V)}\frac{1}{\kappa(s)^n}\left[\mathbb{E}e^{2s\sigma(G_n,x)}\right]^{1/2} \leqslant e^{c_2 n},$$

where $c_2 > 0$ is a constant satisfying $c_2 < c_0/4$ with c_0 given in Lemma 3.8. This, together with (3.16) and Lemma 3.8, concludes the proof of (3.15) with $c_1 = c_0/4$.

Step 2. The invariance of π_s under the operator Q_s^n says that for any non-negative Borel measurable function φ on $\mathbb{P}(V)$, we have

$$\pi_s(\varphi) = \int_{\mathbb{P}(V)}\int_{\mathbb{G}^n}\varphi(g_n\dots g_1 x)\mathbb{Q}_{s,n}^x(dg_1,\dots,dg_n)\pi_s(dx),$$

where $\mathbb{Q}_{s,n}^x$ is a probability measure on $\mathbb{G}^n$ defined by (3.13). This, together with (3.15), gives that uniformly in $s \in (-s_0, s_0)$ and $y \in \mathbb{P}(V^*)$,

$$\pi_s\left(\{x \in \mathbb{P}(V) : \delta(y, x) \leqslant e^{-\varepsilon n}\}\right) = \int_{\mathbb{P}(V)}\mathbb{Q}_{s,n}^x(\delta(y, G_n x) \leqslant e^{-\varepsilon n})\pi_s(dx) \leqslant e^{-c_1 n}. \quad (3.17)$$

We denote $B_n := \{x \in \mathbb{P}(V) : e^{-\varepsilon(n+1)} \leqslant \delta(y, x) \leqslant e^{-\varepsilon n}\}$. Choosing $\alpha \in (0, c_1/\varepsilon)$, we deduce from (3.17) that uniformly in $s \in (-s_0, s_0)$ and $y \in \mathbb{P}(V^*)$,

$$\begin{aligned}&\int_{\mathbb{P}(V)}\frac{1}{\delta(y,x)^\alpha}\pi_s(dx) \\ &= \int_{\{x\in\mathbb{P}(V):\delta(y,x)>e^{-\varepsilon n_0}\}}\frac{1}{\delta(y,x)^\alpha}\pi_s(dx) + \sum_{n=n_0}^{\infty}\int_{B_n}\frac{1}{\delta(y,x)^\alpha}\pi_s(dx) \\ &\leqslant e^{\varepsilon n_0\alpha} + \sum_{n=n_0}^{\infty}e^{\varepsilon\alpha(n+1)}e^{-c_1 n} < +\infty.\end{aligned}$$

This proves (3.11) by the relation $\pi_s(\varphi) = \frac{\nu_s(\varphi r_s)}{\nu_s(r_s)}$ for any $\varphi \in \mathcal{C}(\mathbb{P}(V))$. The inequality (3.12) is a direct consequence of (3.11) by the Markov inequality. □

Let s_0 be small enough. For any real s such that $|s| < s_0$, any $y \in \mathbb{P}(V^*)$ and bounded measurable function φ on $\mathbb{P}(V)$ denote

$$\nu_s^y(\varphi) = \int_{\mathbb{P}(V)}\varphi(x)\delta(x,y)^s\nu_s(dx).$$

Corollary 3.9. *There exists $s_0 > 0$ such that for any $s \in (-s_0, 0)$, the mapping $y \in \mathbb{P}(V^*) \mapsto \nu_s^y \in \mathcal{C}(\mathbb{P}(V))'$ is continuous for the total variation norm on $\mathcal{C}(\mathbb{P}(V))'$.*

Proof. Let s_0 be small enough and $s \in (-s_0, 0)$. Let $y, y' \in \mathbb{P}(V^*)$. Since both ν_s^y and $\nu_s^{y'}$ are absolutely continuous with respect to ν_s, we have

$$\begin{aligned}\|\nu_s^y - \nu_s^{y'}\|_{TV} &= \int_{\mathbb{P}(V)} \left|\delta(x,y)^s - \delta(x,y')^s\right| \nu_s(dx) \\ &= \int_{\mathbb{P}(V)} \frac{|\delta(x,y')^{-s} - \delta(x,y)^{-s}|}{\delta(x,y)^{-s}\delta(x,y')^{-s}} \nu_s(dx).\end{aligned}$$

Applying the Hölder inequality gives

$$\begin{aligned}\|\nu_s^y - \nu_s^{y'}\|_{TV}^3 \leqslant &\int_{\mathbb{P}(V)} |\delta(x,y')^{-s} - \delta(x,y)^{-s}|^3 \nu_s(dx) \\ &\times \int_{\mathbb{P}(V)} \delta(x,y)^{3s}\nu_s(dx) \int_{\mathbb{P}(V)} \delta(x,y')^{3s}\nu_s(dx).\end{aligned}$$

As $y' \to y$, the first term converges to 0 by the dominated convergence theorem, whereas the other two terms remain bounded by Proposition 3.6. $\square$

3.3. **The explicit form of the eigenfunction for negative s.** We apply the results of the previous two sections to give explicit forms of the eigenfunctions of the operators P_s and P_s^* for $s < 0$. Let us recall the corresponding results for $s > 0$ which have been established in [21]: under conditions **A1** and **A2**, for any $s \in I_\mu^+$, the functions

$$r_s(x) = \int_{\mathbb{P}(V^*)} \delta(x,y)^s \nu_s^*(dy), \quad r_s^*(y) = \int_{\mathbb{P}(V)} \delta(x,y)^s \nu_s(dx) \tag{3.18}$$

are the unique (up to a scaling constant) non-negative eigenfunctions of the operators P_s and P_s^*, respectively. The proof of these expressions for $s < 0$ is quite different from that in the case $s > 0$; it requires the Hölder regularity of the eigenmeasures ν_s and ν_s^*, which has been established in Proposition 3.6.

First we state the cohomological equation (see [5]) which will also be useful later on: for any $g \in \mathbb{G}$, $x = \mathbb{R}v \in \mathbb{P}(V)$ and $y = \mathbb{R}f \in \mathbb{P}(V^*)$,

$$\log \delta(y, gx) + \sigma(g, x) = \log \delta(x, g^*y) + \sigma(g^*, y). \tag{3.19}$$

For the ease of the reader we include a short proof of (3.19). By elementary transformations,

$$\log \frac{|\langle f, gv\rangle|}{|f||v|} = \log \frac{|\langle f, gv\rangle|}{|f||gv|} + \log \frac{|gv|}{|v|} = \log \delta(y, gx) + \sigma(g, x)$$

and, in the same way,

$$\log \frac{|\langle v, g^*f\rangle|}{|f||v|} = \log \frac{|\langle v, g^*f\rangle|}{|g^*f||v|} + \log \frac{|g^*f|}{|f|} = \log \delta(x, g^*y) + \sigma(g^*, y).$$

By the definition of the automorphism g^*, we have $\langle f, gv\rangle = \langle v, g^*f\rangle$, hence the identity (3.19) follows.

Lemma 3.10. *Assume conditions **A1**, **A2** and **A3**. Then, there exists a constant $s_0 > 0$ such that for any $s \in (-s_0, 0)$, the eigenfunctions r_s and r_s^* are defined (up to a scaling constant) as follows: for $x \in \mathbb{P}(V)$ and $y \in \mathbb{P}(V^*)$,*

$$r_s(x) = \int_{\mathbb{P}(V^*)} \delta(x,y)^s \nu_s^*(dy), \quad r_s^*(y) = \int_{\mathbb{P}(V)} \delta(x,y)^s \nu_s(dx). \tag{3.20}$$

Moreover, $\kappa^*(s) = \kappa(s)$ for any $s \in (-s_0, 0)$.

Proof. Let $x \in \mathbb{P}(V)$. By Proposition 3.6, there exists $s_0 > 0$ such that for any $s \in (-s_0, 0)$,

$$\phi_s(x) = \int_{\mathbb{P}(V^*)} \delta(x,y)^s \nu_s^*(dy)$$

is well-defined and positive. By Corollary 3.9 (applied to the dual situation), the function ϕ_s is continuous on $\mathbb{P}(V)$ when s_0 is small enough. We claim that $P_s\phi_s = \kappa^*(s)\phi_s$ with κ^* from Lemma 3.5. Indeed, since ϕ_s is uniformly bounded on $\mathbb{P}(V)$, using the cohomological identity (3.19) and Fubini's theorem we get

$$\begin{aligned} P_s\phi_s(x) &= \int_{\mathbb{G}} e^{s\sigma(g,x)} \left(\int_{\mathbb{P}(V^*)} \delta(gx,y)^s \nu_s^*(dy) \right) \mu(dg) \\ &= \int_{\mathbb{P}(V^*)} \int_{\mathbb{G}} e^{s\sigma(g^*,y) + s\log\delta(x,g^*y)} \mu(dg) \nu_s^*(dy). \end{aligned} \tag{3.21}$$

Note that the function $y \mapsto \delta(x,y)^s$ belongs to the space $L^1(\nu_s^*)$. As the operator P_s^* is positive and ν_s^* is a probability eigenmeasure of P_s^*, then P_s^* can be extended to be a bounded operator on $L^1(\nu_s^*)$, which we still denote by P_s^*. Therefore, by (3.21),

$$\begin{aligned} P_s\phi_s(x) &= \int_{\mathbb{P}(V^*)} P_s^*(\delta(x,\cdot)^s)(y) \nu_s^*(dy) \\ &= \kappa^*(s) \int_{\mathbb{P}(V^*)} \delta(x,y)^s \nu_s^*(dy) = \kappa^*(s)\phi_s(x). \end{aligned}$$

By Lemma 3.4 we get that $\kappa^*(s) = \kappa(s)$ and $r_s = c\phi_s$ for some constant $c > 0$.

The proof for r_s^* is similar and therefore will not be detailed here. $\square$

To summarize, the same relations between P_s, P_s^*, ν_s, ν_s^*, r_s, r_s^* and $\kappa(s)$ hold for both positive and small negative s: under appropriate conditions there exists $s_0 > 0$ such that for any $s \in (-s_0, 0) \cup I_\mu^+$,

$$P_s\nu_s = \kappa(s)\nu_s, \quad P_s^*\nu_s^* = \kappa(s)\nu_s^* \tag{3.22}$$

and

$$P_s r_s = \kappa(s) r_s, \quad P_s^* r_s^* = \kappa(s) r_s^*. \tag{3.23}$$

3.4. **Harmonicity of the invariant measure.** Assume that $s \in (-s_0, 0) \cup I_\mu^+$, where $s_0 > 0$ is small enough. For any $y \in \mathbb{P}(V^*)$ and bounded measurable function φ on $\mathbb{P}(V)$, denote

$$\upsilon_s^y(\varphi) = \int_{\mathbb{P}(V)} \varphi(x) \frac{\delta(x,y)^s}{r_s^*(y)} \nu_s(dx). \tag{3.24}$$

Due to the regularity of the eigenmeasure ν_s we have $0 < \delta(x,y) \leqslant 1$, ν_s-a.s. on $\mathbb{P}(V)$ (for large $s > 0$ this follows from Theorem 2.5; for small $s < 0$ it can be deduced from Proposition 3.6). Since the eigenfunction r_s^* is bounded and strictly positive on $\mathbb{P}(V)$, the measures ν_s and υ_s^y are equivalent.

Lemma 3.11. *The following two assertions hold:*

*1. Assume conditions **A1** and **A2**. Then, for any $s \in I_\mu^+$, the mapping $y \in \mathbb{P}(V^*) \mapsto \upsilon_s^y \in \mathcal{C}(\mathbb{P}(V))'$ is continuous in the total variation norm $\|\cdot\|_{\mathrm{TV}}$.*

*2. Assume conditions **A1**, **A2** and **A3**. Then, there exists a constant $s_0 > 0$ such that for any $s \in (-s_0, 0)$, the mapping $y \in \mathbb{P}(V^*) \mapsto \upsilon_s^y \in \mathcal{C}(\mathbb{P}(V))'$ is continuous in the total variation norm $\|\cdot\|_{\mathrm{TV}}$.*

Proof. For positive $s > 0$ the assertion of the lemma is easily proved due to the continuity of the mapping $(x,y) \mapsto \delta(x,y)^s$ (see Lemma 3.5 of [21]). For $s < 0$ the assertion of the lemma follows from Corollary 3.9. $\square$

For any $g \in \mathbb{G}$ and $y \in \mathbb{P}(V^*)$, set

$$q_s^*(g,y) = \frac{e^{s\sigma(g^*,y)}}{\kappa(s)} \frac{r_s^*(g^*y)}{r_s^*(y)}. \tag{3.25}$$

By (3.23), for any $y \in \mathbb{P}(V^*)$, the function $q_s^*(g,y)$ is a positive density on $\mathbb{G}$, i.e.

$$\int_{\mathbb{G}} q_s^*(g,y)\mu(dg) = 1, \tag{3.26}$$

and for any $g \in \mathbb{G}$,

$$q_s^*(g,y) > 0. \tag{3.27}$$

For any $g \in \mathbb{G}$, $y \in \mathbb{P}(V^*)$ and bounded measurable function φ on $\mathbb{P}(V)$, define

$$g\upsilon_s^y(\varphi) = \int_{\mathbb{P}(V)} \varphi(gx)\upsilon_s^y(dx), \tag{3.28}$$

where υ_s^y is a probability measure given by (3.24). The following lemma can be viewed as a generalization of the stationary property $\pi_s = \pi_s Q_s$. For $s > 0$ it has been obtained in [21, Lemma 3.6].

Lemma 3.12. *Assume either conditions **A1**, **A2** and $s \in I_\mu^+$, or conditions **A1**, **A2**, **A3** and $s \in (-s_0, 0)$ with small enough $s_0 > 0$. Then, for any $y \in \mathbb{P}(V^*)$ and bounded measurable function φ on $\mathbb{P}(V)$, we have*

$$\upsilon_s^y(\varphi) = \int_{\mathbb{G}} g\upsilon_s^{g^*y}(\varphi) q_s^*(g,y)\mu(dg).$$

Proof. For short we denote $\psi_s^y(x) = \varphi(x)e^{s\log\delta(x,y)}/r_s^*(y)$. By the definition (3.24) of the measure υ_s^y, we have

$$\upsilon_s^y(\varphi) = \int_{\mathbb{P}(V)} \varphi(x)\frac{e^{s\log\delta(x,y)}}{r_s^*(y)}\nu_s(dx) = \int_{\mathbb{P}(V)} \psi_s^y(x)\nu_s(dx).$$

From the identity $P_s \nu_s(\psi_s^y) = \kappa(s)\nu_s(\psi_s^y)$, it follows that

$$\begin{aligned}\upsilon_s^y(\varphi) &= \frac{1}{\kappa(s)} \int_{\mathbb{P}(V)} P_s \psi_s^y(x) \nu_s(dx) \\ &= \frac{1}{\kappa(s) r_s^*(y)} \int_{\mathbb{P}(V)} \left(\int_{\mathbb{G}} \varphi(gx) e^{s \log \delta(gx,y) + s\sigma(g,x)} \mu(dg) \right) \nu_s(dx).\end{aligned}$$

Using the cohomological identity (3.19), Fubini's theorem and (3.28), we get

$$\begin{aligned}\upsilon_s^y(\varphi) &= \frac{1}{\kappa(s) r_s^*(y)} \int_{\mathbb{G}} \left(\int_{\mathbb{P}(V)} \varphi(gx) e^{s \log \delta(x,g^*y) + s\sigma(g^*,y)} \nu_s(dx) \right) \mu(dg) \\ &= \int_{\mathbb{G}} \frac{e^{s\sigma(g^*,y)}}{\kappa(s) r_s^*(y)} \left(\int_{\mathbb{P}(V)} \varphi(gx) e^{s \log \delta(x,g^*y)} \nu_s(dx) \right) \mu(dg) \\ &= \int_{\mathbb{G}} \frac{e^{s\sigma(g^*,y)}}{\kappa(s)} \frac{r_s^*(g^*y)}{r_s^*(y)} g\upsilon_s^{g^*y}(\varphi) \mu(dg) \\ &= \int_{\mathbb{G}} q_s^*(g,y) g\upsilon_s^{g^*y}(\varphi)\mu(dg),\end{aligned}$$

as desired. □

4. Auxiliary statements

4.1. **Stationary measures on finite extensions.** Let G be a locally compact subgroup of $\mathbb{G}$. Assume that $H < G$ is a closed subgroup of finite index, which means that the quotient G/H is a finite set. Let μ be a probability measure on G. Denote by Ω the set of infinite sequences $(g_1, g_2, \ldots)$ and equip it with the measure $\mu^{\otimes \mathbb{N}^*}$. For any $\omega \in \Omega$, set

$$\tau(\omega) = \min\{k \geqslant 1 : \ g_k \ldots g_1 \in H\}.$$

The stopping time τ is $\mu^{\otimes \mathbb{N}^*}$-a.s. finite, see Lemma 5.5 of [5]. Define the mapping $\mathfrak{f} : \omega \in \Omega \mapsto g_{\tau(\omega)} \ldots g_1 \in H$. Let μ_H be the image of the measure $\mu^{\otimes \mathbb{N}^*}$ by the mapping $\mathfrak{f}$. We call μ_H the probability measure induced by μ on the subgroup H. From Lemma 5.7 of [5] we have the following assertion.

Lemma 4.1. *Assume that the probability measure $\mathfrak{m}$ is μ-stationary on $\mathbb{P}(V)$. Then $\mathfrak{m}$ is also a μ_H-stationary probability measure.*

4.2. **Determination of the support of the stationary measure.** Since we will use facts from complex algebraic geometry, we now introduce some basic notions on complex algebraic sets. We write $V_{\mathbb{C}}$ for the complexification of V. For example, one can define $V_{\mathbb{C}}$ as the space V^2 equipped with the complex vector space structure defined by $(a+ib)(v,w) = (av - bw, aw + bv)$ for $a, b \in \mathbb{R}$ and $v, w \in V$. One then identifies V with the set $V \times \{0\} \subset V^2$. We recall that a subset X in $V_{\mathbb{C}}$ is algebraic if there exist complex polynomial functions $p_1, \ldots, p_k$ on $V_{\mathbb{C}}$ such that $X = \{v \in V_{\mathbb{C}} : p_1(v) = \ldots = p_k(v) = 0\}$. We say that X is homogeneous if for every $t \in \mathbb{C}$ and $v \in X$ it holds that $tv \in X$. The topological space X is called Noetherian if every non-increasing sequence of closed subsets is eventually constant.

We also need the projective space of $V_{\mathbb{C}}$, which is defined in a similar way: $\mathbb{P}(V_{\mathbb{C}})$ is the set of elements $x = \mathbb{C}v$, where $v \in V_{\mathbb{C}} \setminus \{0\}$. A subset Y of $\mathbb{P}(V_{\mathbb{C}})$ is algebraic if there exists an algebraic homogeneous subset X whose projective image is Y. With this notion, a set X in V is (real) algebraic if and only if there exists a complex algebraic set X' such that $X = X' \cap V$. In the same way, a set Y in $\mathbb{P}(V)$ is (real) algebraic if and only if there exists a complex algebraic set Y' such that $Y = Y' \cap \mathbb{P}(V)$.

From the Noetherian property of the ring of polynomial functions it follows that the sets defined as zeros of infinitely many polynomials are also algebraic. This implies that the algebraic sets are precisely the closed sets of the Zariski topology. The set $\mathrm{GL}(V_{\mathbb{C}})$ is a Zariski open subset of the complex vector space $\mathrm{End}(V_{\mathbb{C}})$ of complex endomorphisms of $V_{\mathbb{C}}$, so we can equip it with the Zariski topology of $\mathrm{End}(V_{\mathbb{C}})$. The closure of a set Y in the Zariski topology is denoted by $\mathrm{Zc}(Y)$.

The limit set of the semigroup Γ_μ is a subset of $\mathbb{P}(V_{\mathbb{C}})$ defined as follows:

$$\Lambda(\Gamma_\mu) = \overline{\{x_g^+ : g \in \Gamma_\mu,\ g \text{ is proximal}\}}.$$

It is well known that, under conditions **A1** and **A2**, $\Lambda(\Gamma_\mu)$ is the smallest nonempty closed Γ_μ-invariant set in the projective space $\mathbb{P}_{\mathbb{C}}^{d-1}$.

Lemma 4.2. *Assume either conditions **A1**, **A2** and $s \in I_\mu^+$, or conditions **A1**, **A2**, **A3** and $s \in (-s_0, 0)$ with small enough $s_0 > 0$. Then the support of ν_s is $\Lambda(\Gamma_\mu)$.*

Proof. If we apply Lemma 3.1 with $T = P_s$ and $X = \Lambda(\Gamma_\mu)$, then we obtain that P_s admits an eigenmeasure which is concentrated on $\Lambda(\Gamma_\mu)$. By the uniqueness of the eigenmeasure ν_s (see (2.5) and Lemma 3.5), we get $\nu_s(\Lambda(\Gamma_\mu)) = 1$. This proves that the support of ν_s is contained in $\Lambda(\Gamma_\mu)$.

Conversely, let us prove that for any nonempty open subset $U \subset \Lambda(\Gamma_\mu)$ it holds that $\nu_s(U) > 0$. To show this we fix some point $x \in \mathrm{supp}(\nu_s)$. Since the orbit $\Gamma_\mu x$ is dense in $\Lambda(\Gamma_\mu)$, we can find an integer $n \in \mathbb{N}$ and $g \in \mathrm{supp}(\mu)^n$ such that $gx \in U$. As $\mathrm{supp}(\mu)^n$ is the support of the measure μ^{*n}, we get

$$\mathbb{P}(G_n x \in U) > 0.$$

This gives

$$P_s^n \mathbb{1}_U(x) = \mathbb{E}\left(e^{s\sigma(G_n,x)} \mathbb{1}_{\{G_n x \in U\}}\right) > 0.$$

Since the function $P_s^n(\mathbb{1}_U)$ is upper semi-continuous, the set

$$V = \{x' \in \mathbb{P}(V) : P_s^n \mathbb{1}_U(x') > 0\}$$

is open. By assumption $V \cap \mathrm{supp}(\nu_s)$ is nonempty, hence $\nu_s(V) > 0$ and $\nu_s(P_s^n \mathbb{1}_U) > 0$. It follows that $\nu_s(U) = \kappa(s)^{-n} \nu_s(P_s^n \mathbb{1}_U) > 0$, as required. ☐

Remark 4.3. Let Γ_μ^* be the adjoint semigroup of Γ_μ, which is a subsemigroup of $\mathrm{GL}(V^*)$. Then assumptions **A1** and **A2** hold for Γ_μ^* if and only if they hold for Γ_μ. In particular, there exists a smallest nonempty closed Γ_μ^*-invariant subset in the dual projective space $\mathbb{P}(V^*)$, which will be called limit set and denoted by $\Lambda(\Gamma_\mu^*)$.

Let $H_\mu = \mathrm{Zc}(\Gamma_\mu)$ be the Zariski closure of Γ_μ in $\mathrm{GL}(V_{\mathbb{C}})$. Define $X_\mu = \mathrm{Zc}(\Lambda(\Gamma_\mu)) = \mathrm{Zc}(\mathrm{supp}\,\nu)$. Note that X_μ is an algebraic subset of $\mathbb{P}(V_{\mathbb{C}})$.

Lemma 4.4. *Assume conditions **A1** and **A2**. The algebraic set X_μ is the unique closed H_μ-orbit in $\mathbb{P}(V_{\mathbb{C}})$. In particular, every nonempty Zariski closed H_μ-invariant subset contains X_μ.*

Proof. It is a general fact in algebraic geometry (we refer to Borel [7, Corollary 1.8, p.53]) that there exists $x_0 \in \mathbb{P}(V)$ such that $Y_0 = H_\mu x_0$ is a Zariski closed orbit. We shall prove that $X_\mu = Y_0$.

The orbit Y_0 is also analytically closed and, moreover, it is Γ_μ-invariant (if $g \in \Gamma_\mu$ then $gY_0 = gH_\mu x_0 = \{gg'x_0 : g' \in H_\mu\} \subset Y$, as $gg' \in H_\mu$). Therefore, the minimal closed Γ_μ-invariant set $\Lambda(\Gamma_\mu)$ is contained in Y_0: $\Lambda(\Gamma_\mu) \subset Y_0$. This implies that $X_\mu = \mathrm{Zc}(\Lambda(\Gamma_\mu)) \subset \mathrm{Zc}(Y_0) = Y_0$, since Y_0 is Zariski closed. So we have showed that $X_\mu \subset Y_0$.

Now we prove the converse inclusion $Y_0 \subset X_\mu$. Since X_μ is a Γ_μ-invariant Zariski closed subset, by the continuity of the map $g \in \mathrm{GL}(V_\mathbb{C}) \mapsto gx \in \mathbb{P}(V_\mathbb{C})$ in the Zariski topology for any $x \in \mathbb{P}(V_\mathbb{C})$, the set X_μ is also H_μ-invariant. This means that $H_\mu X_\mu \subset X_\mu$. We know that for any $x \in Y_0$ it holds that $H_\mu x = Y_0$. Since $X_\mu \subset Y_0$, we have that $H_\mu x = Y_0$ for any $x \in X_\mu$. This implies that $Y_0 = H_\mu x \subset X_\mu$, which concludes the proof. □

To avoid any confusion with the notion of strong irreducibility of the set Γ_μ introduced before, let us recall the notion of irreducibility of an algebraic subset. We say that an algebraic subset Y is irreducible if Y cannot be represented as the union of two proper closed algebraic subsets of Y.

Lemma 4.5. *Assume conditions* ***A1*** *and* ***A2****. Then, the algebraic set X_μ is irreducible.*

Proof. Let H_μ^0 be the Zariski connected component of H_μ which contains the unit element of $\mathbb{G}$. Note that H_μ^0 is a finite index Zariski closed subgroup of H_μ.

Let μ_0 be the measure on H_μ^0 induced by the measure μ through the mapping $\mathfrak{f}$ as explained in Section 4.1. By Lemma 4.1, the probabilty measure ν defined by (2.1) is μ_0-stationary. By the construction of μ_0 we have $\Gamma_{\mu_0} = \Gamma_\mu \cap H_\mu^0$. Since any finite index subgroup of a strongly irreducible group is still strongly irreducible, it follows that Γ_{μ_0} is strongly irreducible (we note that the irreducibility is not necessarily preserved). It is also proximal since any positive power of an element of Γ_μ is also proximal. By the result of Furstenberg [13] the measure μ_0 has a unique stationary probability measure ν_0 on $\mathbb{P}(V)$ which (by uniqueness) coincides with ν. This implies that $X_\mu = X_{\mu_0} = \mathrm{Zc}(\operatorname{supp}\nu_0) = \mathrm{Zc}(\Lambda(\Gamma_{\mu_0}))$.

Applying Lemma 4.4 with μ_0 instead of μ, we conclude that $X_\mu = X_\mu^0$ is an H_μ^0-orbit in $\mathbb{P}(V)$. Since H_μ^0 is connected (and therefore irreducible as an algebraic set), we conclude that X_μ is irreducible as the image of H_μ^0 by the Zariski continuous mapping $g \in H_\mu^0 \mapsto gx_0 \in X_\mu$. □

4.3. **The maximum principle.** We shall use repeatedly the following simple fact which we call maximum principle.

Lemma 4.6. *Let $\phi : (X, P) \mapsto \mathbb{R}$ be a measurable function on the measurable space X equipped with the probability measure P which satisfies $\phi(x) \leqslant \beta$, P-a.s. and $\int \phi(x)P(dx) = \beta$, where $\beta \in \mathbb{R}$ is a real number. Then $\phi = \beta$, P-a.s.*

5. Proof of Theorem 2.2

We shall prove the following statement which implies Theorem 2.2. All over this section we assume conditions **A1** and **A2**. First we recall that by the definition of X_μ (see Section 4.2) we have $X_\mu = \mathrm{Zc}(\operatorname{supp} \nu)$, so that $\nu(X_\mu) = 1$.

Proposition 5.1. *For any proper algebraic subset Y of X_μ, it holds $\nu(Y) = 0$.*

We need to recall some elementary notions from algebraic geometry (see, for instance Borel [7]). We say that a topological space X is irreducible if and only if X cannot be written as $X = X_1 \cup X_2$, where X_1 and X_2 are proper closed subsets of X. Recall that a topological space X is Noetherian if every non-increasing sequence of closed subsets is eventually constant. Any Noetherian topological space can be written as a finite union of closed irreducible subsets.

The (combinatorial) dimension of a Noetherian topological space X is the maximum length l of sequences $X_0 \subsetneq \ldots \subsetneq \ldots X_l$ of distinct irreducible closed sets in X:

$$\dim(X) = \sup\{l : X_0 \subsetneq \ldots \subsetneq \ldots X_l \subsetneq X\} \in \mathbb{N} \cup \{\infty\}.$$

From this definition it is obvious that if X is irreducible and has finite dimension then any proper closed subset of X has strictly smaller dimension.

It is known that the projective space $\mathbb{P}(V_\mathbb{C})$ is irreducible and Noetherian for the Zariski topology with $\dim(\mathbb{P}(V_\mathbb{C})) = d - 1$. In the following by the dimension of an algebraic subset of $\mathbb{P}(V)$ we mean the combinatorial dimension.

Denote by $\mathfrak{A}(X_\mu)$ the set of irreducible algebraic subsets Y of X_μ. Set

$$d_0 = \min\{1 \leqslant r \leqslant d : Y \in \mathfrak{A}(X_\mu),\ \dim(Y) = r,\ \nu(Y) > 0\}.$$

It is easy to see that $d_0 \leqslant \dim(X_\mu) \leqslant d - 1$.

Lemma 5.2. *Let Y be an algebraic subset of $\mathbb{P}(V_\mathbb{C})$ with $\dim(Y) < d_0$. Then $\nu(Y) = 0$.*

Proof. Write Y as a union of irreducible algebraic subsets $Y_1, \ldots, Y_r$ of $\mathbb{P}(V_\mathbb{C})$: $Y = Y_1 \cup \ldots \cup Y_r$ (see Proposition 1.5 of Hartshorne [23]). As each Y_k has dimension strictly less that d_0, we have $\nu(Y_k) = 0$, by the definition of d_0. So $\nu(Y) = 0$. □

We will establish the following assertion, which is the key point in the proof of Theorem 2.2.

Proposition 5.3. *The dimension of X_μ is d_0: $\dim(X_\mu) = d_0$.*

The proof of this proposition requires some additional assertions. For any $c > 0$, set

$$\mathcal{W}_0(c) = \{Y \in \mathfrak{A}(X_\mu) :\ \dim(Y) = d_0,\ \nu(Y) \geqslant c\}.$$

We start by showing that this set is finite.

Lemma 5.4. *For any $c > 0$, the set $\mathcal{W}_0(c)$ is finite. Moreover* $\operatorname{card} \mathcal{W}_0(c) \leqslant c^{-1}$.

Proof. Let $Y_1, \cdots, Y_r$ be two by two distinct elements of $W_0(c)$. For any $1 \leqslant i < j \leqslant r$, the intersection $Y_i \cap Y_j$ is an algebraic subset of dimension strictly smaller than d_0. By Lemma 5.2 we have $\nu(Y_i \cap Y_j) = 0$. This implies that $\nu(Y_1 \cup \cdots \cup Y_r) \geqslant cr$, so $r \leqslant c^{-1}$. □

Set

$$\beta = \sup\{\nu(Y) : Y \in \mathfrak{A}(X_\mu),\ \dim(Y) = d_0\}. \tag{5.1}$$

In the following it is important to show that the supremum in (5.1) is attained.

Lemma 5.5. *The following maximum is attained,*

$$\beta = \max\{\nu(Y) : Y \in \mathfrak{A}(X_\mu),\ \dim(Y) = d_0\}.$$

Proof. As $\beta > 0$, we have

$$\beta = \sup\{\nu(Y) : Y \in \mathcal{W}_0(\beta/2)\}.$$

By Lemma 5.4, the set $\mathcal{W}_0(\beta/2)$ is finite. Therefore, the supremum is attained. □

All the algebraic sets $Y \in \mathfrak{A}(X_\mu)$ for which the maximum β is realized are collected in the set

$$\mathcal{W} = \{Y \in \mathfrak{A}(X_\mu) :\ \dim(Y) = d_0,\ \nu(Y) = \beta\},$$

which is finite by Lemma 5.4 (as a subset of $\mathcal{W}_0(\beta)$).

Lemma 5.6. *The set $\mathcal{W}$ is Γ_μ-invariant, which means that for any $g \in \Gamma_\mu$ and any $Y \in \mathcal{W}$ we have that $gY \in \mathcal{W}$.*

Proof. Let Y be an element of $\mathcal{W}$. Since the measure ν is μ-stationary, we have

$$\nu(Y) = \int_{\Gamma_\mu} \nu(g^{-1}Y)\mu(dg).$$

For any $g \in \Gamma_\mu$, we have that $g^{-1}Y \in \mathfrak{A}(X_\mu)$ and $g^{-1}Y$ has dimension d_0, hence $\nu(g^{-1}Y) \leqslant \beta$. Since

$$\beta = \nu(Y) = \int_{\Gamma_\mu} \nu(g^{-1}Y)\mu(dg),$$

by the maximum principle (see Lemma 4.6), we get that $\nu(g^{-1}Y) = \beta$ for μ-almost all $g \in \Gamma_\mu$. Since $\mathcal{W}$ is finite (by Lemma 5.4), we get that $g^{-1}\mathcal{W} = \mathcal{W}$ for μ-almost all $g \in \Gamma_\mu$, and that the set $\{g \in \Gamma_\mu : g^{-1}\mathcal{W} = \mathcal{W}\}$ is closed. We have shown that it has μ measure 1, therefore it contains the $\operatorname{supp}\mu$ (by the definition of the latter). This means that for any $g \in \operatorname{supp}\mu$ we have $g^{-1}\mathcal{W} \subset \mathcal{W}$. Since the cardinality of $\mathcal{W}$ is finite, we get that $g^{-1}\mathcal{W} = \mathcal{W}$ for any $g \in \operatorname{supp}\mu$. By the definition of the support of the measure μ, the last statement implies that $g^{-1}\mathcal{W} = \mathcal{W}$ for all $g \in \Gamma_\mu$. Since $g^{-1}\mathcal{W} = \mathcal{W}$ is equivalent to $g\mathcal{W} = \mathcal{W}$ for $g \in \Gamma_\mu$, the assertion follows. □

Proof of Proposition 5.3. By Lemma 5.6, the set $\mathcal{W}$ is Γ_μ-invariant ($\Gamma_\mu\mathcal{W} \subset \mathcal{W}$), and by Lemma 5.4, the set $\mathcal{W}$ is finite. Therefore, the set $Z = \bigcup_{Y\in\mathcal{W}} Y$ is a Zariski closed Γ_μ-invariant algebraic subset in X_μ. As Γ_μ is Zariski dense in $H_\mu = \mathrm{Zc}(\Gamma_\mu)$, the algebraic set Z is H_μ-invariant. By Lemma 4.4, we have that $X_\mu \subset Z$. On the other hand $Z \subset X_\mu$, so $Z = X_\mu$, which reads as $X_\mu = \bigcup_{Y\in\mathcal{W}} Y$. According to Lemma 4.5, X_μ is irreducible, hence we get $X_\mu = Y$ for some Y in $\mathcal{W}$. In particular, $\dim(Y) = d_0 = \dim(X_\mu)$, which concludes the proof of Proposition 5.3. □

Now we show that Proposition 5.3 implies Proposition 5.1.

Proof of Proposition 5.1. Let Y be an algebraic subset of X_μ with $\nu(Y) > 0$ and let us show that $Y = X_\mu$. By Proposition 1.5 of Hartshorne [23], we can decompose the set Y into a union of Zariski closed and Zariski irreducible subsets $Z_1, \dots, Z_r$ of X_μ: $Y = Z_1 \cup \dots \cup Z_r$. Then, there exists k such that $\nu(Z_k) > 0$, and hence $\dim(Z_k) \geqslant d_0 = \dim(X_\mu)$, by Proposition 5.3. It follows that $Z_k = X_\mu$, hence $Y = X_\mu$ as claimed by our assertion. □

Now we prove that Proposition 5.1 implies Theorem 2.2.

Proof of Theorem 2.2. Let Y be an algebraic subset of the projective space $\mathbb{P}(V_\mathbb{C})$. If $X_\mu \subset Y$, we have $\nu(Y) \geqslant \nu(X_\mu) = 1$. Otherwise, $Y \cap X_\mu$ is a proper algebraic subset of X_μ. Since $\nu(X_\mu) = 1$, we have $\nu(Y) = \nu(Y \cap X_\mu) = 0$, by Proposition 5.1. The assertion of the theorem now follows from the fact that an algebraic subset of $\mathbb{P}(V)$ is the intersection of an algebraic subset of the projective space $\mathbb{P}(V_\mathbb{C})$ with $\mathbb{P}(V)$. □

6. Proof of Theorems 2.6 and 2.8

We shall first prove the following:

Proposition 6.1. *Assume conditions* ***A1*** *and* ***A2****. Then, for any $s \in I_\mu^+$ and for any proper algebraic subset Y of X_μ, it holds $\nu_s(Y) = 0$.*

Proposition 6.2. *Assume conditions* ***A1****,* ***A2*** *and* ***A3****. Then, there exists a constant $s_0 > 0$ such that for any $s \in [-s_0, 0)$ and for any proper algebraic subset Y of X_μ, it holds $\nu_s(Y) = 0$.*

We will see at the end of this section that Propositions 6.1 and 6.2 imply Theorems 2.6 and 2.8, respectively.

To establish Propositions 6.1 and 6.2 we stick to the proof of Guivarc'h and Le Page [21]. The proofs that we give below will work in both cases $s \in I_\mu^+$ and $s \in [-s_0, 0)$, where $s_0 > 0$ is small enough.

We need a series of auxiliary statements. We use the notation $\mathfrak{A}(X_\mu)$ from Section 5 and υ_s^y from Section 3.4. For any $y \in \mathbb{P}(V^*)$ define

$$d_0 = \min\{1 \leqslant r \leqslant d : Y \in \mathfrak{A}(X_\mu),\ \dim(Y) = r,\ \upsilon_s^y(Y) > 0\}.$$

Since the measures υ_s^y and ν_s are equivalent, there is no dependence on y in the above definition of d_0, so that

$$d_0 = \min\{1 \leqslant r \leqslant d : Y \in \mathfrak{A}(X_\mu),\ \dim(Y) = r,\ \nu_s(Y) > 0\}.$$

It easy to see that $d_0 \leqslant \dim(X_\mu) \leqslant d - 1$.

Lemma 6.3. *Let Y be an algebraic subset of $\mathbb{P}(V_\mathbb{C})$ with $\dim(Y) < d_0$. Then $\nu_s(Y) = 0$.*

Proof. The proof being similar to that of Lemma 5.2 is left to the reader. □

In the sequel we are going to prove the following assertion.

Proposition 6.4. *It holds that $\dim(X_\mu) = d_0$.*

We shall show below that Proposition 6.4 implies our Propositions 6.1 and 6.2. The proof of Proposition 6.4 is based on several lemmas. For any $c > 0$, set

$$\mathcal{W}_0^y(c) = \{Y \in \mathfrak{A}(X_\mu) : \dim(Y) = d_0,\ \upsilon_s^y(Y) \geqslant c\}.$$

Lemma 6.5. *Let $c > 0$ be a constant. Then, for any $y \in (\mathbb{P}^{d-1})^*$, the set $\mathcal{W}_0^y(c)$ is finite. Moreover* $\operatorname{card} \mathcal{W}_0^y(c) \leqslant c^{-1}$.

Proof. Let $Y_1, \cdots, Y_r$ be two by two distinct elements of $W_0^y(c)$. For any $1 \leqslant i < j \leqslant r$, the intersection $Y_i \cap Y_j$ is an algebraic set of dimension strictly smaller than d_0, hence by Lemma 6.3, $\upsilon_s^y(Y_i \cap Y_j) = 0$. This implies that $\upsilon_s^y(Y_1 \cup \cdots \cup Y_r) \geqslant cr$, so $r \leqslant c^{-1}$. □

Proceeding as in the proof of Lemma 5.5, from Lemma 6.5 it follows that for $y \in \mathbb{P}(V^*)$, the number

$$h(y) = \max \left\{ \upsilon_s^y(Y) : Y \in \mathfrak{A}(X_\mu),\ \dim(Y) = d_0 \right\} \tag{6.1}$$

is well-defined and the max is attained. By Lemma 3.11, the function $h : \mathbb{P}(V^*) \to \mathbb{R}_+$ is continuous and thus attains its maximum on $\mathbb{P}(V^*)$. Therefore, we can define

$$\beta = \max_{y \in \mathbb{P}(V^*)} h(y). \tag{6.2}$$

Lemma 6.6. *If $h(y) = \beta$ for some $y \in \mathbb{P}(V^*)$, then $h(g^* y) = \beta$ for all $g \in \Gamma_\mu$. In particular $h(y) = \beta$ for all $y \in \Lambda(\Gamma_\mu^*)$.*

Proof. Assume that $y \in \mathbb{P}(V^*)$ and $h(y) = \beta$. Using (6.1) and (6.2), we see that there exists $Y \in \mathfrak{A}(X_\mu)$ such that $\upsilon_s^y(Y) = \beta$. Note that $\mathbb{1}_Y(gx) = \mathbb{1}_{g^{-1}Y}(x)$ for $x \in \mathbb{P}(V)$. By Lemma 3.12, it holds that

$$\beta = \upsilon_s^y(Y) = \int_{\mathbb{G}} \upsilon_s^{g^* y}(g^{-1}Y) q_s^*(g, y) \mu(dg). \tag{6.3}$$

Since $g^{-1}Y$ is an irreducible algebraic subset of dimension d_0, we have that for any $g^{-1} \in \Gamma_\mu$,

$$\upsilon_s^{g^* y}(g^{-1}Y) \leqslant \beta. \tag{6.4}$$

From (6.3), (6.4) and the fact that $q_s^*(g, y)\mu(dg)$ is a probability measure, by the maximum principle (Lemma 4.6), it follows that $\upsilon_s^{g^* y}(g^{-1}Y) = \beta$ for $q_s^*(\cdot, y) d\mu$ almost all $g \in \mathbb{G}$. Using the fact that the measures $q_s^*(\cdot, y) d\mu$ and μ are equivalent, we obtain that $\upsilon_s^{g^* y}(g^{-1}Y) = \beta$ for μ-a.s. $g \in \mathbb{G}$. Therefore, $h(g^* y) = \beta$ for μ-a.s. $g \in \mathbb{G}$. Since the function h is continuous on $\mathbb{P}(V^*)$, we conclude that $h(g^* y) = \beta$ for all $g \in \operatorname{supp} \mu$, hence, by iteration, for all $g \in \Gamma_\mu$.

To prove the second assertion, define

$$Z = \{y \in \mathbb{P}(V^*) : \ h(y) = \beta\}. \tag{6.5}$$

The set Z is nonempty and, since the function h is continuous, it is closed. From the first assertion of the lemma we have $\Gamma_\mu^* Z \subset Z$. Since $\Lambda(\Gamma_\mu^*)$ is the smallest nonempty closed Γ_μ^*-invariant set (by Remark 4.3), we get $\Lambda(\Gamma_\mu^*) \subset Z$. Therefore, by the definition (6.5) of Z, we conclude that $h(y) = \beta$ for all $y \in \Lambda(\Gamma_\mu^*)$. □

For any $y \in \mathbb{P}(V^*)$, we collect all the algebraic sets $Y \in \mathfrak{A}(X_\mu)$ for which $h(y)$ defined by (6.1) is realized, in the set

$$\mathcal{W}(y) = \Big\{ Y \in \mathfrak{A}(X_\mu) : \ \dim(Y) = d_0, \ \upsilon_s^y(Y) = h(y) \Big\}. \tag{6.6}$$

In the same way as in the proof of Lemma 5.5, using Lemma 6.5 one can show that $\mathcal{W}(y)$ is a finite set: $\mathrm{card}(\mathcal{W}(y)) < \infty$. For any $y \in \mathbb{P}(V^*)$, set

$$n(y) = \mathrm{card}(\mathcal{W}(y)). \tag{6.7}$$

Note that for any $Y \in \mathcal{W}(y)$ we have $\upsilon_s^y(Y) = h(y) = \beta$. So, by Lemma 6.5 we get that $n(y) \leqslant \beta^{-1}$ for any $y \in \mathbb{P}(V^*)$. Set

$$r = \max \left\{ n(y) : \ y \in \mathbb{P}(V^*), \ h(y) = \beta \right\}.$$

Lemma 6.7. *Suppose that $y \in \mathbb{P}(V^*)$ is such that $h(y) = \beta$ and $n(y) = r$. Then $h(g^*y) = \beta$ and $n(g^*y) = r$ for all $g \in \Gamma_\mu$. In particular $h(y) = \beta$ and $n(y) = r$ for all $y \in \Lambda(\Gamma_\mu^*)$.*

Proof. Define for any $y \in \mathbb{P}(V^*)$,

$$h_r(y) = \max \Big\{ \upsilon_s^y(Y) : \ Y = Y_1 \cup \cdots \cup Y_r, \ Y_k \in \mathfrak{A}(X_\mu), \dim(Y_k) = d_0, 1 \leqslant k \leqslant r \Big\}.$$

By Lemma 3.11, the function h_r is continuous on $\mathbb{P}(V^*)$. We claim that for any $y \in \mathbb{P}(V^*)$, we have $h_r(y) \leqslant \beta r$ and equality holds if and only if $h(y) = \beta$ and $n(y) = r$. Indeed, for $Y_1, \ldots, Y_r$ from the definition of h_r, (6.1) and (6.2) imply that $\upsilon_s^y(Y_k) \leqslant \beta$, thus $h_r(y) \leqslant \beta r$. Moreover, if $h_r(y) = \beta r$, then there exist $Y_1, \ldots, Y_r \in \mathfrak{A}(X_\mu)$ with $\dim(Y_k) = d_0$ for $1 \leqslant k \leqslant r$ such that $\upsilon_s^y(Y) = \beta r$, where $Y = Y_1 \cup \cdots \cup Y_r$. Then, necessarily the sets $Y_1, \cdots, Y_r$ are two by two distinct and $\upsilon_s^y(Y_i) = \beta$ for any $1 \leqslant i \leqslant r$. Thus $h(y) = \beta$ and $n(y) = r$. The converse statement is obvious.

Therefore, to prove the first assertion of the lemma we need to show that the set $\{y \in \mathbb{P}(V^*) : h_r(y) = \beta r\}$ is Γ_μ^*-invariant. For any y in this set, let $Y_1, \ldots, Y_r \in \mathfrak{A}(X_\mu)$ be such that $\dim(Y_k) = d_0$ for $1 \leqslant k \leqslant r$ and $\upsilon_s^y(Y) = \beta r$, where $Y = Y_1 \cup \cdots \cup Y_r$. Using Lemma 3.12, we get

$$\beta r = \upsilon_s^y(Y_1 \cup \cdots \cup Y_r) = \int_{\mathbb{G}} \upsilon_s^{g^*y} \Big(g^{-1}(Y_1 \cup \cdots \cup Y_r) \Big) q_s^*(g, y) \mu(dg).$$

Since $\upsilon_s^y(g^{-1}(Y_1 \cup \cdots \cup Y_r)) \leqslant \beta r$ for any $g \in \mathbb{G}$ and $y \in \mathbb{P}(V^*)$, by the maximum principle (Lemma 4.6), it follows that $q_s^*(\cdot, y) d\mu$-a.s.

$$\upsilon_s^{g^*y} \Big(g^{-1}(Y_1 \cup \cdots \cup Y_r) \Big) = \beta r.$$

This means that $h_r(g^*y) = \beta r$ for $q_s^*(\cdot, y) d\mu$ almost all $g \in \mathbb{G}$. Since the measures $q_s^*(\cdot, y) d\mu$ and μ are equivalent, we deduce that $h_r(g^*y) = \beta r$ holds for μ almost all $g \in \mathbb{G}$. Thus we have proved that $h_r(y) = \beta r$ implies $h_r(g^*y) = \beta r$ μ-a.s. for μ almost all $g \in \mathbb{G}$. Since the function h is continuous on $\mathbb{P}(V^*)$, it follows that $h_r(g^*y) = \beta r$ for all $g \in \operatorname{supp} \mu$ and hence for $g \in \Gamma_\mu$. This means that $\Gamma_\mu^* Z_r \subset Z_r$, where

$$Z_r = \left\{ y \in \mathbb{P}(V^*) : \ h_r(y) = \beta r \right\}.$$

By Lemma 6.6, we know that $\Gamma_\mu^* Z \subset Z$, where Z is defined by (6.5). Therefore $\Gamma_\mu^*(Z_r \cap Z) \subset (Z_r \cap Z)$. Noting that

$$Z_r \cap Z = \{y \in \mathbb{P}(V^*) : \ h(y) = \beta, \ n(y) = r\},$$

we get the first assertion of the lemma.

Now we prove the second assertion. Since the function h_r is continuous, the set Z_r is nonempty and closed. We have seen in the proof of Lemma 6.6 that the set Z is also nonempty and closed. Recalling that $\Lambda(\Gamma_\mu^*)$ is the smallest closed Γ_μ^*-invariant set (by Remark 4.3), we obtain $\Lambda(\Gamma_\mu^*) \subset Z_r \cap Z$, which is precisely the second assertion. □

Lemma 6.8. *The mapping $y \in \Lambda(\Gamma_\mu^*) \mapsto \mathcal{W}(y)$ is locally constant. In particular, the number of distinct values of this mapping is finite, i.e.*

$$\operatorname{card}\Big\{\mathcal{W}(y) : \ y \in \Lambda(\Gamma_\mu^*)\Big\} < \infty.$$

Proof. For any $y \in \mathbb{P}(V^*)$, set

$$\varepsilon(y) = \max\{\upsilon_s^y(Y) : \ Y \in \mathfrak{A}(X_\mu), \ \dim(Y) = d_0, \ \upsilon_s^y(Y) < \beta\},$$

where the maximum is attained (this can be established using Lemma 6.5 in the same way as in the proof of Lemma 5.5). Equip the space $\mathcal{C}(\mathbb{P}(V))'$ of complex valued Borel measures on $\mathbb{P}(V)$ with the total variation norm $\|\cdot\|_{\mathrm{TV}}$. Since $\varepsilon(y) < \beta$, by the continuity of the mapping $y \in \mathbb{P}(V^*) \mapsto \upsilon_s^y \in \mathcal{C}(\mathbb{P}(V))'$ (see Lemma 3.11), we get

$$\varepsilon_0 = \sup_{y \in \mathbb{P}(V^*)} \varepsilon(y) < \beta.$$

Note that any algebraic subset Y has the following property:

$$\upsilon_s^y(Y) = \beta \quad \text{or} \quad \upsilon_s^y(Y) \leqslant \varepsilon_0. \tag{6.8}$$

On the compact set $\Lambda(\Gamma_\mu^*)$ there exists a neighborhood $U_y \subset \Lambda(\Gamma_\mu)$ of y such that for any $y' \in U_y$,

$$\|\upsilon_s^y - \upsilon_s^{y'}\|_{\mathrm{TV}} < \beta - \varepsilon_0. \tag{6.9}$$

Let $Y_1, \ldots, Y_r$ be the elements of $\mathcal{W}(y)$ that realize β, i.e. $\upsilon_s^y(Y_k) = \beta$, $k = 1, \ldots, r$. In particular, this implies that

$$\upsilon_s^y(Y_1 \cup \ldots \cup Y_r) = \beta r. \tag{6.10}$$

Then, by (6.9) and (6.10),

$$\Big|\upsilon_s^y(Y_1 \cup \ldots \cup Y_r) - \upsilon_s^{y'}(Y_1 \cup \ldots \cup Y_r)\Big| = \Big|\beta r - \upsilon_s^{y'}(Y_1 \cup \ldots \cup Y_r)\Big| < \beta - \varepsilon_0. \tag{6.11}$$

By (6.8) we have either $\upsilon_s^{y'}(Y_k) = \beta$ or $\upsilon_s^{y'}(Y_k) \leqslant \varepsilon_0$, for $k = 1, \ldots, r$. If there exists Y_k such that $\upsilon_s^{y'}(Y_k) \leqslant \varepsilon_0$, then this will lead to a contradiction. Hence we get that $\upsilon_s^{y'}(Y_k) = \beta$ for $k = 1, \ldots, r$, so that $\mathcal{W}(y') \subset \mathcal{W}(y)$. In turn, since $\operatorname{card}\mathcal{W}(y) = \operatorname{card}\mathcal{W}(y') = r$, we obtain that $\mathcal{W}(y) = \mathcal{W}(y')$ for any $y' \in U_y$, which means that the mapping $y \mapsto \mathcal{W}(y)$ is locally constant on $\Lambda(\Gamma_\mu^*)$.

Since $\Lambda(\Gamma_\mu^*)$ is a closed set of the projective space $\mathbb{P}(V^*)$, it is also compact. We know that any locally constant mapping on a compact set has a finite range.

Therefore, the mapping $y \mapsto \mathcal{W}(y)$ on $\Lambda(\Gamma_\mu^*)$ takes only finitely many values. This proves the second assertion. □

Lemma 6.9. *For any $y \in \Lambda(\Gamma_\mu^*)$, we have $\mathcal{W}(g^* y) = g^{-1}\mathcal{W}(y)$ for μ-a.s. all $g \in \Gamma_\mu$.*

Proof. Let $y \in \Lambda(\Gamma_\mu^*)$ and $Y_1, \dots, Y_r$ be two by two distinct elements of $\mathcal{W}(y)$ that realize β, i.e. such that $Y_k \in \mathfrak{A}(X_\mu)$, $\dim(Y_k) = d_0$ and $\upsilon_s^y(Y_k) = \beta$ for any $k = 1, \dots, r$. Since the sets $Y_1, \cdots, Y_r$ are two by two distinct, for any $1 \leqslant i < j \leqslant r$, the intersection $Y_i \cap Y_j$ is an algebraic set of dimension strictly smaller than d_0, in view of Lemma 6.3 it holds that $\upsilon_s^y(Y_i \cap Y_j) = 0$. This implies $\beta r = \upsilon_s^y(Y_1 \cup \cdots \cup Y_r)$. Taking into account Lemma 3.12, we have

$$\beta r = \upsilon_s^y(Y_1 \cup \cdots \cup Y_r) = \int_{\mathbb{G}} \upsilon_s^{g^* y}\left(g^{-1}(Y_1 \cup \cdots \cup Y_r)\right) q_s^*(g, y)\mu(dg).$$

Since $\upsilon_s^y(g^{-1}(Y_1 \cup \cdots \cup Y_r)) \leqslant \beta r$ for any $y \in \mathbb{P}(V^*)$, by the maximum principle (Lemma 4.6), it follows that μ-a.s. on $\mathbb{G}$, and therefore following the same reasoning as in the proof of Lemma 6.7 for all $g \in \operatorname{supp}\mu$,

$$\upsilon_s^{g^* y}\left(g^{-1}(Y_1 \cup \cdots \cup Y_r)\right) = \beta r.$$

Note that $\upsilon_s^{g^* y}(g^{-1}Y_i) \leqslant \beta$, so that for all $g \in \operatorname{supp}\mu$,

$$\beta r = \upsilon_s^{g^* y}\left(g^{-1}(Y_1 \cup \cdots \cup Y_r)\right) \leqslant \sum_{i=1}^r \upsilon_s^{g^* y}(g^{-1}Y_i) \leqslant \beta r.$$

This implies that $\upsilon_s^{g^* y}(g^{-1}Y_i) = \beta$ for all $g \in \operatorname{supp}\mu$ and $i = 1, \cdots, r$. Thus $g^{-1}\mathcal{W}(y) \subset \mathcal{W}(g^* y)$ for all $g \in \operatorname{supp}\mu$. Noticing that both sets have the same cardinality r, we obtain that $g^{-1}\mathcal{W}(y) = \mathcal{W}(g^* y)$ for all $g \in \operatorname{supp}\mu$. The desired assertion follows. □

Proof of Proposition 6.4. The proof is similar to that of Proposition 5.3. Set

$$\mathcal{W} = \bigcup_{y \in \Lambda(\Gamma_\mu^*)} \mathcal{W}(y).$$

By Lemma 6.9, the set $\mathcal{W}$ is Γ_μ-invariant: $\Gamma_\mu \mathcal{W} = \mathcal{W}$. By Lemma 6.8, the set $\mathcal{W}$ is finite, hence $Z := \bigcup_{Y \in \mathcal{W}} Y$ is a nonempty Γ_μ-invariant algebraic subset in X_μ. Since Γ_μ is Zariski dense in $H_\mu = \mathrm{Zc}(\Gamma_\mu)$, the algebraic set Z is H_μ-invariant. By Lemma 4.4 X_μ is the minimal H_μ invariant algebraic subset, so $X_\mu \subset Z$. On the other hand, we have $Z = \bigcup_{Y \in \mathcal{W}} Y \subset X_\mu$, therefore, we get $Z = X_\mu$. This reads as $X_\mu = \bigcup_{Y \in \mathcal{W}} Y$. From Lemma 4.5, we know that X_μ is irreducible, therefore, we obtain that $X_\mu = Y$ for some Y in $\mathcal{W}$. In particular, it holds that $\dim(Y) = d_0 = \dim(X_\mu)$, which concludes the proof of Proposition 6.4. □

We are now prepared to prove Propositions 6.1 and 6.2. The proof is based on Proposition 6.4 and the arguments already used in Section 5.

Proof of Propositions 6.1 and 6.2. Let Y be an algebraic subset of the set X_μ with $\nu_s(Y) > 0$. As the set Y can be reducible, we decompose it as a finite union of irreducible algebraic subsets. Specifically, according to Proposition 1.5 of [23], there exist irreducible algebraic subsets $Z_1, \dots, Z_r$ such that $Y = Z_1 \cup \dots \cup Z_r$. From $\nu_s(Y) > 0$, it follows that $\nu_s(Z_k) > 0$ for some $1 \leqslant k \leqslant r$. Recalling that d_0 is the minimal dimension of irreducible algebraic subsets U satisfying $\nu_s(U) > 0$, we get

$\dim(Z_k) \geqslant d_0$. By Proposition 6.4, we know that $\dim(X_\mu) = d_0$. Hence we have $\dim(Z_k) \geqslant d_0 = \dim(X_\mu)$. This implies that $Z_k = X_\mu$, hence $Y = X_\mu$. The latter obviously proves the claims of Propositions 6.1 and 6.2. □

We end the section by establishing Theorems 2.6 and 2.8.

Proof of Theorems 2.6 and 2.8. Let Y be an algebraic subset of the projective space $\mathbb{P}(V_{\mathbb{C}})$. If $X_\mu \subset Y$, we have $\nu_s(Y) \geqslant \nu_s(X_\mu) = 1$, by Lemma 4.2. Otherwise, $Y \cap X_\mu$ is a proper algebraic subset of X_μ. Since $\nu_s(X_\mu) = 1$, it follows that $\nu_s(Y) = \nu_s(Y \cap X_\mu) = 0$, according to Propositions 6.1 and 6.2 for $s > 0$ and $s < 0$ respectively. The assertion of both theorems now follows from the fact that an algebraic subset of $\mathbb{P}(V)$ is the intersection of an algebraic subset of the projective space $\mathbb{P}(V_{\mathbb{C}})$ with $\mathbb{P}(V)$. □

7. Proof of local limit theorems for coefficients

In this section we show how to apply the zero-one law for the stationary measure ν to establish a local limit theorem for the coefficients $\langle f, G_n v \rangle$ of the random walk G_n, which to the best of our knowledge cannot be found in the literature.

7.1. **Auxiliary results.**

Let us fix a non-negative density function ρ on $\mathbb{R}$ with $\int_{\mathbb{R}} \rho(u) du = 1$, whose Fourier transform $\widehat{\rho}(t) = \int_{\mathbb{R}} e^{-itu} \rho(u) du$, $t \in \mathbb{R}$, is supported on $[-1, 1]$. For any $0 < \varepsilon < 1$, define the rescaled density function ρ_ε by $\rho_\varepsilon(u) = \frac{1}{\varepsilon}\rho(\frac{u}{\varepsilon})$, $u \in \mathbb{R}$, whose Fourier transform has a compact support on $[-\varepsilon^{-1}, \varepsilon^{-1}]$. Set $\mathbb{B}_\varepsilon(u) = \{u' \in \mathbb{R} : |u' - u| \leqslant \varepsilon\}$. For any non-negative integrable function ψ, we define

$$\psi_\varepsilon^+(u) = \sup_{u' \in \mathbb{B}_\varepsilon(u)} \psi(u') \quad \text{and} \quad \psi_\varepsilon^-(u) = \inf_{u' \in \mathbb{B}_\varepsilon(u)} \psi(u'), \quad u \in \mathbb{R}. \tag{7.1}$$

We need the following smoothing inequality from [17], which gives two-sided bounds for the function ψ.

Lemma 7.1. *Suppose that ψ is a non-negative integrable function and that ψ_ε^+ and ψ_ε^- are measurable for any $\varepsilon > 0$. Then, there exists a positive constant $C_\rho(\varepsilon)$ with $C_\rho(\varepsilon) \to 0$ as $\varepsilon \to 0$, such that*

$$\psi_\varepsilon^- * \rho_{\varepsilon^2}(u) - \int_{|w| \geqslant \varepsilon} \psi_\varepsilon^-(u - w) \rho_{\varepsilon^2}(w) dw \leqslant \psi(u) \leqslant (1 + C_\rho(\varepsilon)) \psi_\varepsilon^+ * \rho_{\varepsilon^2}(u), \quad u \in \mathbb{R}.$$

Define the perturbed operator P_{it} as follows: for any $t \in \mathbb{R}$ and $\varphi \in \mathcal{C}(\mathbb{P}(V))$,

$$P_{it}(\varphi)(x) = \int_{\mathbb{G}} e^{it(\sigma(g,x) - \lambda)} \varphi(gx) \mu(dg), \quad x \in \mathbb{P}(V). \tag{7.2}$$

The following proposition which is taken from [29] will be used in the proof of Theorem 2.9. Recall that ν is the unique stationary measure of the Markov chain $(G_n x)_{n \geqslant 0}$ on the projective space $\mathbb{P}(V)$. Let φ be a γ-Hölder continuous function on $\mathbb{P}(V)$. Assume that $\psi : \mathbb{R} \mapsto \mathbb{C}$ is a continuous function with compact support in $\mathbb{R}$, and moreover, ψ is differentiable in a small neighborhood of 0 on the real line.

Proposition 7.2. *Assume conditions **A1**, **A2** and **A3**. Then, there exist constants $\delta > 0$, $c > 0$, $C > 0$ such that for all $x \in \mathbb{P}(V)$, $|l| \leqslant \frac{1}{\sqrt{n}}$, $\varphi \in \mathcal{B}_\gamma$ and $n \geqslant 1$,*

$$\left| \sigma\sqrt{n}\, e^{\frac{nl^2}{2\sigma^2}} \int_{\mathbb{R}} e^{-itln} P_{it}^n(\varphi)(x)\psi(t)dt - \sqrt{2\pi}\nu(\varphi)\psi(0) \right|$$
$$\leqslant \frac{C}{\sqrt{n}}\|\varphi\|_\gamma + \frac{C}{n}\|\varphi\|_\gamma \sup_{|t|\leqslant\delta}\left(|\psi(t)| + |\psi'(t)|\right) + Ce^{-cn}\|\varphi\|_\gamma \int_{\mathbb{R}} |\psi(t)|dt.$$

The explicit dependence of the bound on the target function φ as well as the rate of convergence established in Proposition 7.2 will be used in the proof of Theorem 2.9.

7.2. **Proof of Theorem 2.9.** The goal of this subsection is to establish Theorem 2.9 using Theorems 2.1 and 2.2, Lemma 3.8 and Proposition 7.2.

Proof of Theorem 2.9. The basic idea is to decompose the logarithm of the coefficient $\log|\langle f, G_n v\rangle|$ as a sum of the norm cocycle $\sigma(G_n, x)$ and of $\log\delta(y, G_n x)$, where $x = \mathbb{R}v \in \mathbb{P}(V)$ and $y = \mathbb{R}f \in \mathbb{P}(V^*)$ with $|v| = 1$ and $|f| = 1$, see (1.3). Specifically, we have the decomposition:

$$\begin{aligned} J :&= \sigma\sqrt{2\pi n}\,\mathbb{P}\Big(\log|\langle f, G_n v\rangle| - n\lambda \in [a_1, a_2]\Big) \\ &= \sigma\sqrt{2\pi n}\,\mathbb{P}\Big(S_n^v + \log\delta(y, G_n x) \in [a_1, a_2]\Big), \end{aligned} \tag{7.3}$$

where

$$S_n^v = \log|G_n v| - n\lambda.$$

For any fixed small constant $0 < \eta < 1$, we denote

$$I_k := (-\eta k, -\eta(k-1)], \quad k \in \mathbb{N}^*.$$

In the sequel, let $\lfloor a \rfloor$ denote the integral part of $a \in \mathbb{R}$. To apply limit theorems established for the norm cocycle $\log|G_n v|$, we are led to discretize the function $x \mapsto \log\delta(y, x)$ to obtain that: with a sufficiently large constant $C_1 > 0$,

$$\begin{aligned} J &= \sigma\sqrt{2\pi n}\,\mathbb{P}\Big(S_n^v + \log\delta(y, G_n x) \in [a_1, a_2],\ \log\delta(y, G_n x) \leqslant -\eta\lfloor C_1 \log n\rfloor\Big) \\ &\quad + \sigma\sqrt{2\pi n} \sum_{k=1}^{\lfloor C_1 \log n\rfloor} \mathbb{P}\Big(S_n^v + \log\delta(y, G_n x) \in [a_1, a_2],\ \log\delta(y, G_n x) \in I_k\Big) \\ &=: J_1 + J_2. \end{aligned}$$

Upper bound of J_1. The term J_1 is easily handled by using the fact that, with very small probability, the Markov chain $(G_n x)_{n\geqslant 0}$ stays close to the hyperplane $\ker f$. Indeed, by Lemma 3.8, we get that there exists a constant $c_\eta > 0$ such that

$$J_1 \leqslant \sigma\sqrt{2\pi n}\,\mathbb{P}\Big(\log\delta(y, G_n x) \leqslant -\eta\lfloor C_1 \log n\rfloor\Big) \leqslant \sigma\sqrt{2\pi n}\, e^{-c_\eta\lfloor C_1 \log n\rfloor} \to 0, \tag{7.4}$$

as $n \to \infty$, since the constant $C_1 > 0$ is sufficiently large.

Upper bound of J_2. Note that on the set $\{\log \delta(y, G_n x) \in I_k\}$, we have $0 < \log \delta(y, G_n x) + \eta k \leqslant \eta$ and hence

$$J_2 \leqslant \sigma\sqrt{2\pi n} \sum_{k=1}^{\lfloor C_1 \log n \rfloor} \mathbb{E}\Big(\mathbb{1}_{\{S_n^v - \eta k \in [a_1, a_2 + \eta]\}} \mathbb{1}_{\{\log \delta(y, G_n x) \in I_k\}} \Big).$$

We denote $\psi_1(u) = \mathbb{1}_{\{u \in [a_1, a_2+\eta]\}}$, $u \in \mathbb{R}$, and $\psi_\varepsilon^+(u) = \sup_{u' \in \mathbb{B}_\varepsilon(u)} \psi_1(u')$ as in (7.1), for $0 < \varepsilon < 1$. Using the upper bound in Lemma 7.1 gives

$$J_2 \leqslant (1 + C_\rho(\varepsilon))\sigma\sqrt{2\pi n} \sum_{k=1}^{\lfloor C_1 \log n \rfloor} \mathbb{E}\left[(\psi_\varepsilon^+ * \rho_{\varepsilon^2})(S_n^v - \eta k) \mathbb{1}_{\{\log \delta(y, G_n x) \in I_k\}} \right]. \tag{7.5}$$

For small enough constant $\varepsilon_1 > 0$, we define the density function $\bar{\rho}_{\varepsilon_1}$ by setting $\bar{\rho}_{\varepsilon_1}(u) := \frac{1}{\varepsilon_1}(1 - \frac{|u|}{\varepsilon_1})$ for $u \in [-\varepsilon_1, \varepsilon_1]$, and $\bar{\rho}_{\varepsilon_1}(u) = 0$ otherwise. For any $k \in \mathbb{N}^*$, with the notation $\chi_k(u) := \mathbb{1}_{\{u \in I_k\}}$ and $\chi_{k,\varepsilon_1}^+(u) = \sup_{u' \in \mathbb{B}_{\varepsilon_1}(u)} \chi_k(u')$, one can check that

$$\chi_k(u) \leqslant (\chi_{k,\varepsilon_1}^+ * \bar{\rho}_{\varepsilon_1})(u) \leqslant \chi_{k,2\varepsilon_1}^+(u), \quad u \in \mathbb{R}. \tag{7.6}$$

For $y \in \mathbb{P}(V^*)$, we denote for short $\varphi_k^y(x) = (\chi_{k,\varepsilon_1}^+ * \bar{\rho}_{\varepsilon_1})(\log \delta(y, x))$, $x \in \mathbb{P}(V)$, which is Hölder continuous on $\mathbb{P}(V)$. Using (7.6) leads to

$$J_2 \leqslant (1 + C_\rho(\varepsilon))\sigma\sqrt{2\pi n} \sum_{k=1}^{\lfloor C_1 \log n \rfloor} \mathbb{E}\left[\varphi_k^y(G_n x)(\psi_\varepsilon^+ * \rho_{\varepsilon^2})(S_n^v - \eta k) \right]. \tag{7.7}$$

Denote by $\widehat{\psi}_\varepsilon^+$ the Fourier transform of ψ_ε^+, then by the Fourier inversion formula,

$$\psi_\varepsilon^+ * \rho_{\varepsilon^2}(u) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{itu} \widehat{\psi}_\varepsilon^+(t) \widehat{\rho}_{\varepsilon^2}(t) dt, \quad u \in \mathbb{R}.$$

Substituting $u = S_n^v - \eta k$ and using Fubini's theorem, we obtain

$$J_2 \leqslant (1 + C_\rho(\varepsilon))\sigma\sqrt{\frac{n}{2\pi}} \sum_{k=1}^{\lfloor C_1 \log n \rfloor} \int_{\mathbb{R}} e^{-it\eta k} P_{it}^n(\varphi_k^y)(x) \widehat{\psi}_\varepsilon^+(t) \widehat{\rho}_{\varepsilon^2}(t) dt, \tag{7.8}$$

where P_{it} is the perturbed operator defined by (7.2). We shall apply Proposition 7.2 to deal with each integral in (7.8) for fixed $k \geqslant 1$. Note that $e^{\frac{Ck^2}{n}} \to 1$ as $n \to \infty$, uniformly in $1 \leqslant k \leqslant \lfloor C_1 \log n \rfloor$. Since the function $\widehat{\psi}_\varepsilon^+ \widehat{\rho}_{\varepsilon^2}$ is compactly supported in $\mathbb{R}$, using Proposition 7.2 with $\varphi = \varphi_k^y$, $\psi = \widehat{\psi}_\varepsilon^+ \widehat{\rho}_{\varepsilon^2}$ and $l = \frac{\eta k}{n}$, we obtain that for any fixed $k \geqslant 1$, as $n \to \infty$, uniformly in $f \in V^*$ and $v \in V$ with $|f| = 1$ and $|v| = 1$,

$$\left| \sigma\sqrt{\frac{n}{2\pi}} \int_{\mathbb{R}} e^{-it\eta k} P_{it}^n(\varphi_k^y)(x) \widehat{\psi}_\varepsilon^+(t) \widehat{\rho}_{\varepsilon^2}(t) dt - \widehat{\psi}_\varepsilon^+(0) \widehat{\rho}_{\varepsilon^2}(0) \nu(\varphi_k^y) \right| \leqslant \frac{C}{\sqrt{n}} \|\varphi_k^y\|_\gamma.$$

Note that $\widehat{\rho}_{\varepsilon^2}(0) = 1$ and

$$\widehat{\psi}_\varepsilon^+(0) = \int_{\mathbb{R}} \sup_{y' \in \mathbb{B}_\varepsilon(u)} \psi_\varepsilon^+(u') du = \int_{\mathbb{R}} \mathbb{1}_{\{u \in [a_1 - \varepsilon, a_2 + \eta + \varepsilon]\}} du = a_2 - a_1 + \eta + 2\varepsilon.$$

One can calculate that γ-Hölder norm $\|\varphi_k^y\|_\gamma$ is dominated by $C\frac{e^{\eta\gamma k}}{(1 - e^{-2\varepsilon_1})^\gamma}$, uniformly in $y \in \mathbb{P}(V^*)$. Taking sufficiently small $\gamma > 0$, we obtain that the series

$\frac{C}{\sqrt{n}} \sum_{k=1}^{\lfloor C_1 \log n \rfloor} \frac{e^{\eta\gamma k}}{(1-e^{-2\varepsilon_1})^\gamma}$ converges to 0 as $n \to \infty$. Consequently, we are allowed to interchange the limit as $n \to \infty$ and the sum over k in (7.8) to obtain that, uniformly in $f \in V^*$ and $v \in V$ with $|f| = 1$ and $|v| = 1$,

$$\limsup_{n\to\infty} J_2 \leqslant (1 + C_\rho(\varepsilon))(a_2 - a_1 + \eta + 2\varepsilon) \sum_{k=1}^{\infty} \nu(\varphi_k^y). \tag{7.9}$$

Observe that for any $x \in \mathbb{P}(V)$,

$$\varphi_k^y(x) \leqslant \mathbb{1}_{\left\{ \log\delta(y,\cdot)\in I_k \right\}}(x) + \mathbb{1}_{\left\{ \log\delta(y,\cdot)\in I_{k,\varepsilon_1} \right\}}(x), \tag{7.10}$$

where $I_{k,\varepsilon_1} = \big(-\eta k - 2\varepsilon_1, -\eta k \big] \cup \big(-\eta(k-1), -\eta(k-1) + 2\varepsilon_1 \big]$. For the first part in (7.10), we have that for any $y \in \mathbb{P}(V^*)$,

$$\sum_{k=1}^{\infty} \nu\Big(x \in \mathbb{P}(V) : \log\delta(y,x) \in I_k \Big) = 1. \tag{7.11}$$

For the second part in (7.10), we need to apply Theorem 2.1 and the zero-one law for the stationary measure ν established in Theorem 2.2. By the Lebesgue dominated convergence theorem, we get

$$\begin{aligned}
E &:= \lim_{\varepsilon_1\to 0} \sum_{k=1}^{\infty} \nu\Big(x \in \mathbb{P}(V) : \log\delta(y,x) \in I_{k,\varepsilon_1} \Big) \\
&= \sum_{k=1}^{\infty} \nu\Big(x \in \mathbb{P}(V) : \log\delta(y,x) = -\eta k \Big) + \sum_{k=1}^{\infty} \nu\Big(x \in \mathbb{P}(V) : \log\delta(y,x) = -\eta(k-1) \Big) \\
&= 2 \sum_{k=1}^{\infty} \nu\Big(x \in \mathbb{P}(V) : \log\delta(y,x) = -\eta k \Big),
\end{aligned}$$

where in the last equality we used Theorem 2.1. We are going to apply Theorem 2.2 to prove that $E = 0$. In fact, for any $y \in \mathbb{P}(V^*)$ and any set $Y_{y,t} = \{x \in \mathbb{P}(V) : \log\delta(y,x) = t\}$ with $t \in (-\infty, 0)$, by Theorem 2.2 it holds that either $\nu(Y_{y,t}) = 0$ or $\nu(Y_{y,t}) = 1$. If $\nu(Y_{y,t}) = 0$ for all $y \in \mathbb{P}(V^*)$ and $t \in (-\infty, 0)$, then clearly we get that $E = 0$. If $\nu(Y_{y_0,t_0}) = 1$ for some $y_0 \in \mathbb{P}(V^*)$ and $t_0 \in (-\infty, 0)$, then we can always choose $0 < \eta < 1$ in such a way that $-\eta k \neq t_0$ for all $k \geqslant 1$, so that we also obtain $E = 0$ for all $y \in \mathbb{P}(V^*)$. Therefore, combining this with (7.9), (7.10) and (7.11), letting first $\varepsilon \to 0$ and then $\eta \to 0$, and noting that $C_\rho(\varepsilon) \to 0$ as $\varepsilon \to 0$, we obtain the upper bound: uniformly in $f \in V^*$ and $v \in V$ with $|f| = 1$ and $|v| = 1$,

$$\limsup_{n\to\infty} J_2 \leqslant a_2 - a_1. \tag{7.12}$$

Lower bound of J_2. Since $0 < \log\delta(y, G_n x) + \eta k \leqslant \eta$ on the set $\{\log\delta(y, G_n x) \in I_k\}$, we have

$$J_2 \geqslant \sigma\sqrt{2\pi n} \sum_{k=1}^{\lfloor C_1 \log n \rfloor} \mathbb{E}\Big[\mathbb{1}_{\{S_n^v - \eta k \in [a_1+\eta, a_2-\eta]\}} \mathbb{1}_{\{\log\delta(y,G_n x)\in I_k\}} \Big].$$

We denote $\psi_2(u) = \mathbb{1}_{\{u\in[a_1,a_2+\eta]\}}$, $u \in \mathbb{R}$, and recall that $\psi_\varepsilon^-(u) = \inf_{u'\in\mathbb{B}_\varepsilon(u)} \psi_2(u')$ is defined by (7.1), for $0 < \varepsilon < 1$. By Lemma 7.1, we get

$$\begin{aligned} J_2 \geqslant\ & \sigma\sqrt{2\pi n} \sum_{k=1}^{\lfloor C_1 \log n\rfloor} \mathbb{E}\left[(\psi_\varepsilon^- * \rho_{\varepsilon^2})(S_n^v - \eta k) \mathbb{1}_{\{\log\delta(y,G_n x)\in I_k\}} \right] \\ & - \sigma\sqrt{2\pi n} \sum_{k=1}^{\lfloor C_1 \log n\rfloor} \int_{|w|\geqslant\varepsilon} \mathbb{E}\left[\psi_\varepsilon^-(S_n^v - \eta k - w) \mathbb{1}_{\{\log\delta(y,G_n x)\in I_k\}} \right] \rho_{\varepsilon^2}(w) dw \\ =:\ & J_3 - J_4. \end{aligned} \tag{7.13}$$

For any $k \in \mathbb{N}$, define $\chi_k(u) := \mathbb{1}_{\{u\in I_k\}}$ and $\chi_{k,\varepsilon_1}^-(u) = \inf_{u'\in\mathbb{B}_{\varepsilon_1}(u)} \chi_k(u')$. It is easy to verify that

$$\chi_k(u) \geqslant (\chi_{k,\varepsilon_1}^- * \bar\rho_{\varepsilon_1})(u) \geqslant \chi_{k,2\varepsilon_1}^-(u), \quad u\in\mathbb{R}, \tag{7.14}$$

where $\bar\rho_{\varepsilon_1}$ is the density function introduced in (7.6). For short, we denote $\tilde\varphi_k^y(x) = (\chi_{k,\varepsilon_1}^- * \bar\rho_{\varepsilon_1})(\log\delta(y,x))$, $x \in \mathbb{P}(V)$, which is Hölder continuous on $\mathbb{P}(V)$.

Lower bound of J_3. Using (7.14), we get

$$J_3 \geqslant \sigma\sqrt{2\pi n} \sum_{k=1}^{\lfloor C_1 \log n\rfloor} \mathbb{E}\left[\tilde\varphi_k^y(G_n x)(\psi_\varepsilon^- * \rho_{\varepsilon^2})(S_n^v - \eta k) \right].$$

In an analogous way as in the proof of (7.9), we obtain

$$\liminf_{n\to\infty} J_3 \geqslant (a_2 - a_1 - 2\eta - 2\varepsilon) \sum_{k=1}^{\infty} \nu(\tilde\varphi_k^y). \tag{7.15}$$

Proceeding in a similar way as in the proof of the upper bound (7.12) for J_2, using Theorem 2.2, we can obtain the lower bound for J_3: uniformly in $f \in V^*$ and $v \in V$ with $|f| = 1$ and $|v| = 1$,

$$\liminf_{\eta\to 0} \liminf_{\varepsilon\to 0} \liminf_{n\to\infty} J_3 \geqslant a_2 - a_1. \tag{7.16}$$

Upper bound of J_4. Note that $\psi_\varepsilon^- \leqslant \psi$, then it follows from Lemma 7.1 that $\psi_\varepsilon^- \leqslant (1+C_\rho(\varepsilon))\widehat{\psi}_\varepsilon^+ \widehat{\rho}_{\varepsilon^2}$. Moreover, using (7.6), we get $\mathbb{1}_{\{\log\delta(y,G_n x)\in I_k\}} \leqslant (\chi_{k,\varepsilon_1}^+ * \bar\rho_{\varepsilon_1})(G_n x)$. Similarly to (7.8), we have that J_4 defined in (7.13) is bounded from above by

$$(1 + C_\rho(\varepsilon))\sigma\sqrt{\frac{n}{2\pi}} \sum_{k=1}^{\lfloor C_1 \log n\rfloor} \int_{|w|\geqslant\varepsilon} \left[\int_{\mathbb{R}} e^{-it(\eta k + w)} P_{it}^n(\varphi_k^y)(x) \widehat{\psi}_\varepsilon^+(t) \widehat{\rho}_{\varepsilon^2}(t) dt \right] \rho_{\varepsilon^2}(w) dw.$$

Applying Proposition 7.2 with $\varphi = \varphi_k^y$ and $\psi = \widehat{\psi}_\varepsilon^+ \widehat{\rho}_{\varepsilon^2}$, it follows from the Lebesgue dominated convergence theorem that

$$\liminf_{n\to\infty} J_4 \leqslant (1 + C_\rho(\varepsilon)) \sum_{k=1}^{\lfloor C_1 \log n\rfloor} \nu(\varphi_k^y) \widehat{\psi}_\varepsilon^+(0) \widehat{\rho}_{\varepsilon^2}(0) \int_{|w|\geqslant\varepsilon} \rho_{\varepsilon^2}(w) dw \to 0,$$

as $\varepsilon \to 0$. Combining this with (7.13) and (7.16), we get the lower bound for J_2: uniformly in $f \in V^*$ and $v \in V$ with $|f| = 1$ and $|v| = 1$,

$$\liminf_{n\to\infty} J_2 \geqslant a_2 - a_1. \tag{7.17}$$

Putting together (7.4), (7.12) and (7.17), we conclude the proof of Theorem 2.9. $\square$

References

[1] Aoun R.: Transience of algebraic varieties in linear groups-applications to generic Zariski density *Annales de l'Institut Fourier*, 63(5): 2049-2080, 2013.

[2] Benoist Y., Quint J. F.: Stationary measures and invariant subsets of homogeneous spaces (II). *Journal of the American Mathematical Society*, 26(3): 659-734, 2013.

[3] Benoist Y., Quint J. F.: Random walks on projective spaces. *Composito Mathematica*, 150(9): 1579-1606, 2014.

[4] Benoist Y., Quint J. F.: Central limit theorem for linear groups. *The Annals of Probability*, 44(2): 1308-1340, 2016.

[5] Benoist Y., Quint J. F.: Random walks on reductive groups. *Springer International Publishing*, 2016.

[6] Bonsall F. F.: Lectures on some fixed point theorems of functional analysis, *Bombay: Tata Institute of Fundamental Research*, 1962.

[7] Borel A.: Linear algebraic groups. *Springer Science Business Media*, 1991.

[8] Borovkov A. A., Borovkov K. A.: Asymptotic analysis of random walks. *Cambridge University Press*, 2008.

[9] Bougerol P., Lacroix J.: Products of random matrices with applications to Schrödinger operators. *Birkhäuser Boston*, 1985.

[10] Bourgain J., Furman A., Lindenstrauss E., Mozes S.: Stationary measures and equidistribution for orbits of nonabelian semigroups on the torus. *Journal of the American Mathematical Society*, 24(1): 231-280, 2011.

[11] Breuillard E.: Distributions diophantiennes et théorème limite local sur $\mathbb{R}^d$. *Probability Theory and Related Fields*, 132(1): 13-38, 2005.

[12] Furstenberg H.: Noncommuting random products. *Transactions of the American Mathematical Society*, 108(3): 377-428, 1963.

[13] Furstenberg H.: Boundary theory and stochastic processes on homogeneous spaces. *Proc. Symp. Pure Math.*, 26: 193-229, 1973.

[14] Furstenberg H., Kesten H.: Products of random matrices. *The Annals of Mathematical Statistics*, 31(2): 457-469, 1960.

[15] Gnedenko B. V.: On a local limit theorem of the theory of probability. *Uspekhi Matematicheskikh Nauk*, 3(3): 187-194, 1948.

[16] Goldsheid I. Y., Guivarc'h Y.: Zariski closure and the dimension of the Gaussian law of the product of random matrices. *Probability Theory and Related Fields*, 105(1): 109-142, 1996.

[17] Grama I., Lauvergnat R., Le Page É.: Conditioned local limit theorems for random walks defined on finite Markov chains. *Probability Theory and Related Fields*, 176(1-2): 669-735, 2020.

[18] Guivarc'h Y.: Produits de matrices aléatoires et applications aux propriétés géométriques des sous-groupes du groupe linéaire. *Ergodic Theory and Dynamical Systems*, 10(3): 483-512, 1990.

[19] Guivarc'h Y.: Spectral gap properties and limit theorems for some random walks and dynamical systems. *Proc. Sympos. Pure Math*, 89: 279-310, 2015.

[20] Guivarc'h Y., Le Page É.: Simplicité de spectres de Lyapunov et propriété d'isolation spectrale pour une famille d'opérateurs de transfert sur l'espace projectif. Kaimanovich, Vadim A. (ed.), *Random walks and geometry.* Proceedings of a workshop at the Erwin Schrödinger Institute, Vienna, June 18 – July 13, 2001. In collaboration with Klaus Schmidt and Wolfgang Woess. Collected papers. Berlin: de Gruyter (ISBN 3-11-017237-2/hbk). 181-259 (2004).

[21] Guivarc'h Y., Le Page É.: Spectral gap properties for linear random walks and Pareto's asymptotics for affine stochastic recursions. *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques*, 52(2): 503-574, 2016.

[22] Guivarc'h Y., Raugi A.: Frontiere de Furstenberg, propriétés de contraction et théorèmes de convergence. *Probability Theory and Related Fields*, 69(2): 187-242, 1985.

[23] Hartshorne, R.: Algebraic geometry. Vol. 52. *Springer Science and Business Media*, 2013.

[24] Hennion H., Hervé L.: Limit theorems for Markov chains and stochastic properties of dynamical systems by quasi-compactness. Vol. 1766 of Lecture Notes in Mathematics. *Springer-Verlag, Berlin,* 2001.

[25] Le Page É.: Théorèmes limites pour les produits de matrices aléatoires. In Probability measures on groups. *Springer Berlin Heidelberg*, 258-303, 1982.

[26] Mattila P.: Geometry of Sets and Measures in Euclidean Spaces: Fractals and Rectifiability (Cambridge Studies in Advanced Mathematics) *Cambridge University Press*, 2015.

[27] Stone C.: A local limit theorem for nonlattice multi-dimensional distribution functions. *The Annals of Mathematical Statistics*, 36(2): 546-551, 1965.

[28] Xiao H., Grama I., Liu Q.: Precise large deviation asymptotics for products of random matrices. *Stochastic Processes and their Applications*, 130(9): 5213-5242, 2020.

[29] Xiao H., Grama I., Liu Q.: Berry-Esseen bound and precise moderate deviations for products of random matrices. *Journal of the European Mathematical Society*, to appear, see also arXiv:1907.02438, 2019.

[30] Xiao H., Grama I., Liu Q.: Large deviation expansions for the coefficients of random walks on the general linear group. *arXiv preprint*, arXiv:2010.00553, 2020.

[31] Xiao H., Grama I., Liu Q.: Law of large numbers and moderate deviations for the coefficients of random walks on the general linear group. *In preparation*, 2020.

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