Examples of unique ergodicity of algebraic flows

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Contents

Chapter 1

Continuous time dynamics

In this chapter, we will see how to extend to continuous time dynamical systems the notions that have been introduced for discrete time dynamical systems in Professor Le Calvez's course. This will be an occasion for revising these notions!

1.1 Continuous flows

Let X be a (Hausdorff) topological space.

Definition 1.1.1. A continuous flow on X is a family $(\varphi_t)_{t\in\mathbb{R}}$ of homeomorphisms of X such that the map

$$\begin{aligned} \mathbb{R} \times X &\to X \\ (t, x) &\mapsto \varphi_t(x) \end{aligned}$$

is continuous and that, for any t and s in $\mathbb{R}$, one has $\varphi_{t+s} = \varphi_t \circ \varphi_s$.

In other words, a flow $(\varphi_t)_{t\in\mathbb{R}}$ is a homomorphism from the additive group $(\mathbb{R}, +)$ to the group of homeomorphisms of X such that the map $(t, x) \mapsto \varphi_t(x)$ is continuous. One could say that a flow is a continuous action of the group $(\mathbb{R}, +)$ on the topological space X.

In particular, one has $\varphi_0 = \varphi_0 \circ \varphi_0$ so that $\varphi_0 = \mathrm{Id}_X$ and, for any t in $\mathbb{R}$, φ_{-t} is the inverse of φ_t.

Example 1.1.2. *(i)* Recall that, if $U \subset \mathbb{R}^d$ is an open subset, a $\mathcal{C}^1$ vector field $F : U \to \mathbb{R}^d$ is said to be complete if, for any x in U, there exists a $\mathcal{C}^1$ curve $\gamma : \mathbb{R} \to U$ such that $\gamma' = F(\gamma)$ and $\gamma(0) = x$. Such a curve is necessarily unique by Cauchy-Lipschitz theorem. Then, if $\varphi_t(x)$ denotes its value at t, the family (φ_t) is a flow on U.

(ii) Suppose now $U = \mathbb{R}^d$ and take F to be of the form $x \mapsto Ax$, where A is a square matrix of size d. This vector field is complete and the associate flow φ_t satisfies $\varphi_t(x) = \exp(tA)x$, $t \in \mathbb{R}$, $x \in \mathbb{R}^d$, where, for any square matrix B, its exponential $\exp B$ is defined by

$$\exp B = \sum_{n=0}^{\infty} \frac{1}{n!} B^n.$$

Such flows are called linear flows. In particular, if A is an antisymmetric matrix, for any t in $\mathbb{R}$, $\exp(tA)$ is an orthogonal matrix, so that for any t, one has $\exp(tA)(\mathbb{S}^{d-1}) \subset \mathbb{S}^{d-1}$.

(iii) Let y be a vector in $\mathbb{R}^d$. For t in $\mathbb{R}$ and x in the torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$, set $\varphi_t(x) = x + ty$ (that is $\varphi_t(x)$ is the image in $\mathbb{T}^d$ of $\tilde{x} + ty$, where $\tilde{x}$ is any vector in $\mathbb{R}^d$ which image in $\mathbb{T}^d$ is x). Then (φ_t) is a continuous flow on $\mathbb{T}^d$. Such flows are called translation flows.

Let us fix some continuous flow $(\varphi_t)_{t\in\mathbb{R}}$ on X. For x in X, let us say that the orbit of x under the flow $(\varphi_t)_{t\in\mathbb{R}}$ is the set $\{\varphi_t(x) | t \in \mathbb{R}\}$.

The usual notions that have been introduced in the case of continuous maps $X \to X$ extend to flows. We recall the most important ones.

Definition 1.1.3. Let x be in X. The point x is said to be recurrent if there exists a sequence $(t_k)_{k\in\mathbb{N}}$ in $\mathbb{R}_+$ going to infinity such that $\varphi_{t_k}(x) \xrightarrow[k\to\infty]{} x$; otherwise, x is called non recurrent. The point x is is called non wandering if, for every open set U containing x and for any t_0 in $\mathbb{R}$, there exists $t \geq t_0$ such that $\varphi_{-t}(U) \cap U \neq \emptyset$; otherwise, x is called wandering.

In other words, a point is recurrent if it is an accumulation point of its orbit. A recurrent point is non wandering. The set of recurrent point may behave badly. The set of non wandering points is closed.

Let us now recall the notion of topological transitivity:

Definition 1.1.4. The flow $(\varphi_t)_{t\in\mathbb{R}}$ is said to be topollogically transitive if, for any nonempty open subsets U and V of X, there exists a t in $\mathbb{R}$ such that $U \cap \varphi_{-t}(V) \neq \emptyset$. The flow $(\varphi_t)_{t\in\mathbb{R}}$ is said to be topologically mixing if, for any nonempty open subsets U and V of X, there exists a t_0 in $\mathbb{R}$ such that, for any $t \geq t_0$, one has $U \cap \varphi_{-t}(V) \neq \emptyset$.

A topologically mixing flow is topologically transitive (this is often a very simple way to prove topological transitivity). We shall soon see on an example that the converse is not true.

Recall that a set $\mathcal{U}$ of open subsets of X is said to be a basis for the topology of X if, for any open subset V of X and for any x in V, there exists U in $\mathcal{U}$ such that $x \in U \subset V$. In the definition of topological transitivity and topological mixing, it is sufficient to verify the criterion for U and V in a basis of the topology of X. We say that X has a countable basis if there exists such a $\mathcal{U}$ which is countable. The space $\mathbb{R}^d$, his open subsets and his close subsets have countable basis.

Finally, let us recall that X is said to be a Baire space if, for every countable family $(U_i)_{i\in\mathbb{N}}$ of dense open subsets of X, the set $\bigcap_{i\in\mathbb{N}} U_i$ is dense in X. A locally compact space and a complete metric space are Baire spaces.

As in the case of discrete time dynamics, the pertinence of the definition of transitivity comes from the

Proposition 1.1.5. *Suppose X is Baire with countable basis. Then the following are equivalent:*

(i) the flow $(\varphi_t)_{t\in\mathbb{R}}$ is transitive.

(ii) there exists a point x in X with dense orbit.

(iii) the set of points with dense orbit is a dense G_δ subset of X.

Finally let us define minimal flows:

Definition 1.1.6. The flow $(\varphi_t)_{t\in\mathbb{R}}$ is said to be minimal if and only if every orbit is dense.

Example 1.1.7. Let $y = (y_1, \ldots, y_d)$ be a vector in $\mathbb{R}^d$. Then the associate translation flow on the torus $\mathbb{T}^d$ is transitive if and only if, for any non zero $\alpha = (\alpha_1, \ldots, \alpha_d)$ in $\mathbb{Q}^d$, one has $\alpha_1 y_1 + \ldots + \alpha_d y_d \neq 0$. Then, this flow is in fact minimal. It is never mixing.

Say that a closed subset Y of X is invariant by the flow if, for any t in $\mathbb{R}$, one has $\varphi_t(Y) = Y$. Then, (φ_t) induces a continuous flow on Y. As in the case of discrete time dynamics, one deduces from an application of Zorn lemma the

Proposition 1.1.8. *Suppose X to be compact. There exists a closed subset Y of X which is stable by the flow and such that the restriction of (φ_t) to Y is minimal.*

1.2 Measure preserving flows

Let now $(X, \mathcal{A}, \mu)$ be a probability space, that is X is a set, $\mathcal{A}$ is a σ-algebra of subsets of X and μ is a probability measure defined on $\mathcal{A}$. We will always suppose $\mathcal{A}$ to be complete with respect to μ, that is, if B is a subset of X such that there exists A in $\mathcal{A}$ with $B \subset A$ and $\mu(A) = 0$, one has $B \in \mathcal{A}$. This can be achieved by adding to $\mathcal{A}$ the necessary subsets of X.

Definition 1.2.1. A measurable flow on $(X, \mathcal{A}, \mu)$ is a family $(\varphi_t)_{t\in\mathbb{R}}$ of measure preserving measurable automorphisms of $(X, \mathcal{A}, \mu)$ such that the map

$$\begin{aligned} \mathbb{R} \times X &\to X \\ (t, x) &\mapsto \varphi_t(x) \end{aligned}$$

is measurable and that, for any t and s in $\mathbb{R}$, one has $\varphi_{t+s} = \varphi_t \circ \varphi_s$.

In other words, a measurable flow is a measurable action by measure preserving automorphisms of the group $(\mathbb{R}, +)$ on $(X, \mathcal{A}, \mu)$.

Example 1.2.2. *(i)* Let U be a bounded open subset of $\mathbb{R}^d$ and let $F : U \to \mathbb{R}^d$ be a complete $\mathcal{C}^1$ vector field. Suppose the divergence of F is equal to 0 everywhere on U. Then the flow associated to F preserves the restriction to U of the Lebesgue measure of $\mathbb{R}^d$.

(ii) Let A be an antisymmetric square matrix of size d. Then the associate flow on the unit sphere $\mathbb{S}^{d-1}$ preserves the Riemannian volume of the sphere.

(iii) Let y be a vector in $\mathbb{R}^d$. Then the associate translation flow on the torus $\mathbb{T}^d$ preserves the Haar measure of the torus.

In the sequel, we will make a supplementary assumption on the probability space $(X, \mathcal{A}, \mu)$. We will suppose the space $\mathrm{L}^1(X)$ to be separable: it is the case in all examples above and, more generally, as soon as X is a separable locally compact space, μ is a Radon measure on X and $\mathcal{A}$ is the σ-algebra of Borel subsets of X, completed with respect to μ. We shall not encounter other probability spaces.

Let us fix a measurable flow $(\varphi_t)_{t\in\mathbb{R}}$ on $(X, \mathcal{A}, \mu)$. We will study the associate action of $(\mathbb{R}, +)$ on the spaces of functions on X. Note that, for any t in $\mathbb{R}$ and $1 \leq p \leq \infty$, the map $f \mapsto f \circ \varphi_t$ is a norm preaserving linear automorphism of $\mathrm{L}^p(X, \mathcal{A}, \mu)$.

Lemma 1.2.3. *For any $1 \leq p < \infty$, the map*

$$\begin{aligned} \mathbb{R} \times \mathrm{L}^p(X, \mathcal{A}, \mu) &\to \mathrm{L}^p(X, \mathcal{A}, \mu) \\ (t, f) &\mapsto f \circ \varphi_t \end{aligned}$$

is continuous for the norm topology on $\mathrm{L}^p(X, \mathcal{A}, \mu)$.

Remark 1.2.4. When $p = \infty$, this map is only continuous for the weak-$*$ topology on $\mathrm{L}^\infty(X, \mathcal{A}, \mu)$, viewed as the dual space to $\mathrm{L}^1(X, \mathcal{A}, \mu)$.

The proof relies on a lemma on $\mathbb{R}$. Let λ denote the Lebesgue measure of $\mathbb{R}$. The measure λ is regular, that is, for any Borel subset B of $\mathbb{R}$, we have

$$\lambda(B) = \sup_{\substack{K \subset B \subset \mathbb{R} \\ K \text{ compact}}} \lambda(K) = \inf_{\substack{B \subset U \subset \mathbb{R} \\ U \text{ open}}} \lambda(U).$$

Lemma 1.2.5. *Let B be a Borel subset of $\mathbb{R}$ such that $\lambda(B) > 0$. Then 0 in an interior point of the set $B - B$.*

Proof. By regularity of the measure λ, we can find a compact set K of $\mathbb{R}$ such that $K \subset B$ and $\lambda(K) > 0$: it suffices to prove the lemma for K. Let U be an open subset of $\mathbb{R}$ such that $K \subset U$ and $\lambda(U) < 2\lambda(K)$. As K is compact, there exists a neighborhood V of 0 in $\mathbb{R}$ such that $K + V \subset U$. Let us show that we have $V \subset K - K$. Indeed, if t belongs to V, one has $K + t \subset U$ and $\lambda(K + t) = \lambda(K) > \frac{1}{2}\lambda(U)$. Therefore, we have $(K + t) \cap K \neq \emptyset$, that is there exists r and s in K such that $r + t = s$. Thus, $t = s - r$ belongs to $K - K$ and the lemma is proved. $\square$

Proof of lemma 1.2.3. First, let us prove that, if $p = 1$ and f belongs to $\mathrm{L}^1(X, \mathcal{A}, \mu)$, the map

$$\begin{aligned} \mathbb{R} &\to \mathrm{L}^1(X, \mathcal{A}, \mu) \\ t &\mapsto f \circ \varphi_t \end{aligned}$$

is continuous at 0. Indeed, let $(g_k)_{k \in \mathbb{N}}$ be a dense sequence in $\mathrm{L}^1(X, \mathcal{A}, \mu)$. For $\varepsilon > 0$, for any g in $\mathrm{L}^1(X, \mathcal{A}, \mu)$, there exists k in $\mathbb{N}$ with $\|g - g_k\|_1 \leq \varepsilon$. For k in $\mathbb{N}$, the map

$$\begin{aligned} \mathbb{R} \times X &\to \mathbb{R} \\ (t, x) &\mapsto |f(\varphi_t(x)) - g_k(x)| \end{aligned}$$

is Borel. Therefore, by Fubini theorem, the map

$$\begin{aligned} \mathbb{R} &\to \mathbb{R} \\ t &\mapsto \|f \circ \varphi_t - g_k\|_1 \end{aligned}$$

is Borel and the set

$$B_k = \{t \in \mathbb{R} |\, \|f \circ \varphi_t - g_k\|_1 \leq \varepsilon\}$$

is a Borel subset of $\mathbb{R}$. As we have

$$\mathbb{R} = \bigcup_{k \in \mathbb{N}} B_k,$$

we can find a k such that $\lambda(B_k) > 0$. By lemma 1.2.5, there exists a neighborhood V of 0 in $\mathbb{R}$ such that $V \subset B_k - B_k$. Then, if t belongs to V, there exists r and s in B_k with $t = r - s$ and we have

$$\begin{aligned} \|f \circ \varphi_t - f\|_1 &= \|(f \circ \varphi_r - f \circ \varphi_s) \circ \varphi_{-s}\|_1 \\ &= \|f \circ \varphi_r - f \circ \varphi_s\|_1 \\ &\leq \|f \circ \varphi_r - g_k\|_1 + \|f \circ \varphi_s - g_k\|_1 \leq 2\varepsilon, \end{aligned}$$

since r and s belong to B_k. Hence the map $t \mapsto f \circ \varphi_t$ is continuous at 0.

Let now p be in $[1, \infty[$ and let us prove that, if f belongs to $\mathrm{L}^p(X, \mathcal{A}, \mu)$, the map

$$\begin{aligned} \mathbb{R} &\to \mathrm{L}^p(X, \mathcal{A}, \mu) \\ t &\mapsto f \circ \varphi_t \end{aligned}$$

is continuous at 0. Suppose first that f belongs to $\mathrm{L}^\infty(X, \mathcal{A}, \mu)$. Then, for t in $\mathbb{R}$, we have

$$\begin{aligned} \|f \circ \varphi_t - f\|_p &\leq \|f \circ \varphi_t - f\|_\infty^{\frac{p-1}{p}} \|f \circ \varphi_t - f\|_1^{\frac{1}{p}} \\ &\leq (2\|f\|_\infty)^{\frac{p-1}{p}} \|f \circ \varphi_t - f\|_1^{\frac{1}{p}} \end{aligned}$$

so that the map $t \mapsto f \circ \varphi_t$ is continuous at 0. Finally, if f belongs to $\mathrm{L}^p(X, \mathcal{A}, \mu)$, for $\varepsilon > 0$, there exists g in $\mathrm{L}^\infty(X, \mathcal{A}, \mu)$ such that $\|f - g\|_p \leq \varepsilon$ and, for any t in $\mathbb{R}$, we have

$$\begin{aligned} \|f \circ \varphi_t - f\|_p &\leq \|(f - g) \circ \varphi_t\|_p + \|g \circ \varphi_t - g\|_p + \|f - g\|_p \\ &\leq 2\varepsilon + \|g \circ \varphi_t - g\|_p, \end{aligned}$$

so that the map $t \mapsto f \circ \varphi_t$ is continuous at 0.

Let t_0 be in $\mathbb{R}$ and f_0 be in $\mathrm{L}^p(X, \mathcal{A}, \mu)$. We will conclude the proof by proving continuity at (t_0, f_0) of the map

$$\begin{aligned} \mathbb{R} \times \mathrm{L}^p(X, \mathcal{A}, \mu) &\to \mathrm{L}^p(X, \mathcal{A}, \mu) \\ (t, f) &\mapsto f \circ \varphi_t. \end{aligned}$$

For any t in $\mathbb{R}$ and f in $\mathrm{L}^p(X, \mathcal{A}, \mu)$, one has

$$\begin{aligned} \|f \circ \varphi_t - f_0 \circ \varphi_{t_0}\|_p \leq \|(f - f_0) \circ \varphi_t\|_p &+ \|f_0 \circ \varphi_t - f_0 \circ \varphi_{t_0}\|_p \\ &= \|f - f_0\|_p + \|f_0 \circ \varphi_{t-t_0} - f_0\|_p , \end{aligned}$$

so that continuity at (t_0, f_0) of $(t, f) \mapsto f \circ \varphi_t$ follows from continuity at 0 of $t \mapsto f_0 \circ \varphi_t$. □

We will now study invariant functions and Birkhoff theorem for flows. We need to overcome a technical difficulty with the

Lemma 1.2.6. *Let f be a measurable function on X such that, for any t in $\mathbb{R}$, one has $f \circ \varphi_t = f$ almost everywhere on X. Then, there exists a function g on X such that, for any t in $\mathbb{R}$, $g \circ \varphi_t = g$ everywhere on X and $g = f$ almost everywhere on X.*

Proof. For all t in $\mathbb{R}$, the set

$$\{x \in X | f(\varphi_t(x)) = f(x)\}$$

has full measure in X. By Fubini theorem, applied to the product measure $\lambda \otimes \mu$ on $\mathbb{R} \times X$, if A denotes the set of x in X such that the set

$$\{t \in \mathbb{R} | f(\varphi_t(x)) = f(x)\}$$

has full measure in $\mathbb{R}$, one has $\mu(A) = 1$. *A fortiori,* the set B of points x in X such that the Borel function $t \mapsto f(\varphi_t(x))$ is essentially constant on $\mathbb{R}$ has full measure in X. As the Lebesgue measure of $\mathbb{R}$ is invariant under translations, for any t in $\mathbb{R}$, one has $\varphi_t(B) = B$. For x in B, define $g(x)$ to be the essential value of $t \mapsto f(\varphi_t(x))$, and, for $x \notin B$, set $g(x) = 0$. Then, for any t in $\mathbb{R}$, one has $g \circ \varphi_t = g$ and, by construction, for any x in A, one has $g(x) = f(x)$ so that the function g satisfies the requirements of the lemma. □

We will now focus on the Birkhoff theorem for flows. To construct the analogue of Birkhoff sums, we need the

Lemma 1.2.7. *Let $1 \leq p \leq \infty$ and let f be in $\mathrm{L}^p(X, \mathcal{A}, \mu)$. Then, for almost every x in X, the Borel function $t \mapsto f(\varphi_t(x))$ is locally integrable on $\mathbb{R}$. For any $T > 0$, the function $x \mapsto \frac{1}{T}\int_0^T f(\varphi_t(x))\mathrm{d}t$ belongs to $\mathrm{L}^p(X, \mathcal{A}, \mu)$ and has norm $\leq \|f\|_p$.*

Proof. Let n be an integer. Then, by Fubini theorem, we have

$$\begin{aligned}\int_{[-n,n]\times X} |f(\varphi_t(x))| \,\mathrm{d}t\mathrm{d}\mu(x) &= \int_{-n}^{n}\left(\int_X |f(\varphi_t(x))|\,\mathrm{d}\mu(x)\right)\mathrm{d}t\\ &= \int_{-n}^{n}\left(\int_X |f(x)|\,\mathrm{d}\mu(x)\right)\mathrm{d}t\\ &= 2n\,\|f\|_1\,,\end{aligned}$$

so that, again by Fubini theorem, for almost every x in X, one has

$$\int_{-n}^{n} |f(\varphi_t(x))|\,\mathrm{d}t < \infty.$$

As this is true for every n, there exists a set Y in $\mathcal{A}$ with $\mu(Y) = 1$ such that, for any x in Y, for any compact K in $\mathbb{R}$, one has $\int_K |f(\varphi_t(x))|\,\mathrm{d}t < \infty$.

Let $T > 0$. If $p = \infty$, for any t in $\mathbb{R}$, as φ_t preserves the measure μ, for almost every x, we have $|f(\varphi_t(x))| \leq \|f\|_\infty$, so that, by Fubini theorem, for almost every x, one has $|f(\varphi_t(x))| \leq \|f\|_\infty$ almost everywhere on $[0, T]$ and, therefore, $\left\|\frac{1}{T}\int_0^T f\circ\varphi_t\mathrm{d}t\right\|_\infty \leq \|f\|_\infty$. If $p < \infty$, as the function $x \mapsto x^p$ is convex on $\mathbb{R}_+$, by Jensen inequality, we have, for almost every x in X, $\left|\frac{1}{T}\int_0^T f(\varphi_t(x))\mathrm{d}t\right|^p \leq \frac{1}{T}\int_0^T |f(\varphi_t(x))|^p\,\mathrm{d}t$, and, again by Fubini theorem,

$$\left\|\frac{1}{T}\int_0^T f\circ\varphi_t\mathrm{d}t\right\|_p^p \leq \frac{1}{T}\int_0^T\left(\int_X |f(\varphi_t(x))|^p\,\mathrm{d}\mu(x)\right)\mathrm{d}t = \|f\|_p^p\,,$$

what should be proved. □

Let $\mathcal{B}$ be a complete sub-σ-algebra of $\mathcal{A}$. For any f in $\mathrm{L}^1(X, \mathcal{A}, \mu)$, the map $B \mapsto \int_B f\mathrm{d}\mu$ defines a finite measure on $\mathcal{B}$ which is absolutely continuous with respect to μ. By Radon-Nikodym theorem, there exists an unique $\bar{f}$ in

$\mathrm{L}^1(X, \mathcal{B}, \mu)$ such that, for any B in $\mathcal{B}$, one has $\int_{\mathcal{B}} f \mathrm{d}\mu = \int_{\mathcal{B}} \bar{f} \mathrm{d}\mu$. We call $\bar{f}$ the conditional expectation of f with respect to $\mathcal{B}$ and we denote it by $\mathbb{E}(f|\mathcal{B})$. For any $1 \leq p \leq \infty$, the map $f \mapsto \mathbb{E}(f|\mathcal{B})$ induces a norm 1 projection of $\mathrm{L}^p(X, \mathcal{A}, \mu)$ onto $\mathrm{L}^p(X, \mathcal{B}, \mu)$. For $p = 2$, this projection is the orthogonal projection.

Let $\mathcal{I}$ denote the sub-algebra of sets I in $\mathcal{A}$ such that, for any t in $\mathbb{R}$, one has $I = \varphi_{-t}(I)$ almost everywhere, that is $\mu(I \Delta \varphi_{-t}(I)) = 0$, where Δ denotes symmetric difference. Then, we have a Birkhoff theorem for flows:

Theorem 1.2.8. *Let f be in $\mathrm{L}^1(X, \mathcal{A}, \mu)$. Then, one has*

$$\frac{1}{T} \int_0^T f(\varphi_t(x)) \mathrm{d}t \xrightarrow[T \to \infty]{} \mathbb{E}(f|\mathcal{I})(x)$$

for almost every x in X. If $1 \leq p < \infty$ and f belongs to $\mathrm{L}^p(X, \mathcal{A}, \mu)$, one has

$$\frac{1}{T} \int_0^T f \circ \varphi_t \mathrm{d}t \xrightarrow[T \to \infty]{} \mathbb{E}(f|\mathcal{I})$$

in $\mathrm{L}^p(X, \mathcal{A}, \mu)$.

We don't give the proof of this theorem which is obtained exactly the same way as in the discrete time case.

Definition 1.2.9. The flow (φ_t) is said to be ergodic if any invariant set is trivial, that is if, for A in $\mathcal{A}$, if for any t in $\mathbb{R}$ one has $\varphi_t(A) = A$ almost everywhere, then $\mu(A) \in \{0, 1\}$.

According to Birkhoff theorem, we immediately get the

Proposition 1.2.10. *The following are equivalent:*

(i) the flow (φ_t) is ergodic.

(ii) for any A, B in $\mathcal{A}$, one has $\frac{1}{T} \int_0^T \mu(A \cap \varphi_{-t}(B)) \mathrm{d}t \xrightarrow[T \to \infty]{} \mu(A)\mu(B)$.

(iii) for any f in $\mathrm{L}^1(X, \mathcal{A}, \mu)$, one has $\frac{1}{T} \int_0^T f(\varphi_t(x)) \mathrm{d}t \xrightarrow[T \to \infty]{} \int_X f \mathrm{d}\mu$ for almost every x in X.

Example 1.2.11. Let $y = (y_1, \ldots, y_d)$ be in $\mathbb{R}^d$ and let us study the associate translation flow on $\mathbb{T}^d$ with respect to the Lebesgue measure μ of $\mathbb{T}^d$. Suppose that, for any non zero $\alpha = (\alpha_1, \ldots, \alpha_d)$ in $\mathbb{Q}^d$, one has $\alpha_1 y_1 + \ldots + \alpha_d y_d \neq 0$. Let $\mathcal{I}$ be the σ-algebra of invariant subsets for the flow on $\mathcal{T}^d$. We will prove that, for any f in $\mathrm{L}^1(\mathbb{T}^d)$, one has $\mathbb{E}(f|\mathcal{I}) = \int_{\mathbb{T}^d} f \mathrm{d}\mu$ almost everywhere. By density, we can suppose f to be of the form $x \mapsto e^{2i\pi\langle x,\alpha\rangle}$ where $\langle ., . \rangle$ denotes the usual scalar product and α belongs to $\mathbb{Z}^d$. Suppose $\alpha \neq 0$. Then, since $\langle y, \alpha\rangle \neq 0$, for any t in $\mathbb{R}$ and x in $\mathbb{T}^d$, we have

$$\frac{1}{T}\int_0^T f(x+ty)\mathrm{d}t = \frac{1}{T}\frac{e^{2i\pi\langle x,\alpha\rangle}}{2i\pi\langle y,\alpha\rangle}\left(e^{2i\pi T\langle y,\alpha\rangle} - 1\right) \xrightarrow[T\to\infty]{} 0,$$

so that, by Birkhoff theorem, $\mathbb{E}(f|\mathcal{I}) = 0 = \int_{\mathbb{T}^d} f\mathrm{d}\mu$. If $\alpha = 0$, one has $f = 1$ and $\mathbb{E}(f|\mathcal{I}) = 1 = \int_{\mathbb{T}^d} f\mathrm{d}\mu$. Therefore, the translation flow associate to y is ergodic. In fact, this proof shows that this flow preserves an unique Borel probability measure on $\mathbb{T}^d$. Conversely, if there exists a non zero α in $\mathbb{Q}^d$ such that $\langle y, \alpha\rangle = 0$, one can show that this flow is not ergodic.

As in the discrete time case, we have notions of mixing for dynamical systems.

Definition 1.2.12. The flow (φ_t) is said to be weakly mixing if, for any A, B in $\mathcal{A}$, one has $\frac{1}{T}\int_0^T |\mu(A\cap\varphi_{-t}(B)) - \mu(A)\mu(B)|\,\mathrm{d}t \xrightarrow[T\to\infty]{} 0$. The flow (φ_t) is said to be strongly mixing if, for any A, B in $\mathcal{A}$, one has $\mu(A\cap\varphi_{-t}(B)) \xrightarrow[t\to\infty]{} \mu(A)\mu(B)$.

Remark 1.2.13. By Cesaro phenomenon, strong mixing implies weak mixing, but the converse is not true.

In most concrete mixing examples, we shall have strong mixing. However, the most relevant notion for abstract purposes is the one of weak mixing, as shown by the

Proposition 1.2.14. *Let (φ_t) be ergodic. The following are equivalent:*

(i) the flow (φ_t) is weakly mixing.

(ii) the diagonal product flow on $(X \times X, \mathcal{A}\otimes\mathcal{A}, \mu\otimes\mu)$ is ergodic.

(iii) for any ergodic measure preserving flow (ψ_t) on a probability space $(Y, \mathcal{B}, \nu)$, the diagonal product flow on $(X\times Y, \mathcal{A}\otimes\mathcal{B}, \mu\otimes\nu)$ is ergodic.

(iv) for any $t \neq 0$ in $\mathbb{R}$, the measure preserving map φ_t on $(X, \mathcal{A}, \mu)$ is ergodic.

(v) the flow (φ_t) does not have eigenvectors: for any α in $\mathbb{R}$, if there exists a non zero function f on X such that, for any t in $\mathbb{R}$, $f \circ \varphi_t = e^{2i\pi\alpha t} f$ almost everywhere on X, then one has $\alpha = 0$ and f is constant almost everywhere.

Remark 1.2.15. Last condition of the proposition can be read as: the system does not admit a one-dimensional translation flow as a measurable factor.

Remark 1.2.16. There is an analogue notion of weak mixing for measure preserving transformations: a measure preserving transformation T of $(X, \mathcal{A}, \mu)$ if weakly mixing if and only if, for any A, B in $\mathcal{A}$, one has

$$\frac{1}{n} \sum_{k=0}^{n-1} \left| \mu(A \cap T^{-k} B) - \mu(A)\mu(B) \right| \xrightarrow[n \to \infty]{} 0.$$

One can prove a result similar to proposition 1.2.14, except for condition *(iv)* of this proposition: for example, all the non trivial powers of an irrational rotation of the circle $\mathcal{T}$ are ergodic, but it is not weakly mixing. As we shall see in the proof of the proposition, this difference between the discrete time case and the continuous time case is due to the fact that there exists injective homomorphisms $\mathbb{Z} \to \mathbb{T}$ (precisely those of the form $n \mapsto \alpha^n$ where α is irrational) whereas every continuous homomorphism $\mathbb{R} \to \mathbb{T}$ has a kernel.

In the course of the proof, we shall use the

Lemma 1.2.17. *Let F be in $\mathrm{L}^2(X \times X, \mathcal{A} \otimes \mathcal{A}, \mu \otimes \mu)$. For any f in $\mathrm{L}^2(X, \mathcal{A}, \mu)$ set, for almost every y in X,*

$$T_F(f)(y) = \int_X f(x) F(x, y) \mathrm{d}\mu(x).$$

Then T_F defines a bounded compact operator of $\mathrm{L}^2(X, \mathcal{A}, \mu)$ with norm $\leq \|F\|_2$. Its adjoint operator is $T_{F^\vee}$, where $F^\vee$ is the function $(x, y) \mapsto \overline{F(y, x)}$. If $F \neq 0$, then $T_F \neq 0$.

Proof. Let f be in $\mathrm{L}^2(X, \mathcal{A}, \mu)$. By Fubini theorem, $T_F(f)$ is defined almost everywhere on X and, by Cauchy-Schwarz inequality,

$$\begin{aligned}\int_X |T_F(f)|^2 \mathrm{d}\mu \leq \int_X \left(\int_X |F(x, y)|^2 \mathrm{d}\mu(x) \right) \left(\int_X |f(x)|^2 \mathrm{d}\mu(x) \right) \mathrm{d}\mu(y) \\ = \|F\|_2^2 \|f\|_2^2,\end{aligned}$$

so that T_F is bounded with norm $\leq \|F\|_2$. For any g in $\mathrm{L}^2(X,\mathcal{A},\mu)$, we have, by Fubini theorem,

$$\int_X \overline{g} T_F(f)\mathrm{d}\mu = \int_{X\times X} \overline{g(y)} f(x)F(x,y)\mathrm{d}\mu(x)\mathrm{d}\mu(y) = \int_X \overline{T_{F^\vee}(g)} f\mathrm{d}\mu,$$

that is $T_{F^\vee}$ is the adjoint operator to T_F.

Let us suppose now F to be of the form $(x,y)\mapsto h(x)k(y)$ where h and k are in $\mathrm{L}^2(X,\mathcal{A},\mu)$. Then, for any f in $\mathrm{L}^2(X,\mathcal{A},\mu)$, one has

$$T_F(f) = \left(\int_X hf\mathrm{d}\mu\right)k$$

and T_F, being a rank one operator, is compact. As the set of functions of this form spans a dense subset of $\mathrm{L}^2(X\times X,\mathcal{A}\otimes\mathcal{A},\mu\otimes\mu)$, and as the map $F\mapsto T_F$ is linear and continuous, for any F, T_F is compact.

Finally, let us suppose $T_F = 0$. Then for any f and g in $\mathrm{L}^2(X,\mathcal{A},\mu)$, one has

$$\int_{X\times X} f(x)g(y)F(x,y)\mathrm{d}\mu(x)\mathrm{d}\mu(y) = \int_X T_F(f)(y)g(y)\mathrm{d}\mu(y) = 0.$$

Again, as the set of functions of the form $(x,y)\mapsto f(x)g(y)$ spans a dense subset of $\mathrm{L}^2(X\times X,\mathcal{A}\otimes\mathcal{A},\mu\otimes\mu)$, one has $F=0$. □

Proof of proposition 1.2.14. Note that we have clearly *(iii)⇒(ii)*. We shall first prove *(i)⇒(iii)* and *(ii)⇒(i)*, then *(iv)⇒(v)* and *(v)⇒(iv)* and, lastly, *(ii)⇒(v)* and *(v)⇒(ii)*.

(i)⇒(iii). Let A,B be in $\mathcal{A}$ and C,D be in $\mathcal{B}$. For any $T>0$, we have

$$\begin{aligned}\frac{1}{T}\int_0^T &(\mu\otimes\nu)((A\cap\varphi_{-t}(B))\times(C\cap\psi_{-t}(D)))\mathrm{d}t\\ &= \frac{1}{T}\int_0^T (\mu(A\cap\varphi_{-t}(B)) - \mu(A)\mu(B))\nu(C\cap\psi_{-t}(D))\mathrm{d}t\\ &\qquad + \frac{1}{T}\mu(A)\mu(B)\int_0^T \nu(C\cap\psi_{-t}(D))\mathrm{d}t,\end{aligned}$$

so that

$$\begin{aligned}
&\left|\frac{1}{T}\int_0^T (\mu\otimes\nu)((A\cap\varphi_{-t}(B))\times(C\cap\psi_{-t}(D)))\mathrm{d}t-\mu(A)\mu(B)\nu(C)\nu(D)\right|\\
&\qquad\le \frac{1}{T}\int_0^T |\mu(A\cap\varphi_{-t}(B))-\mu(A)\mu(B)|\,\mathrm{d}t\\
&\qquad\quad+\left|\frac{1}{T}\mu(A)\mu(B)\left(\int_0^T \nu(C\cap\psi_{-t}(D))\mathrm{d}t-\nu(C)\nu(D)\right)\right|.
\end{aligned}$$

Thus, as the flow (φ_t) is weakly mixing and (ψ_t) is ergodic,

$$\frac{1}{T}\int_0^T (\mu\otimes\nu)((A\cap\varphi_{-t}(B))\times(C\cap\psi_{-t}(D)))\mathrm{d}t \xrightarrow[T\to\infty]{} \mu(A)\mu(B)\nu(C)\nu(D).$$

By density of the characteristic functions of product subsets, it implies that, for any f and g in $\mathrm{L}^2(X\times Y,\mathcal{A}\otimes\mathcal{B},\mu\otimes\nu)$, one has

$$\frac{1}{T}\int_0^T\int_{X\times Y} f(g\circ(\varphi_t\otimes\psi_t))\mathrm{d}(\mu\otimes\nu)\mathrm{d}t \xrightarrow[T\to\infty]{} \int_{X\times Y} f\mathrm{d}(\mu\otimes\nu)\int_{X\times Y} g\mathrm{d}(\mu\otimes\nu)$$

Therefore, by Birkhoff theorem, the product flow is ergodic.

(ii)⇒(i). Let A, B be in $\mathcal{A}$. By Jensen inequality, we have

$$\begin{aligned}
&\left(\frac{1}{T}\int_0^T |\mu(A\cap\varphi_{-t}(B))-\mu(A)\mu(B)|\,\mathrm{d}t\right)^2\\
&\qquad\le \frac{1}{T}\int_0^T (\mu(A\cap\varphi_{-t}(B))-\mu(A)\mu(B))^2\mathrm{d}t\\
&\qquad\quad= \frac{1}{T}\int_0^T \mu(A\cap\varphi_{-t}(B))^2\mathrm{d}t\\
&\qquad\quad-\frac{2}{T}\mu(A)\mu(B)\int_0^T \mu(A\cap\varphi_{-t}(B))\mathrm{d}t+\mu(A)^2\mu(B)^2.
\end{aligned}$$

As the diagonal product flow is ergodic, we have

$$\frac{1}{T}\int_0^T \mu(A\cap\varphi_{-t}(B))^2\mathrm{d}t \xrightarrow[T\to\infty]{} \mu(A)^2\mu(B)^2$$

and, as the flow is ergodic,

$$\frac{1}{T}\int_0^T \mu(A\cap\varphi_{-t}(B))\mathrm{d}t \xrightarrow[T\to\infty]{} \mu(A)\mu(B).$$

Therefore, we have

$$\frac{1}{T}\int_0^T (\mu(A\cap\varphi_{-t}(B)) - \mu(A)\mu(B))^2 \mathrm{d}t \xrightarrow[T\to\infty]{} 0$$

and the flow is weakly mixing.

(iv)⇒(v). Let f and α be as in the setting. If $\alpha \neq 0$, we have $f\circ\varphi_{\frac{1}{\alpha}} = f$. As the transformation $\varphi_{\frac{1}{\alpha}}$ is ergodic, f is constant. As $f\circ\varphi_{\frac{1}{2\alpha}} = -f$, we have $f = -f$, so that $f = 0$.

(v)⇒(iv). To simplify notations, we will suppose $t = 1$. Let f be in $\mathrm{L}^1(X, \mathcal{A}, \mu)$ such that $f\circ\varphi_1 = f$. We will show that, for any t in $\mathbb{R}$, we have $f\circ\varphi_t = f$. As the flow (φ_t) is ergodic, this will imply that f is constant. By changing f on a set of measure 0, we can suppose that, for every x in X, we have $f(\varphi_1(x)) = f(x)$. For k in $\mathbb{Z}$ and x in X set

$$f_k(x) = \int_0^1 e^{-2i\pi ks} f(\varphi_s(x))\mathrm{d}s.$$

This function is defined almost everywhere and belongs to $\mathrm{L}^1(X, \mathcal{A}, \mu)$ with norm $\leq \|f\|_1$. As $f\circ\varphi_1 = f$ everywhere, the set of x in X such that the Borel function $[0,1]\to\mathbb{C}, s\mapsto e^{-2i\pi ks} f(\varphi_s(x))$ is integrable is φ_t-invariant for every t in $\mathbb{R}$ so that, for almost every x in X, for every t in $\mathbb{R}$, we have

$$f_k(x) = \int_t^{t+1} e^{-2i\pi ks} f(\varphi_s(x))\mathrm{d}s.$$

In particular, we have $f_k\circ\varphi_t = e^{2i\pi kt} f_k$, that is f_k is an eigenvector. Therefore, we have $f_k = 0$ for $k \neq 0$. Let g be in $\mathrm{L}^\infty(X, \mathcal{A}, \mu)$. The Borel bounded function $\theta_g : t\mapsto \int_X g(f\circ\varphi_t)\mathrm{d}\mu$ is 1-periodic on $\mathbb{R}$ and, by Fubini theorem, for any k in $\mathbb{Z}$, one has

$$\int_0^1 e^{-2i\pi ks}\theta_g(s)\mathrm{d}s = \int_X g f_k \mathrm{d}\mu.$$

Therefore, all the Fourier coefficients of θ_g are zero, except eventually the one corresponding to $k = 0$. Thus, θ_g is constant, that is, for any t in $\mathbb{R}$,

$$\int_X g(f\circ\varphi_t)\mathrm{d}\mu = \int_X g f \mathrm{d}\mu.$$

As it is true for any g in $\mathrm{L}^\infty(X, \mathcal{A}, \mu)$, we have $f\circ\varphi_t = f$. Hence, as the flow (φ_t) is ergodic, f is constant, what should be proved.

(ii)⇒(v). Suppose α is a real number and f is a non zero α-eigenvector for φ_t, that is f is a non zero measurable function on X such that, for any t, $f \circ \varphi_t = e^{2i\pi\alpha t} f$ almost everywhere. Then, consider the function $(x, y) \mapsto f(x)\overline{f(y)}$ on $X \times Y$. It is invariant by the diagonal product flow. Hence, it is constant, so that f is constant. In particular, $\alpha = 0$.

(v)⇒(ii). Let F be an invariant function in $\mathrm{L}^2(X \times X, \mathcal{A} \otimes \mathcal{A}, \mu \otimes \mu)$. We shall prove that F is constant. After having replaced F by $F - \int_{X \times X} F \mathrm{d}(\mu \otimes \mu)$, we can suppose F to have 0 mean. Let us denote, as in lemma 1.2.17, by $F^\vee$ the function $(x, y) \mapsto \overline{F(y, x)}$. Then, by studying the functions $F + F^\vee$ and $i(F - F^\vee)$, we can suppose that the operator T_F of lemma 1.2.17 is self-adjoint. Suppose $F \neq 0$. Then, by lemma 1.2.17, T_F is a non zero compact self-adjoint operator of $\mathrm{L}^2(X, \mathcal{A}, \mu)$. Therefore (see theorem A.4.4, there exists a real number $\rho \neq 0$ such that the eigenspace

$$H = \{f \in \mathrm{L}^2(X, \mathcal{A}, \mu) | T_F(f) = \rho f\}$$

is a non zero finite dimensional subspace of $\mathrm{L}^2(X, \mathcal{A}, \mu)$. Let t be in $\mathbb{R}$. As F is invariant, for any f in $\mathrm{L}^2(X, \mathcal{A}, \mu)$, we have, for almost every y in X,

$$\begin{aligned} T_F(f)(\varphi_t(y)) = \int_X f(x) F(x, \varphi_t(y)) \mathrm{d}\mu(x) &= \int_X f(x) F(\varphi_{-t}(x), y) \mathrm{d}\mu(x) \\ &= \int_X f(\varphi_t(x)) F(x, y) \mathrm{d}\mu(x) = T_F(f \circ \varphi_t)(y), \end{aligned}$$

that is T_F commutes with the action of φ_t on $\mathrm{L}^2(X, \mathcal{A}, \mu)$. Therefore, the eigenspace H is stable by φ_t. The family $(\varphi_t)_{t \in \mathbb{R}}$ induces a commutative family of unitary automorphisms of the finite dimensional Hilbert space H. Therefore, these unitary automorphisms are simulatenously diagonalizable. Hence, there exists a non zero f in H and a homomorphism $\chi : \mathbb{R} \to \mathbb{T}$ such that, for any t in $\mathbb{R}$, we have $f \circ \varphi_t = \chi(t) f$. By lemma 1.2.3, the homomorphism χ is continuous. Therefore, there exists α in $\mathbb{R}$ such that, for any t, $\chi(t) = e^{2i\pi\alpha t}$. By hypothesis, we have $\alpha = 0$ and f is constant. But, as $T_F(f) = \rho f$, we have

$$\int_X f \mathrm{d}\mu = \frac{1}{\rho} \int_{X \times X} f(x) F(x, y) \mathrm{d}\mu(x) \mathrm{d}\mu(y) = 0,$$

since F has zero mean and f is constant. Thus, $f = 0$, a contradiction. □

Remark 1.2.18. The rather intricate last part of this proof could be simplified by using the following *spectral theorem*: let $t \mapsto u_t$ be a homomorphism from $\mathbb{R}$ to the group of unitary automorphisms of a Hilbert space H such that the map $\mathbb{R} \times H \to H, (t, v) \mapsto u_t(v)$ is continuous. Suppose there exists a vector v_0 in H such that the set $\{u_t(v) | t \in \mathbb{R}\}$ spans a dense subspace of H. Then, there exists a unique Radon measure ν on $\mathbb{R}$ such that there exists an isometry $\Phi : H \to \mathrm{L}^2(\mathbb{R}, \nu)$ which sends v_0 to the constant function 1 and such that, for any t in $\mathbb{R}$ and v in H, $\Phi(u_t(v))$ is the function $\xi \mapsto e^{2i\pi t\xi}\Phi(v)(\xi)$. In fact, the compact operator trick we used in the last part of the proof of proposition 1.2.14 is used in the proof of the spectral theorem too.

1.3 Invariant measures for continuous flows

We shall conclude this review of the basic dynamical properties of flows by proving, as in the discrete time case, the

Proposition 1.3.1. *Let (φ_t) be a continuous flow on the (Hausdorff) compact metric space X. Then, there exists a Borel probability measure μ on X which is preaserved by φ_t, for any t in $\mathbb{R}$.*

Proof. As in the discrete time case, identify, thanks Riesz representation theorem, the space $\mathcal{P}$ of probability Borel measures on X with the set of positive linear form with norm one on the space of continuous functions on X. Then, equipped with the weak-$*$ topology, $\mathcal{P}$ is a compact space, by Banach-Alaoglu theorem. If ν is any element of $\mathcal{P}$, the limits points as T goes to infinity of $\frac{1}{T}\int_0^T (\varphi_t)_* \nu \mathrm{d}t$ are invariant measures for the flow. □

Let $\mathcal{P}_\varphi$ be the convex set of invariant probability Borel measures, equipped with the weak-$*$ topology. As in the discrete time case, an invariant measure is ergodic if and only if it is an extreme point of $\mathcal{P}$. By Krein-Millman theorem, we therefore get the

Corollary 1.3.2. *There exists ergodic measures for (φ_t).*

Definition 1.3.3. Let (φ_t) be a continuous flow on the (Hausdorff) compact metric space X. We say that (φ_t) is uniquely ergodic if it possesses an unique invariant Borel probability measure, which is then necessarily ergodic.

As a corollary of the proof of proposition 1.3.1, we get the

Proposition 1.3.4. *Let (φ_t) be a continuous flow on the (Hausdorff) compact metric space X. The following are equivalent*

(i) (φ_t) is uniquely ergodic.

(ii) there exists a Borel probability measure μ on X such that, for any continuous function f on X, one has

$$\frac{1}{T}\int_0^T f(\varphi_t(x))\mathrm{d}t \xrightarrow[T\to\infty]{} \int_X f\mathrm{d}\mu$$

uniformly in $x \in X$.

Example 1.3.5. Let $y = (y_1, \ldots, y_d)$ be in $\mathbb{R}^d$ such that, for any non zero $\alpha = (\alpha_1, \ldots, \alpha_d)$ in $\mathbb{Q}^d$, one has $\alpha_1 y_1 + \ldots + \alpha_d y_d \neq 0$. Then the associate translation flow on $\mathbb{T}^d$ is uniquely ergodic.

1.4 Exercices

1.4.1 Continuous morphisms

Prove that every continuous morphism $\mathbb{R} \to \mathbb{R}$ is of the form $t \mapsto \alpha t$ for some $\alpha \in \mathbb{R}$. Deduce from it that every continuous morphism $\mathbb{R} \to \mathbb{T}$ is of the form $t \mapsto e^{2i\pi\alpha t}$ for some $\alpha \in \mathbb{R}$.

1.4.2 Quotients by proper actions

Recall that, if X is a topological space, $\sim$ an equivalence relation on X and $\pi : X \to X/\sim$ the associate quotient map, the quotient topology on $X/\sim$ is the topology for which a subset U of $X/\sim$ is open if and only if $\pi^{-1}(U)$ is open in X. This is the weakest topology that makes π continuous.

Let X be a locally compact Hausdorff topological space X. Let Γ be a group acting continuously on X, that is, for any γ in Γ, the map $x \mapsto \gamma x, X \to X$ is a homeomorphism of X. We say that the action of Γ is proper if, for any compact subset K of X, the set $\{\gamma \in \Gamma | \gamma K \cap K \neq \emptyset\}$ is a finite subset of Γ.

Suppose Γ acts properly continuously on X.

1. Prove that the quotient space $\Gamma\backslash X$, equipped with the quotient topology, is a locally compact Hausdorff space.

2. Let $\mathcal{C}_c^0(X)$ be the space of continuous compactly supported functions on X. For any φ in $\mathcal{C}_c^0(X)$, for any x in X, set

$$\overline{\varphi}(x) = \sum_{\gamma \in \Gamma} \varphi(\gamma x)$$

and consider $\overline{\varphi}$ as a function on $\Gamma \backslash X$. Prove that $\overline{\varphi}$ belongs to $\mathcal{C}_c^0(\Gamma \backslash X)$ and that the map $\varphi \mapsto \overline{\varphi}$ induces a positive surjective linear map $\mathcal{C}_c^0(X) \to \mathcal{C}_c^0(\Gamma \backslash X)$ (use partitions of identity).

3. Let μ be a Γ-invariant Radon measure on X. Prove that there exists an unique Radon measure $\overline{\mu}$ on $\Gamma \backslash X$ such that, for any φ in $\mathcal{C}_c^0(X)$, one has $\int_X \varphi \mathrm{d}\mu = \int_{\Gamma \backslash X} \overline{\varphi} \mathrm{d}\overline{\mu}$. Prove that the map $\mu \mapsto \overline{\mu}$ establishes a one-to-one correspondance between the set of Γ-invariant Radon measures on X and the set of Radon measures on $\Gamma \backslash X$.

1.4.3 Suspension flows

Let X be a compact Hausdorff topological space, $T : X \to X$ a homeomorphism of X and $f : X \to \mathbb{R}_+^*$ a continuous positive function. Denote by $\tilde{X}$ the space $X \times \mathbb{R}$ and, for any (x, s) in $\tilde{X}$, set $\tilde{T}(x, s) = (Tx, s - f(x))$.

1. Prove that $\tilde{T}$ is a homeomorphism of $\tilde{X}$ and that the $\mathbb{Z}$ action on $\tilde{X}$ generated by $\tilde{T}$ is proper.

We let $\overline{X}$ be the quotient of $\tilde{X}$ by this action and we denote by $\pi : \tilde{X} \to \overline{X}$ the natural projection.

2. Prove that there exists an unique continuous flow (φ_t) on $\overline{X}$ such that, for any t in $\mathbb{R}$ and (x, s) in $\tilde{X}$, one has $\varphi_t(\pi(x, s)) = \pi(x, t + s)$.

3. Prove that there exists an injective continuous map $\iota : X \to \overline{X}$ such that, for any x in X, the set $\{t > 0 | \varphi_t(\iota(x)) \in \iota(X)\}$ admits $f(x)$ as its smallest element. What are the other elements of this set ? Prove that, for any y in $\overline{X}$, there exists $0 \leq t < \max_X f$ such that $\varphi_t(y)$ belongs to $\iota(X)$ (don't forget to draw a picture).

We call (φ_t) the suspension flow of T with deck function f.

1.4.4 Diophantine and Liouville numbers

Let ρ be a real number. We denote by E_ρ the set of real numbers α such that there exists an infinity of integers p in $\mathbb{Z}$ and q in $\mathbb{N}^*$ with

$$\left| \alpha - \frac{p}{q} \right| < \frac{1}{q^\rho}.$$

1. Denote by (β) the fractional part of a real number β, that is the unique element of $[0,1)$ for which there exists an integer p with $\beta = (\beta) + p$. Let q be a positive integer. By considering the partition

$$[0,1) = \bigcup_{r=0}^{q-1} \left[\frac{r}{q}, \frac{r+1}{q}\right),$$

prove that, for any α in $\mathbb{R}$, there exists $0 \leq q_1 < q_2 \leq q$ with

$$|(q_1\alpha) - (q_2\alpha)| < \frac{1}{q}.$$

2. Prove that $E_2 = \mathbb{R}$.
3. Let $\rho > 2$ and set, for any integer $q > 0$,

$$F_\rho^q = \left\{\alpha \in \mathbb{R} \middle| \exists p \in \mathbb{Z} \quad \left|\alpha - \frac{p}{q}\right| < \frac{1}{q^\rho}\right\}.$$

Prove that one has

$$E_\rho = \bigcap_{q_0 \in \mathbb{N}*} \bigcup_{q \geq q_0} F_\rho^q.$$

Let λ be the Lebesgue measure of $\mathbb{R}$. Prove that, for any integer $q > 0$, one has

$$\lambda\left(F_\rho^q \cap [0,1]\right) \leq 2\frac{q+1}{q^\rho}.$$

Deduce from it that, for any $\rho > 2$, one has $\lambda(E_\rho) = 0$.

The real numbers that don't belong to any of the E_ρ, $\rho > 2$, are called diophantine numbers.

4. Prove that, for any ρ in $\mathbb{R}$, the set E_ρ contains a dense G_δ subset of $\mathbb{R}$.

The real numbers that belong to some E_ρ, $\rho > 2$, are called Liouville numbers.

1.4.5 Suspended rotations

Let α be an irrational number, $r_\alpha : \mathbb{T} \to \mathbb{T}$ be the rotation of angle α and $f : \mathbb{T} \to \mathbb{R}_+^*$ be a continuous positive function on $\mathbb{T}$.

1. Prove that the suspension flow of $(\mathbb{T}, r_\alpha)$ with deck function f is uniquely ergodic.

2. Let l be a nonnegative integer and let h be a $\mathcal{C}^l$ function on $\mathbb{T}$. For any integer k set

$$\hat{h}(k) = \int_0^1 e^{-2i\pi kt} h(t)\mathrm{d}t$$

the k-th Fourier coefficient of h. Prove that there exists a real number $C \geq 0$ such that, for any integer k, one has $\left|\hat{h}(k)\right| \leq C(1+|k|)^{-l}$.

Suppose α is diophantine.

3. Prove that, if h is a $\mathcal{C}^3$ function on $\mathbb{T}$ with $\int_0^1 h(t)\mathrm{d}t = 0$, there exists a continuous function g on $\mathbb{T}$ such that $h = g - g \circ r_\alpha$. What are the other solutions of this equation ?

4. Suppose f is $\mathcal{C}^3$. Prove that the suspension flow of $(\mathbb{T}, r_\alpha)$ with deck function f is topologically conjugated to a translation flow on the torus $\mathbb{T}^2$.

1.4.6 Translation flows on tori

Recall the following theorem from the algebra course: if M is a free abelian group of finite rank r and N is a subgroup of M, there exists $s \leq r$, a basis $(e_1, e_2, \ldots, e_r)$ and positive integers $n_1|n_2|\ldots|n_s$ such that the family $(n_1e_1, n_2e_2, \ldots, n_se_s)$ is a basis of N.

Let y be in $\mathbb{R}^d$.

1. Prove that there exists a smallest sub vector space V of $\mathbb{R}^d$ such that V is spanned by vectors with rational coefficients and that y belongs to V.

Let V be this subspace and let $\Gamma = V \cap \mathbb{Z}^d$.

2. Prove that Γ is cocompact in V, that is the quotient V/Γ is compact.

3. Prove that there exists a subgroup Λ of $\mathbb{Z}^d$ such that, as abelian groups, one has $\mathbb{Z}^d = \Gamma \oplus \Lambda$.

Let us fix such a Λ and let W be the $\mathbb{R}$-vector subspace of $\mathbb{R}$ spanned by Λ.

4. Prove that Λ is cocompact in W and that the decomposition $\mathbb{R}^d = V \oplus W$ induces a topological group isomorphism $\mathbb{T}^d \to (V/\Gamma) \times (W/\Lambda)$.

5. Prove that the space of ergodic invariant measures of the translation flow associated to y is homeomorphic to W/Λ.

Chapter 2

Topological groups and lattices

In this chapter, we will introduce objects which will be used in the sequel to extend the notion of translations and translations of tori which appeared as basic examples of dynamical systems.

2.1 Topological groups

Definition 2.1.1. A topological group is a group G equipped with a topology such that the product map $G \times G \to G$ and the inverse map $G \to G$ are continuous.

We shall always deal with locally compact topological groups.

Example 2.1.2. The groups $\mathbb{R}^d$ and $\mathbb{T}^d$ are locally compact abelian topological groups. The group $\mathrm{GL}_d(\mathbb{R})$, viewed as an open subset of the space of square matrices of size d, and the group $\mathrm{SL}_d(\mathbb{R})$, viewed as a closed subgroup of $\mathrm{GL}_d(\mathbb{R})$, are locally compact topological groups (they are non-abelian as soon as $d \geq 2$). The group $\mathrm{O}(d)$ of orthogonal matrices is compact.

If G is a topological group and g is an element of G, we denote by L_g the map $G \to G, h \mapsto gh$ and by R_g the map $G \to G, h \mapsto hg$. We call L_g the left translation by g and R_g the right translation by G. Both are homeomorphisms of G. In particular, if V is a neighborhood of e, gV and Vg are neighborhoods of g.

Definition 2.1.3. Let X be a topological space and G be a topological group. A continuous action of G on X is an action of G on the set X such that the action map $G \times X \to X$ is continuous.

Example 2.1.4. The classical linear action of $\mathrm{GL}_d(\mathbb{R})$ on $\mathbb{R}^d$ is continuous.

Definition 2.1.5. Let X be a locally compact topological space and G be a locally compact topological group acting on X. The action is said to be proper if, for any compact subset K of X, the set

$$\{g \in G | gK \cap K \neq \emptyset\}$$

is compact.

Example 2.1.6. The action of a group on itself by left translations is proper. The action of $\mathrm{GL}_d(\mathbb{R})$ on $\mathbb{R}^d$ is not proper, since every point has a non compact stabilizer.

Proper actions have nice quotients:

Proposition 2.1.7. *Let X be a locally compact topological space, G be a locally compact topological group acting continuously on X and $G\backslash X$ be the quotient space of this action, equipped with the quotient topology. Then the projection map $\pi : X \to G\backslash X$ is open. If the action is proper, the space $G\backslash X$ is Hausdorff and locally compact.*

Proof. Let us first show that π is open. Let $U \subset X$ be en open subset. Then one has

$$\pi^{-1}\pi(U) = \{x \in X | Gx \cap U \neq \emptyset\} = \bigcup_{g \in G} gU.$$

As G acts by homeomorphisms, this is an union of open sets, hence open. Thus, $\pi(U)$ is open, by definition of the quotient topology.

Suppose now the action is proper. Let us show that $G\backslash X$ is Hausdorff. Let x and y be in X such that $\pi(x) \neq \pi(y)$, that is $y \notin Gx$. As the quotient map is open, we have to exhibit neighborhoods $\overline{U}$ of x and $\overline{V}$ of y such that $\overline{U} \cap G\overline{V} = \emptyset$. Let U and V be compact neighborhoods of x and y. As the action is proper, the set $K = \{g \in G | U \cap gV \neq \emptyset\}$ is compact. For any k in K, since $x \neq ky$, there exists a neighborhood U_k of x and a neighborhood W_k of kx such that $U_k \cap W_k \neq \emptyset$. As the action is continuous, there exists a neighborhood V_k of y in X and a neighborhood A_k of k in G such that $A_k V_k \subset W_k$. As the set K is compact, we can pick points $k_1, \ldots, k_n$ in K such that $K \subset \bigcup_{p=1}^n A_{k_i}$. Then, set

$$\overline{U} = U \cap \bigcap_{p=1}^n U_{k_i} \text{ and } \overline{V} = V \cap \bigcap_{p=1}^n V_{k_i}.$$

For any g in G, we have $\overline{U} \cap g\overline{V} = \emptyset$. Indeed, if g does not belong to K, we have $U \cap gV = \emptyset$. Else, there exists some p such that g belongs to A_{k_p} and then we have $g\overline{V} \subset gV_{k_p} \subset W_{k_p}$ and, by definition, $W_{k_p} \cap U = \emptyset$. Thus, $\pi(\overline{U}) \cap \pi(\overline{V}) = \emptyset$ and $G \backslash X$ is Hausdorff.

Finally, let U be a compact neighborhood of some x in X. Then, as π is open, $\pi(U)$ is a compact neighborhood of $\pi(x)$, so that $G \backslash X$ is locally compact. □

Corollary 2.1.8. *Let G be a locally compact group and H be a closed subgroup of G. Then the quotient space G/H, equipped with the quotient topology, is a locally compact Hausdorff space. If H is normal, it is a locally compact topological group.*

Example 2.1.9. The space $\mathbb{R}^d - \{0\}$ may be seen as the quotient of $\mathrm{GL}_d(\mathbb{R})$ by the closed subgroup which is the stabilizer of $(1, 0, \ldots, 0)$. The torus $\mathbb{T}^d$ is the quotient of $\mathbb{R}^d$ by the closed subgroup $\mathbb{Z}^d$.

Proof. Consider the action of H on G by right translations: for any h in H, we let h act by $R_{h^{-1}}$ (the inverse makes the action a left action). For any compact subset K of G, we have

$$\{h \in H | R_{h^{-1}}K \cap K \neq \emptyset\} = H \cap (KK^{-1}),$$

which is a compact subset of H. The corollary follows now from proposition 2.1.7, since G/H is exactly the quotient of G by this action. □

2.2 Haar measure

In this section, we shall associate to any locally compact group a natural measure on it, which plays the role of the Lebesgue measure on $\mathbb{R}$ or the counting measure on $\mathbb{Z}$.

Let G be a locally compact topological group.

Theorem 2.2.1. *There exists a non zero Radon measure μ on G which is invariant by all left translations of G. Any other left-invariant Radon measure on G is proportional to μ.*

Definition 2.2.2. The left Haar measure of G is the (up to multiplication by a scalar) unique left-invariant Radon measure on G.

Usually, when having fixed a Haar measure μ on G, we shall write, for any μ-integrable function φ on G, $\int_G \varphi(g)\mathrm{d}g$ for $\int_G \varphi(g)\mathrm{d}\mu(g)$.

Example 2.2.3. The Haar measure of $\mathbb{R}^d$ is the Lebesgue measure. The Haar measure of $\mathbb{T}^d$ is the image in $\mathbb{T}^d$ of the restriction of the Lebesgue measure to $[0,1]^d$. The Haar measure of $\mathrm{GL}_d(\mathbb{R})$ is $\chi\lambda$ where λ is the restriction to $\mathrm{GL}_d(\mathbb{R})$ of the Lebesgue measure of the space of square matrices and χ is the function $g \mapsto |\det g|^{-d}$. In particular the Haar measure of the group $\mathbb{R}^*$ is the restriction of the Lebesgue measure of $\mathbb{R}$, multiplied by the function $x \mapsto \frac{1}{|x|}$.

Theorem 2.2.1 is, in some sense, unuseful: indeed, as seen in the examples, for every concrete locally compact group, its Haar measure is explicit. We shall however give its proof. The idea of the this proof is to estimate the measure of a compact set K by the minimal number of left translates of a small fixed compact set U with nonempty interior that are necessary to cover K. By normalizing conveniently and letting U shrink, we get the measure. Let us study more precisely this covering process.

Lemma 2.2.4. *Let U be compact subset of G with nonempty interior. For any compact subset K of G, define $(K:U)$ to be the minimal integer p such that there exists $g_1,\ldots,g_p$ in G with $K \subset g_1U \cup \ldots \cup g_pU$. Then we have,*

(i) for any compact set K in G and for any g in G, $(gK:U) = (K:U)$.

(ii) for any compact sets $K \subset L$ in G, $(K:U) \leq (L:U)$.

(iii) for any compact sets K and L in G, $(K \cup L:U) \leq (K:U)+(L:U)$ and, if $(KU^{-1}) \cap (LU^{-1}) = \emptyset$, $(K \cup L:U) = (K:U)+(L:U)$.

(iv) for any compact subset V of G with nonempty interior, $(K:U) \leq (K:V)(V:U)$.

Proof. The first point comes from the fact that the definition of $(K:U)$ is made up to translation by elements of G. The second point is clear, since a covering of L is *a fortiori* a covering of K.

For the third point, since the concatenation of a covering of K and of a covering of L is a covering of $K \cup L$, we have the inequality. Now, note that

$$KU^{-1} = \{g \in G | gU \cap K \neq \emptyset\} \text{ and } LU^{-1} = \{g \in G | gU \cap L \neq \emptyset\},$$

so that, if $(KU^{-1}) \cap (LU^{-1}) = \emptyset$, for any, $g_1, \ldots, g_p$ in G with $K \cup L \subset g_1U \cup \ldots \cup g_pU$, the sets

$$Q = \{1 \leq q \leq p | g_qU \cap K \neq \emptyset\} \text{ and } R = \{1 \leq r \leq p | g_rU \cap L \neq \emptyset\},$$

are disjoint. As $K \subset \bigcup_{q\in Q}(g_qU \cap K)$ and $L \subset \bigcup_{r\in R}(g_rU \cap L)$, we have $\sharp Q \geq (K : U)$ and $\sharp R \geq (L : U)$, so that

$$p \geq \sharp Q + \sharp R \geq (K : U) + (L : U),$$

what should be proved.

Finally, for the fourth point, suppose we have $K \subset g_1V \cup \ldots \cup g_pV$ and $V \subset h_1U \cup \ldots \cup h_qU$, then

$$K \subset \bigcup_{\substack{1\leq i\leq q\\1\leq j\leq r}} g_ih_jU,$$

so that $pq \geq (K : U)$, what should be proved. □

We can separate compact sets in G:

Lemma 2.2.5. *Let K and L be compact subsets of G with $K \cap L = \emptyset$. There exists a neighborhood U of e in G such that $(KU) \cap (LU) = \emptyset$.*

Proof. The set $L^{-1}K$ is a compact subset of G that does not contain e. Therefore, there exists a neighborhood V of e such that $V \cap L^{-1}K = \emptyset$. There exists a neighborhood U of e such that $UU^{-1} \subset V$, so that $UU^{-1} \cap L^{-1}K = \emptyset$, which is equivalent to $(KU) \cap (LU) = \emptyset$. □

Proof of theorem 2.2.1. Let us prove the existence of the measure. Recall the following version of Riesz representation theorem: if X is a locally compact space and μ is a finite nonnegative function on the set of compact subsets of X such that

(i) for any compact sets $K \subset L$ in X, $\mu(K) \leq \mu(L)$,

(ii) for any compact sets K and L in X, $\mu(K \cup L) \leq \mu(K) + \mu(L)$ and, if $K \cap L = \emptyset$, $\mu(K \cup L) = \mu(K) + \mu(L)$, then there exists an unique Radon measure on X which extends μ. By lemma 2.2.4 the functions $K \mapsto (K : U)$ almost satisfy these hypothesis, except for the additivity assumption. To satisfy this additivity assumption, we have to let U shrink and to use lemma 2.2.5.

Let us be more precise. Let us fixe once for all a compact subset V of G with nonempty interior. Let $\mathcal{K}$ be the set of compact subsets of G and let $\mathcal{M}$ be the set of the functions $m : \mathcal{K} \to \mathbb{R}$ such that,

(i) for any K in $\mathcal{K}$ and for any g in G, $m(gK) = m(K)$,

(ii) for any K, L in $\mathcal{K}$, with $K \subset L$, $m(K) \leq m(L)$,

(iii) for any K, L in $\mathcal{K}$, $m(K \cup L) \leq m(K) + m(L)$,

(iv) for any K in $\mathcal{K}$, $0 \leq m(K) \leq (K : V)$,

equipped with the product topology. By Tychonoff theorem, $\mathcal{M}$ is compact. For any compact neighborhood U of e, let us denote by m_U the function

$$\begin{aligned} \mathcal{K} &\to \mathbb{R}_+ \\ K &\mapsto \frac{(K : U)}{(V : U)}. \end{aligned}$$

One has $m_U(V) = 1$ and, by lemma 2.2.4, $m_U \in \mathcal{M}$. Set $M_U = \{m_W | W \subset U\}$. For any compact neighborhoods $U_1, \ldots, U_p$ of e, one has

$$M_{U_1 \cap \ldots \cap U_p} \subset M_{U_1} \cap \ldots \cap M_{U_p}.$$

Therefore, by compactness, the intersection, as U ranges over all the neighborhood of e, of the closures of the sets M_U in $\mathcal{M}$ is nonempty. Let μ be a point in this intersection. One has $\mu(V) = 1$, so that μ is nonzero. Moreover, if K and L are compact subsets of G with $K \cap L = \emptyset$, by lemma 2.2.5, there exists a neighborhood U of e such that $(KU^{-1}) \cap (LU^{-1}) = \emptyset$, so that, by lemma 2.2.4, for any neighborhood $W \subset U$, $m_W(K \cup L) = m_W(K) + m_W(L)$. Hence, we have $\mu(K \cup L) = \mu(K) + \mu(L)$ and, by Riesz representation theorem, there exists an unique Radon measure which extends μ. As μ is invariant by left translations, so does this extension, by uniqueness.

The uniqueness part of the theorem relies on a combinatorial trick. Let us fix a non zero left-invariant Radon measure μ on G. Note that, if K is a compact subset of G and U an open subset, there exists $g_1, \ldots, g_p$ in

G such that $K \subset g_1U \cup \ldots \cup g_pU$ so that $\mu(K) \leq p\mu(U)$. In particular, we have $\mu(U) \neq 0$ since $\mu \neq 0$. So, if ψ is a fixed non zero nonnegative function on G with compact support, the function $\theta : h \mapsto \int_G \psi(gh)\mathrm{d}\mu(g)$ takes only positive values on G. Moreover, this function is continuous, by Lebesgue continuity theorem. Now, let ν be an other left-invariant Radon measure on G. By Fubini theorem, the measure $\mu \otimes \nu$ on $G \times G$ is invariant by the transformations $(g, h) \mapsto (g, gh)$ and $(g, h) \mapsto (h^{-1}g, g)$, so that it is invariant by the transformation $(g, h) \mapsto (h^{-1}, gh)$, which is the composite of both. Hence, if φ is a continuous function with compact support on G, we have, by Fubini theorem,

$$\begin{aligned}\int_G \varphi(g)\mathrm{d}\mu(g) \int_G \psi(h)\mathrm{d}\nu(h) &= \int_{G\times G} \varphi(h^{-1})\psi(gh)\mathrm{d}\mu(g)\mathrm{d}\nu(h) \\ &= \int_G \varphi(h^{-1})\theta(h)\mathrm{d}\nu(h).\end{aligned}$$

Replacing φ by the continuous function $h \mapsto \varphi(h^{-1})\theta(h)$, we get, for every continuous function φ with compact support,

$$\int_G \frac{\varphi(g^{-1})}{\theta(g^{-1})}\mathrm{d}\mu(g) \int_G \psi(h)\mathrm{d}\nu(h) = \int_G \varphi(h)\mathrm{d}\nu(h).$$

Applying this to the case where $\nu = \mu$, we also get,

$$\int_G \frac{\varphi(g^{-1})}{\theta(g^{-1})}\mathrm{d}\mu(g) = \frac{1}{\int_G \psi(h)\mathrm{d}\mu(h)} \int_G \varphi(h)\mathrm{d}\mu(h),$$

so that, finally, for any ν, for every continuous function φ with compact support, we have

$$\int_G \varphi(h)\mathrm{d}\nu(h) = \frac{\int_G \psi(h)\mathrm{d}\nu(h)}{\int_G \psi(h)\mathrm{d}\mu(h)} \int_G \varphi(h)\mathrm{d}\mu(h),$$

that is the measures are proportional. $\square$

From the beginning of our study of invariant measures, we have only been dealing with left-invariant ones. Let us study what happens when letting G act on the right on a left-invariant measure.

Proposition 2.2.6. *There exists an unique continuous homomorphism $\Delta_G : G \to \mathbb{R}_+^*$ such that, for any left-invariant Haar measure μ on G, for any g in G, one has $(R_{g^{-1}})_*\mu = \Delta_G(g)\mu$, that is, in other words, for any continuous function φ with compact support on G,*

$$\int_G \varphi(hg^{-1})\mathrm{d}h = \Delta_G(g) \int_G \varphi(h)\mathrm{d}h.$$

Moreover, for any continuous function φ with compact support, one has

$$\int_G \varphi(g^{-1})\mathrm{d}g = \int_G \frac{\varphi(g)}{\Delta_G(g)}\mathrm{d}g.$$

Definition 2.2.7. The homomorphism Δ_G is called the modular function of G. The group G is said to be unimodular if $\Delta_G = 1$, that is if G admits left and right-invariant measures.

When there are no ambiguities, we shall write Δ for Δ_G.

Example 2.2.8. Abelian groups are clearly unimodular. Discrete groups are unimodular: their Haar measures are counting measures which are invariant under any bijection. Compact groups are unimodular: indeed, if G is compact, if μ is a left-invariant Haar measure on G, μ is finite and, for any g in G, one has $\mu(G) = \mu(Gg) = \mu(R_g(G))$ so that $\Delta_G(g) = 1$. By the exact computation of its Haar measure, the group $\mathrm{GL}_d(\mathbb{R})$ is unimodular. However, there exists non unimodular groups: let us consider the group P of matrices of the form

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix},$$

with $a, c \neq 0$ and b in $\mathbb{R}$. Then, in this systems of coordinates, the measure $\frac{1}{|a^2c|}\mathrm{d}a\mathrm{d}b\mathrm{d}c$ is left-invariant, but it is not right-invariant. One checks that, for $a, c \neq 0$ and b in $\mathbb{R}$, one has

$$\Delta_P \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \left|\frac{c}{a}\right|.$$

Proof of proposition 2.2.6. For any g in G, $(R_{g^{-1}})_*\mu$ is a non zero left-invariant measure on G. By theorem 2.2.1, it is of the form $\Delta_G(g)\mu$, for some uniquely defined $\Delta_G(g) > 0$. By uniqueness, the map Δ_G is a homomorphism

$G \to \mathbb{R}$. Finally, let us pick some non zero nonnegative function φ with compact support on G. We have, for any h in G,

$$\Delta_G(g) = \frac{\int_G \varphi(hg^{-1})\mathrm{d}h}{\int_G \varphi(h)\mathrm{d}h},$$

so that, by Lebesgue continuity theorem, Δ_G is continuous.

For the second formula, let us recall that, from the uniqueness part of the proof of theorem 2.2.1, for any continuous functions φ and ψ with compact support on G, we have

$$\int_G \varphi(g)\mathrm{d}g \int_G \psi(h)\mathrm{d}h = \int_{G\times G} \varphi(h^{-1})\psi(gh)\mathrm{d}g\mathrm{d}h.$$

Therefore, by Fubini theorem and the definition of Δ_G, we have

$$\begin{aligned}\int_G \varphi(g)\mathrm{d}g \int_G \psi(h)\mathrm{d}h &= \int_G \varphi(h^{-1}) \left(\int_G \psi(gh)\mathrm{d}g\right) \mathrm{d}h \\ &= \int_G \psi(g)\mathrm{d}g \int_G \varphi(h^{-1})\Delta_G(h)^{-1}\mathrm{d}h,\end{aligned}$$

which proves the formula. □

We shall now finish this section by studying the measures on quotient spaces of G. Let H be a closed subgroup of G and let us fix some left-invariant Haar measures on G and H. For any continuous function φ on G with compact support, set, for any g,

$$\overline{\varphi}(g) = \int_H \varphi(gh)\mathrm{d}h.$$

By construction, this function is right-H-invariant on G, so that it may be seen as a continuous function with compact support on the homogeneous space G/H. Moreover, let us note that, for any h in H, we have

$$\overline{\varphi \circ R_{h^{-1}}}(g) = \Delta_H(h)\overline{\varphi}(g).$$

Proposition 2.2.9. *The map*

$$\begin{gathered}\varphi \mapsto \overline{\varphi} \\ \mathcal{C}_c^0(G) \to \mathcal{C}_c^0(G/H)\end{gathered}$$

is surjective. If ν is a Radon measure on G such that, for any h in H, $(R_{h^{-1}})_*\nu = \Delta_H(h)\nu$, there exists an unique Radon measure $\overline{\nu}$ on G/H such that, for any φ in $\mathcal{C}_c^0(G)$, one has

$$\int_G \varphi \mathrm{d}\nu = \int_{G/H} \overline{\varphi} \mathrm{d}\overline{\nu}.$$

This correspondance establishes a bijection between the set of Radon measures on G/H and the set of Radon measures ν on G such that, for any h in H, $(R_{h^{-1}})_*\nu = \Delta_H(h)\nu$.

One more time, we shall use a topological lemma:

Lemma 2.2.10. *Let K be a compact subset of G/H. There exists a nonnegative continuous function with compact support Φ on G such that, for any x in K, there exists g in G such that $gH = x$ and $\Phi(g) > 0$.*

Proof. Let $\pi : G \to G/H$ be the canonical projection and let V be a compact neighborhood of e in G. We have $K = \bigcup_{\pi(g)\in K} \pi(gV)$. Therefore, there exists $g_1, \ldots, g_p$ in G such that $K \subset \pi(g_1V) \cup \ldots \cup \pi(g_pV)$. Let Ψ be a nonnegative continuous function with compact support on G which is > 0 on V and set, for any g in G,

$$\Phi(g) = \sum_{i=1}^{p} \Psi(g_i^{-1}g).$$

Then, for any x in K, there exists g in G and $1 \leq i \leq p$ such that $\pi(x) = g$ and $g \in g_iV$, so that $\Phi(g) \geq \Psi(g_i^{-1}g) > 0$. □

Proof of proposition 2.2.9. Let us prove that the map $\varphi \mapsto \overline{\varphi}$ is surjective. Let ψ be in $\mathcal{C}_c^0(G/H)$. Let K be the support of ψ and let Φ be as in lemma 2.2.10. By construction, the compactly supported function $g \mapsto \Phi(g)\psi(gH)$ has its support contained in the open set $\{g \in G | \overline{\Phi}(g) > 0\}$. For any g in G, set

$$\varphi(g) = \frac{\Phi(g)}{\overline{\Phi}(g)}\psi(gH)$$

if $\overline{\Phi}(g) > 0$ and $\varphi(g) = 0$ else. Then φ is a continuous function with compact support on G and $\overline{\varphi} = \psi$.

Let λ be a Radon measure on G/H and let $\tilde{\lambda}$ be the Radon measure on G such that, for every φ in $\mathcal{C}_c^0(G)$, one has $\int_G \varphi \mathrm{d}\tilde{\lambda} = \int_{G/H} \overline{\varphi} \mathrm{d}\lambda$. For any h in

H and φ in $\mathcal{C}_c^0(G)$, we have

$$\int_G \varphi \mathrm{d}((R_{h^{-1}})_*\tilde{\lambda}) = \int_G \varphi \circ R_{h^{-1}} \mathrm{d}\tilde{\lambda} = \int_{G/H} \overline{\varphi \circ R_{h^{-1}}} \mathrm{d}\lambda$$
$$= \Delta_H(h) \int_{G/H} \overline{\varphi} \mathrm{d}\lambda = \Delta_H(h) \int_G \varphi \mathrm{d}\tilde{\lambda}.$$

Moreover, as the map $\varphi \mapsto \overline{\varphi}$ is surjective, the map $\lambda \mapsto \tilde{\lambda}$ is injective. To finish the proof, we have to prove that it is surjective. Let us pick some Radon measure ν on G such that, for any h in H, $(R_{h^{-1}})_*\nu = \Delta_H(h)\nu$ and let us show that there exists a Radon measure λ on G/H such that $\tilde{\lambda} = \nu$. For any ψ in $\mathcal{C}_c^0(G/H)$, we want to set $\int_{G/H} \psi \mathrm{d}\lambda = \int_G \varphi \mathrm{d}\nu$ where φ is in $\mathcal{C}_c^0(G)$ with $\overline{\varphi} = \psi$. Thus, we only have to check that, for any φ in $\mathcal{C}_c^0(G)$ with $\overline{\varphi} = 0$, we have $\int_G \varphi \mathrm{d}\nu = 0$. We shall more or less repeat the trick we used to prove the uniqueness of Haar measure. Indeed, if φ and ψ belong to $\mathcal{C}_c^0(G)$, we have, by Fubini theorem,

$$\begin{aligned}
\int_G \overline{\varphi}(g)\psi(g)\mathrm{d}\nu(g) &= \int_H \left(\int_G \varphi(gh)\psi(g)\mathrm{d}\nu(g) \right) \mathrm{d}h \\
&= \int_H \frac{1}{\Delta_H(h)} \left(\int_G \varphi(g)\psi(gh^{-1})\mathrm{d}\nu(g) \right) \mathrm{d}h \\
&= \int_G \varphi(g) \left(\int_H \frac{\psi(gh^{-1})}{\Delta_H(h)} \mathrm{d}h \right) \mathrm{d}\nu(g) \\
&= \int_G \varphi(g)\overline{\psi}(g)\mathrm{d}\nu(g).
\end{aligned}$$

Now, if $\overline{\varphi} = 0$, we have, for any ψ, $\int_G \varphi\overline{\psi}\mathrm{d}\nu = 0$. In particular, as the map $\psi \mapsto \overline{\psi}$ is surjective, we can find ψ in $\mathcal{C}_c^0(G)$ such that $\overline{\psi} = 1$ on the support of φ. We then get $\int_G \varphi \mathrm{d}\nu = 0$, what should be proved. □

Corollary 2.2.11. *There exists a non zero G-invariant Radon measure on G/H if and only if, for any h in H, one has $\Delta_G(h) = \Delta_H(h)$. If it is the case, this measure ν is unique up to multiplication by a positive scalar and can be normalized in such a way that, for any φ in $\mathcal{C}_c^0(G)$, one has*

$$\int_G \varphi(g)\mathrm{d}g = \int_{G/H} \overline{\varphi}\mathrm{d}\nu.$$

Proof. Let $\nu \mapsto \overline{\nu}$ be the correspondance of proposition 2.2.9. For g in G, let us still denote by L_g the translation by g acting on the left in G/H. Then, for any Radon measure ν on G, we have $\overline{(L_g)_*\nu} = (L_g)_*\overline{\nu}$. Therefore, $\overline{\nu}$ is G-invariant if and only if ν is a Haar measure on G. But in this case, we have, for any h in H, $(R_{h^{-1}})_*\nu = \Delta_G(h)\nu$. Therefore, by proposition 2.2.9, such a ν exists if and only if $\Delta_G = \Delta_H$ on H. If this is the case, we have the formula by definition of $\overline{\nu}$. □

Corollary 2.2.12. *Suppose G and H are unimodular. Then the quotient space G/H possesses an (up to scalar multiplication) unique non zero G-invariant Radon measure.*

Example 2.2.13. Let P be as in example 2.2.8. Then the quotient $\mathrm{GL}_2(\mathbb{R})/P$ is the projective line $\mathbb{P}^1_{\mathbb{R}}$. One easily checks that the action of $\mathrm{GL}_2(\mathbb{R})$ on $\mathbb{P}^1_{\mathbb{R}}$ does not preserve a non zero Radon measure.

2.3 Lattices

Let G be a locally compact topological group.

Proposition 2.3.1. *Let Γ be a subgroup of G. Then Γ is discrete for the induced topology if and only if there exists a neighborhood U of e in G such that, for any $\gamma \neq \eta$ in Γ, $\gamma U \cap \eta U = \emptyset$. If this is the case, Γ is closed in G.*

Remark 2.3.2. Note that in general a discrete subset of a topological space is not closed: think, for example, to the set $\{\frac{1}{n} | n \in \mathbb{N}^*\}$ in $\mathbb{R}$.

Proof. If such a neighborhood exists, then Γ is discrete since, for any γ in Γ, we have $\Gamma \cap \gamma U = \{\gamma\}$. Reciprocally, suppose Γ is discrete. Then, there exists a neighborhood V of e in G such that $\Gamma \cap V = \{e\}$. Let U be a neighborhood of e in G such that $UU^{-1} \subset V$. For any γ, η in Γ, if u and v are in U and $\gamma u = \eta v$, we have $\eta^{-1}\gamma = vu^{-1} \in V$, so that $\eta = \gamma$, what should be proved.

Let Γ be discrete, let still U be such a neighborhood and let g be in G, with $g \notin \Gamma$. We want to find a neighborhood of g that does not encounter Γ. If $g \notin \Gamma U$, we have $gU^{-1} \cap \Gamma = \emptyset$. If $g \in \Gamma U$, there exists an unique γ in Γ such that $g \in \gamma U$. As $g \neq \gamma$, there exists a neighborhood W of e such that $\gamma \notin gW$. We get $g(U^{-1} \cap W) \cap \Gamma = \emptyset$, what should be proved. □

Example 2.3.3. Let $e_1, \dots, e_d$ be the canonical basis of $\mathbb{R}^d$. For any $0 \leq r \leq d$, the group $\mathbb{Z}e_1 \oplus \dots \oplus \mathbb{Z}e_r$ is discrete in $\mathbb{R}^d$. The group $\mathrm{GL}_d(\mathbb{Z})$ is discrete in $\mathrm{GL}_d(\mathbb{R})$.

Definition 2.3.4. A discrete subgroup Γ of G is cocompact (or uniform) if the quotient space G/Γ is compact.

Example 2.3.5. For $0 \leq r \leq d$, the group $\mathbb{Z}e_1 \oplus \dots \oplus \mathbb{Z}e_r$ is cocompact in $\mathbb{R}^d$ if and only if $r = d$. The group $\mathrm{GL}_d(\mathbb{Z})$ is not cocompact in $\mathrm{GL}_d(\mathbb{R})$. Let H be the Heisenberg group, that is the group of 3 by 3 matrices of the form

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$$

with x, y, z in $\mathbb{R}$, and let Λ be the subgroup of H constituted of those elements for which x, y, z are in $\mathbb{Z}$. Then Λ is cocompact in H.

Proposition 2.3.6. *Suppose G admits a cocompact discrete subgroup Γ. Then G is unimodular and the quotient space G/Γ possesses an unique G-invariant Radon probability measure.*

Proof. Note that, by corollary 2.2.11, if such a probability measure exists, it is necessarily unique.

Let ν be a *right-invariant* Haar measure on G (for example, chose a left-invariant Haar measure μ and set, for any Borel set B in G, $\nu(B) = \mu(B^{-1})$). Then, for any g in G, one checks easily that $(L_g)_*\nu = \Delta(g)\nu$, where Δ is the modular function of G. Since ν is right-invariant, for any γ in Γ, one has $(R_{\gamma^{-1}})_*\nu = \nu$. As Γ is unimodular (since it is discrete), by proposition 2.2.9, there exists an unique Radon measure $\overline{\nu}$ on G/Γ such that, for any φ in $\mathcal{C}^0_c(G)$, one has $\int_{G/\Gamma} \overline{\varphi} \mathrm{d}\overline{\nu} = \int_G \varphi \mathrm{d}\nu$, where we set, for g in G,

$$\overline{\varphi}(g) = \sum_{\gamma \in \Gamma} \varphi(g\gamma).$$

Now, by uniqueness, for any g in G, we have $(L_g)_*\overline{\nu} = \Delta(g)\overline{\nu}$. But, as G/Γ is compact, $\overline{\nu}$ is finite and we also have $\overline{\nu}(G/\Gamma) = ((L_g)_*\overline{\nu})(G/\Gamma)$, so that $\Delta(g) = 1$. Thus, G is unimodular and $\overline{\nu}$ is G-invariant. $\square$

Definition 2.3.7. A discrete subgroup Γ of G is a lattice if and only if the quotient space G/Γ possesses a finite G-invariant Radon measure.

Example 2.3.8. A cocompact discrete subgroup is a lattice, by proposition 2.3.6. We shall see later that, for $d \geq 2$, the group $\mathrm{SL}_d(\mathbb{Z})$ is a lattice in $\mathrm{SL}_d(\mathbb{R})$, but it is not cocompact. In fact, the existence of this example is the major reason why we will deal with general lattices rather than with cocompact ones.

Remark 2.3.9. If Γ is a lattice in G, the invariant Radon probability measure of G/Γ is necessarily unique, by proposition 2.2.11.

Proposition 2.3.10. *Suppose G possesses a lattice. Then G is unimodular.*

Proof. Let Γ be a lattice in G and let Δ be the modular function of G. As a discrete group, Γ is unimodular. Therefore, by proposition 2.2.9, we have $\Delta = 1$ on Γ, so that, if N is the kernel of the continuous morphism $\Delta : G \to \mathbb{R}_+^*$, we have $\Gamma \subset N$. Let ν be the invariant Radon probability measure on G/Γ and let μ be its image on G/N by the natural map $G/\Gamma \to G/N$. Then μ is invariant by the left action of G on G/N. But, as N is normal in G, G/N has a natural structure of locally compact topological group and G acts on the left on G/N via the action on the left of G/N on itself. Therefore, the probability Radon measure μ on G/N is invariant by the left translations of G/N. Hence, μ is the Haar measure of G/N. As it is finite, G/N is compact. Since Δ factors through a continuous morphism $G/N \to \mathbb{R}_+^*$, the set $\Delta(G)$ is a compact subgroup of $\mathbb{R}_+^*$. Thus, we have $\Delta(G) = 1$ and G is unimodular. $\square$

Example 2.3.11. The group P of examples 2.2.8 and 2.2.13 does not admit lattices.

Remark 2.3.12. If G is a locally compact topological group and Γ a lattice in G, for any g in G, the left translation by g is a measure preserving transformation of the space G/Γ, equipped with its unique G-invariant probability measure. In the sequel of the course, we shall be concerned with the study of this kind of transformations from a dynamical point of view.

2.4 Exercices

2.4.1 Connected and open subgroups

Let G be a topological group.

1. Prove that the connected component of e in G is a normal subgroup of G.

2. Prove that every open subgroup of G is closed. Prove that every closed subgroup with finite index in G is open.

3. Suppose G is connected. Prove that every neighborhood of e in G spans G.

2.4.2 Projective limits

Let I be a set with an order relation $\prec$ such that, for every i, j in I, there exists k in I such that $i \prec k$ and $j \prec k$. Suppose we are given a family $(G_i)_{i \in I}$ of groups and, for any $i \prec j$ in I, a surjective morphism $\varphi_{i,j} : G_j \to G_i$ such that, for any $i \prec j \prec k$ in I, one has $\varphi_{i,j} \circ \varphi_{j,k} = \varphi_{i,k}$. Then, we say that $((G_i)_{i \in I}, (\varphi_{i,j})_{i \prec j})$ is a projective system of groups.

1. Let p be a prime integer. Equip $\mathbb{N}$ with the usual order relation and, for any n in $\mathbb{N}$, set $P_n = \mathbb{Z}/p^n\mathbb{Z}$ and, for any $n \leq m$, denote by $\varphi_{n,m}$ the natural morphism $\mathbb{Z}/p^m\mathbb{Z} \to \mathbb{Z}/p^n\mathbb{Z}$. Prove that $((P_n)_{n \in \mathbb{N}}, (\varphi_{n,m})_{n \leq m})$ is a projective system of groups. We call this system the p-adic system.

2. Equip $\mathbb{N}^*$ with the division order relation, that is, for any positive integers n and m, set $n|m$ if $\frac{m}{n}$ is an integer. Then, for any n in $\mathbb{N}^*$, denote by Q_n the group $\mathbb{Z}/n\mathbb{Z}$ and, for any $n|m$, denote by $\varphi_{n,m}$ the natural morphism $\mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$. Prove that $((Q_n)_{n \in \mathbb{N}^*}, (\varphi_{n,m})_{n|m})$ is a projective system of groups. We call this system the Prüfer system.

If $((G_i)_{i \in I}, (\varphi_{i,j})_{i \prec j})$ is a projective system of groups, we define the projective limit of the system $((G_i)_{i \in I}, (\varphi_{i,j})_{i \prec j})$ as the group G of the elements $(g_i)_{i \in I}$ of the product group $\prod_{i \in I} G_i$ such that, for any $i \prec j$ in I, one has $\varphi_{i,j}(g_j) = g_i$. We write

$$G = \varprojlim_{i \in I} G_i.$$

Suppose now each of the G_i is a compact topological group and the morphisms $(\varphi_{i,j})_{i \prec j}$ are continuous. Then, we equip G with the topology induced by the product topology on $\prod_{i \in I} G_i$.

3. Prove that G is a compact topological group and that, for any i in I, the coordinate morphism $\psi_i : G \to G_i$ is surjective (use the finite intersection property in compact spaces). Prove that, for any $i \prec j$ in I, one has $\varphi_{i,j} \circ \psi_j = \psi_i$.

4. Let H be a compact topological group and, for any i in I, let θ_i be a continuous morphism $H \to G_i$ such that, for any $i \prec j$ in I, one

has $\varphi_{i,j} \circ \theta_j = \theta_i$. Prove that there exists an unique continuous morphism $\theta : H \to G$ such that, for any i in I, one has $\theta_i = \psi_i \circ \theta$.

5. We equip each of the $\mathbb{Z}/p^n\mathbb{Z}$, $n \in \mathbb{N}$, with the discrete topology and we denote by $\mathbb{Z}_p$ the projective limit of the p-adic system. We call $\mathbb{Z}_p$ the group of p-adic integers. Prove that there exists an unique morphism $\theta : \mathbb{Z} \to \mathbb{Z}_p$ such that, for any n in $\mathbb{N}$, the component in $\mathbb{Z}/p^n\mathbb{Z}$ of $\theta(1)$ is the image of 1 in $\mathbb{Z}/p^n\mathbb{Z}$. Prove that θ is injective and has dense image.

6. We equip each of the $\mathbb{Z}/n\mathbb{Z}$, $n \in \mathbb{N}^*$, with the discrete topology and we denote by $\hat{\mathbb{Z}}$ the projective limit of the Prüfer system. We call $\hat{\mathbb{Z}}$ the group of Prüfer integers. Prove that there exists an unique morphism $\theta : \mathbb{Z} \to \hat{\mathbb{Z}}$ such that, for any n in $\mathbb{N}^*$, the component in $\mathbb{Z}/n\mathbb{Z}$ of $\theta(1)$ is the image of 1 in $\mathbb{Z}/n\mathbb{Z}$. Prove that θ is injective and has dense image.

In the sequel, we shall always consider $\mathbb{Z}$ as a subgroup of $\mathbb{Z}_p$ and $\hat{\mathbb{Z}}$.

7. Prove that, as topological groups, one has

$$\hat{\mathbb{Z}} \simeq \prod_{p \text{ prime}} \mathbb{Z}_p$$

(use the chinese remainder lemma !)

2.4.3 Kronecker systems

Let G be a compact abelian group and let x be an element of G.

1. Suppose that the subgroup spanned by x is dense in G. Prove that the translation $L_x : y \mapsto x + y, G \to G$ is minimal and uniquely ergodic on G and that its unique invariant measure is the Haar measure of G.

2. In the general case, denote by H the closure in G of the subgroup spanned by x. Prove that the ergodic invariant measures of the translation L_x are the measures of the form $(L_y)_*\nu$ where y is an element of G and ν is the Haar measure of H.

4. Let α be an irrational element of $\mathbb{R}$. Prove that the rotation associated to α on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ is minimal and uniquely ergodic.

3. Prove that the map $x \mapsto x + 1, \mathbb{Z}_p \to \mathbb{Z}_p$ is minimal and uniquely ergodic. Prove that the map $x \mapsto x + 1, \hat{\mathbb{Z}} \to \hat{\mathbb{Z}}$ is minimal and uniquely ergodic.

2.4.4 Dynamics of isometries

Let X be a compact metric space and let G be the group of isometries of X, that is the group of all the homeomorphisms g of X such that, for any x and y in X, one has $d(gx, gy) = d(x, y)$. We equip G with the uniform distance, that is, for any g and h in G, we set

$$d_\infty(g, h) = \max_{x \in X} d(gx, hx).$$

1. Prove that the topology associate to the distance d_∞ makes G a compact topological group (use Ascoli theorem).

2. Let g be in G. Prove that, for any x in X, the action of g on the closure of the g-orbite of x is minimal.

2.4.5 Subgroups of $\mathbb{R}^d$

Let Γ be a closed subgroup of $\mathbb{R}^d$.

1. Suppose Γ is discrete. Fix a norm $\|.\|$ on $\mathbb{R}^d$ and let x be an element of Γ with minimal norm. Prove that the image of Γ in the quotient vector space $\mathbb{R}^d/(\mathbb{R}x)$ is discrete. Deduce from it, by induction on d, that there exists $r \leq d$ and a $\mathbb{R}$-free system $e_1, \ldots, e_r$ in Γ with

$$\Gamma = \mathbb{Z}e_1 \oplus \ldots \oplus \mathbb{Z}e_r.$$

2. Suppose Γ is not discrete. Prove that Γ contains a vector line.

3. In the general case, prove that there exists nonnegative integers r, s with $r + s \leq d$ and a $\mathbb{R}$-free system $e_1, \ldots, e_{r+s}$ in Γ such that

$$\Gamma = \mathbb{Z}e_1 \oplus \ldots \oplus \mathbb{Z}e_r \oplus \mathbb{R}e_{r+1} \oplus \ldots \oplus \mathbb{R}e_{r+s}.$$

4. Prove that Γ is a lattice in $\mathbb{R}^d$ if and only if it is discrete and cocompact and that, in this case, there exists g in $\mathrm{GL}_d(\mathbb{R})$ with $\Gamma = g\mathbb{Z}^d$.

2.4.6 Lattices in subgroups

Let G be an unimodular locally compact group. Let H be a closed unimodular subgroup of G and Γ a discrete subgroup of G such that $H \cap \Gamma$ is a lattice in H. We equip the space $H/H \cap \Gamma$ with its unique H-invariant Borel probability measure κ and we chose G-invariant Radon measures λ, μ and ν on the spaces $G/(H \cap \Gamma)$, G/H and G/Γ.

1. Suppose $H \cap \Gamma$ is a cocompact lattice in H. Prove that the natural map $\Gamma/H \cap \Gamma \to G/H$ is proper.

2. In the general case, prove that, after an eventual normalization of the measures, for any nonnegative continuous compactly supported functions φ on G/H and ψ on G/Γ, one has

$$\int_{G/\Gamma} \psi(x) \left(\sum_{\gamma \in \Gamma/H\cap\Gamma} \varphi(x\gamma) \right) \mathrm{d}\nu(x) = \int_{G/H\cap\Gamma} \varphi\psi \mathrm{d}\lambda$$
$$= \int_{G/H} \varphi(y) \left(\int_{H/H\cap\Gamma} \psi(yz) \mathrm{d}\kappa(z) \right) \mathrm{d}\mu(y).$$

In particular, prove that, for ν-almost x in G/Γ, one has

$$\sum_{\gamma \in \Gamma/H\cap\Gamma} \varphi(x\gamma) < \infty.$$

3. Prove that the natural maps $\Gamma/H \cap \Gamma \to G/H$ and $H/H \cap \Gamma \to G/\Gamma$ are proper.

2.4.7 Lattices and normal subgroups

Let G be a locally compact group and N be a closed normal subgroup of G. Set $\overline{G} = G/N$ and denote by $\pi : G \to \overline{G}$ the natural map.

1. Let μ be a Haar measure of N. Prove that there exists a continuous homomorphism $\chi : G \to \mathbb{R}_+^*$ such that, for any g in G, one has $(\mathrm{Ad}_g)_*\mu = \chi(g)\mu$, where Ad_g is the map $h \mapsto ghg^{-1}$. Prove that $\Delta_G = \chi(\Delta_{\overline{G}} \circ \pi)$.

Let Γ be a discrete subgroup of G such that $N \cap \Gamma$ is a lattice in N and that $\overline{\Gamma} = \pi(\Gamma)$ is discrete and is a lattice of $\overline{G}$. We will prove that Γ is a lattice in G.

2. Prove that G is unimodular.

3. Prove that the natural map $N/N \cap \Gamma \to G/\Gamma$ is proper.

We let μ be the image under this map of the N-invariant Borel probability measure of $N/N \cap \Gamma$. Let φ be a continuous function with compact support on G/Γ. For any g in G, set

$$\overline{\varphi}(g) = \int_{G/\Gamma} \varphi \circ L_g \mathrm{d}\mu.$$

4. Prove that $\overline{\varphi}$ is right-ΓN-invariant.

Let ν be the $\overline{G}$-invariant Borel probability measure of $\overline{G}/\overline{\Gamma}$ and λ be the unique Borel probability measure on G/Γ such that, for any φ in $\mathcal{C}_c^0(G/\Gamma)$,

$$\int_{G/\Gamma} \varphi \mathrm{d}\lambda = \int_{\overline{G}/\overline{\Gamma}} \overline{\varphi} \mathrm{d}\nu.$$

5. Prove that λ is G-invariant and that Γ is a lattice in G.

6. Prove that, if $N \cap \Gamma$ is cocompact in N and $\overline{\Gamma} = \pi(\Gamma)$ is cocompact in $\overline{G}$, Γ is cocompact in G.

2.4.8 Lattices in solvable groups

Let G and H be a locally compact topological groups. A continuous action of G on H is a homomorphism θ from G into the group of automorphisms group of H such that the associate map $G \times H \to H, (g, h) \mapsto \theta_g(h)$ is continuous. Given such an action, define the associate semi-direct product $G \ltimes_\theta H$ as the set $G \times H$ equipped with the product topology and the map

$$\begin{aligned} (G \times H) \times (G \times H) &\to G \times H \\ ((g_1, h_1), (g_2, h_2)) &\mapsto \left(g_1 g_2, \theta_{g_2^{-1}}(h_1) h_2\right). \end{aligned}$$

1. Prove that $G \ltimes_\theta H$ is a locally compact topological group and that the maps $G \to G \ltimes_\theta H, g \mapsto (g, e)$ and $H \to G \ltimes_\theta H, h \mapsto (e, h)$ are proper injective continuous morphisms. In the sequel we shall consider G and H as closed subgroups of $G \ltimes_\theta H$. Prove that H is a normal subgroup of $G \ltimes_\theta H$, that $GH = G \ltimes_\theta H$ and that, for any g in G and h in H, one has $ghg^{-1} = \theta_g(h)$.

Let A be a hyperbolic matrix in $\mathrm{SL}_2(\mathbb{R})$.

2. Prove that there exists an unique continuous action of $\mathbb{R}$ on $\mathbb{R}^2$ such that 1 acts on $\mathbb{R}^2$ via the matrix A.

For any t in $\mathbb{R}$, we denote by A^t the linear automorphism of $\mathbb{R}^2$ associate to t by this action. The semi-direct product coming from the action is denoted by $\mathbb{R} \ltimes_A \mathbb{R}^2$.

3. Prove that the Haar measure of $\mathbb{R} \ltimes_A \mathbb{R}^2$ is the product of the Lebesgue measure of $\mathbb{R}$ and of the Lebesgue measure of $\mathbb{R}^2$. Prove that $\mathbb{R} \ltimes_A \mathbb{R}^2$ is unimodular.

Suppose now A belongs to $\mathrm{SL}_2(\mathbb{Z})$ and denote by $\mathbb{Z} \ltimes_A \mathbb{Z}^2$ the semi-direct product associate to the action of the integer powers of A on $\mathbb{Z}^2$. Consider $\Gamma = \mathbb{Z} \ltimes_A \mathbb{Z}^2$ as a subgroup of $G = \mathbb{R} \ltimes_A \mathbb{R}^2$, via the natural embedding.

4. Prove that Γ is a cocompact lattice of G and that there exists a surjective continuous map $\varpi : G/\Gamma \to \mathbb{T}$ such that, for any x in $\mathbb{T}$, $\varpi^{-1}(x)$ is homeomorphic to a 2-dimensional torus.

5. Prove that $(L_{A^t})_{t\in\mathbb{R}}$ is a continuous flow on G/Γ and that this flow is topologically conjugate to the suspension of the action of A on $\mathbb{T}^2$ with deck function 1.

6. Prove that the G-invariant measure of G/Γ is invariant and ergodic for the flow $(L_{A^t})_{t\in\mathbb{R}}$.

7. Prove that the set of ergodic invariant measures on G/Γ for the flow $(L_{A^t})_{t\in\mathbb{R}}$ is uncountable.

Chapter 3

Dynamics on Heisenberg quotients

In all this chapter, we let H be the Heisenberg group, that is the group of 3 by 3 matrices of the form

$$h_{x,y,z} = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$$

with x, y, z in $\mathbb{R}$. We will study the action of one-parameter subgroups of H in the quotients H/Γ, where Γ is a lattice in H.

3.1 Structure of the Heisenberg group

Let us write precisely some algebraic relations in H:

Lemma 3.1.1. *Let x, y, z and x', y', z' be in $\mathbb{R}$. One has*

(i) $h_{x,y,z}h_{x',y',z'} = h_{x+x',y+y',z+z'+xy'}$.

(ii) $h^{-1}_{x,y,z} = h_{-x,-y,xy-z}$.

(iii) $h_{x,y,z}h_{x',y',z'}h^{-1}_{x,y,z} = h_{x',y',z'+xy'-x'y}$.

(iv) $h_{x,y,z}h_{x',y',z'}h^{-1}_{x,y,z}h^{-1}_{x',y',z'} = h_{0,0,xy'-x'y}$.

Proof. By direct computations. $\square$

Corollary 3.1.2. *The image of the Lebesgue measure of $\mathbb{R}^3$ under the map $(x,y,z) \mapsto h_{x,y,z}$ is a Haar measure on H. In particular, H is unimodular.*

Proof. Let θ be in $\mathcal{C}_c^0(H)$. By lemma 3.1.1, for any x', y', z' in $\mathbb{R}$, one has

$$\begin{aligned}\int_{\mathbb{R}^3} \theta(h_{x',y',z'}h_{x,y,z})\mathrm{d}x\mathrm{d}y\mathrm{d}z &= \int_{\mathbb{R}^3} \theta(h_{x'+x,y'+y,z'+z+x'y})\mathrm{d}x\mathrm{d}y\mathrm{d}z \\ &= \int_{\mathbb{R}^3} \theta(h_{x,y,z})\mathrm{d}x\mathrm{d}y\mathrm{d}z \\ &= \int_{\mathbb{R}^3} \theta(h_{x,y,z}h_{x',y',z'})\mathrm{d}x\mathrm{d}y\mathrm{d}z,\end{aligned}$$

so that the image of the Lebesgue measure of $\mathbb{R}^3$ is left and right-invariant. The corollary follows. □

The map $z \mapsto h_{0,0,z}$ is a group homomorphism from $\mathbb{R}$ into H. Let Z be its image. We immediately get the

Corollary 3.1.3. *The subgroup Z is the center of H. The quotient H/Z is isomorphic to the group $\mathbb{R}^2$.*

Proof. Let x, y, z be in $\mathbb{R}$. Then, by lemma 3.1.1, $h_{x,y,z}$ is central if and only if, for any x', y' in $\mathbb{R}$, one has $xy' - x'y = 0$, that is if and only if $x = 0$ and $y = 0$.

Let π be the natural map $H \to H/Z$. Then, for any x, y, z and x', y', z' in $\mathbb{R}$, one has $\pi(h_{x,y,z}) = \pi(h_{x',y',z'})$ if and only if there exists z'' in $\mathbb{R}$ with $h_{x,y,z} = h_{x',y',z'}h_{0,0,z''}$. By lemma 3.1.1, this happens if and only if $x = x'$ and $y = y'$. Therefore, the map $\theta : (x,y) \mapsto \pi(h_{x,y,0})$ is a homeomorphism from $\mathbb{R}^2$ onto G/Z. Still by lemma 3.1.1, for any x, y and x', y' in $\mathbb{R}$, one has $h_{x,y,0}h_{x',y',0} = h_{x+x',y+y',0}h_{0,0,xy'}$, so that θ is a group isomorphism. □

In abstract terms, we would say we have an exact sequence

$$\{e\} \to Z \to H \to H/Z \to \{e\},$$

where Z is isomorphic to $\mathbb{R}$ and H/Z is isomorphic to $\mathbb{R}^2$. We shall always denote by $\pi : H \to H/Z$ the natural projection.

Let us make an other easy observation:

Lemma 3.1.4. *Let g be in H. Then there exists an unique continuous morphism*

$$\begin{aligned} t &\mapsto g^t \\ \mathbb{R} &\to H \end{aligned}$$

which value at 1 is g. It is proper. If g does not belong to Z, the centralizer Z_g of g in H is exactly the group of elements of the form $g^t z$, where t is in $\mathbb{R}$ and z is in Z, and the map

$$\begin{aligned} \mathbb{R} \times Z &\to Z_g \\ (t, z) &\mapsto g^t z \end{aligned}$$

is a continuous isomorphism. The group Z_g is normal in H and the quotient H/Z_g is isomorphic to $\mathbb{R}$.

In the sequel, we will always denote this morphism by $t \mapsto g^t$ and the centralizer of g by Z_g.

Proof. Let x, y, z be such that $g = h_{x,y,z}$. We set, for any t,

$$g^t = h_{tx,ty,tz+\frac{1}{2}t(t-1)xy}$$

and one verifies easily that the map $t \mapsto g^t$ is a proper continuous group homomorphism. Let $\varphi : \mathbb{R} \to H$ be an other continuous group homomorphism with $\varphi(1) = g$.

Suppose first g belongs to Z. Then, $\psi = \pi \circ \varphi$ is a continous homomorphism $\mathbb{R} \to H/Z \simeq \mathbb{R}^2$ and, for any p in $\mathbb{Z}$, we have $\psi(p) = \pi(g^p) = e$. Hence we have $\psi = e$, that is, for any t, $\varphi(t) \in Z$. As Z is isomorphic to $\mathbb{R}$, unicity follows.

Suppose now $g \notin Z$, that is $(x, y) \neq (0, 0)$, and let us first describe the centralizer Z_g of g. Let x', y', z' be in $\mathbb{R}$ such that g and $h_{x',y',z'}$ commute. From lemma 3.1.1, we get $xy' - x'y = 0$, that is there exists t in $\mathbb{R}$ such that $x' = tx$ and $y' = ty$. Thus, we get $h_{x',y',z'} \in g^t Z$. As such elements clearly commute with g, we have $Z_g = g^{\mathbb{R}} Z$. As, for any t, g^t does not belong to Z, the map $(t, z) \mapsto g^t z$ is clearly an isomorphism. In particular, Z_g is isomorphic to $\mathbb{R}^2$. Now, for any t, g and $\varphi(t)$ commute. Therefore, $\varphi(t)$ belongs to Z_g. Once more, uniqueness follows from the description of continuous morphisms $\mathbb{R} \to \mathbb{R}^2$. Finally, as Z_g contains Z and as every subgroup of H/Z is normal, Z_g is normal in H and $H/Z_g \simeq (H/Z)/(Z_g/Z) \simeq \mathbb{R}$. $\square$

Corollary 3.1.5. *Let F be a closed connected subgroup of H. Then either F is H, Z or $\{e\}$ or there exists a non central g in H with $F = g^{\mathbb{R}}$ or $F = Z_g$. In particular, for any g in F and t in $\mathbb{R}$, we have $g^t \in F$.*

Proof. Let F be different from Z or $\{e\}$ and let g be a non central element of H belongin to F. We will show that, if H is neither $g^{\mathbb{R}}$ nor Z_g, we have $F = H$. First, let us note that, by lemma 3.1.1, for any h in H, we have $ghg^{-1}h^{-1} \in Z$. Consider the continuous map $\theta : F \to Z, h \mapsto ghg^{-1}h^{-1}$. As F is connected, $\theta(F)$ is a connected subset of Z, which is contained in $Z \cap F$. As $F \not\subset Z_g$, we have $\theta(F) \neq \{e\}$. Therefore, $\theta(F)$ has nonempty interior and the group $Z \cap F$ is open in Z. As Z is connected, we have $Z \subset F$. Now $\pi(F)$ is a closed connected subgroup of $H/Z \simeq \mathbb{R}^2$ which is not reduced to a vector line, that is $\pi(F) = H/Z$ and $F = H$. □

3.2 Lattices in the Heisenberg group

As for the case of closed connected subgroups, we have a complete description of lattices of H:

Proposition 3.2.1. *Let Γ be a discrete subgroup of H. Then Γ is a lattice in H if and only if there does not exists a closed connected strict subgroup F of H with $\Gamma \subset F$. If Γ is a lattice in H, then Γ is cocompact in H, $\Gamma \cap Z$ is cocompact in Z and $\Gamma/(\Gamma \cap Z) = (\Gamma Z)/Z$ is discrete and cocompact in H/Z.*

Proof. By lemma 3.1.5, every closed connected subgroup of H has infinite covolume and, hence, it can not contain a lattice of H. Conversely, let Γ be a discrete subgroup of H that is not contained in any closed connected strict subgroup. Let γ be an element of Γ that is not central in H and let η be an element of Γ that does not belong to Z_γ. Then, by lemma 3.1.1, $\gamma\eta\gamma^{-1}\eta^{-1}$ is a non zero element of Z. Therefore $\Gamma \cap Z$ is a cocompact lattice in Z. Hence, the map $\Gamma/(\Gamma \cap Z) \to H/Z$ is proper and $\Gamma/(\Gamma \cap Z)$ is a discrete subgroup of H/Z which is not contained in any closed connected strict subgroup of H/Z. As H/Z is isomorphic to $\mathbb{R}^2$, this implies that $\Gamma/(\Gamma \cap Z)$ is a cocompact lattice in H/Z. Thus, Γ is a cocompact lattice in H. □

Remark 3.2.2. Let Γ be a lattice in H. Then, the space

$$H/(\Gamma Z) = (H/Z)/(\Gamma/(\Gamma \cap Z))$$

is homeomorphic to a two-dimensional torus $\mathbb{T}^2$. For any x in $H/(\Gamma Z)$, the inverse image of x under the natural map $H/\Gamma \to H/(\Gamma Z)$ identifies with $Z/(\Gamma \cap Z)$ which is homeomorphic to a one-dimensional torus $\mathbb{T}$.

Example 3.2.3. For any n in $\mathbb{N}^*$, let Λ_n be the set of elements of H of the form

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$$

with x, y in $\mathbb{Z}$ and z in $\frac{1}{n}\mathbb{Z}$. Then Λ_n is a lattice in H. For any lattice Γ in H, there exists an unique n in $\mathbb{N}^*$ such that there exists a continuous automorphism θ of H with $\theta(\Gamma) = \Lambda_n$.

3.3 Unique ergodicity on compact quotients

Let us fix a lattice Γ in H. We will prove the

Theorem 3.3.1. *Let g be a non central element of H. Then the following are equivalent*

- *(i) the group $\Gamma \cap Z_g$ is not a lattice in Z_g.*
- *(ii) the flow $(L_{g^t})_{t\in\mathbb{R}}$ is uniquely ergodic on H/Γ.*
- *(iii) the flow $(L_{g^t})_{t\in\mathbb{R}}$ is uniquely ergodic on $H/(\Gamma Z)$.*

In particular, if they are satisfied, the unique invariant measure of $(L_{g^t})_{t\in\mathbb{R}}$ on H/Γ is the H-invariant measure.

Remark 3.3.2. Note that, with the notations of the theorem, the space $H/(\Gamma Z)$ is homeormorphic to a torus $\mathbb{T}^2$ and the flow $(L_{g^t})_{t\in\mathbb{R}}$ is conjugate to a translation flow.

The proof uses a general version of Birkhoff theorem, uniformized by Egorov theorem:

Lemma 3.3.3. *Let $(\varphi_t)_{t\in\mathbb{R}}$ be a continuous flow on the locally compact and separable topological space X. Let μ be a Borel invariant ergodic probability for $(\varphi_t)_{t\in\mathbb{R}}$. Then, for any $0 < \varepsilon \leq 1$, there exists a Borel subset A in X with $\mu(A) \geq 1 - \varepsilon$ such that, for any θ in $\mathcal{C}_c^0(X)$, one has*

$$\frac{1}{T}\int_0^T \theta(\varphi_t(x))\mathrm{d}t \xrightarrow[T\to\infty]{} \int_X \theta \mathrm{d}\mu,$$

uniformly for x in A.

Proof. Let us first prove the lemma for a fixed continuous compactly supported function θ. Fix an integer $p \geq 1$. Then, by Birkhoff theorem, we have

$$\mu\left(\bigcup_{S\in\mathbb{N}^*}\bigcap_{\substack{T\in\mathbb{R}\\T\geq S}}\left\{x\in X \,\middle|\, \left|\frac{1}{T}\int_0^T \theta(\varphi_t(x))\mathrm{d}t - \int_X \theta \mathrm{d}\mu\right| \leq \frac{1}{p}\right\}\right) = 1.$$

Note that, as θ is continous, the function $T \mapsto \frac{1}{T}\int_0^T \theta(\varphi_t(x))\mathrm{d}t$ is continuous, so that, for any S,

$$\begin{aligned}\bigcap_{\substack{T\in\mathbb{R}\\T\geq S}}&\left\{x\in X \,\middle|\, \left|\frac{1}{T}\int_0^T \theta(\varphi_t(x))\mathrm{d}t - \int_X \theta \mathrm{d}\mu\right| \leq \frac{1}{p}\right\}\\ &= \bigcap_{\substack{T\in\mathbb{Q}\\T\geq S}}\left\{x\in X \,\middle|\, \left|\frac{1}{T}\int_0^T \theta(\varphi_t(x))\mathrm{d}t - \int_X \theta \mathrm{d}\mu\right| \leq \frac{1}{p}\right\}\end{aligned}$$

is a measurable set. As this family is increasing in S, we can find S_p in $\mathbb{N}^*$ such that

$$\mu\left(\bigcap_{T\geq S_p}\left\{x\in X \,\middle|\, \left|\frac{1}{T}\int_0^T \theta(\varphi_t(x))\mathrm{d}t - \int_X \theta \mathrm{d}\mu\right| \leq \frac{1}{p}\right\}\right) \geq 1-\varepsilon 2^{-p}.$$

We therefore get

$$\begin{aligned}\mu\left(\bigcap_{p\in\mathbb{N}^*}\bigcap_{T\geq S_p}\left\{x\in X \,\middle|\, \left|\frac{1}{T}\int_0^T \theta(\varphi_t(x))\mathrm{d}t - \int_X \theta \mathrm{d}\mu\right| \leq \frac{1}{p}\right\}\right) &\geq 1-\sum_{p=1}^{\infty}\varepsilon 2^{-p}\\ &= 1-\varepsilon.\end{aligned}$$

In other words, there exists a Borel subset A_θ in X with $\mu(A_\theta) \geq 1-\varepsilon$ such that

$$\frac{1}{T}\int_0^T \theta(\varphi_t(x))\mathrm{d}t \xrightarrow[T\to\infty]{} \int_X \theta\mathrm{d}\mu,$$

uniformly for x in A_θ.

As X is separable, we can find a sequence (θ_n) of elements of $\mathcal{C}_c^0(X)$ which is dense for the topology of uniform convergence. For any n, fix a Borel set A_{θ_n} in X such that $\mu(A_{\theta_n}) \geq 1 - \varepsilon 2^{-n-1}$ and

$$\frac{1}{T}\int_0^T \theta_n(\varphi_t(x))\mathrm{d}t \xrightarrow[T\to\infty]{} \int_X \theta_n \mathrm{d}\mu,$$

uniformly for x in A_{θ_n}. Set $A = \bigcap_{n\in\mathbb{N}} A_{\theta_n}$, so that

$$\mu(A) \geq 1 - \sum_{n=0}^{\infty} \varepsilon 2^{-n-1} = 1 - \varepsilon$$

and let us prove that A satisfies the requirements of the lemma. Indeed, if θ is in $\mathcal{C}_c^0(X)$, for any $\eta > 0$, there exists n in $\mathbb{N}$ such that $|\theta - \theta_n| \leq \eta$ everywhere on X. There exists $S > 0$ such that, for any x in $A \subset A_{\theta_n}$ and $T \geq S$, one has

$$\left|\frac{1}{T}\int_0^T \theta_n(\varphi_t(x))\mathrm{d}t - \int_X \theta_n \mathrm{d}\mu\right| \leq \eta$$

and, hence,

$$\left|\frac{1}{T}\int_0^T \theta(\varphi_t(x))\mathrm{d}t - \int_X \theta \mathrm{d}\mu\right| \leq 3\eta.$$

Thus, $\frac{1}{T}\int_0^T \theta(\varphi_t(x))\mathrm{d}t \xrightarrow[T\to\infty]{} \int_X \theta\mathrm{d}\mu$ uniformly on A, what should be proved. □

Now we will give a criterium for a measure on H/Γ to be the H-invariant measure:

Lemma 3.3.4. *Let Γ be a lattice in H and let μ be a Borel probability measure on H/Γ. Suppose μ is Z-invariant and the image of μ by the natural projection $H/\Gamma \to H/\Gamma Z$ is H-invariant. Then μ is the H-invariant measure of H/Γ.*

Proof. As Γ is unimodular, there exists a left-H-equivariant bijection between the set of Radon measures on H/Γ and the set of right-Γ-invariant measures on H. As Z is central in H, this bijection sends left-Z-invariant measures on H/Γ to right-ΓZ-invariant measures on H. As ΓZ is unimodular (since, for example, it admits Γ as a lattice), there is a left-H-equivariant bijection between the set of right-ΓZ-invariant measures on H and the set of measures on $H/\Gamma Z$. Hence, there is a left-H-equivariant bijection between the set of left-Z-invariant measures on H/Γ and the set of measures on $H/\Gamma Z$. One verifies that, by construction, this bijection is, up to a constant multiple, the map that sends a measure to its image by the natural map $H/\Gamma \to H/\Gamma Z$. The lemma follows. □

We can now proceed to the

Proof of theorem 3.3.1. First, note that *(i)* and *(iii)* are equivalent. Indeed, as $\Gamma \cap Z$ is cocompact in Z, the map $(\Gamma Z)/Z = \Gamma/(\Gamma \cap Z) \to H/Z$ is proper and $\Gamma \cap Z_g$ is a lattice in Z_g if and only if $((\Gamma Z)/Z) \cap (Z_g/Z)$ is a lattice in Z_g/Z and, by classical settings in the study of translation flows on tori, this is equivalent to the fact that the the flow $(L_{g^t})_{t\in\mathbb{R}}$ is uniquely ergodic on $H/(\Gamma Z)$. In particular, if this is true, the unique invariant measure of this flow is the H-invariant measure of $H/(\Gamma Z)$.

Now, as the action of $(L_{g^t})_{t\in\mathbb{R}}$ in $H/(\Gamma Z)$ is a factor of its action on H/Γ, if the action in H/Γ is uniquely ergodic, the one on $H/(\Gamma Z)$ is *a fortiori* uniquely ergodic, so that *(ii)* implies *(iii)*.

What we have to prove is the strong converse statement, that is *(iii)* implies *(ii)*. So, suppose the translation flow $(L_{g^t})_{t\in\mathbb{R}}$ on $H/(\Gamma Z)$ is uniquely ergodic and fix an invariant ergodic probability measure for $(L_{g^t})_{t\in\mathbb{R}}$ on H/Γ. The projection of μ on $H/(\Gamma Z)$ is H-invariant, so that, by lemma 3.3.4, we only have to prove that μ is Z-invariant.

First, let us remark that, if A is a Borel subset of H/Γ with $\mu(A) > 0$, there exists a point x in A and a sequence (k_n) of elements of $H - Z_g$ such that $k_n \xrightarrow[n\to\infty]{} e$ and that, for any n, $k_n x$ belongs to A. Indeed, if this not true, let K be a compact subset in A such that $\mu(K) > 0$. Then, for every x in K, there exists a neighborhood V_x of e in H with $V_x x \cap K \subset Z_g x$. As K is compact, there exists $x_1, \ldots, x_p$ in K with $K \subset V_{x_1}x_1 \cup \ldots \cup V_{x_p}x_p$ and, hence, $K \subset Z_g x_1 \cup \ldots \cup Z_g x_p$. In particular, there exists $1 \le i \le p$ with $\mu(Z_g x_i) > 0$, which is not possible, since the projection of μ on $H/(\Gamma Z)$ is the H-invariant measure.

By lemma 3.3.3, there exists a Borel set A in H/Γ with $\mu(A) > 0$ such that, for any continuous function θ on H/Γ,

$$\frac{1}{T}\int_0^T \theta(g^t x)\mathrm{d}t \xrightarrow[T\to\pm\infty]{} \int_X \theta \mathrm{d}\mu,$$

uniformly for x in A. Note that, for any $a < b$ we have, for $T > 0$,

$$\frac{1}{(b-a)T}\int_{aT}^{bT}\theta(g^t x)\mathrm{d}t = \frac{1}{(b-a)T}\left(\int_0^{bT}\theta(g^t x)\mathrm{d}t - \int_0^{aT}\theta(g^t x)\mathrm{d}t\right),$$

so that

$$\frac{1}{(b-a)T}\int_{aT}^{bT}\theta(g^t x)\mathrm{d}t \xrightarrow[T\to\infty]{} \int_X \theta\mathrm{d}\mu,$$

uniformly for x in A.

Now, fix a point x in A and a sequence (k_n) of elements of $H - Z_g$ such that $k_n \xrightarrow[n\to\infty]{} e$ and that, for any n, $k_n x$ belongs to A. By lemma 3.1.1, there exists a sequence (z_n) of non zero real numbers such that $z_n \xrightarrow[n\to\infty]{} 0$ and that, for any n, for any real number t, one has $g^t k_n = k_n g^t h_{0,0,z_n t}$. Let θ be a continuous function on H/Γ and a be a real number. We will prove that we have $\int_{H/\Gamma}\theta\mathrm{d}\mu = \int_{H/\Gamma}\theta(h_{0,0,a}y)\mathrm{d}\mu(y)$. Fix $\varepsilon > 0$. By uniform continuity, there exists $\eta > 0$ and a neighborhood V of e in H such that, for any $|z| \leq \eta$ and k in V, for any x in H/Γ, one has $|\theta(h_{0,0,z}ky) - \theta(y)| \leq \varepsilon$. Let $b = a + \eta$. Observe that, for any n in $\mathbb{N}$ and $T \geq 0$, we have

$$\frac{1}{(b-a)T}\int_{aT}^{bT}\theta(g^t k_n x)\mathrm{d}t = \frac{1}{(b-a)T}\int_{aT}^{bT}\theta(h_{0,0,z_n t}k_n g^t x)\mathrm{d}t.$$

Therefore, setting $T = \frac{1}{z_n}$, we get, for sufficiently large n,

$$\left|\frac{z_n}{(b-a)}\int_{\frac{a}{z_n}}^{\frac{b}{z_n}}\theta(g^t k_n x)\mathrm{d}t - \frac{z_n}{(b-a)}\int_{\frac{a}{z_n}}^{\frac{b}{z_n}}\theta(h_{0,0,a}g^t x)\mathrm{d}t\right| \leq \varepsilon.$$

But, as the $(k_n x)$ belong to A, we have

$$\frac{z_n}{(b-a)}\int_{\frac{a}{z_n}}^{\frac{b}{z_n}}\theta(g^t k_n x)\mathrm{d}t \xrightarrow[n\to\infty]{} \int_X \theta\mathrm{d}\mu$$

and, in the same way, as x belongs to A,

$$\frac{z_n}{(b-a)}\int_{\frac{a}{z_n}}^{\frac{b}{z_n}}\theta(h_{0,0,a}g^t x)\mathrm{d}t \xrightarrow[n\to\infty]{} \int_X \theta\circ L_{h_{0,0,a}}\mathrm{d}\mu.$$

Thus, we have $\left|\int_X \theta\circ L_{h_{0,0,a}}\mathrm{d}\mu - \int_X\theta\mathrm{d}\mu\right| \leq \varepsilon$ and, as this is true for any $\varepsilon > 0$, the measure μ is $L_{h_{0,0,a}}$-invariant. Hence μ is Z-invariant and, by lemma 3.3.4, it is H-invariant. Therefore, the flow $(L_{g^t})_{t\in\mathbb{R}}$ is uniquely ergodic on H/Γ and its unique invariant measure is the H-invariant measure, what should be proved. $\square$

3.4 Exercices

3.4.1 Classification of invariant measures

Let Γ be a lattice in the Heisenberg group H and let g be in H. Describe all the invariant probability measures for the action of the translation flow $(L_{g^t})_{t\in\mathbb{R}}$ in H/Γ.

3.4.2 Lattice classification in the Heisenberg group

1. Let g and h be two non-commutating elements of H. Prove that there exists an unique continuous automorphism θ of H such that $\theta(g) = h_{1,0,0}$ and $\theta(h) = h_{0,1,0}$.

Let Γ be a lattice in H.

2. Prove that there exists g and h in Γ such that, if Δ is the subgroup of Γ spanned by g and h, one has $\Gamma/(\Gamma \cap Z) = \Delta/(\Delta \cap Z)$.

For any n in $\mathbb{N}^*$, let Λ_n be the set of the elements of H of the form

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$$

with x, y in $\mathbb{Z}$ and z in $\frac{1}{n}\mathbb{Z}$. If G is a group, denote by $Z(G)$ the center of G and by $[G, G]$ the subgroup of G spanned by the elements of the form $[g, h] = ghg^{-1}h^{-1}$ where g and h belong to G.

3. Prove that, for any n in $\mathbb{N}^*$, Λ_n is a lattice in H and that the index of $[\Lambda_n, \Lambda_n]$ in $Z(\Lambda_n)$ is n.

4. Prove that there exists an unique n in $\mathbb{N}^*$ such that there exists a continuous automorphism θ of H with $\theta(\Gamma) = \Lambda_n$.

3.4.3 Higher Heisenberg groups

Let $d \geq 1$ and let H_d be the group of matrices of the form

$$h_{x,z} = \begin{pmatrix} 1 & x & iz - \frac{\|x\|^2}{2} \\ 0 & I_d & -\overline{x}^t \\ 0 & 0 & 1 \end{pmatrix},$$

where x is a vector in $\mathbb{C}^d$, z a real number, $x \mapsto \overline{x}$ denotes complex conjugation, ^{t} denotes matrix transposition and $\|.\|$ is the usual hermitian norm on $\mathbb{C}^d$.

1. Prove that H_d is a closed subgroup of $\mathrm{GL}_{d+2}(\mathbb{C})$ and that H_1 is topologically isomorphic to the Heisenberg group.

The group H_d is called the d-th Heisenberg group.

2. Prove that the center Z of H_d is topologically isomorphic to $\mathbb{R}$ and that H_d/Z is topologically isomorphic to $\mathbb{R}^{2d}$.

3. Let g be in H_d. Prove that there exists an unique continuous morphism $g \mapsto g^t$ from $\mathbb{R}$ into H_d such that $g^1 = g$. Prove that, if g is not central in H_d, the centralizer Z_g of g in H is normal in H and isomorphic to $\mathbb{R}^{2d}$ and that one has $H/Z_g \simeq \mathbb{R}$.

4. Exhibit a lattice in H_d.

5. Prove that lattices in H_d are cocompact.

6. Let Γ be a lattice in H_d and let g be an element of H_d such that the translation flow $(L_{g^t})_{t\in\mathbb{R}}$ on $H_d/(\Gamma Z)$ is uniquely ergodic. Prove that $(L_{g^t})_{t\in\mathbb{R}}$ is uniquely ergodic on H_d/Γ and that its unique invariant measure is the H_d-invariant measure.

3.4.4 Kronecker factor of Heisenberg minimal flows

Let $(X, \mathcal{A}, \mu)$ be a probability space with $\mathrm{L}^1(X, \mu)$ separable and let (φ_t) be an ergodic measure preserving flow on X. Let λ be a non zero real number. We will suppose (φ_t) admits a non zero λ-eigenvector, that is there exists a non zero measurable function ρ on X such that, for any t in $\mathbb{R}$, one has $\rho \circ \varphi_t = e^{2i\pi\lambda t}\rho$.

1. Prove that ρ has constant modulus and that every other λ-eigenvector is of the form $c\rho$ for some almost constant function c.

In the sequel, we will assume $|\rho| = 1$ almost everywhere and consider ρ indifferently as a function $X \to \mathbb{C}$ or $X \to \mathbb{T} = \mathbb{R}/\mathbb{Z}$.

2. Equip $\mathbb{T}$ with its Lebesgue measure ν. Prove that the product flow

$$\begin{gathered}\tilde{\varphi}_t : (x, u) \mapsto (\varphi_t(x), u + \lambda t)\\ X \times \mathbb{T} \to X \times \mathbb{T}\end{gathered}$$

is not ergodic for the product measure $\mu \otimes \nu$. Let $\mathcal{I}$ be the σ-algebra of $\mu \otimes \nu$-almost invariant measurable subsets of $X \times \mathbb{T}$. Prove that, for any f

in $\mathrm{L}^1(X \times \mathbb{T}, \mu \otimes \nu)$, for $\mu \otimes \nu$-almost (x, u) in $X \times \mathbb{T}$, one has

$$\mathbb{E}(f|\mathcal{I})(x,u) = \int_X f(y, u - \rho(x) + \rho(y))\mathrm{d}\mu(y).$$

3. Let f be in $\mathrm{L}^1(X, \mu)$. Prove that, for μ-almost every x in X, one has

$$\frac{1}{T}\int_0^T f(\varphi_t(x))e^{-2i\pi\lambda t}\mathrm{d}t \xrightarrow[T\to\infty]{} \rho(x)\int_X f\overline{\rho}\mathrm{d}\mu.$$

4. Let H be the Heisenberg group, Z its center, Γ a lattice in H and g an element of H such that the translation flow (L_{g^t}) on H/Γ is uniquely ergodic. Let ρ be a non zero λ-eigenvector for this flow with respect to its unique invariant measure μ. Prove that ρ is Z-invariant. Conversely, prove that the space of Z-invariant functions in $\mathrm{L}^1(H/\Gamma, \mu)$ is spanned by eigenvectors.

Chapter 4

Dynamics on $\mathrm{SL}_2(\mathbb{R})$ quotients

In this chapter, we will denote by G the group $\mathrm{SL}_2(\mathbb{R})$, that is the group of 2 by 2 real matrices of the form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with $ad - bc = 1$. As we shall see in the exercices, this group admits both cocompact and non cocompact lattices. We will study the action of one-parameter subgroups of G in the quotients G/Γ, where Γ is a lattice.

4.1 Structure of $\mathrm{SL}_2(\mathbb{R})$

We will have to introduce classical notations for remarkable subgroups of G.

First, for all s in $\mathbb{R}$, we set

$$a_s = \begin{pmatrix} e^s & 0 \\ 0 & e^{-s} \end{pmatrix}.$$

The map $s \mapsto a_s$ is a proper continuous homomorphism from $\mathbb{R}$ into G. We denote its image by A: in other words, A is the group of diagonal matrices with positive coefficients and determinant 1.

For t in $\mathbb{R}$, we set

$$u_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \text{ and } v_t = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}.$$

Again, the maps $t \mapsto u_t$ and $t \mapsto v_t$ are proper continuous homomorphisms from $\mathbb{R}$ into G. Let $U = \{u_t | t \in \mathbb{R}\}$ and $V = \{v_t | t \in \mathbb{R}\}$: U (resp. V) is the group of upper-triangular (resp. lower-triangular) matrices which all eigenvalues are equal to 1. Note that, for any s, t in $\mathbb{R}$, we have

$$a_s u_t a_{-s} = u_{e^{2s}t} \text{ and } a_s v_t a_{-s} = v_{e^{-2s}t}.$$

This relations will play a crucial role in the sequel. As U (resp. V) is the stabilizer of $(1,0)$ (resp. $(0,1)$) for the canonical linear action of G on $\mathbb{R}^2$, the homogeneous space G/U (resp. G/V) identifies with $\mathbb{R}^2 - \{0\}$.

Let us use these groups to prove the

Lemma 4.1.1. *The group G is unimodular.*

Proof. We shall in fact prove much more, that is every (abstract) morphism from G to an abelian group is trivial, so that the modular function of G is trivial. Indeed, if A is an abelian group and $\varphi : G \to A$ is a morphism, for any g and h in G, we have $\varphi(ghg^{-1}h^{-1}) = e$. For any s and t in $\mathbb{R}$, we have $a_s u_t a_{-s} u_{-t} = u_{(e^{2s}-1)t}$. Thus, for any t in $\mathbb{R}$, $\varphi(u_t) = e$. In the same way, $\varphi(V) = e$.

To conclude, we will prove that U and V span G has a group. Let H be the subgroup of G spanned by U and V and let us study the action of H on $\mathbb{R}^2$. Let (x,y) be in $\mathbb{R}^2 - \{(0,0)\}$. If $y \neq 0$, we have $u_{\frac{1-x}{y}}(x,y) = (1,z)$ for some z in $\mathbb{R}$. Thus, $v_{-z} u_{\frac{1-x}{y}}(x,y) = (1,0)$ and $(1,0) \in H(x,y)$. If $y = 0$, for any $t \neq 0$, $v_t(x,0) = (x,tx)$ has non zero second coordinate, so that, once more, $(1,0) \in H(x,y)$. Hence H acts transitively on $\mathbb{R}^2 - \{0\} = G/U$, so that $G = HU$. As U is contained in H, we get $G = H$. In particular, $\varphi = e$, what should be proved. □

Let M be the group $\{\pm e\}$ in G. The group M is the center of G. As A normalizes U, the set AU is a subgroup of G. We set $P = MAU$: P is the group of upper-triangular matrices with determinant 1. As P is the stabilizer of $\mathbb{R}(1,0)$ for the canonical projective action of G on the projective line $\mathbb{P}^1_{\mathbb{R}}$, the homogeneous space G/P identifies with $\mathbb{P}^1_{\mathbb{R}}$.

Finally, for any θ in $\mathbb{R}$, we set

$$k_\theta = \begin{pmatrix} \cos(2\pi\theta) & -\sin(2\pi\theta) \\ \sin(2\pi\theta) & \cos(2\pi\theta) \end{pmatrix}.$$

The map $\theta \mapsto k_\theta$ induces a continuous injective morphism from $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ into G. Its image is the group $\mathrm{SO}(2)$, that we shall denote by K in the sequel.

The structure results about G we will use are summarized in the

Proposition 4.1.2. *The group G admits the following decompositions:*

(i) Bruhat decomposition: the map

$$\begin{aligned} V \times M \times A \times U &\to G \\ (v, m, a, u) &\mapsto vmau \end{aligned}$$

induces a homeomorphism from $V \times M \times A \times U$ onto its image which is the dense open set of elements g in G with upper left coefficient $g_{1,1} \neq 0$. The complement of this open set is the coset $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} P$.

(ii) Iwasawa decomposition: the map

$$\begin{aligned} K \times A \times U &\to G \\ (k, a, u) &\mapsto kau \end{aligned}$$

is a homeomorphism.

(iii) polar decomposition: let S be the set of positive definite matrices with determinant 1, then the map

$$\begin{aligned} K \times S &\to G \\ (k, \sigma) &\mapsto k\sigma \end{aligned}$$

is a homeomorphism.

(iv) Cartan decomposition: let $A^+ = \{a_s | s \geq 0\}$, then the map

$$\begin{aligned} K \times A^+ \times K &\to G \\ (k, a, l) &\mapsto kal \end{aligned}$$

is proper and onto and, for any g in G, there exists an unique $s \geq 0$ with $g \in K a_s K$.

For the study of polar decomposition, we shall use the

Lemma 4.1.3. *Let S be the set of positive definite matrices with determinant 1. The map $\sigma \mapsto \sigma^2$ is a homeomorphism of S.*

Proof. Let us prove that this map is bijective. Let σ be in S. As σ is diagonalizable in an orthonormal basis and as it is positive definite with determinant 1, there exists k in K ans s in $\mathbb{R}_+$ with $\sigma = ka_sk^{-1}$, so that $\sigma^2 = ka_{2s}k^{-1}$. In particular, if $\sigma^2 = 1$, one has $\sigma = 1$. Let $\tau = ka_{\frac{1}{2}s}k^{-1}$. We have $\tau^2 = \sigma$, so that the square map is onto. Suppose ρ is an other element of S with $\rho^2 = \sigma$. Then, if $\sigma = 1$, one has $\rho^2 = 1$ and, as noted before, this implies $\rho = 1$. If $\sigma \neq 1$, σ and ρ commute, that is $k^{-1}\rho k$ commutes with a_s and $s \neq 0$. Hence $k^{-1}\rho k$ is diagonal. As $(k^{-1}\rho k)^2 = a_s$ and $k^{-1}\rho k$ is positive definite, we have $k^{-1}\rho k = a_{\frac{1}{2}s}$, that is $\rho = \tau$ and the square map is injective.

As it is clearly continuous, to prove that it is a homeomorphism, we need to prove that it is proper. Let us describe the compact subsets of S. Let $\|.\|$ be the usual euclidean norm on $\mathbb{R}^2$ and let still denote by $\|.\|$ the associate operator norm. For any $s \geq 0$ and k in K, we have $\|ka_sk^{-1}\| = e^s$ so that $L_s = \{\sigma \in S | \, \|\sigma\| \leq e^s\} = \{ka_tk^{-1} | 0 \leq t \leq s, k \in K\}$ is a compact subset of S. Moreover, if L is a compact subset of S, the norm function is bounded on L, so that there exists $s \geq 0$ with $L \subset L_s$. Finally, we have immediately $L_s^2 = L_{2s}$ and the map $\sigma \mapsto \sigma^2$ is proper. Thus, it is an homeomorphism and the lemma is proved. $\square$

Proof of proposition 4.1.2. Bruhat decomposition for elements g of G with $g_{1,1} \neq 0$ is a version of Gauss elimination. If $g_{1,1} = 0$, one has $g(1,0) \in \mathbb{R}^*(0,1) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbb{R}^*(1,0)$, so that $g \in \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} P$.

Iwasawa decomposition is a matrix translation of Gram-Schmidt orthogonalisation process.

Let us prove polar decomposition. Let g be in G. The matrix g^tg (where $g \mapsto g^t$ denotes matrix transposition) belongs to S. By lemma 4.1.3, there exists an unique σ in S with $\sigma^2 = g^tg$ and σ depends continuously on g. We have $(g\sigma^{-1})^t(g\sigma^{-1}) = 1$, that is $k = g\sigma^{-1}$ belongs to $\mathrm{O}(2)$. As k has determinant 1, k belongs to $K = \mathrm{SO}(2)$ and we have $g = k\sigma$ and k and σ depend continuously on g. Uniqueness of the decomposition follows from uniqueness of the square root in S of an element of S.

Finally, for Cartan decomposition, we use first polar decomposition $g = k\sigma$ and then diagonalize σ in an orthonormal basis. We thus get existence. As K is compact, the map $K \times A^+ \times K \to G$ is clearly proper. Finally, for any $s \geq 0$, if g belongs to Ka_sK, we have $\|g\| = e^s$ and s is uniquely determined by g. $\square$

From polar (or Iwasawa) decomposition, we perfectly now the topology of G:

Corollary 4.1.4. *The group G is homeomorphic to $\mathbb{R}^2 \times \mathbb{T}$. In particular, it is connected.*

4.2 Unitary representations of $\mathrm{SL}_2(\mathbb{R})$

Let us define some abstract notions.

Definition 4.2.1. Let F be a locally compact topological group and let H be a Hilbert space. An unitary representation of F in H is a homomorphism of F into the group of unitary automorphisms of H such that the associate map $F \times H \to H$ is continuous.

Example 4.2.2. *(i)* The natural action by left or by right translations of F on the space $\mathrm{L}^2(F)$ of square-integrable functions on F with respect to Haar measure is an unitary representation. If Γ is a lattice in F, the action of F on the space $\mathrm{L}^2(F/\Gamma)$ of square-summable functions on F/Γ with respect to the F-invariant measure is an unitary representation. Given a measure preserving action of F on a measure space $(X, \mathcal{A}, \mu)$ (that is an action such that each element F preserves the measure and that the map $F \times X \to X$ is Borel), reasoning as in lemma 1.2.3, one proves that the associate action on $\mathrm{L}^2(X, \mathcal{A}, \mu)$ is an unitary representation (as soon as $\mathrm{L}^2(X, \mathcal{A}, \mu)$ is separable).

(ii) An unitary representation of $\mathbb{Z}$ in H is just the data of an unitary automorphism of H. For example, if ν is a Radon measure on $\mathbb{T}$, the map that sends each function f in $\mathrm{L}^2(\mathbb{T}, \nu)$ to the function $\xi \mapsto e^{2i\pi\xi} f(\xi)$ is an unitary automorphism and induces an unitary representation of $\mathbb{Z}$ in $\mathrm{L}^2(\mathbb{T}, \nu)$. The *spectral theorem* asserts that every representation of $\mathbb{Z}$ is an orthogonal direct sum of representations of this form.

(iii) Let ν be a Radon measure on $\mathbb{R}$. For any t in $\mathbb{R}$, let φ_t be the map that sends any f in $\mathrm{L}^2(\mathbb{R}, \nu)$ to the function $\xi \mapsto e^{2i\pi t\xi} f(\xi)$. Then $t \mapsto \varphi_t$ is an unitary representation of $\mathbb{R}$ in $\mathrm{L}^2(\mathbb{R}, \nu)$. The $\mathbb{R}$-version of the spectral theorem asserts that every representation of $\mathbb{R}$ is an orthogonal direct sum of representations of this form.

(iv) Let λ be the Lebesgue measure on $\mathbb{R}$. With the notations of chapter 3, define, for x, y, z in $\mathbb{R}$, $\rho_{h_{x,y,z}}$ as the map that sends any f in $\mathrm{L}^2(\mathbb{R}, \lambda)$ to the function $\xi \mapsto e^{2i\pi(z+x\xi)} f(y+\xi)$. Then ρ is an unitary representation of H in $\mathrm{L}^2(\mathbb{R}, \lambda)$.

In this section, we will be interested with unitary representations of $G = \mathrm{SL}_2(\mathbb{R})$. There is an analogue of the spectral theorem for this group, but we will need a different information which is provided by the Howe-Moore theorem:

Theorem 4.2.3. *Let H be a Hilbert space equipped with an unitary representation of G. Suppose the space of invariant vectors*

$$H^G = \{v \in H | \forall g \in G \quad gv = v\}$$

is reduced to $\{0\}$. Then, for any v and w in H, one has

$$\langle gv, w \rangle \xrightarrow[g \to \infty]{} 0,$$

that is, for every $\varepsilon > 0$, there exists a compact subset L of G such that, for any g in $G - L$, one has $|\langle gv, w \rangle| \leq \varepsilon$.

Remark 4.2.4. Let $(X, \mathcal{A}, \mu)$ be a probability space (with $\mathrm{L}^1(X, \mathcal{A}, \mu)$ separable), equipped with a measure preserving action of G. Suppose the action is ergodic, that is, if $A \in \mathcal{A}$ is G-invariant, one has $\mu(A) \in \{0, 1\}$. Then the theorem asserts that the one-parameter subgroups $(a_s)_{s \in \mathbb{R}}$ and $(u_t)_{t \in \mathbb{R}}$ act on $(X, \mathcal{A}, \mu)$ as mixing measure preserving flows! In particular, the theorem is very specific of G: we know that the groups $\mathbb{Z}$ and $\mathbb{R}$ possess a lot of non-mixing ergodic actions.

Fix a Hilbert space H, equipped with an unitary representation of G. Recall that the weak topology of H is the weakest topology of H making the scalar products by a fixed vector continuous (see section A.3 of the appendix). In particular, the conclusion of the theorem can be written as follows: for any v in H, one has $gv \xrightarrow[g \to \infty]{} 0$ weakly in H. Now, for any g, we have $\|gv\| = \|v\|$ so that the map $g \mapsto gv$ has bounded image. By Banach-Alaoglu theorem, bounded sets of H are weakly compact. Hence, to prove the theorem, we will study the limit points of Gv. We shall need two lemmas.

Lemma 4.2.5. *Let (v_n) be a sequence of vectors in H, weakly converging to a vector v. Let (T_n) and T be bounded operators of H such that, for any w in H, the sequence $(T_n^* w)$ strongly converges to $T^* w$ (where T^* stands for the adjoint operator of T). Then, the sequence $(T_n v_n)$ weakly converges to Tv.*

Proof. For any w in H, we have, for any n,

$$\langle T_n v_n, w\rangle = \langle v_n, T_n^* w\rangle = \langle v_n, T^* w\rangle + \langle v_n, T_n^* w - T^* w\rangle.$$

As (v_n) weakly converges to v, we have $\langle v_n, T^* w\rangle \xrightarrow[n\to\infty]{} \langle v, T^* w\rangle = \langle Tv, w\rangle$. Moreover, by Banach-Steinhaus theorem, (v_n) is bounded. Hence, as

$$\|T_n^* w - T^* w\| \xrightarrow[n\to\infty]{} 0,$$

we have $\langle v_n, T_n^* w - T^* w\rangle \xrightarrow[n\to\infty]{} 0$ and the lemma is proved. □

Lemma 4.2.6. *Let (v_n) be a sequence of vectors in H, strongly converging to a vector v. Let (T_n) be a bounded sequence of bounded operators such that $(T_n v_n)$ weakly converges to some vector u in H. Then, the sequence $(T_n v)$ weakly converges to u.*

Proof. For any w in H, we have

$$\langle T_n v, w\rangle = \langle T_n v_n, w\rangle + \langle T_n(v - v_n), w\rangle.$$

As $(T_n v_n)$ weakly converges to u, we have $\langle T_n v_n, w\rangle \xrightarrow[n\to\infty]{} \langle u, w\rangle$. As

$$\|v - v_n\| \xrightarrow[n\to\infty]{} 0,$$

and the sequence (T_n) is bounded, we have $\langle T_n(v - v_n), w\rangle \xrightarrow[n\to\infty]{} 0$ and the lemma is proved. □

Proof of theorem 4.2.3. As G is separable, it suffices to prove that, for any sequence (g_n) in G going to infinity (that is leaving every compact subset of G), for any v and w in H, if $(g_n v)$ weakly converges to w, one has $w = 0$ (see the proof of corollary A.3.11 in the appendix). Fix these notations and write, for any n, $g_n = k_n a_n l_n$ a Cartan decomposition of g_n, that is k_n and l_n belong to K and a_n belongs to A^+. The sequence (a_n) goes to infinity in A^+. As K is compact, after eventually extracting a subsequence, we can suppose there exists k, l in K with $k_n \xrightarrow[n\to\infty]{} k$ and $l_n \xrightarrow[n\to\infty]{} l$.

For any n, we have $a_n l_n v = k_n^{-1} g_n v$. As $(g_n v)$ weakly converges to w and $k_n \xrightarrow[n\to\infty]{} k$ we get, by lemma 4.2.5, $a_n l_n v \xrightarrow[n\to\infty]{} k^{-1} w$ weakly. In the same way, the sequence $(l_n v)$ strongly converges to lv and the sequence (a_n) is bounded for the operator norm, so that, by lemma 4.2.6, we get $a_n l v \xrightarrow[n\to\infty]{} k^{-1}w$ weakly. In other words, after having replaced v by lv and w by $k^{-1}w$, we can suppose that, for any n, $g_n = a_n$ belongs to A^+.

Now, let us prove that w is U-invariant. Indeed, for u in U, for any n, we have $ua_n v = a_n(a_n^{-1}ua_n)v$. As $a_n^{-1}ua_n \xrightarrow[n\to\infty]{} e$ in G, we have $(a_n^{-1}ua_n)v \xrightarrow[n\to\infty]{} v$ strongly in H. On the other hand, as u acts as a continuous operator on H, we have $ua_n v \xrightarrow[n\to\infty]{} uw$ weakly in H, so that, by lemma 4.2.6, we get $a_n v \xrightarrow[n\to\infty]{} uw$ weakly in H and, thus, $uw = w$.

We will now prove that, as w is U-invariant, it is necessary G-invariant: by the hypothesis, this will imply $w = 0$ and we will be done. Consider the continuous function

$$\begin{aligned} f : G &\to \mathbb{C} \\ g &\mapsto \langle gw, w\rangle. \end{aligned}$$

For any u, u' in U and g in G, we have

$$f(ugu') = \langle ugu'w, w\rangle = \langle gu'w, u^{-1}w\rangle = \langle gw, w\rangle = f(g)$$

as w is U-invariant. In other words, the function f is left and right-G-invariant. We can consider it as a continuous U-invariant function on $G/U \simeq \mathbb{R}^2 - \{0\}$. For any (x, y) in $\mathbb{R}^2$ with $y \neq 0$, the U-orbit of (x, y) is the set $\mathbb{R} \times \{y\}$. Hence, f is constant on each of these sets. As it is continuous, it is constant on $\mathbb{R}^* \times \{0\} = MA(1, 0)$. Thus, for any g in ma, we have

$$\langle gw, w\rangle = \langle w, w\rangle.$$

As $\|gw\| = \|w\|$, by Cauchy-Schwarz inequality, this implies $gw = w$. Hence, w is $P = MAU$-invariant and the function f is left and right-P-invariant. Thus, we may consider f as a continuous P-invariant function on $G/P \simeq \mathbb{P}^1_{\mathbb{R}}$. But in $\mathbb{P}^1_{\mathbb{R}}$ the P-orbit of the line $\mathbb{R}(0, 1)$ is $\mathbb{P}^1_{\mathbb{R}} - \{\mathbb{R}(1, 0)\}$. Therefore, f is constant, that is, for any g in G,

$$\langle gw, w\rangle = \langle w, w\rangle$$

and, again by Cauchy-Schwarz inequality, w is G-invariant. Hence $w = 0$, what should be proved. □

As we shall see in the exercices, the group $\mathrm{SL}_2(\mathbb{Z})$ is a non cocompact lattice in $G = \mathrm{SL}_2(\mathbb{R})$. Hence, from Howe-Moore theorem, we immediately deduce the

Corollary 4.2.7. *The flows* $(L_{a_s})_{s\in\mathbb{R}}$ *and* $(L_{u_t})_{t\in\mathbb{R}}$ *on* $\mathrm{SL}_2(\mathbb{R})/\mathrm{SL}_2(\mathbb{Z})$ *are mixing with respect to the* $\mathrm{SL}_2(\mathbb{R})$*-invariant probability measure.*

Remark 4.2.8. For geometric reasons this flows are respectively called *geodesic* and *horocyclic* flow.

Remark 4.2.9. From the relations

$$a_s u_t a_{-s} = u_{e^{2s}t} \text{ and } a_s v_t a_{-s} = v_{e^{-2s}t},$$

we can see the flow $(L_{a_s})_{s\in\mathbb{R}}$ as a continuous analogue of a hyperbolic transformation. Indeed, there is a notion of an Anosov flow and the standard example of such a flow is the action of $(L_{a_s})_{s\in\mathbb{R}}$ in quotients G/Γ, where Γ is a cocompact lattice in G. For such flows, there is a theory of Markov partitions. In particular this flows are very chaotic and possess a lot of closed invariant subsets and invariant measures. We shall now see that the flow $(L_{u_t})_{t\in\mathbb{R}}$ is much more rigid.

4.3 Unique ergodicity of the horocyclic flow on compact quotients of $\mathrm{SL}_2(\mathbb{R})$

The group $\mathrm{SL}_2(\mathbb{Z})$ is a lattice in G. As it contains u_1, the flow $(L_{u_t})_{t\in\mathbb{R}}$ has a periodic orbit in $\mathrm{SL}_2(\mathbb{R})/\mathrm{SL}_2(\mathbb{Z})$. Moreover, since A permutes the periodic orbits of $(L_{u_t})_{t\in\mathbb{R}}$, there are infinitely many periodic orbits for the action of $(L_{u_t})_{t\in\mathbb{R}}$ in $\mathrm{SL}_2(\mathbb{R})/\mathrm{SL}_2(\mathbb{Z})$.

The case of cocompact lattices is different, in view of the following theorem by Fürstenberg:

Theorem 4.3.1. *Let* Γ *be a cocompact lattice in* G*. Then the flow* $(L_{u_t})_{t\in\mathbb{R}}$ *is uniquely ergodic on* G/Γ*.*

Remark 4.3.2. In fact, a more general result is true: if Γ is any lattice in G, for any x in G/Γ, either x is a fixed point for some L_{u_t}, $t \in \mathbb{R}$, or, for any compactly supported continuous function θ on G/Γ, one has

$$\frac{1}{T}\int_0^T \theta(u_t x)\mathrm{d}t \xrightarrow[T\to\infty]{} \int_{G/\Gamma} \theta \mathrm{d}\mu,$$

where μ is the G-invariant probability on G/Γ.

We shall need to describe the Haar measure of G, viewed through the Bruhat decomposition:

Lemma 4.3.3. *Equip G with a Haar measure. Then, up to a renormalization, for any compactly supported continuous function θ on VAU, we have*

$$\int_{VAU} \theta(g)\mathrm{d}g = \int_{\mathbb{R}^3} e^{2s}\theta(v_r a_s u_t)\mathrm{d}r\mathrm{d}s\mathrm{d}t.$$

Proof. Let Q be the closed subgroup VA of G. By a direct computation, the left Haar measure of Q may be normalized in such a way that, for any θ in $\mathcal{C}_c^0(Q)$, one has

$$\int_Q \theta(q)\mathrm{d}q = \int_{\mathbb{R}^2} e^{2s}\theta(v_r a_s)\mathrm{d}r\mathrm{d}s.$$

Now, consider the action of the group $Q\times U$ on VAU such that, for any (q,u) in $Q\times U$, (q,u) acts by the map $g\mapsto qgu^{-1}$. By Bruhat decomposition, this action is simply transitive, that is, taking $g=e$ as a base point, it identifies VAU and $Q\times U$. Now G is unimodular, so that the action of $Q\times U$ on VAU leaves the restriction of the Haar measure of G to VAU invariant. Hence, through the identification between VAU and $Q\times U$, the Haar measure of G becomes the Haar measure of $Q\times U$. Therefore, for any θ in $\mathcal{C}_c^0(VAU)$, we get, after renormalization,

$$\int_{VAU} \theta(g)\mathrm{d}g = \int_{\mathbb{R}^3} e^{2s}\theta(v_r a_s u_{-t})\mathrm{d}r\mathrm{d}s\mathrm{d}t = \int_{\mathbb{R}^3} e^{2s}\theta(v_r a_s u_t)\mathrm{d}r\mathrm{d}s\mathrm{d}t,$$

what should be proved. □

We will also need an uniform version of mixing:

Lemma 4.3.4. *Let H be a Hilbert space, equipped with an unitary representation of a locally compact group F. Let w be a vector of H such that, for any v in H, one has*

$$\langle gv, w\rangle \xrightarrow[g\to\infty]{} 0.$$

Then, for any compact subset L of H, the convergence is uniform for v in L.

Proof. Let L be a compact subset of H. Fix $\varepsilon>0$. Then, there exists $v_1,\dots,v_p$ in L with $L\subset B(v_1,\varepsilon)\cup\ldots\cup B(v_p,\varepsilon)$. Now, for any v in L, there exists $1\le i\le p$ such that $\|v-v_i\|\le\varepsilon$ and we get, for any g in F,

$$|\langle gv,w\rangle| \le |\langle g(v-v_i),w\rangle| + |\langle gv_i,w\rangle| \le \varepsilon\,\|w\| + \max_{1\le j\le p}|\langle gv_j,w\rangle|\,.$$

The lemma follows. □

Finally, we shall use a natural formula for the computation of certain integrals on G/Γ:

Lemma 4.3.5. *Let F be a locally compact topological group and let Λ be a lattice in F. Fix a Haar measure on F and equip F/Λ with the F-invariant measure μ associate with the Haar measure of F and the counting measure on Λ. Then, for any x in F/Λ, if B is a Borel subset of F such that the map $B \to F/\Lambda, g \mapsto gx$ is injective, we have, for any continuous compactly supported function θ on F/Λ,*

$$\int_{Bx} \theta \mathrm{d}\mu = \int_B \theta(gx)\mathrm{d}g.$$

Proof. First let us prove the result for $x = \Lambda$, the image of e in F/Λ. In this case, by the hypothesis on B, we have, for any g in F, if $y = g\Lambda$,

$$\sum_{\gamma\in\Lambda} \mathbf{1}_B(g\gamma)\theta(y) = \mathbf{1}_{B\Lambda}(y)\theta(y),$$

so that the formula comes from the definition of the measure μ.

Now, in the general case, chose h in F with $x = h\Lambda$ and note that, as μ is F-invariant, one has

$$\int_{Bx} \theta \mathrm{d}\mu = \int_{h^{-1}Bx} \theta(hy)\mathrm{d}\mu(y).$$

Since the map $B \to F/\Lambda, g \mapsto gx$ is injective, the map $h^{-1}Bh \to F/\Lambda, g \mapsto g\Lambda$ is injective and we get, from the case $x = \Lambda$,

$$\int_{Bx} \theta \mathrm{d}\mu = \int_{h^{-1}Bh} \theta(hg\Lambda)\mathrm{d}g = \int_{h^{-1}Bh} \theta(hgh^{-1}x)\mathrm{d}g.$$

The lemma follows since, F being unimodular, its Haar measure is invariant under conjugation by h. □

Proof of theorem 4.3.1. Let μ be the G-invariant probability of G/Γ. We will prove directly the equidistribution of orbits, that is, for any continuous function θ on G/Γ,

$$\frac{1}{T}\int_0^T \theta(u_t x)\mathrm{d}t \xrightarrow[T\to\infty]{} \int_{G/\Gamma} \theta \mathrm{d}\mu,$$

uniformly for x in G/Γ. Fix θ in $\mathcal{C}^0(\Gamma)$ and $\varepsilon > 0$. By uniform continuity, there exists a neighborhood W of e in G such that, for any x in G/Γ, for any w in W, one has $|\theta(wx) - \theta(x)| \leq \varepsilon$.

Now we claim that there exists a neighborhood W'' of e in G such that, for any x in G/Γ, the map $w \mapsto wx, W'' \to G/\Gamma$ is injective. Indeed, as the group Γ is discrete, there exists a neighborhood W' of e such that, for any γ in Γ, $\gamma \neq e$, one has $W' \cap W'\gamma = \emptyset$. Then, chose a compact set L in G such that the natural map $L \to G/\Gamma$ is surjective and set $W'' = \bigcap_{g \in L} gW'g^{-1}$. As L is compact, it is still a neighborhood of e. Let us prove that it is convenient. Let x be in G/Γ and chose g in L such that $x = g\Gamma$. Let w and w' be in W'' with $wx = w'x$. There exists γ in Γ such that $wg = w'g\gamma$, that is $g^{-1}wg = g^{-1}w'g\gamma$. As $g^{-1}wg$ and $g^{-1}w'g$ belong to W', we get $\gamma = e$ and $w = w'$, what should be proved.

Let us now remark that, for any Borel subset B of W'', for any $1 \leq p < \infty$, the map $\beta_p : G/\Gamma \to \mathrm{L}^p(G/\Gamma, \mu)$ which sends an element x of G/Γ to the characteristic function $\mathbf{1}_{Bx}$ of the set Bx is continuous. Indeed, since the right action of G on $\mathrm{L}^1(G)$ is continuous, the map $\tilde{\beta}_1 : G \to \mathrm{L}^1(G), g \mapsto \mathbf{1}_{Bg}$ is continuous. For any φ in $\mathrm{L}^1(G)$, for any $x = g\Gamma$ in G/Γ, set $\varpi(\varphi)(x) = \sum_{\gamma\in\Gamma} \varphi(g\gamma)$: the sum does not depend on the choice of g. By the construction of the measure μ, $\varpi(\varphi)(x)$ is defined for μ-almost every x in G/Γ and one has $\|\varpi(\varphi)\|_1 \leq \|\varphi\|_1$, so that the map $\varpi : \mathrm{L}^1(G) \to \mathrm{L}^1(G/\Gamma, \mu)$ is continuous. Now, as for any g in G and $\gamma \neq e$ in Γ one has $Bg\gamma \cap Bg = \emptyset$, we have $\beta_1(g\Gamma) = \varpi(\tilde{\beta}_1(g))$, so that the map β_1 is continuous. Finally, for any x in G/Γ, we have $\|\mathbf{1}_{Bx}\|_\infty \leq 1$, so that we get the continuity of β_p from the inequality $\|\psi\|_p \leq \|\psi\|_1^{\frac{1}{p}} \|\psi\|_\infty^{\frac{p-1}{p}}$, $\psi \in \mathrm{L}^\infty(G/\Gamma, \mu)$.

For any $\eta > 0$, set $U_\eta = \{u_t | 0 \leq t \leq \eta\}$, $V_\eta = \{v_t | 0 \leq t \leq \eta\}$ and $A_\eta = \{a_s | 0 \leq s \leq \eta\}$ and chose η sufficiently small so that $B_\eta = V_\eta A_\eta U_\eta \subset W \cap W''$. As $B_\eta \subset W''$, for any x in G/Γ, by lemma 4.3.5, $\mu(B_\eta x)$ is, up to a constant multiple, equal to the Haar measure of B_η. Set $\varphi_x = \frac{1}{\mu(B_\eta x)} \mathbf{1}_{B_\eta x}$, so that the map $x \mapsto \varphi_x, G/\Gamma \to \mathrm{L}^2(G/\Gamma, \mu)$ is continuous. As the image of this map is compact, since G/Γ is compact, we get, by Howe-Moore theorem and by lemma 4.3.4,

$$\int_{G/\Gamma} \varphi_x(a_{-s}y)\theta(y)\mathrm{d}\mu(y) \xrightarrow[s\to\infty]{} \int_{G/\Gamma} \theta \mathrm{d}\mu$$

uniformly for x in G/Γ. To finish the proof, we will now compute the integral $\int_{G/\Gamma} \varphi_x(a_{-s}y)\theta(y)\mathrm{d}\mu(y)$ and prove that it is very close to an integral of the form $\frac{1}{T}\int_0^T \theta(u_t a_s x)\mathrm{d}t$. Indeed, for any s in $\mathbb{R}$ and x in G/Γ, we have

$$\begin{aligned}\int_{G/\Gamma} \varphi_x(a_{-s}y)\theta(y)\mathrm{d}\mu(y) &= \int_{G/\Gamma} \varphi_x(y)\theta(a_s y)\mathrm{d}\mu(y)\\ &= \frac{1}{\mu(B_\eta x)}\int_{B_\eta x} \theta(a_s y)\mathrm{d}\mu(y).\end{aligned}$$

As the map $B_\eta \to G/\Gamma, g \mapsto gx$ is injective, by lemma 4.3.5, we get

$$\frac{1}{\mu(B_\eta x)}\int_{B_\eta x} \theta(a_s y)\mathrm{d}\mu(y) = \frac{1}{\int_{B_\eta}\mathrm{d}g}\int_{B_\eta} \theta(a_s g x)\mathrm{d}g.$$

As G is unimodular, its Haar measure is invariant under conjugation by a_s, so that

$$\frac{1}{\int_{B_\eta}\mathrm{d}g}\int_{B_\eta} \theta(a_s g x)\mathrm{d}g = \frac{1}{\int_{B_\eta}\mathrm{d}g}\int_{a_s B_\eta a_{-s}} \theta(g a_s x)\mathrm{d}g.$$

Now, we have $a_s B_\eta a_{-s} = V_{e^{-2s}\eta} A_\eta U_{e^{2s}\eta}$, so that, by lemma 4.3.3, we get

$$\begin{aligned}&\frac{1}{\int_{B_\eta}\mathrm{d}g}\int_{a_s B_\eta a_{-s}} \theta(g a_s x)\mathrm{d}g\\ &\qquad = \frac{1}{\eta^2\int_0^\eta e^{2\sigma}\mathrm{d}\sigma}\int_0^{e^{-2s}\eta}\int_0^\eta\int_0^{e^{2s}\eta} e^{2\sigma}\theta(v_r a_\sigma u_t a_s x)\mathrm{d}r\mathrm{d}\sigma\mathrm{d}t.\end{aligned}$$

For $s \geq 0$, we have $V_{e^{-2s}\eta}A_\eta \subset W$, so that, for any $0 \leq r \leq e^{-2s}\eta$, $0 \leq \sigma \leq \eta$ and $0 \leq t \leq e^{2s}\eta$, $|\theta(v_r a_\sigma u_t a_s x) - \theta(u_t a_s x)| \leq \varepsilon$ and hence, for any $s \geq 0$ and x in G/Γ,

$$\left|\int_{G/\Gamma} \varphi_x(a_{-s}y)\theta(y)\mathrm{d}\mu(y) - \frac{1}{e^{2s}\eta}\int_0^{e^{2s}\eta} \theta(u_t a_s x)\mathrm{d}t\right| \leq \varepsilon$$

Letting $s \to \infty$, we thus get

$$\limsup_{T\to\infty}\left|\int_{G/\Gamma} \theta\mathrm{d}\mu - \frac{1}{T}\int_0^T \theta(u_t x)\mathrm{d}t\right| \leq \varepsilon$$

uniformly for x in G/Γ. As this is true for any $\varepsilon > 0$, the proof is complete. $\square$

4.4 Exercices

4.4.1 Regular representations

Let G be a locally compact topological group and H be a closed subgroup of G such that the quotient space G/H admits an invariant Radon measure μ. Prove that the natural action of G in $\mathrm{L}^2(G/H)$ is an unitary representation.

4.4.2 The lattice $\mathrm{SL}_2(\mathbb{Z})$

Equip $\mathbb{R}^2$ with the usual scalar product and determinant.

1. Let Λ be a lattice in $\mathbb{R}^2$ and let (x, y) be a basis of Λ. Prove that $|\det(x, y)|$ does not depend on (x, y).

The quantity $|\det(x, y)|$, where (x, y) is a basis of Λ, is called the covolume of Λ.

2. Prove that the covolume of Λ is the total measure of $\mathbb{R}^2/\Lambda$ for the $\mathbb{R}^2$-invariant measure associate to the usual Lebesgue measure of $\mathbb{R}^2$.

3. Prove that the map $g \mapsto g\mathbb{Z}^2$ identifies $\mathrm{SL}_2(\mathbb{R})/\mathrm{SL}_2(\mathbb{Z})$ and the set of covolume 1 lattices in $\mathbb{R}^2$.

4. Let x be in $\mathbb{R}^2$ with $\|x\| > \left(\frac{4}{3}\right)^{\frac{1}{4}}$ and let y be in $\mathbb{R}^2$ with $\det(x, y) = 1$. Prove that there exists n in $\mathbb{Z}$ with $\|y + nx\| < \|x\|$.

For α, θ in $\mathbb{R}_+^*$, set

$$A_\alpha = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \middle| \, 0 < a \leq \alpha \right\} \text{ and } U_\theta = \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \middle| \, 0 \leq t \leq \theta \right\}.$$

5. Prove that $U\mathrm{SL}_2(\mathbb{Z}) = U_1\mathrm{SL}_2(\mathbb{Z})$ and that, for $\alpha = \left(\frac{4}{3}\right)^{\frac{1}{4}}$, one has $\mathrm{SL}_2(\mathbb{R}) = KA_\alpha U\mathrm{SL}_2(\mathbb{Z}) = KA_\alpha U_1\mathrm{SL}_2(\mathbb{Z})$.

6. Prove the following integral formula for the Iwasawa decomposition: after a suitable renormalization of the Haar measures on K and G, one has, for every θ in $\mathcal{C}_c^0(G)$,

$$\int_G \theta(g)\mathrm{d}g = \int_{K\times\mathbb{R}\times\mathbb{R}} e^{2s}\theta(ka_s u_t)\mathrm{d}k\mathrm{d}s\mathrm{d}t.$$

7. A topological space X is said to be σ-compact if there exists a sequence $(K_n)_{n\in\mathbb{N}}$ of compact subsets of X with $X = \bigcup_{n\in\mathbb{N}} K_n$. Let F be a unimodular locally compact and σ-compact topological group and Λ be a

discrete subgroup of F. Prove that Λ is a lattice of F if and only if there exists a Borel subset B of F with finite Haar measure such that $F = B\Lambda$.

8. Prove that $\mathrm{SL}_2(\mathbb{Z})$ is a lattice of $\mathrm{SL}_2(\mathbb{R})$.

9. Prove that, for every $\varepsilon > 0$, there exists x in $\mathrm{SL}_2(\mathbb{R})/\mathrm{SL}_2(\mathbb{Z})$ with $u_\varepsilon x = x$.

10. Prove that $\mathrm{SL}_2(\mathbb{Z})$ is not cocompact in $\mathrm{SL}_2(\mathbb{R})$.

4.4.3 A cocompact lattice in $\mathrm{SL}_2(\mathbb{R})$

1. Let X, Y, Z be in $\mathbb{Q}$ such that $X^2 - 2Y^2 - 3Z^2 = 0$. Prove that $X = Y = Z = 0$ (suppose first X, Y, Z are in $\mathbb{Z}$ and relatively prime, then prove that 3 divides both X and Y).

For any $v = (x, y, z, t)$ in $\mathbb{R}^4$, set

$$g_v = \begin{pmatrix} x + y\sqrt{2} & \sqrt{3}(z - t\sqrt{2}) \\ \sqrt{3}(z + t\sqrt{2}) & x - y\sqrt{2} \end{pmatrix}.$$

2. Let $v = (x, y, z, t)$ be in $\mathbb{R}^4$. Prove that, if $\det(g_v) = 0$, one has $X^2 - 2Y^2 - 3Z^2 = 0$, with $X = xz + 2yt$, $Y = xt + yz$ and $Z = z^2 - 2t^2$ (multiply by $z^2 - 2t^2$).

Let $A = \{g_v | v \in \mathbb{Z}^4\}$ and $\Gamma = \{a \in A | \det(a) = 1\}$.

3. Prove that, for any $v \neq 0$ in $\mathbb{Q}^4$, g_v belongs to $\mathrm{GL}_2(\mathbb{R})$, that A is stable by matrix product and that Γ is a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$.

4. Minkowski theorem. Let λ be the Lebesgue measure on $\mathbb{R}^d$. Prove that, if Λ is a lattice of covolume 1 in $\mathbb{R}^d$ and if C is a symmetric closed convex subset of $\mathbb{R}^d$ with $\lambda(C) > 2^d$, then C contains a non zero element of Λ (note that the projection map $\frac{1}{2}C \to \mathbb{R}^d/\Lambda$ is not injective).

5. Prove that there exists a compact subset C of the space $\mathcal{M}_2(\mathbb{R})$ of real 2 by 2 square matrices such that, for any g in $\mathrm{SL}_2(\mathbb{R})$, there exists a in A, $a \neq 0$, with $ga \in C$.

6. Let g be in $\mathcal{M}_d(\mathbb{Z})$ with $\det(g) = n \neq 0$. Prove that there exists γ in $\mathrm{SL}_d(\mathbb{Z})$ such that $h = g\gamma$ satisfies $h_{1,j} = 0$ for any $2 \leq j \leq d$ (note that, for any $1 \leq i \neq j \leq d$, the matrix $1 + e_{i,j}$ belongs to $\mathrm{SL}_d(\mathbb{Z})$, where $(e_{i,j})$ is the canonical basis of $\mathcal{M}_d(\mathbb{Z})$). Prove, by induction on $d \geq 1$, that there exists η in $\mathrm{SL}_d(\mathbb{Z})$ such that $k = g\eta$ satisfies $k_{i,j} = 0$ for any $1 \leq i < j \leq d$ and $|k_{i,j}| < |k_{i,i}|$ for any $1 \leq j < i \leq d$.

7. Prove that, for any n in $\mathbb{Z}$, there exists a finite subset F of $A_n = \{a \in A | \det(a) = n\}$ with $A_n = F\Gamma$ (consider multiplication by a fixed element of A_n as an endomorphism of the abelian group A, which is isomorphic to $\mathbb{Z}^4$).

8. Prove that Γ is a cocompact lattice in $\mathrm{SL}_2(\mathbb{R})$.

4.4.4 Ratner theorem for $\mathrm{SL}_2(\mathbb{R})$

In this exercice, we will prove Ratner theorem for the classification of invariant measures of unipotent actions in the case of $\mathrm{SL}_2(\mathbb{R})$, that is, we will prove that, if Γ is a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$, if μ is an ergodic invariant probability measure for the flow $(L_{u_t})_{t\in\mathbb{R}}$ on $\mathrm{SL}_2(\mathbb{R})/\Gamma$, either there exists x in $\mathrm{SL}_2(\mathbb{R})/\Gamma$ which is periodic for $(L_{u_t})_{t\in\mathbb{R}}$ and such that $\mu(Ux)=1$, or Γ is a lattice and μ is the $\mathrm{SL}_2(\mathbb{R})$-invariant probability.

1. Let $g=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$ be in $\mathrm{SL}_2(\mathbb{R})$. Prove that, for any t such that $\alpha+t\gamma\neq 0$, one has the Bruhat decomposition

$$u_t g = v_{\frac{\gamma}{\alpha+t\gamma}} a_{\alpha+t\gamma} u_{\frac{\beta+t\delta}{\alpha+t\gamma}}.$$

2. Suppose $\gamma\neq 0$. Let p and q be real numbers such that $-\alpha\notin[p,q]$. Prove that, for φ in $\mathcal{C}^0(\mathbb{R})$, one has

$$\int_{\frac{p}{\gamma}}^{\frac{q}{\gamma}}\varphi\left(\frac{\beta+t\delta}{\alpha+t\gamma}\right)\mathrm{d}t=\int_{\frac{\beta+p\frac{\delta}{\gamma}}{\alpha+p}}^{\frac{\beta+q\frac{\delta}{\gamma}}{\alpha+q}}\varphi(u)\frac{\mathrm{d}u}{(\gamma u-\delta)^2}.$$

Let Γ be a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$ and let μ be an ergodic invariant probability measure for the flow $(L_{u_t})_{t\in\mathbb{R}}$ on $\mathrm{SL}_2(\mathbb{R})/\Gamma$.

3. Suppose there exists x in $\mathrm{SL}_2(\mathbb{R})/\Gamma$ with $\mu(Ux)\neq 0$. Prove that x is U periodic and that μ is the unique $(L_{u_t})_{t\in\mathbb{R}}$-invariant probability measure with $\mu(Ux)=1$

We suppose now that, for any x in $\mathrm{SL}_2(\mathbb{R})/\Gamma$, we have $\mu(Ux)=0$ and we will prove that μ is $\mathrm{SL}_2(\mathbb{R})$-invariant.

3. Prove that, for any x in $\mathrm{SL}_2(\mathbb{R})/\Gamma$, we have $\mu(Px)=0$.

4. Let X be a Borel subset of $\mathrm{SL}_2(\mathbb{R})/\Gamma$ such that $\mu(X)>0$. Prove that there exists x in X and a sequence $\left(g_n=\begin{pmatrix}\alpha_n & \beta_n\\ \gamma_n & \delta_n\end{pmatrix}\right)$ of elements of $\mathrm{SL}_2(\mathbb{R})$ such that $g_n \xrightarrow[n\to\infty]{} e$ and that, for any n, one has $g_n x\in X$ and $\gamma_n\neq 0$.

5. Prove that μ is A-invariant (keep in mind the proof of unique ergodicity for Heisenberg flows).

7. Prove that the flow $(L_{a_s})_{s\in\mathbb{R}}$ is ergodic with respect to μ (remind the proof of Howe-Moore theorem).

For θ in $\mathcal{C}_c^0(\mathrm{SL}_2(\mathbb{R})/\Gamma)$ we set, for any x in $\mathrm{SL}_2(\mathbb{R})/\Gamma$,

$$\hat{\theta}(x) = \liminf_{S\to\infty} \frac{1}{S}\int_0^S \theta(a_{-s}x)\mathrm{d}s.$$

8. Prove that, for any θ in $\mathcal{C}_c^0(\mathrm{SL}_2(\mathbb{R})/\Gamma)$, t in $\mathbb{R}$ and x in $\mathrm{SL}_2(\mathbb{R})/\Gamma$, one has $\hat{\theta}(v_t x) = \hat{\theta}(x)$.

9. Let X be a Borel subset of $\mathrm{SL}_2(\mathbb{R})/\Gamma$ such that $\mu(X) = 1$. Prove that, for μ-almost every x in $\mathrm{SL}_2(\mathbb{R})/\Gamma$, for Lebesgue-almost every (s,t) in $\mathbb{R}^2$, one has $a_s u_t x \in X$.

We let λ be a Haar measure on $\mathrm{SL}_2(\mathbb{R})$ and $\overline{\lambda}$ be the associate $\mathrm{SL}_2(\mathbb{R})$-invariant measure on $\mathrm{SL}_2(\mathbb{R})/\Gamma$.

10. Let X be a Borel subset of $\mathrm{SL}_2(\mathbb{R})$ such that, for any t in $\mathbb{R}$, $v_t X = X$. Suppose there exists g in X such that, for Lebesgue-almost every (s,t) in $\mathbb{R}^2$, one has $a_s u_t g \in X$. Prove that $\lambda(VAUg - X) = 0$.

11. Prove that, for any θ in $\mathcal{C}_c^0(\mathrm{SL}_2(\mathbb{R})/\Gamma)$, there exists x in $\mathrm{SL}_2(\mathbb{R})/\Gamma$ such that $\hat{\theta}$ is constant $\overline{\lambda}$-almost everywhere on $VAUx$.

Recall Fatou lemma: if $(M, \mathcal{A}, \nu)$ is a measure space and (f_n) a sequence of positive measurable functions on M, one has

$$\int_M \liminf_{n\to\infty} f_n \mathrm{d}\nu \leq \liminf_{n\to\infty} \int_M f_n \mathrm{d}\nu.$$

11. Prove that there exists x in $\mathrm{SL}_2(\mathbb{R})/\Gamma$ such that $\overline{\lambda}(VAUx) < \infty$. Deduce that Γ is a lattice in G.

After normalizing, we suppose now $\overline{\lambda}(\mathrm{SL}_2(\mathbb{R})/\Gamma) = 1$.

12. Prove that, for any θ in $\mathcal{C}_c^0(\mathrm{SL}_2(\mathbb{R})/\Gamma)$, one has

$$\hat{\theta} = \int_{\mathrm{SL}_2(\mathbb{R})/\Gamma} \theta \mathrm{d}\overline{\lambda}$$

$\overline{\lambda}$-almost everywhere on $\mathrm{SL}_2(\mathbb{R})/\Gamma$.

13. Prove that $\mu = \overline{\lambda}$.

Appendix A

Basic elements of functional analysis

We will remind there the notions of functional analysis that will be necessary to the full understanding of the course.

All the vector spaces are assumed to be complex vector spaces.

A.1 Compact operators

If X is a metric space space, x belongs to X and r is a positive real number, we shall denote by $B_X(x, r)$ (or $B(x, r)$ when there are no ambiguities) the closed ball with center x and radius r in X.

Definition A.1.1. Let E and F be Banach spaces. A continuous linear operator $T : E \to F$ is said to be compact if the set $TB_E(0, 1)$ is relatively compact in F, that is if its closure $\overline{TB_E(0, 1)}$ in F is compact.

Example A.1.2. As the unit ball of a finite dimensional Banach space is compact, if T has finite rank it is compact. Equip the spaces $\mathcal{C}^0([0, 1])$ and $\mathcal{C}^1([0, 1])$ with their natural structures of Banach spaces. Then, by Ascoli theorem, the natural map $\mathcal{C}^1([0, 1]) \to \mathcal{C}^0([0, 1])$ is a compact operator.

Proposition A.1.3. *Let E and F be Banach spaces and let $\mathcal{L}(E, F)$ be the Banach space of continuous linear operators $E \to F$. Then the set $\mathcal{K}(E, F)$ of compact operators $E \to F$ is a closed sub-vector space of $\mathcal{L}(E, F)$.*

The proof that $\mathcal{K}(E, F)$ is closed relies on the

Lemma A.1.4. *Let (X, d) be a complete metric space. Suppose, for any $r > 0$, X is a finite union of ball with radius r, that is there exists $x_1, \ldots, x_p$ in X with $X = B(x_1, r) \cup \ldots \cup B(x_p, r)$. Then X is compact.*

Proof. We use the Bolzano-Weierstrass criterion for compacity, so that we have to prove that every sequence $(y_n)_{n\in\mathbb{N}}$ of elements of X has a converging subsequence. Let $(y_n)_{n\in\mathbb{N}}$ be such a sequence. We will construct, by induction on k, a increasing sequence $(n_k)_{k\in\mathbb{N}}$ of nonnegative integers with the following properties:

(i) for any $0 \leq l \leq k$, one has $d(y_{n_k}, y_{n_l}) \leq \frac{1}{l+1}$.

(ii) for any $k \geq 0$, the set of n in $\mathbb{N}$ with, for any $0 \leq l \leq k$, $d(y_{n_l}, y_n) \leq \frac{1}{l+1}$ is infinite.

Suppose this subsequence is constructed. Then, for any $0 \leq l \leq k$, we have $d(y_{n_k}, y_{n_l}) \leq \frac{1}{l+1}$, so that the sequence $(y_{n_k})_{k\in\mathbb{N}}$ is a Cauchy sequence. As X is complete, this sequence is convergent, that is $(y_n)_{n\in\mathbb{N}}$ admits a convergent subsequence and we are done.

Let us now construct $(n_k)_{k\in\mathbb{N}}$. For $k = 0$, chose $x_1, \ldots, x_p$ in X such that one has $X = B(x_1, \frac{1}{2}) \cup \ldots \cup B(x_p, \frac{1}{2})$. Then, there exists $1 \leq i \leq p$ such that the set of $n \in \mathbb{N}$ with $d(y_n, x_i) \leq \frac{1}{2}$ is infinite. Let n_0 be any integer with $y_{n_0} \in B(x_i, \frac{1}{2})$. Then, for any n with $y_n \in B(x_i, \frac{1}{2})$, we have $d(y_{n_0}, y_n) \leq d(y_{n_0}, x_i) + d(x_i, y_n) \leq 1$, so that the set of $n \in \mathbb{N}$ with $d(y_{n_0}, y_n) \leq 1$ is infinite and n_0 satisfies our requirements. Now suppose $n_0, \ldots, n_k$ are constructed. Pick $x_1, \ldots, x_p$ in X with $X = B\left(x_1, \frac{1}{2(k+1)}\right) \cup \ldots \cup B\left(x_p, \frac{1}{2(k+1)}\right)$. By induction, the set of n in $\mathbb{N}$ with, for any $0 \leq l \leq k$, $d(y_{n_l}, y_n) \leq \frac{1}{l+1}$ is infinite. Hence, there exists some $1 \leq i \leq p$ such that the set of n in $\mathbb{N}$ with $d(y_n, x_i) \leq \frac{1}{2(k+2)}$ and, for any $0 \leq l \leq k$, $d(y_{n_l}, y_n) \leq \frac{1}{l+1}$ is infinite. Let n_{k+1} be any element which is $> n_k$ in this set. Then n_{k+1} satisfies the requirements of the induction. The construction follows. □

Proof of proposition A.1.3. First, let us prove that $\mathcal{K}(E, F)$ is a sub-vector space. As the multiplication of a compact operator by a scalar is clearly a compact operator, it suffices to prove that the sum of two compact operators is a compact operator. So, let T and S be compact operators $E \to F$. As the sum $F \times F \to F$ is a continuous map, we have

$$\overline{(T+S)B_E(0,1)} \subset \overline{TB_E(0,1) + SB_E(0,1)} \subset \overline{TB_E(0,1)} + \overline{SB_E(0,1)}$$

and, still by continuity of the addition in F, the set $\overline{TB_E(0,1)} + \overline{SB_E(0,1)}$ is compact in F. Now, $\overline{(T+S)(B_E(0,1))}$, being a closed subset of a compact set, is compact and $T+S$ is a compact operator.

Let now T belong to the closure of $\mathcal{K}(E,F)$ and let us prove that T is compact. By lemma A.1.4, it suffices to prove that, for any $r>0$, there exists $y_1, \ldots, y_p$ in $\overline{TB_E(0,1)}$ with

$$\overline{TB_E(0,1)} \subset B_F(y_1,r) \cup \ldots \cup B_F(y_p,r).$$

Let $r>0$ be given. Then, there exists a compact operator $S : E \to F$ with $\|T-S\| \leq r$. As S is compact, there exists $y_1, \ldots, y_p$ in $\overline{SB_E(0,1)}$ with

$$\overline{SB_E(0,1)} \subset B_F(y_1,r) \cup \ldots \cup B_F(y_p,r).$$

For any $1 \leq i \leq p$, there exists x_i in $B_E(0,1)$ with $\|Sx_i - y_i\| \leq r$, so that we have

$$\overline{SB_E(0,1)} \subset B_F(Sx_1,2r) \cup \ldots \cup B_F(Sx_p,2r).$$

Now, let x be in $B_E(0,1)$ and pick $1 \leq i \leq p$ with $\|Sx - Sx_i\| \leq 2r$. We have

$$\|Tx - Tx_i\| \leq \|Tx - Sx\| + \|Sx - Sx_i\| + \|Sx_i - Tx_i\| \leq 4r.$$

Hence, we get

$$TB_E(0,1) \subset B_F(Tx_1,4r) \cup \ldots \cup B_F(Tx_p,4r).$$

As the latter is a finite union of closed balls, it is a closed set, so that we have

$$\overline{TB_E(0,1)} \subset B_F(Tx_1,4r) \cup \ldots \cup B_F(Tx_p,4r)$$

and we are done. □

A.2 Weak-* topology

We shall use the following fact from general topology:

Lemma A.2.1. *Let X be a set and let Y be a topological space. If $\mathcal{F}$ is a set of maps $X \to Y$, there exists a smallest topology on X that makes all the elements of $\mathcal{F}$ continuous. If Z is another topological space, a map $g : Z \to X$ is continuous for this topology if and only if, for any f in $\mathcal{F}$, the map $f \circ g : Z \to Y$ is continuous.*

Proof. Let $\mathcal{T}$ be the set of subsets U of X which may be written

$$U = f_1^{-1}V_1 \cap \ldots \cap f_p^{-1}V_p$$

where $f_1, \ldots, f_p$ belong to $\mathcal{F}$ and $V_1, \ldots, V_p$ are open subsets of Y. Then, one checks easily that $\mathcal{T}$ is a topology on X. As any topology $\mathcal{S}$ on X making the elements of $\mathcal{F}$ continuous necessarily contains the sets of this form, $\mathcal{T}$ is the smallest topology making the elements of $\mathcal{F}$ continuous. The last setting is easy. $\square$

If E is a Banach space, we let E^* be its (topological) dual space, that is the space of continuous linear forms on E, equipped with its natural structure of a Banach space. If F is another Banach space and $T : E \to F$ a continuous linear map, we denote by T^* its adjoint, that is the continuous linear map $F^* \to E^*, \varphi \mapsto \varphi \circ T$.

Definition A.2.2. Let E be a Banach space. The weak-* topology on E^* is the smallest topology on E^* that makes the maps $E^* \to \mathbb{R}, \varphi \mapsto \varphi(x)$, $x \in E$, continuous.

Let us state some elementary properties of this topology.

Lemma A.2.3. *Let E be a Banach space. Then the weak-* topology is a Hausdorff topology on E^*, that makes all the operations of this vector space continuous. If F is another Banach space and $T : E \to F$ a continuous linear map, the adjoint map $T^* : F^* \to E^*$ is continuous with respect to the weak-* topologies of E^* and F^*.*

Proof. Let φ and ψ be in E^* with $\varphi \neq \psi$. Then, there exists x in E with $\varphi(x) \neq \psi(x)$. Set

$$U = \left\{ \theta \in E^* \middle| |\theta(x) - \varphi(x)| < \frac{1}{2} |\varphi(x) - \psi(x)| \right\}$$

and

$$V = \left\{ \theta \in E^* \middle| |\theta(x) - \psi(x)| < \frac{1}{2} |\varphi(x) - \psi(x)| \right\}.$$

Then U and V are disjoint weak-* open subsets of E^* and one has $\varphi \in U$ and $\psi \in V$. Thus, the weak-* topology is Hausdorff.

Consider now the addition map $E^* \times E^* \to E^*$. For any x in E, the map

$$\begin{aligned} E^* \times E^* &\to E^* \\ (\varphi, \psi) &\mapsto (\varphi + \psi)(x) = \varphi(x) + \psi(x) \end{aligned}$$

is continuous for the product weak-* topology on $E^* \times E^*$. Hence, the addition map is weak-* continuous $E^* \times E^* \to E^*$. In the same way, we prove continuity of the scalar multiplication map $\mathbb{C} \times E^* \to E^*$.

Lastly, let F and T be as in the setting. Then, for any x in E, the map

$$\begin{aligned} F^* &\to \mathbb{C} \\ \varphi &\mapsto (T^*\varphi)(x) = \varphi(Tx) \end{aligned}$$

is continuous, so that $T^* : F^* \to E^*$ is weak-* continuous. $\square$

Lemma A.2.4. *Let E be a separable Banach space. Then the restriction of the weak-* topology of E^* to the unit ball $B_{E^*}(0,1)$ of E^* is metrizable.*

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a dense sequence in the unit ball $B_E(0,1)$ of E. Then, for any φ and ψ in $B_{E^*}(0,1)$, set

$$d(\varphi, \psi) = \sum_{n=0}^{\infty} 2^{-n} |\varphi(x_n) - \psi(x_n)|$$

(the series converges since its general term is bounded by $2^{-n} \|\varphi - \psi\|$). We claim that this is a distance function that induces the weak-* topology on $B_{E^*}(0,1)$.

Indeed, the symmetry and the triangle identity are evident. For the separation, suppose that φ and ψ in $B_{E^*}(0,1)$ are such that $d(\varphi, \psi) = 0$. Then, for any n, one has $\varphi(x_n) = \psi(x_n)$. As the $(x_n)_{n\in\mathbb{N}}$ are dense in $B(0,1)$, one has therefore $\varphi = \psi$ on $B(0,1)$ and, hence, $\varphi = \psi$, what should be proved.

Finally, let us prove that the topology $\mathcal{T}$ induced by this distance is the weak-* topology of $B_{E^*}(0,1)$. First, let us prove that $\mathcal{T}$ is more fine than the weak-* topology. Let x be in $B_E(0,1)$ and chose $\varepsilon > 0$. There exists n in $\mathbb{N}$ with $\|x - x_n\| \le \varepsilon$. Let φ and ψ be in $B_{E^*}(0,1)$ such that $d(\varphi, \psi) \le \varepsilon 2^{-n}$. We then have

$$|\varphi(x) - \psi(x)| \le |\varphi(x - x_n)| + |\varphi(x_n) - \psi(x_n)| + |\psi(x_n - x)| \le 3\varepsilon,$$

so that the map $B_{E^*}(0,1) \to \mathbb{C}, \varphi \mapsto \varphi(x)$ is continuous for the topology $\mathcal{T}$. Hence, the weak-* topology is smaller than $\mathcal{T}$. Conversely, to prove that the topology $\mathcal{T}$ is smaller than the weak-* topology, we have to prove that, for any φ in $B_{E^*}(0,1)$, the function $\psi \mapsto d(\varphi, \psi), B_{E^*}(0,1) \to \mathbb{R}_+$ is weakly continuous. But, for the weak-* topology, this function is the sum of a series of continuous functions which converges normally on $B_{E^*}(0,1)$. Hence it is continuous and the result follows. □

The main result we shall use on the weak-* topology is the Banach-Alaoglu theorem:

Theorem A.2.5. *Let E be a Banach space. Then the unit ball $B_{E^*}(0,1)$ is compact for the weak-* topology of E.*

Proof. Consider the product topological space $P = \prod_{x \in E}[-\|x\|, \|x\|]$. By Tychonoff theorem, this is a compact space. The natural map $B_{E^*}(0,1) \to P, \varphi \mapsto (\varphi(x))_{x \in E}$ induces a bijection between $B_{E^*}(0,1)$ and the closed subset of P

$$Q = \{(p_x)_{x \in E} \in P | \forall x, y \in E \quad \forall \lambda \in \mathbb{C} \quad p_{x+\lambda y} = p_x + \lambda p_y\}.$$

One easily checks that this bijection is an homeomorphism, when $B_{E^*}(0,1)$ is equipped with the weak-* topology and Q with the topology induced by the product topology. The theorem follows. □

A.3 Hilbert spaces

Let us now recall the basic tools of Hilbert spaces theory.

Definition A.3.1. Let E be a vector space. A hermitian sesquilinear form on E is a map $E \times E \to \mathbb{C}, (x, y) \mapsto \langle x, y \rangle$ such that, for any x, y, z in E and λ in $\mathbb{C}$, one has

(i) $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$.

(ii) $\langle x, y + z \rangle = \langle x, y \rangle + \langle x, z \rangle$.

(iii) $\langle \lambda x, y \rangle = \overline{\lambda} \langle x, y \rangle$.

(iv) $\langle x, \lambda y \rangle = \lambda \langle x, y \rangle$.

(v) $\langle y, x\rangle = \overline{\langle x, y\rangle}$.

It is said to be a hermitian scalar product if it is positive definite, that is if, for any x in E, $x \neq 0$, one has $\langle x, x\rangle > 0$.

One immediately gets the Cauchy-Schwarz inequality:

Proposition A.3.2. *Let E be a vector space equipped with a hermitian scalar product. For any x, y in E, one has*

$$|\langle x, y\rangle|^2 \leq \langle x, x\rangle\langle y, y\rangle,$$

where equality holds if and only if x and y are colinear vectors.

Proof. Chose α in $\mathbb{C}$ with $|\alpha| = 1$ such that $\langle x, \alpha y\rangle \in \mathbb{R}$ and note that, after replacing y by αy, it suffices to prove the proposition when $\langle x, y\rangle$ is a real number. We can also suppose that y is not zero, the case $y = 0$ being trivial.

In this case, for any t in $\mathbb{R}$, one has

$$0 \leq \langle x + ty, x + ty\rangle = \langle x, x\rangle + 2t\langle x, y\rangle + t^2\langle y, y\rangle,$$

so that this quadratic polynomial has a nonpositive discriminant, that is

$$4\langle x, y\rangle^2 - 4\langle x, x\rangle\langle y, y\rangle \leq 0,$$

which proves the inequality. In the equality case, the polynomial has a root, that is there exists t with $\langle x + ty, x + ty\rangle = 0$ and, thus, $x = -ty$ and x and y are colinear, what should be proved. □

From Cauchy-Schwarz inequality, we deduce Minkowski inequality:

Corollary A.3.3. *For any x, y in E, one has*

$$\sqrt{\langle x + y, x + y\rangle} \leq \sqrt{\langle x, x\rangle} + \sqrt{\langle y, y\rangle},$$

where equality holds if and only if x and y are positively colinear vectors.

Proof. From Cauchy-Schwarz inequality, one gets

$$\langle x + y, x + y\rangle = \langle x, x\rangle + 2\Re(\langle x, y\rangle) + \langle y, y\rangle \leq \left(\sqrt{\langle x, x\rangle} + \sqrt{\langle y, y\rangle}\right)^2$$

and equality holds if and only if one has both equality in Cauchy-Schwarz inequality and $\langle x, y\rangle \in \mathbb{R}_+$, that is, if and only if x and y are positively colinear. □

By Minkowski inequality, the map $x \mapsto \sqrt{\langle x, x\rangle}$ is a norm on E. We shall now denote it by $\|.\|$. By Cauchy-Schwarz inequality, the scalar product is a continuous map $E \times E \to E$ for the topologies induced by the norm $\|.\|$.

Definition A.3.4. A Hilbert space is a vector space H equipped with a hermitian scalar product such that the associate norm makes H a Banach space.

Example A.3.5. If $(X, \mathcal{A}, \mu)$ is a measure space, the usual hermitian product induces a Hilbert space structure on $\mathrm{L}^2(X, \mathcal{A}, \mu)$.

The basic properties of Hilbert spaces rely on the

Proposition A.3.6. *Let H be a Hilbert space and let $C \subset H$ be a closed convex set. Then, for any x in H, there exists an unique y in C with $\|x - y\| = \inf_{z\in C} \|x - z\|$.*

Proof. To simplify the proof, let us suppose $x = 0$. Let $(y_n)_{n\in\mathbb{N}}$ be any sequence of elements of C such that $\|y_n\| \xrightarrow[n\to\infty]{} \inf_{z\in C} \|z\|$. We will prove that $(y_n)_{n\in\mathbb{N}}$ is Cauchy: this implies both existence and uniqueness of y. By a direct computation, we have, for any n and m,

$$\|y_n - y_m\|^2 + \|y_n + y_m\|^2 = 2\|y_n\|^2 + 2\|y_m\|^2,$$

so that, as $\frac{y_n+y_m}{2}$ belongs to C,

$$\begin{aligned}\|y_n - y_m\|^2 &= 2\|y_n\|^2 + 2\|y_m\|^2 - 4\left\|\frac{y_n + y_m}{2}\right\|^2 \\ &\le 2\|y_n\|^2 + 2\|y_m\|^2 - 4\inf_{z\in C}\|z\|^2.\end{aligned}$$

Hence the sequence is Cauchy and the proposition follows. □

From this proposition, we get a result about orthogonal decompositions:

Corollary A.3.7. *Let H be a Hilbert space and let K be a closed subspace of H. Then one has $H = K \oplus K^\perp$, where $K^\perp$ is the orthogonal space to K, that is the space of y in H such that $\langle x, y\rangle = 0$ for any x in K.*

Proof. Let x be in H and let y be, as in proposition A.3.6, the unique element of K such that $\|x - y\| = \inf_{z\in K} \|x - z\|$. We will prove that $x - y$ belongs to $K^\perp$: this will imply the result. Indeed, for any z in K, we have

$$\|x - y\|^2 \le \|x - (y - z)\|^2 = \|x - y\|^2 + 2\Re\langle x - y, z\rangle + \|z\|^2,$$

so that

$$2\Re\langle x - y, z\rangle + \|z\|^2 \geq 0.$$

Replacing z by tz, we get, for any real number t,

$$2t\Re\langle x - y, z\rangle + t^2 \|z\|^2 \geq 0.$$

Hence, $\Re\langle x-y, z\rangle = 0$. As we also have $\Re\langle x-y, iz\rangle = 0$, we get $\langle x-y, z\rangle = 0$, what should be proved. □

This last result implies in particular the Riesz theorem on linear forms:

Corollary A.3.8. *Let H be a Hilbert space. The map that associates to each x in H the continuous linear form $y \mapsto \langle x, y\rangle$ on H is a continuous anti-linear isomorphism from H onto its topological dual space H^*.*

In the sequel, we shall always identify a Hilbert space with its topological dual space through this map.

Proof. Since, for any $x \neq 0$ in H, one has $\langle x, x\rangle > 0$, this map is injective. Let us prove that it is surjective. Let φ be a non zero continuous linear form on H and let K be its kernel. As K is a closed subspace of H, we have $H = K \oplus K^\perp$. As K has codimension one, we have $K^\perp = \mathbb{C}x$ for some $x \neq 0$ in H. As the linear forms φ and $y \mapsto \langle x, y\rangle$ have the same kernel, they are proportional, so that there exists λ in $\mathbb{C}$ such that φ is the linear form $y \mapsto \langle \lambda x, y\rangle$, what should be proved. □

To conclude this introduction to Hilbert spaces, let us say a few words about weak topologies.

Definition A.3.9. Let H be a Hilbert space. Then the weak topology of H is the smallest topology making the linear forms $y \mapsto \langle x, y\rangle$, for x in H, continuous.

Of course, one can define weak topologies in the general context of Banach spaces as the smallest topologies making all the continuous linear forms continuous. But they would be more delicate to handle. For example, to prove that they are Hausdorff, we would need to use the Hahn-Banach theorem. In the context of Hilbert spaces, as the space identifies with its dual, the weak topology may be viewed as a weak-* topology, so that we get, by the preceding section, the

Proposition A.3.10. *Let H be a Hilbert space. Then the weak topology of H is Hausdorff and makes the addition and the scalar multiplication of H continuous. The closed unit ball of H is compact for the weak topology and, if H is separable, it is metrizable.*

Proof. This is the translation of the results of the section on weak-* topologies, thanks to the identification of H with its topological dual space. $\square$

We shall use this result in the following form.

Corollary A.3.11. *Let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence of elements of H. Then $(x_n)_{n\in\mathbb{N}}$ admits a subsequence which is weakly converging.*

Example A.3.12. Let $\ell^2(\mathbb{N})$ be the Hilbert space of square summable sequences of integers. For any n in $\mathbb{N}$, denote by $\mathbf{1}_n$ the element of $\ell^2(\mathbb{N})$ with value 1 at n and 0 anywhere else. Then the sequence $(\mathbf{1}_n)_{n\in\mathbb{N}}$ weakly converges to 0.

Proof. Consider the closed subspace K of H spanned by the $(x_n)_{n\in\mathbb{N}}$. This is a separable Hilbert space and the sequence $(x_n)_{n\in\mathbb{N}}$ is contained in a closed ball B of K. As B is weakly compact and metrizable, the sequence $(x_n)_{n\in\mathbb{N}}$ admits a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ which is weakly converging to some x in K. Now, every y in H may be written $y = u + v$ where u belongs to K and v is orthogonal to K. Thus we get,

$$\langle x_{n_k}, y\rangle = \langle x_{n_k}, u\rangle \xrightarrow[k\to\infty]{} \langle x, u\rangle = \langle x, y\rangle,$$

that is $(x_{n_k})_{k\in\mathbb{N}}$ weakly converges to x in H. $\square$

A.4 Compact self-adjoint operators

We shall use all the general results we have proved in this introduction to establish the result on compact self-adjoint operators that was used in the proof of proposition 1.2.14.

Definition A.4.1. Let H and K be Hilbert spaces and let $T : H \to K$ be a continuous linear operator. Then the adjoint of T is the unique continuous linear operator $T^* : K \to H$ such that, for any x in H and y in K, one has $\langle Tx, y\rangle = \langle x, T^*y\rangle$.

The existence of the adjoint follows directly from the identification of the spaces with their topological dual spaces, thanks to Riesz theorem.

Example A.4.2. Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be σ-finite measure spaces and let F be in $\mathrm{L}^2(X \times Y, \mathcal{A} \otimes \mathcal{B}, \mu \otimes \nu)$. For any f in $\mathrm{L}^2(X, \mathcal{A}, \mu)$ and for ν-almost every y in Y, set

$$Tf(y) = \int_X f(x)F(x,y)\mathrm{d}\mu(x).$$

Then the adjoint operator T^* of T satisfies, for any g in $\mathrm{L}^2(Y, \mathcal{B}, \nu)$ and for μ-almost every x in X,

$$T^*g(x) = \int_Y g(y)\overline{F(x,y)}\mathrm{d}\nu(y).$$

The operator T is compact (see lemma 1.2.17).

Definition A.4.3. Let H be a Hilbert space and let T be a continuous linear operator $H \to H$. Then T is said to be self-adjoint if $T^* = T$, that is if, for any x, y in H, one has $\langle Tx, y\rangle = \langle x, Ty\rangle$.

Note that, if T is self-adjoint, for any x, one has $\langle Tx, x\rangle \in \mathbb{R}$. In particular, the eigenvalues of T belong to $\mathbb{R}$ (more generally, one can prove that the spectrum of T is contained in $\mathbb{R}$).

Theorem A.4.4. *Let H be a Hilbert space and let T be a compact self-adjoint continuous linear operator $H \to H$. Then one has the Hilbert sum decomposition*

$$H = \bigoplus_{\lambda \in \mathbb{R}} \ker(T - \lambda)$$

(that is the subspaces are mutually orthogonal and their algebraic direct sum is dense in H). For any $\varepsilon > 0$, T has but a finite number of eigenvalues in $\mathbb{R} - [-\varepsilon, \varepsilon]$ and the associate eigenspaces are finite-dimensional.

The core of the proof is the

Lemma A.4.5. *Let H be a non zero Hilbert space and let T be a compact self-adjoint continuous linear operator $H \to H$. Then T admits a non zero eigenvector in H.*

Proof. Let us first prove that there exists a unitary vector x in H with $\|Tx\| = \|T\|$. We can suppose that we have $T \neq 0$. There exists a sequence $(x_n)_{n\in\mathbb{N}}$ of unitary vectors of H with $\|Tx_n\| \xrightarrow[n\to\infty]{} \|T\|$. As this sequence is bounded, we can suppose it admits a weak limit x with $\|x\| \leq 1$. In the same way, as T is compact, we can suppose the sequence $(Tx_n)_{n\in\mathbb{N}}$ admits a norm limit y. As T is continuous, it is weakly continuous and, therefore, we have $Tx = y$ and, hence, $\|T\| = \|Tx\| \leq \|T\| \|x\|$ so that $\|x\| = 1$ and we are done.

Now let us prove that x is an eigenvector for T^2 (which is also a self-adjoint operator), that is T^2x is proportional to x, or T^2x is orthogonal to every vector y that is orthogonal to x. Indeed, for such a y, we have, for any real number t,

$$\|T(x+ty)\|^2 \leq \|T\|^2 \|x+ty\|^2 = \|T\|^2 (1 + t^2 \|y\|^2).$$

By expanding and using the fact that T is self-adjoint, we get

$$\begin{aligned}\|T(x+ty)\|^2 &= \|Tx\|^2 + 2\Re\langle Tx, T(ty)\rangle + \|T(ty)\|^2 \\ &= \|T\|^2 + 2t\Re\langle T^2x, y\rangle + t^2 \|Ty\|^2,\end{aligned}$$

so that, finally, we have, for any t,

$$2t\Re\langle T^2x, y\rangle \leq t^2(\|T\|^2 \|y\|^2 - \|Ty\|^2).$$

As usual, this implies $\Re\langle T^2x, y\rangle = 0$. As this is also true when we replace y by iy, we get, for any y that is orthogonal to x, $\langle T^2x, y\rangle = 0$. Thus, x is an eigenvector for T^2.

We now know that, for some real number λ, the space $\ker(T^2 - \lambda)$ is not zero. Note (but this not really important) that as, for any y in H, we have $\langle T^2y, y\rangle = \|Ty\|^2 \geq 0$, λ is necessarily nonnegative. As T and T^2 commute to each other, the space $\ker(T^2 - \lambda)$ is stable by T, so that we can suppose $H = \ker(T^2 - \lambda)$. If $\lambda > 0$, we then have $T^2 - \lambda = 0$, that is $(T - \sqrt{\lambda})(T + \sqrt{\lambda}) = 0$ and, by the kernel lemma from linear algebra, this implies $H = \ker(T - \sqrt{\lambda}) \oplus \ker(T + \sqrt{\lambda})$, so that T has a non zero eigenvector. If $\lambda = 0$, we have $T^2 = 0$, that is, for any y in H, $\|Ty\|^2 = \langle T^2y, y\rangle = 0$ and $T = 0$. $\square$

We can now proceed to the

Proof of theorem A.4.4. First, note that the different eigenspaces of T are mutually orthogonal. Indeed, if $\lambda \neq \mu$ are two real numbers and x and y are vectors in H with $Tx = \lambda x$ and $Ty = \mu y$, we have

$$\lambda\langle x, y\rangle = \langle Tx, y\rangle = \langle x, Ty\rangle = \mu\langle x, y\rangle,$$

so that $\langle x, y\rangle = 0$.

Let K be the Hilbert sum $\bigoplus_{\lambda\in\mathbb{R}} \ker(T-\lambda)$ (that is the closure of the algebraic direct sum). Then K is stable by T and, hence, its orthogonal space $K^\perp$ is stable by T. Indeed, for any x in $K^\perp$ and y in K, we have $\langle Tx, y\rangle = \langle x, Ty\rangle = 0$, so that Tx belongs to $K^\perp$.

The restriction of T to the Hilbert space $K^\perp$ is a compact self-adjoint operator that, by definition, does not admit any eigenvector. Hence, by lemma A.4.5, we have $K^\perp = \{0\}$, that is $H = K = \bigoplus_{\lambda\in\mathbb{R}} \ker(T-\lambda)$.

Now, to finish the proof, let us fix some $\varepsilon > 0$ and let us prove that T has but a finite number of eigenvalues in $\mathbb{R} - [-\varepsilon, \varepsilon]$ and that the associate eigenspaces have finite dimension. This amounts to say that the space $L_\varepsilon = \bigoplus_{|\lambda|>\varepsilon} \ker(T-\lambda)$ has finite dimension. To prove this, we will prove that the closed ball $B_{L_\varepsilon}(0, \varepsilon)$ is compact: this implies that L_ε is finite dimensional by the famous Riesz theorem. Let x belong to $\ker(T-\lambda)$ for some λ with $|\lambda| > \varepsilon$ and set $S_\varepsilon x = \lambda^{-1}x$. We have $\|S_\varepsilon x\| \leq \varepsilon^{-1}\|x\|$. Extend S_ε by linearity to an endomorphism of the algebraic direct sum of the $\ker(T-\lambda)$, $|\lambda| > \varepsilon$. As these spaces are mutually orthogonal, we get, for any x in this algebraic direct sum, $\|S_\varepsilon x\| \leq \varepsilon^{-1}\|x\|$, so that S_ε extends to a continuous endomorphism of L_ε with norm $\leq \varepsilon^{-1}$. As we have $TS_\varepsilon = S_\varepsilon T = 1$ on a dense subspace of L_ε, T is therefore invertible on L_ε and its inverse has norm $\leq \varepsilon^{-1}$. Hence, we have $B_{L_\varepsilon}(0, \varepsilon) \subset TB_{L_\varepsilon}(0, 1) \subset TB_H(0, 1)$ and, as T is compact, $B_{L_\varepsilon}(0, \varepsilon)$ is compact, what should be proved. $\square$

Bibliography

[1] E. Ghys, Dynamique des flots unipotents sur les espaces homogènes, *Astérisque* **206** (1992), 93-136.

[2] P. Halmos, *Measure Theory*, Van Nostrand, New York, 1950.

[3] G.A. Margulis, G.M. Tomanov, Invariant measures for actions of unipotent groups over local fields on homogeneous spaces, *Inventiones mathematicae* **116** (1994), 347-392.

[4] M. Ratner, Invariant measures and orbit closures for unipotent actions on homogeneous spaces, *Geometric and functional analysis* **2** (1994), 236-257.

[5] A. Weil, *L'intégration dans les groupes topologiques et ses applications*, Hermann, Paris, 1940.

[6] R. Zimmer, *Ergodic theory and semisimple groups*, Brikhäuser, Boston, 1984.