ADDITIVE REPRESENTATIONS OF TREE LATTICES 2. RADICAL PSEUDOFIELDS AND LIMIT METRICS

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Abstract. In this article, we continue the systematic study of algebraic structures associated to spaces of representations of tree lattices started in [5]. We answer natural questions that were unsolved in that paper.

Contents

1. Introduction

1.1. **Motivations and objectives.** Let X be a tree and Γ be a cofinite lattice of X, that is, Γ is a discrete group of automorphisms of X and the quotient $\Gamma\backslash X$ is finite. In [5], we have introduced constructions of unitary representations of Γ that rely on geometric properties of the action of Γ on X. These constructions lead us to define several families of finite-dimensional vector spaces which parametrize objects relied to these representations. These vector spaces may be seen as analogues of spaces of sections of vector bundles over the quotient space $\Gamma\backslash X_1$, where X_1 is the set of edges of X.

The original objective of the present article was to solve two problems that came out naturally in the formalism of [5].

The first question is a problem of convex geometry. In [5], we defined an injective map from a certain infinite dimensional vector space (the space of symmetric cohomology classes of smooth functions over the geodesic time shift of $\Gamma\backslash X$), towards the space of Γ-invariant symmetric bilinear forms on a Hilbert space H_0^ω, which is equipped with a natural action of Γ. The above mentioned space of cohomology classes is written as an increasing union of finite-dimensional subspaces $\bigcup_{k\geq 2}\mathcal{W}_k$ and the trace on $\mathcal{W}_k$ of the set of non-negative bilinear forms on H_0^ω is a convex cone $\mathcal{W}_k^+ \subset \mathcal{W}_k$. For $k \geq 2$, there is a natural duality, called the weight pairing, which induces an identification between the dual space of $\mathcal{W}_k$ and another space $\mathcal{F}_k$ (the space of Γ-invariant k-quadratic fields). In Theorem 4.1, we give a description of the dual cone of $\mathcal{W}_k^+$ in $\mathcal{F}_k$. In Theorem 5.1, we use this description to show that every cohomology class in $\mathcal{W}_k^+$ contains a non-negative function.

The second question is a question of Hilbert spaces construction. There is a projective system structure on the sequence $(\mathcal{F}_k)_{k\geq 2}$. An important role was played in [5] by a certain open subset $\mathcal{P}_k^{\mathrm{ad}}$ of $\mathcal{F}_k$, which is called the set of admissible Γ-invariant k-Euclidean fields. To the choice of p in $\mathcal{P}_k^{\mathrm{ad}}$, we have associated a coherent family of scalar products on the projective system $(\mathcal{F}_j)_{j\geq k}$, called the weight metrics. In Theorem 8.5, we show that the Euclidean projective limit of this system is actually independent of p, by describing it with objects coming from the theory of Von Neumann algebras.

While trying to solve these problems, we exhibited new related algebraic structures, which will later turn out to play a role in the description of the spectral theory of Euclidean fields in [6]. The present article also contains a study of these new structures, that goes beyond the scope of proving Theorem 4.1 and Theorem 8.5.

I am very thankful to Rémi Boutonnet for clarifying my ideas on traces on operator algebras and for giving me the reference [3].

1.2. **Structure of the article.** All over the article, we use the language of [5]. References to this article are indicated with I.

In Section 2, we formalize certain algebraic operations on pseudokernels (see Section I.8) that already played an implicit role in [5], in particular in Section I.10. We use these operations to give a new interpretation of certain constructions, such as the orthogonal extension of dual kernels from Section I.5.

In Section 3, we define by duality the adjoint operations on quadratic pseudofields (see Section I.10). We then use this language to define the notion of a radical quadratic pseudofield: this is the main new object of this aticle. We describe how radical pseudofields can be used to construct quadratic fields, thanks to a linear map called the shoot map.

These algebraic operations and the notion of a radical pseudofield (and its generalizations) will play a key role in [6].

In Section 4, we prove Theorem 4.1: we show that, for $k \geq 2$, the dual cone of the cone $\mathcal{W}_k^+$ in $\mathcal{F}_k$ is the image by the shoot map of the cone of non-negative radical quadratic k-pseudofields. We call this cone the cone of tight k-quadratic fields.

In Section 5, we use Theorem 4.1 to prove that, for $k \geq 2$, all functions in $\mathcal{W}_k^+$ are cohomologous to functions with non-negative values. This also uses a Livšic type characterization of cohomology classes of non-negative functions inspired by [4]. The proof of the latter result requires us to show that Γ contains sufficiently many hyperbolic elements, which we prove by an equidistribution argument.

In Section 6, we study the null space of the shoot map introduced in Section 3. It turns out that this null space enjoys properties which can be considered as skew symmetric analogues of the properties of quadratic fields. We describe carefully these properties. The elements of the null space of the shoot map are called skew quadratic fields.

In Section 7, we pursue this analogy by defining skew dual kernels which are related to skew quadratic fields as dual kernels are related to quadratic fields. In particular, we introduce a skew weight pairing and a skew weight metric in the spirit of Section I.11.

The results of these last two sections are not used in the proof of neither of our main results. Later, in [6], a version of the skew weight metric will appear in the Plancherel formula for Euclidean fields.

Finally, in Section 8, we begin by describing the projective limit of the projective system of skew quadratic fields, equipped with the skew weight metric. By analogy, this computation will help us to identify the Euclidean projective limit of the projective system of quadratic fields in Theorem 8.5. To proceed to this identification, we relate the weight pairing to traces on group algebras.

Appendix A contains classical results on traces that we formulated in our language in order to use them in the proof of Theorem 8.5. Although the statement of this result relies on an analogy with the case of skew quadratic fields, his proof formally only requires the material of this Appendix and not the rest of the paper.

We use the general notation introduced in Subsection I.1.8 and Subsection I.2.1.

2. The canonical map

We introduce new notation for certain natural operations on pseudokernels. For $k \geq 2$, we use these operations to define and study a new linear map that embeds k-dual kernels inside k-pseudokernels. This will give us a new interpretation of the weight map and of the notion of non-negativity for dual kernels.

2.1. **Operations on pseudokernels.** We start by introducing natural algebraic operations on pseudokernels which will shed a new light on previously introduced constructions. See Subsection I.8.2 for the language of pseudokernels.

For $x \sim y$, the operators I^ℓ_{xy}, $\ell \geq 0$, and J^ℓ_{xy}, $\ell \geq 1$, are defined in Subsection I.4.2.

Definition 2.1. Let $k \geq 1$ and L be a k-pseudokernel.

If k is even, for any $x \sim y$ in X, we set $L^\vee_{xy} = \sum_{\substack{z \sim x \\ z \neq y}} L_{xz}$. We call $L^\vee$ the reversal of L. The map $L \mapsto L^\vee$ is a linear automorphisms on the space of k-pseudokernels.

If k is odd, for any $x \sim y$ in X, we set $L^\vee_{xy} = L_{yx}$. We call $L^\vee$ the inversion of L. The map $L \mapsto L^\vee$ is an involution of the space of k-pseudokernels.

Definition 2.2. We define the direct extension of pseudokernels as follows. Let $k \geq 1$ and L be a k-pseudokernel. We let $L^>$ be the $(k+1)$-pseudokernel with $L^>_{xy} = L_{xy}$, $x \sim y$ in X. In other words, the symmetric bilinear forms $(r^L_{xy})_{x \sim y \in X}$ and $(r^{L^>}_{xy})_{x \sim y \in X}$ are related in the following way.

If k is odd, $k = 2\ell + 1$, $\ell \geq 0$, for any $x \sim y$ in X, we have $r^{L^>}_{xy} = (I^{\ell,*}_{xy})^\star r^L_{xy}$. If k is even, $k = 2\ell$, $\ell \geq 1$, for any $x \sim y$ in X, we have $r^{L^>}_{xy} = (J^{\ell,*}_{xy})^\star r^L_{xy}$.

Direct extension is a linear embedding from the space of k-pseudokernels into the space of $(k+1)$-pseudokernels.

Recall that in Definition I.8.14 and Definition I.8.15, we have introduced orthogonal extension of pseudokernels, which was proved in Proposition I.8.17 to be the restriction to pseudokernels of orthogonal extension of dual kernels. Orthogonal extension may be equivalenltly defined by using reversal, inversion and direct extension: a straightforward use of the definitions gives

Lemma 2.3. *Let $k \geq 1$ and L be a k-pseudokernel. We have $L^+ = L^{>\vee}$.*

Here comes a fundamental commutation property of those maps.

Lemma 2.4. *Let $k \geq 1$ and L be a k-pseudokernel. We have $L^{\vee>>} = L^{>>\vee}$.*

Proof. Assume that k is odd, $k = 2\ell + 1$, $\ell \geq 0$. From Lemma I.4.4, we get, for $x \sim y$ in X, $J_{xy}^{\ell+1} I_{xy}^{\ell} = M_{xy}^{\ell}$, where $M_{xy}^{\ell} = M_{yx}^{\ell}$ is the natural embedding $\overline{V}^{\ell}(xy) \hookrightarrow \overline{V}^{\ell+1}(xy)$. Therefore, if r_{xy}^{L} is the symmetric bilinear form associated to L_{xy} on $V_0^{\ell}(xy)$, the symmetric bilinear form associated with $L^{>>}$ on $V_0^{\ell+1}(xy)$ is $(M_{xy}^{\ell,*})^{\star} r_{xy}^{L}$ and the conclusion follows.

In the same way, if k is even, $k = 2\ell$, $\ell \geq 1$, for any $x \sim y$ in X, the symmetric bilinear form associated with $L^{>>}$ on $V_0^{\ell+1}(x)$ is $(M_x^{\ell,*})^{\star} r_{xy}^{L}$. □

Conversely, we have

Lemma 2.5. *Let $k \geq 1$ and L, M be k-pseudokernels. Assume that we have $L^{>} = M^{>\vee}$.*

If $k = 1$, then $L = M = 0$.

If $k \geq 2$, then there exists a $(k-1)$-pseudokernel N with $L = N^{\vee >}$ and $M = N^{>}$.

The proof is a translation of Lemma I.8.12.

Proof. As usual, we denote by r_{xy}^{L} and r_{xy}^{M}, $x \sim y$ in X, the symmetric bilinear forms associated with L and M.

Assume $k = 1$. Then there exists functions u and v on X_1 such that, for $x \sim y$ in X and f in $V_0^0(xy)$, one has $r_{xy}^{L}(f,f) = u(x,y) f(y)^2$ and $r_{xy}^{M}(f,f) = v(x,y) f(y)^2$. The equation $L^{>} = M^{>\vee}$ reads as

$$u(x,y) f(y)^2 = \sum_{\substack{z \sim x \\ z \neq y}} v(x,z) f(z)^2, \quad f \in V_0^1(x).$$

By taking $f = \mathbf{1}_y - \mathbf{1}_z$ for $z \neq y$, we get $u(x,y) = v(x,z)$. As $d(x) \geq 3$, this implies that $u(x,.)$ and $v(x,.)$ are constant on $S^1(x)$ with the same value. Would this value be non-zero, we would get

$$f(y)^2 = \sum_{\substack{z \sim x \\ z \neq y}} f(z)^2, \quad f \in V_0^1(x).$$

The quadratic form on the right hand-side is positive definite on $V_0^1(x)$, whereas the one on the left hand-side has rank 1. Both can not be equal since $\dim V_0^1(x) = d(x) - 1 \geq 2$. We get $u = v = 0$ as required.

Assume k is odd, $k = 2\ell + 1$, $\ell \geq 1$. Then, for $x \sim y$ in X, we have

$$(I_{xy}^{\ell,*})^{\star} r_{xy}^{L} = \sum_{\substack{z \sim x \\ z \neq y}} (I_{xz}^{\ell,*})^{\star} r_{xz}^{M}.$$

Proposition I.4.5 and Lemma I.8.12 imply that there exists families

$$(s_{xy})_{x \sim y \in X} \text{ and } (t_{xy})_{x \sim y \in X}$$

where, for $x \sim y$ in X, s_{xy} and t_{xy} are symmetric bilinear forms on $V_0^\ell(x)$ and

$$\begin{aligned} r_{xy}^L &= (J_{xy}^{\ell,*})^\star s_{xy} \\ r_{xy}^M &= (J_{xy}^{\ell,*})^\star t_{xy} \\ s_{xy} &= \sum_{\substack{z\sim x \\ z\neq y}} t_{xz}. \end{aligned}$$

The conclusion directly follows, by taking N to be the $(k-1)$-pseudokernel associated with $(t_{xy})_{x\sim y\in X}$.

The proof in the even case is analogous. □

2.2. **The canonical pseudokernel.** We now introduce a natural embedding of the space of k-dual kernels into the one of k-pseudokernels.

Definition 2.6. (k even) Let $k \geq 2$ be an even integer and (K, K^-) be a k-dual kernel. The canonical k-pseudokernel L of (K, K^-) is defined by

$$L_{xy} = K_x - K_{xy}^- \quad x \sim y \in X.$$

Definition 2.7. (k odd) Let $k \geq 3$ be an odd integer and (K, K^-) be a k-dual kernel. The canonical k-pseudokernel L of (K, K^-) is defined by

$$L_{xy} = K_{xy} - K_x^- \quad x \sim y \in X.$$

Dual kernels are defined in Subsection I.5.1.

Remark 2.8. Let us state this definition in an other way. For $k \geq 1$, any k-dual prekernel K may be considered as a k-pseudokernel. Indeed, if k is odd, K may be seen as the k-pseudokernel L with $L_{xy} = K_{xy} = L_{yx}$, $x \sim y \in X$. If k is even, K may be seen as the k-pseudokernel L with $L_{xy} = K_x$, $x \sim y \in X$. Now, if $k \geq 2$ and (K, K^-) is a k-dual kernel, the definitions say that the canonical k-pseudokernel L of (K, K^-) is $L = K - K^{->}$.

We let $C_k : \mathcal{K}_k \to \mathcal{L}_k$ denote the linear map that sends a Γ-invariant k-dual kernel to its canonical k-pseudokernel. We call C_k the canonical map. We can relate the canonical map to the previously introduced operations.

Lemma 2.9. *Let $k \geq 2$ and (K, K^-) be a k-dual kernel with canonical k-pseudokernel L and orthogonal extension (K^+, K). The canonical $(k+1)$-pseudokernel of (K^+, K) is the orthogonal extension L^+ of L.*

Recall from Lemma 2.3 that the orthogonal extension L^+ may be defined by $L^+ = L^{>\vee}$.

Proof. Let M be the canonical $(k+1)$-pseudokernel of (K^+, K). Fix $x \sim y$ in X. If k is even, we have

$$M_{xy} = K^+_{xy} - K_x = (K_x + K_y - K^-_{xy}) - K_x = K_y - K^-_{xy} = L_{yx}.$$

If k is odd, we have

$$\begin{aligned} M_{xy} = K^+_x - K_{xy} &= \sum_{z \sim x} K_{xz} - (d(x) - 1) K^-_x - K_{xy} \\ &= \sum_{\substack{z \sim x \\ z \neq y}} (K_{xz} - K_x) = \sum_{\substack{z \sim x \\ z \neq y}} L_{xz}. \end{aligned}$$

□

In case (K, K^-) is the k-dual kernel associated to a $(k-1)$-pseudokernel, the canonical k-pseudokernel may be described.

Lemma 2.10. *Let $k \geq 2$, M be a $(k-1)$-pseudokernel and (K, K^-) be the k-dual kernel associated to $M^\vee$. Let L be the canonical k-pseudokernel of (K, K^-). We have $L = M^{\vee>\vee} - M^>$.*

See Definition I.8.9 and Definition I.8.10 for the construction of the k-dual kernel associated to a $(k-1)$-pseudokernel.

Proof. Fix $x \sim y$ in X. If k is even, we have

$$L_{xy} = K_x - K^-_{xy} = \sum_{z \sim x} M_{zx} - (M_{xy} + M_{yx}) = \sum_{\substack{z \sim x \\ z \neq y}} M_{zx} - M_{xy}.$$

If k is odd, we have

$$\begin{aligned} L_{xy} = K_{xy} - K^-_x &= M^\vee_{xy} + M^\vee_{yx} - \frac{1}{d(x) - 1} \sum_{z \sim x} M^\vee_{xz} \\ &= M^\vee_{xy} + M^\vee_{yx} - \frac{1}{d(x) - 1} \sum_{\substack{z, t \sim x \\ z \neq t}} M_{xt} \\ &= M^\vee_{xy} + M^\vee_{yx} - \sum_{t \sim x} M_{xt} = M^\vee_{yx} - M_{xy}. \end{aligned}$$

□

The canonical map is injective.

Lemma 2.11. *Let $k \geq 2$ and (K, K^-) be a k-dual kernel. If the canonical k-pseudokernel of (K, K^-) is 0, then $(K, K^-) = 0$.*

Proof. We use the convention established in Remark 2.8, that is, we consider dual prekernels as pseudokernels. We set $L = K^-$ if k is odd and, if k is even, we let L be the $(k-1)$-pseudokernel defined by $L_{xy} = (d(x)-1)K^-_{xy}$, $x \sim y \in X$. In both cases, as by assumption, we have $K = K^{->}$, we get $K^{->\vee} = L^>$.

We prove the result by induction on $k \geq 2$.

If $k = 2$, by Lemma 2.5, since $K^{->\vee} = L^>$, we have $K^- = 0$, hence $K = 0$.

Assume now $k \geq 3$ and the result holds for $k-1$. Still by Lemma 2.5, as $K^{->\vee} = L^>$, there exists a $(k-2)$-pseudokernel H with $K^- = H^>$ and $L = H^{\vee>}$. If k is odd, we have $K^- = L$, hence $H^> = H^{\vee>}$ and therefore $H = H^\vee$, which says that H is a $(k-2)$-dual prekernel. Similarly, if k is even, for $x \sim y$ in X, we have $L_{xy} = (d(x)-1)K^-_{xy}$, hence $H^{\vee>}_{xy} = (d(x)-1)H^>_{xy}$ and $H^\vee_{xy} = (d(x)-1)H_{xy}$, which also says that H is a $(k-2)$-dual prekernel. In both cases, the $(k-1)$-dual kernel (K^-, H) precisely satisfies $K^- = H^>$, that is, it has zero canonical $(k-1)$-pseudokernel. By induction, we get $(K^-, H) = 0$ and, as $K = K^{->}$, $K = 0$, which should be proved. □

From Lemma 2.10 and Lemma 2.11, we deduce

Corollary 2.12. *Let $k \geq 2$, (K, K^-) be a k-dual kernel, L be its canonical k-pseudokernel and M be a $(k-1)$-pseudokernel. Then $L = M^{\vee>\vee} - M^>$ if and only if (K, K^-) is the k-dual kernel associated to $M^\vee$.*

2.3. **A non-negativity criterion.** The canonical map allows to verify whether a dual kernel is non-negative up to a pseudokernel.

Proposition 2.13. *Let $k \geq 2$ and (K, K^-) be a k-dual kernel with canonical k-pseudokernel L.*

If (K, K^-) is non-negative, then L is non-negative.

If (K, K^-) is Γ-invariant and L is non-negative, then there exists a Γ-invariant $(k-1)$-pseudokernel M, with associated k-dual kernel (J, J^-), such that $(J + K, J^- + K^-)$ is non-negative.

See Definition I.5.12 and Definition I.5.13 for the notion of a non-negative dual kernel.

The first statement of the Proposition is obvious. We focus on the second one. As usual, we denote by K^j, $j \geq k-1$, the dual prekernels obtained from (K, K^-) by successive orthogonal extensions. A first step towards the proof is

Lemma 2.14. *Let $k \geq 2$ and (K, K^-) be a k-dual kernel with canonical k-pseudokernel L. If L is non-negative, then, for every $h \geq 0$, the $(h+k)$-pseudokernel $K^{h+k} - K^{->^{h+1}}$ is non-negative.*

Proof. We prove by induction on $h \geq 0$ that the statement is true for any $k \geq 2$.

For $h = 0$, this is the definition of a non-negative k-dual kernel.

If the statement is true for $h \geq 0$, we fix a k-dual kernel (K, K^-) and we consider its orthogonal extension (K^+, K). By Lemma 2.9, the canonical $(k+1)$-pseudokernel of (K^+, K) is L^+, which is non-negative. By the induction assumption, which we apply to (K^+, K), the $(h+k)$-pseudokernel $K^{h+k+1} - K^{>^{h+1}}$ is non-negative. By assumption, $L = K - K^{->}$ is non-negative, hence $L^{>^{h+1}} = K^{>^{h+1}} - K^{->^{h+2}}$ is non-negative. The conclusion follows. $\square$

By going to the limit as $h \to \infty$, this gives

Corollary 2.15. *Let $k \geq 2$ and (K, K^-) be a Γ-invariant k-dual kernel with canonical k-pseudokernel L. Assume that L is non-negative. Let w be a Γ-invariant weight function of (K, K^-) and θ be in H_0^ω.*

If k is even, $k = 2\ell$, $\ell \geq 1$, for any $x \sim y$ in X, we have

$$\Phi_w(\theta, \theta) \geq q_{xy}^{K^-}(N_{xy}^{\ell-1,*}\theta, N_{xy}^{\ell-1,*}\theta),$$

where $q_{xy}^{K^-}$ is the symmetric bilinear form on $V_0^{\ell-1}(xy)$ associated with K^-.

If k is odd, $k = 2\ell+1$, $\ell \geq 1$, for any x in X, we have

$$\Phi_w(\theta, \theta) \geq q_x^{K^-}(N_x^{\ell,*}\theta, N_x^{\ell,*}\theta),$$

where $q_x^{K^-}$ is the symmetric bilinear form on $V_0^\ell(x)$ associated with K^-.

In this Corollary, we have used freely the language of Section I.3 and Section I.5.

Proof. Assume k is even. By Lemma 2.14, for every $h \geq \ell$, we have $q_x^{K^{2h}}(N_x^{h,*}\theta, N_x^{h,*}\theta) \geq q_{xy}^{K^-}(N_{xy}^{\ell-1,*}\theta, N_{xy}^{\ell-1,*}\theta)$. The result now follows from Corollary I.7.9.

The proof in the odd case is analogous. $\square$

The next lemma will show that the right hand-side of the inequalities in Corollary 2.15 can be chosen to be close to 0.

Lemma 2.16. *Let θ be in H_0^ω. Then, as γ leaves finite subsets of Γ, the distribution $\gamma\theta$ converges weakly to 0 in the Hilbert space H_0^ω.*

Proof. Indeed, by construction (see Subsection I.3.1), the space H_0^ω may be seen as a closed subspace of $\ell^2(X_1)$ where the statement clearly holds true. □

We can now conclude.

Proof of Proposition 2.13. By definition, if (K, K^-) is non-negative, then L is non-negative.

Conversely, suppose (K, K^-) is Γ-invariant and L is non-negative. Let us assume for example that k is even, $k = 2\ell$, $\ell \geq 1$. We fix $x \sim y$ in X and θ in H_0^ω. As Φ_w is Γ-invariant, for every γ in Γ, by Corollary 2.15, we have

$$\Phi_w(\theta, \theta) \geq q_{xy}^{K^-}(N_{xy}^{\ell-1,*}\gamma\theta, N_{xy}^{\ell-1,*}\gamma\theta).$$

As the bounded linear map $N_{xy}^{\ell-1,*}$ on H_0^ω has finite-dimensional range, by Lemma 2.16, $N_{xy}^{\ell-1,*}\gamma\theta$ goes to 0 as γ leaves finite subsets of Γ and we get $\Phi_w(\theta, \theta) \geq 0$, that is, Φ_w is non-negative. By Theorem I.7.17, there exists a non-negative k-dual kernel (H, H^-) which admits w as a weight function. By Theorem I.8.32, $(H - K, H^- - K^-)$ is a associated to a $(k-1)$-pseudokernel, which should be proved. □

2.4. **The canonical map and the weight map.** Recall from Section I.8 that, for $k \geq 2$, the weight map $W_k : \mathcal{K}_k \to \mathcal{W}_k$ sends a Γ-invariant k-dual kernel to the cohomology class of its weight functions. We will now use the canonical pseudokernel construction to give an equivalent definition of this map. The key observation is

Lemma 2.17. *Let $k \geq 2$, (K, K^-) be a k-dual kernel and L be its canonical k-pseudokernel. Then the bias function of (K, K^-) is the pseudoweight of L.*

See Definition I.6.12 for the introduction of the bias function of a dual kernel. See Definition I.8.23 for the introduction of the pseudoweight of a pseudokernel.

Proof. By Lemma 2.9, the canonical $(2k-1)$-pseudokernel of the $(2k-1)$-dual kernel (K^{2k-1}, K^{2k-2}) is L^{2k-1}. The result now directly follows from the definitions. □

For $k \geq 1$ and v a function on X_k, the symmetrization of v is the function $(x, y) \mapsto \frac{1}{2}(v(x, y) + v(y, x))$.

Corollary 2.18. *Let $k \geq 2$, (K, K^-) be a k-dual kernel and L be its canonical k-pseudokernel. Let v be the bias function of (K, K^-) which is also the pseudoweight of L. Then the symmetrization of v is a weight function of (K, K^-).*

Proof. It follows from Definition I.6.5 that (K, K^-) admits a unique symmetric compatible function u. Let w be the associated weight function as in Definition I.6.7. By Lemma I.6.14, for (x, y) in X_k, we have

$$v(x, y) = u(x_1, y) - u(x, y_1) + w(x, y),$$

where x_1 and y_1 are the neighbours of x and y on $[xy]$. Summing with the symmetric equality, we get, u and w being symmetric,

$$v(x, y) + v(y, x) = 2w(x, y)$$

as required. □

3. Radical pseudofields

Recall that quadratic pseudofields were introduced in Subsection I.10.2. For $k \geq 1$, the space $\mathcal{M}_k$ of Γ-invariant k-quadratic pseudofields may be seen as the dual space of the space $\mathcal{L}_k$ of Γ-invariant k-pseudokernels. We will introduce a new property for quadratic pseudofields that will be dual to the constructions above.

3.1. Operations on quadratic pseudofields.

First, we start by defining the dual operations to the natural operations on pseudokernels.

Definition 3.1. Let $k \geq 1$ and s be a k-quadratic pseudofield.

If k is even, for any $x \sim y$ in X, we set $s^\vee_{xy} = \sum_{\substack{z \sim x \\ z \neq y}} s_{xz}$. The k-quadratic pseudofield $s^\vee$ is called the reversal of s. The map $s \mapsto s^\vee$ is a linear automorphisms of the space of k-quadratic pseudofields.

If k is odd, for any $x \sim y$ in X, we set $s^\vee_{xy} = s_{yx}$. The k-quadratic pseudofield $s^\vee$ is called the inversion of s. This map $s \mapsto s^\vee$ is an involution of the space of k-quadratic pseudofields.

Definition 3.2. We define the direct restriction of quadratic pseudofields as follows. Let $k \geq 1$ and s be a $(k+1)$-quadratic pseudofield.

If k is odd, $k = 2\ell + 1$, $\ell \geq 0$, for any $x \sim y$ in X, we set $s^<_{xy} = (I^\ell_{xy})^\star s_{xy}$.

If k is even, $k = 2\ell$, $\ell \geq 1$, for any $x \sim y$ in X, we set $s^<_{xy} = (J^\ell_{xy})^\star s_{xy}$.

Remark 3.3. If V is a vector space and W is a subspace of V, the natural restriction map $\mathcal{Q}(V) \to \mathcal{Q}(W)$ is surjective. Hence direct restriction maps $(k+1)$-quadratic pseudofields onto k-quadratic pseudofields.

As in Lemma 2.4, we show

Lemma 3.4. *Let $k \geq 1$ and s be a $(k+2)$-quadratic pseudofield. We have $s^{<<\vee} = s^{\vee<<}$.*

If $k \geq 2$ and s is a k-quadratic pseudofield, we have defined its reduction s^- in Subsection I.10.2. By Lemma I.10.3, reduction of Γ-invariant quadratic pseudofields is the adjoint operation of orthogonal extension of Γ-invariant pseudokernels. A direct computation gives

Lemma 3.5. *Let $k \geq 2$ and s be a k-quadratic pseudofield. Then one has $s^- = s^{\vee<}$.*

As announced, these operations are dual to the operations on quadratic pseudofields.

Lemma 3.6. *Let $k \geq 1$, L be a Γ-invariant k-pseudokernel, s be a Γ-invariant k-quadratic pseudofield and t be a Γ-invariant $(k+1)$-quadratic pseudofield. We have*

$$\langle L^\vee, s\rangle = \langle L, s^\vee\rangle \text{ and } \langle L^>, t\rangle = \langle L, t^<\rangle.$$

The proof follows from the definition of the duality in Subsection I.10.2 and from Lemma I.C.2.

3.2. **Radical quadratic pseudofields and the shoot map.** We now use this formalism to define a subclass of quadratic pseudofields.

Definition 3.7. Let $k \geq 2$ and s be a k-quadratic pseudofield. We say that s is radical if $s^< = s^{\vee<\vee}$. If $k = 1$, by convention, any quadratic pseudofield is said to be radical.

The space of Γ-invariant radical k-quadratic pseudofields is denoted by $\mathcal{M}^1_k$.

The reduction of a radical quadratic pseudofield is again radical.

Lemma 3.8. *Let $k \geq 2$ and s be a radical k-quadratic pseudofield. Then the $(k-1)$-quadratic pseudofield $s^- = s^{\vee<}$ is radical.*

Proof. If $k = 2$, there is nothing to prove. If $k \geq 3$, we get

$$s^{-<} = s^{\vee<<} = s^{<<\vee} = s^{\vee<\vee<\vee} = s^{-\vee<\vee},$$

where we have used Lemma 3.4. The result follows. □

We will now use radical quadratic pseudofields to define quadratic fields.

Lemma 3.9. *Let $k \geq 1$ and s be a radical k-quadratic pseudofield.*

If k is even, for x in X, set $p_x = \sum_{y\sim x} s_{xy}$.

If k is odd, for $x \sim y$, set $p_{xy} = s_{xy} + s_{yx}$.

In both cases, p is a quadratic field.

See Definition I.4.7 and Definition I.4.8 for the precise definition of a quadratic field.

Definition 3.10. Let $k \geq 1$, s be a radical k-quadratic pseudofield and p be as in Lemma 3.9. Then the quadratic field p is called the shoot of s. The linear map $P_k : \mathcal{M}_k^1 \to \mathcal{F}_k$ that sends a Γ-invariant radical k-quadratic pseudofield to its shoot is called the shoot map.

Proof of Lemma 3.9. If $k = 1$, there is nothing to prove, p being a quadratic field only meaning that $p_{xy} = p_{yx}$, $x \sim y \in X$.

If $k \geq 2$, first note that, as s is radical, we have

$$(s + s^\vee)^< = s^< + s^{\vee<} = s^{\vee<\vee} + s^{\vee<} = s^{-\vee} + s^-.$$

If k is even, $k = 2\ell$, $\ell \geq 1$, for $x \sim y$ in X, we have $p_x = (s + s^\vee)_{xy}$, hence,

$$(I_{xy}^{\ell-1})^\star p_x = (I_{xy}^{\ell-1})^\star (s + s^\vee)_{xy} = (s + s^\vee)_{xy}^< = (s^- + s^{-\vee})_{xy} = (I_{yx}^{\ell-1})^\star p_y,$$

that is, p is a quadratic field.

Similarly, if k is odd, $k = 2\ell + 1$, $\ell \geq 1$, for $x \sim y$ in X, we have $p_{xy} = (s + s^\vee)_{xy}$, hence

$$(J_{xy}^\ell)^\star p_{xy} = (J_{xy}^\ell)^\star (s + s^\vee)_{xy} = (s + s^\vee)_{xy}^< = (s^- + s^{-\vee})_{xy} = (J_{xz}^\ell)^\star p_{xz},$$

for any $z \sim x$. Again, this meens that p is a quadratic field. $\square$

The proof also gives

Corollary 3.11. *Let $k \geq 2$, s be a radical k-quadratic pseudofield and p be the shoot of s. Then p^- is the shoot of s^-.*

3.3. **Reduction of radical pseudofields.** As an illustration of the use of the above introduced formalism, we now prove that the reduction map is surjective over radical quadratic pseudofields.

Proposition 3.12. *Let $k \geq 1$. Then the reduction map $s \mapsto s^-$ maps the space $\mathcal{M}_{k+1}^1$ of Γ-invariant radical $(k+1)$-quadratic pseudofields onto the space $\mathcal{M}_k^1$ of Γ-invariant radical k-quadratic pseudofields.*

Proof. We will actually show the dual statement, namely that the adjoint map is injective. We prove this by induction on $k \geq 1$.

If $k = 1$, as every 1-quadratic pseudofield is radical, we must show that, if L and M are in $\mathcal{L}_1$, that is, if L and M are Γ-invariant 1-pseudokernels, and $L^{>\vee} = M^{\vee>\vee} - M^>$, then $L = 0$. Indeed, by Lemma 2.5, we have $M = 0 = M^\vee - L$, hence $L = 0$.

If $k \geq 2$ and the result holds for $k - 1$, we must now show that, if L and M are in $\mathcal{L}_k$, and $L^{>\vee} = M^{\vee>\vee} - M^>$, then $L = N^{\vee>\vee} - N^>$ for some N in $\mathcal{L}_{k-1}$. Indeed, Lemma 2.5 now says that there exists N in $\mathcal{L}_{k-1}$ with

$$M^\vee - L = N^> \text{ and } M = N^{\vee>}$$

and the conclusion follows. $\square$

3.4. **Crossed duality.** We now prove that the shoot map and the canonical map from Section 2 are dual to each other, through the weight pairing of Subsection I.11.2.

Lemma 3.13. *Let $k \geq 2$. Let s in $\mathcal{M}_k^1$ be a Γ-invariant radical k-quadratic pseudofield and set $p = P_k s$ to be the shoot of s. Let (K, K^-) in $\mathcal{K}_k$ be a Γ-invariant k-dual kernel and set $L = C_k(K, K^-)$ to be the canonical k-pseudokernel of (K, K^-). We have*

$$[p, (K, K^-)] = \langle s, L \rangle.$$

Proof. Assume k is even, $k = 2\ell$, $\ell \geq 1$. As usual, for $x \sim y$ in V, denote by q_x, q_{xy}^- and r_{xy} the symmetric bilinear forms associated with K_x, K_{xy}^- and L_{xy}. By Definition 2.6, we have $r_{xy} = q_x - (I_{xy}^{\ell-1,*})^\star q_{xy}^-$. By Theorem I.11.4, we have

$$[p, (K, K^-)] = \sum_{x \in \Gamma \backslash X} \frac{1}{|\Gamma_x|} \langle p_x, q_x \rangle - \frac{1}{2} \sum_{(x,y) \in \Gamma \backslash X_1} \frac{1}{|\Gamma_x \cap \Gamma_y|} \langle p_{xy}^-, q_{xy}^- \rangle.$$

By definition of the shoot map and by Lemma I.9.11, we have

$$\sum_{x \in \Gamma \backslash X} \frac{1}{|\Gamma_x|} \langle p_x, q_x \rangle = \sum_{(x,y) \in \Gamma \backslash X_1} \frac{1}{|\Gamma_x \cap \Gamma_y|} \langle s_{xy}, q_x \rangle.$$

Besides, for $x \sim y$ in X, we have, as $q_{xy}^- = q_{yx}^-$,

$$\begin{aligned} \langle p_{xy}^-, q_{xy}^- \rangle = \langle s_{xy}^< + s_{xy}^{\vee<}, q_{xy}^- \rangle &= \langle s_{xy}^< + s_{xy}^{<\vee}, q_{xy}^- \rangle = \langle s_{xy}^< + s_{yx}^<, q_{xy}^- \rangle \\ &= \langle s_{xy}, (I_{xy}^{\ell-1,*})^\star q_{xy}^- \rangle + \langle s_{yx}, (I_{yx}^{\ell-1,*})^\star q_{xy}^- \rangle, \end{aligned}$$

where the latter follows from the definition of the direct restriction map and Lemma I.C.2. As $q_{xy}^- = q_{yx}^-$, we get

$$\sum_{(x,y) \in \Gamma \backslash X_1} \frac{1}{|\Gamma_x \cap \Gamma_y|} \langle p_{xy}^-, q_{xy}^- \rangle = 2 \sum_{(x,y) \in \Gamma \backslash X_1} \frac{1}{|\Gamma_x \cap \Gamma_y|} \langle s_{xy}, (I_{xy}^{\ell-1,*})^\star q_{xy}^- \rangle$$

and the conclusion follows.

Assume k is odd, $k = 2\ell + 1$, $\ell \geq 1$, and write now, for $x \sim y$ in X, q_{xy}, q_x^- and r_{xy} for the symmetric bilinear forms associated with K_{xy}, K_x^- and L_{xy}. Definition 2.7 gives $r_{xy} = q_{xy} - (J_{xy}^{\ell,*})^\star q_x^-$, whereas Theorem I.11.4 says that

$$[p, (K, K^-)] = \frac{1}{2} \sum_{(x,y) \in \Gamma \backslash X_1} \frac{1}{|\Gamma_x \cap \Gamma_y|} \langle p_{xy}, q_{xy} \rangle - \sum_{x \in \Gamma \backslash X} \frac{d(x) - 1}{|\Gamma_x|} \langle p_x^-, q_x^- \rangle.$$

As above, we have

$$\frac{1}{2} \sum_{(x,y) \in \Gamma \backslash X_1} \frac{1}{|\Gamma_x \cap \Gamma_y|} \langle p_{xy}, q_{xy} \rangle = \sum_{(x,y) \in \Gamma \backslash X_1} \frac{1}{|\Gamma_x \cap \Gamma_y|} \langle s_{xy}, q_{xy} \rangle.$$

Now, we write, by Corollary 3.11 and the definition of the direct restriction map, for x in X,

$$(d(x)-1)\langle p_x^-, q_x^-\rangle = (d(x)-1)\sum_{y\sim x}\langle s_{xy}^{\vee<}, q_x^-\rangle = \sum_{\substack{y,z\sim x\\ y\neq z}}\langle s_{xy}^{\vee<}, q_x^-\rangle$$
$$= \sum_{z\sim x}\langle s_{xz}^{\vee<\vee}, q_x^-\rangle = \sum_{z\sim x}\langle s_{xz}^{<}, q_x^-\rangle = \sum_{z\sim x}\langle (J_{xz}^{\ell})^\star s_{xz}, q_{xz}\rangle$$
$$= \sum_{z\sim x}\langle s_{xz}, (J_{xz}^{\ell,*})^\star q_{xz}\rangle,$$

where we have used the fact that the quadratic pseudofield s is radical. This gives, by Lemma I.9.11,

$$\sum_{x\in\Gamma\backslash X}\frac{d(x)-1}{|\Gamma_x|}\langle p_x^-, q_x^-\rangle = \sum_{(x,y)\in\Gamma\backslash X_1}\frac{1}{|\Gamma_x\cap\Gamma_y|}\langle s_{xy}, (J_{xy}^{\ell,*})^\star q_{xy}\rangle,$$

and we are done. □

Let us translate Lemma 3.13 in terms of our linear maps. Denote by $I_k : \mathcal{M}_k^1 \hookrightarrow \mathcal{M}_k$ the natural inclusion. Recall that $W_k : \mathcal{K}_k \to \mathcal{W}_k$ is the weight map. By Theorem I.11.4, we can use the weight pairing to identify $\mathcal{F}_k$ with the dual space of $\mathcal{W}_k$ and we consider the adjoint operator W_k^* as a linear map from $\mathcal{F}_k$ to the dual space of $\mathcal{K}_k$.

Corollary 3.14. *Let $k \geq 2$. We have $W_k^* P_k = C_k^* I_{k+1}$.*

Proof. Indeed, if s, p, (K, K^-) and L are as in Lemma 3.13, by defintion of the weight pairing (see Definition I.11.5), we have

$$[p, (K, K^-)] = \langle P_k s, W_k(K, K^-)\rangle = \langle W_k^* P_k s, (K, K^-)\rangle.$$

Besides, we have

$$\langle s, L\rangle = \langle I_{k+1}s, C_k(K, K^-)\rangle = \langle C_k^* I_{k+1} s, (K, K^-)\rangle.$$

The claim now follows from Lemma 3.13. □

3.5. **Fibered surjectivity.** From the duality between the shoot map and the canonical map, we get a strengthening of Proposition 3.12.

Proposition 3.15. *Let $k \geq 1$. Then P_k maps $\mathcal{M}_k^1$ onto $\mathcal{F}_k$. If $k \geq 2$, then the map $s \mapsto (P_k s, s^-)$ sends $\mathcal{M}_k^1$ onto the space*

$$\{(p,t) \in \mathcal{F}_k \times \mathcal{M}_{k-1}^1 | p^- = P_{k-1} t\}.$$

Note that the fact that the map of the statement takes values in the considered space follows from Corollary 3.11.

Again, we will prove Proposition 3.15 by establishing a dual statement. This will require us to use the following abstract description of dual spaces of fibered products.

Lemma 3.16. *Let V, V_1, V_2 and W be finite-dimensional vector spaces. Assume we are given linear maps*

$$\begin{array}{ccc} V & \xrightarrow{\pi_2} & V_2 \\ {\scriptstyle \pi_1}\big\downarrow & & \big\downarrow{\scriptstyle \varpi_2} \\ V_1 & \xrightarrow[\varpi_1]{} & W \end{array}$$

such that $\varpi_1\pi_1 = \varpi_2\pi_2$.

Suppose π_1 and ϖ_1 are surjective. Then, ϖ_2 is surjective.

Besides, suppose the following property holds: for any linear functionals φ_1 on V_1 and φ_2 on V_2 with $\pi_1^*\varphi_1 = \pi_2^*\varphi_2$, there exists θ in W^* with $\varpi_1^*\theta = \varphi_1$ and $\varpi_2^*\theta = \varphi_2$. Then, the linear map $\pi = \pi_1 \oplus \pi_2$ maps V onto the fibered product

$$X = \{v_1 + v_2 \in V_1 \oplus V_2 | \varpi_1(v_1) = \varpi_2(v_2)\}.$$

Proof. First, we claim that, if π_1 and ϖ_1 are surjective, then ϖ_2 is surjective. Indeed, if w is in W, as ϖ_1 is surjective, we may find v_1 in V_1 with $\varpi_1 v_1 = w$. As π_1 is surjective, we may find v in V with $\pi_1 v = v_1$. We get $\varpi_2\pi_2 v = \varpi_1\pi_1 v = w$ and we are done.

Assume besides the property of the statement holds and let X be the fibered product of V_1 and V_2 above W. The natural linear map $V_1^* \oplus V_2^* \to X^*$ is surjective and its null space is the space

$$Y = \{\varpi_1^*\theta - \varpi_2^*\theta | \theta \in W^*\} \subset V_1^* \oplus V_2^*.$$

As $\varpi_1\pi_1 = \varpi_2\pi_2$, the map π takes values in X. Therefore, to conclude, it suffices to show that the null space of π^* is Y. The latter is a direct consequence of the assumption. □

To deal with the boundary case $k = 2$ of Proposition 3.15, we shall need the following computation of a weight function.

Lemma 3.17. *Let J be a 1-pseudokernel and K^- be a 1-dual prekernel. Set $K = J^> + J^{>\vee}$, so that K is a 2-dual prekernel. Then a function u on X_1 is (K, K^-)-compatible if and only if, for $x \sim y$ in X, one has*

$$u(x, y) + u(y, x) = K^-_{xy}(x, y)$$

and the associated weight function is then given by, for (x, y) in X_2,

$$w(x, y) = u(x, z) + u(y, z) + J_{zx}(x, z) + J_{zy}(y, z) - K^-_{xz}(x, z) - K^-_{yz}(y, z)$$

where z is the middle point of the segment $[xy]$. In particular, if J and K^- are Γ-invariant, for any p in $\mathcal{F}_2$, we have

$$[p,(K,K^-)] = \frac{1}{2}\sum_{(x,y)\in\Gamma\backslash X_1}\frac{1}{|\Gamma_x\cap\Gamma_y|}\varphi_{p^-}(x,y)(J_{xy}(x,y)+J_{yx}(x,y)-K^-_{xy}(x,y)),$$

where φ_{p^-} is the quadratic type function on X_1 associated to the reduction p^- of p.

See Subsection I.4.2 for the relation between quadratic type functions and quadratic fields.

Proof. The formulae for u and w directly follow from Definition I.6.5 and Definition I.6.7. Now, by Theorem I.11.4, for p in $\mathcal{F}_2$, we have

$$[p,(K,K^-)] = \frac{1}{2}\sum_{(x,y)\in\Gamma\backslash X_2}\frac{1}{|\Gamma_x\cap\Gamma_y|}\varphi_p(x,y)w(x,y),$$

where φ_p is the quadratic type function on X_2 associated with p. Recall from Lemma I.4.12 that, for (x,y) in X_1, we have

$$\varphi_{p^-}(x,y)=\sum_{\substack{z\sim x\\ z\neq y}}\varphi_p(z,y)=\sum_{\substack{z\sim y\\ z\neq x}}\varphi_p(x,z),$$

so that the formula for w together with Lemma I.9.11 give

$$\begin{aligned}[p,(K,K^-)] = {} & \frac{1}{2}\sum_{(x,y)\in\Gamma\backslash X_1}\frac{1}{|\Gamma_x\cap\Gamma_y|}\varphi_{p^-}(x,y)(J_{xy}(x,y)+J_{yx}(x,y)) \\ & + \frac{1}{2}\sum_{(x,y)\in\Gamma\backslash X_1}\frac{1}{|\Gamma_x\cap\Gamma_y|}\varphi_{p^-}(x,y)(2u(x,y)-2K^-_{xy}(x,y)).\end{aligned}$$

As φ_{p^-} is symmetric, we have

$$\begin{aligned}\sum_{(x,y)\in\Gamma\backslash X_1} & \frac{1}{|\Gamma_x\cap\Gamma_y|}2\varphi_{p^-}(x,y)u(x,y) \\ & = \sum_{(x,y)\in\Gamma\backslash X_1}\frac{1}{|\Gamma_x\cap\Gamma_y|}\varphi_{p^-}(x,y)(u(x,y)+u(y,x)) \\ & = \sum_{(x,y)\in\Gamma\backslash X_1}\frac{1}{|\Gamma_x\cap\Gamma_y|}\varphi_{p^-}(x,y)K^-_{xy}(x,y)\end{aligned}$$

and the result follows. □

We will also use the following direct consequence of Lemma I.5.1.

Lemma 3.18. *Let A be a finite set and V_0 be the vector space of functions with zero sum on A. Let q be a symmetric bilinear form on V_0 and set $K(a,b) = q(\mathbf{1}_a - \mathbf{1}_b, \mathbf{1}_a - \mathbf{1}_b)$, $a, b \in A$. Let u be a function on A. Then the following are equivalent:*
(i) *one has $K(a,b) = u(a) + u(b)$, $a, b \in A$.*
(ii) *one has $q(f,g) = \sum_{a\in A} u(a)f(a)g(a)$, f, g in V_0.*

Proof of Proposition 3.15. For $k = 1$, we just need to show that the shoot map is surjective : this is clear since, for p in $\mathcal{F}_1$, we have $P_1(s) = p$, where s is the 2-quadratic pseudofield defined by $s_{xy} = \frac{1}{2}p_{xy}$, $x \sim y \in X$.

For $k = 2$, we apply Lemma 3.16 to $V = \mathcal{M}_2^1$, $V_1 = \mathcal{M}_1^1 = \mathcal{M}_1$, $V_2 = \mathcal{F}_2$ and $W = \mathcal{F}_1$. The assumption hold: indeed, we have $P_2(s)^- = P_1(s^-)$ for s in $\mathcal{M}_2^1$ by Corollary 3.11; the reduction map $\mathcal{M}_2^1 \to \mathcal{M}_1^1$ is surjective by Proposition 3.12 and we have just shown that the shoot map $P_1 : \mathcal{M}_1^1 \to \mathcal{F}_1$ is surjective.

Therefore, by Lemma I.C.5, Lemma 3.13 and Lemma 3.16, it suffices to show that, if (K, K^-) in $\mathcal{K}_2$ and L, M in $\mathcal{L}_1$ are such that, with the notation of Remark 2.8,

$$C_2(K, K^-) = K - K^{->} = L^{>\vee} + M^{\vee>\vee} - M^{>}, \tag{3.1}$$

then L is a 1-predual kernel, that is, $L_{xy} = L_{yx}$, $x \sim y \in X$, and

$$[p, (K, K^-)] = \frac{1}{2} \sum_{(x,y)\in\Gamma\backslash X_1} \varphi_{p^-}(x,y) L_{xy}(x,y), \quad p \in \mathcal{F}_2. \tag{3.2}$$

Indeed if (3.1) holds, the function $(x,y) \mapsto K_z(x,y), X_2 \to \mathbb{R}$ (where z is the middle point of the segment $[xy]$) is split in the sense of Subsection I.8.7, that is, we may write

$$K_z(x,y) = v(x,z) + v(y,z), \quad (x,y) \in X_2,$$

where z is as above and v is a Γ-invariant function on X_1. By Lemma 3.18, there exists a 1-pseudokernel J in $\mathcal{L}_1$ with $K = J^> + J^{>\vee}$. From (3.1), we get

$$J^> + J^{>\vee} - K^{->} = L^{>\vee} + M^{\vee>\vee} - M^>,$$

hence, by Lemma 2.5,

$$J - K^- = -M \text{ and } J = L + M^\vee,$$

which gives in particular $L = K^- - M - M^\vee$ and therefore $L^\vee = L$ as required. Now, we have

$$J + J^\vee = 2K^- - M - M^\vee = K^- + L,$$

hence

$$J + J^{\vee} - K^{-} = L$$

and, as $(K, K^{-}) = (J^{>} + J^{>\vee}, K^{-})$, (3.2) follows from Lemma 3.17.

For $k \geq 2$, we prove the statement by induction on k. We have just shown that it holds for $k = 2$. Now, we assume that $k \geq 3$ and that it has been established for $k - 1$. Again, we will apply Lemma 3.16 to $V = \mathcal{M}_k^1$, $V_1 = \mathcal{M}_{k-1}^1$, $V_2 = \mathcal{F}_k$ and $W = \mathcal{F}_{k-1}$. The assumption still hold: we have $P_k(s)^{-} = P_{k-1}(s^{-})$ for s in $\mathcal{M}_k^1$ by Corollary 3.11; the reduction map $\mathcal{M}_k^1 \to \mathcal{M}_{k-1}^1$ is surjective by Proposition 3.12 and the induction assumption implies that the shoot map $P_{k-1} : \mathcal{M}_{k-1}^1 \to \mathcal{F}_{k-1}$ is surjective.

By Lemma I.C.5, Lemma 3.13 and Lemma 3.16, it now suffices to show that, if (K, K^{-}) in $\mathcal{K}_k$ and L, M in $\mathcal{L}_{k-1}$ are such that, still with the notation of Remark 2.8,

$$C_k(K, K^{-}) = K - K^{->} = L^{>\vee} + M^{\vee>\vee} - M^{>}, \tag{3.3}$$

then there exists a $(k-1)$-dual kernel (J, J^{-}) in $\mathcal{K}_{k-1}$, with orthogonal extension (J^{+}, J), A in $\mathcal{L}_{k-1}$ and N in $\mathcal{L}_{k-2}$ such that

$$C_{k-1}(J, J^{-}) = L + N^{\vee>\vee} - N^{>} \tag{3.4}$$

and (K, K^{-}) is the sum of (J^{+}, J) and the k-dual kernel associated to A. First we note that, with the language of Subsection I.8.7, (3.3) implies that the pseudoweight of (K, K^{-}) is split. Thus, Proposition I.8.28 and Lemma I.8.30 say that there exists (J, J^{-}) in $\mathcal{K}_{k-1}$ and A in $\mathcal{L}_{k-1}$ such that (K, K^{-}) is the sum of the orthogonal extension of (J, J^{-}) and the k-dual kernel associated with A. Let now B be the pseudokernel with $B^{\vee} = A$. By Lemma 2.9 and Lemma 2.10, we get

$$C_k(K, K^{-}) = C_{k-1}(J, J^{-})^{>\vee} + B^{\vee>\vee} - B^{>}.$$

Together with (3.3), this gives

$$C_{k-1}(J, J^{-})^{>\vee} + B^{\vee>\vee} - B^{>} = L^{>\vee} + M^{\vee>\vee} - M^{>}.$$

Thus, Lemma 2.5 says that there exists C in $\mathcal{L}_{k-2}$ with

$$C_{k-1}(J, J^{-}) + B^{\vee} = C^{>} + L + M^{\vee} \text{ and } C^{\vee>} - B = -M.$$

We get $C^{\vee>} = B - M$ and (3.4) holds with $N = -C$. □

3.6. **The root of a quadratic field.** Proposition 3.15 shows in particular that, for any $k \geq 1$, the shoot map $\mathcal{M}_k^1 \to \mathcal{F}_k$ is surjective. We will now show that it actually admits a natural section.

Recall from Lemma I.4.14 that, if A is a finite set and $\overline{V}$ is the quotient of the space of functions on A by the constant ones, then a symmetric bilinear form p on $\overline{V}$ is completely determined by the numbers $p(\mathbf{1}_a, \mathbf{1}_b)$, $a, b \in A$, $a \neq b$.

Definition 3.19. (k even) Let $k \geq 2$ be an even integer, $k = 2\ell$, $\ell \geq 1$, and p be a k-quadratic field and φ_p be the associated quadratic type function on X_k. Define a k-quadratic pseudofield s as follows. For any $x \sim y$ in X and any $z \neq t$ in $S^\ell(x)$, we set

$$\begin{aligned} s_{xy}(\mathbf{1}_z, \mathbf{1}_t) &= 0 \quad \text{if } y \notin [xz] \cup [xt] \\ &= -\frac{1}{2}\varphi_p(z,t) \quad \text{if } [xy] \subset [zt] \\ &= -\varphi_p(z,t) \quad \text{if } y \in [xz] \cap [xt]. \end{aligned}$$

Then s is called the root of p.

Definition 3.20. (k odd) Let $k \geq 1$ be an odd integer, $k = 2\ell + 1$, $\ell \geq 0$, and p be a k-quadratic field and φ_p be the associated quadratic type function on X_k. Define a k-quadratic pseudofield s as follows. For any $x \sim y$ in X and any $z \neq t$ in $S^\ell(xy)$, we set

$$\begin{aligned} s_{xy}(\mathbf{1}_z, \mathbf{1}_t) &= 0 \quad \text{if } y \in [xz] \cap [xt] \\ &= -\frac{1}{2}\varphi_p(z,t) \quad \text{if } [xy] \subset [zt] \\ &= -\varphi_p(z,t) \quad \text{if } y \notin [xz] \cup [xt]. \end{aligned}$$

Then s is called the root of p.

Proposition 3.21. *Let $k \geq 1$, p be a k-quadratic field and s be the root of p. Then s is a radical quadratic pseudofield and p is the shoot of s. If $k \geq 2$, s^- is the root of p^-.*

If p is in $\mathcal{F}_k$, we write $R_k p$ for the root of p. Proposition 3.21 says that R_k sends $\mathcal{F}_k$ into $\mathcal{M}_k^1$ and that $P_k R_k$ is the identity map of $\mathcal{F}_k$.

Proof. Let $x \sim y$ be in X.

Assume k is even, $k = 2\ell$, $\ell \geq 1$. From the definition of s, we get, for $z \neq t$ in $S^\ell(x)$,

$$\begin{aligned} s^\vee_{xy}(\mathbf{1}_z, \mathbf{1}_t) &= -\varphi_p(z,t) \quad \text{if } y \notin [xz] \cup [xt] \\ &= -\frac{1}{2}\varphi_p(z,t) \quad \text{if } [xy] \subset [zt] \\ &= 0 \quad \text{if } y \in [xz] \cap [xt]. \end{aligned} \tag{3.5}$$

In particular, $s_{xy} + s^\vee_{xy} = p_x$. We also get, for $z \neq t$ in $S^{\ell-1}(xy)$, if $y \in [xz] \cap [xt]$,

$$s^{\vee<}_{xy}(\mathbf{1}_z, \mathbf{1}_t) = s^<_{xy}(\mathbf{1}_z, \mathbf{1}_t) = 0,$$

whereas, if $x \in [yz]$ and $y \in [xt]$,

$$s_{xy}^{\vee<}(\mathbf{1}_z, \mathbf{1}_t) = \sum_{\substack{u \sim z \\ u \notin [xz]}} s_{xy}^{\vee}(\mathbf{1}_u, \mathbf{1}_t) = -\frac{1}{2} \sum_{\substack{u \sim z \\ u \notin [xz]}} \varphi_p(u,t) = -\frac{1}{2}\varphi_{p^-}(z,t)$$

and, if $x \in [yz] \cap [yt]$,

$$s_{xy}^{\vee<}(\mathbf{1}_z, \mathbf{1}_t) = \sum_{\substack{u \sim z \\ u \notin [xz]}} \sum_{\substack{v \sim t \\ v \notin [xt]}} s_{xy}^{\vee}(\mathbf{1}_u, \mathbf{1}_v) = -\sum_{\substack{u \sim z \\ u \notin [xz]}} \sum_{\substack{v \sim t \\ v \notin [xt]}} \varphi_p(u,v) = -\varphi_p(z,t).$$

In other words, $s^{\vee<}$ is the root of p^-. Finally, still for $z \neq t$ in $S^{\ell-1}(xy)$, if y belongs to $[xz] \cap [xt]$, we have

$$s_{xy}^{<}(\mathbf{1}_z, \mathbf{1}_t) = -\varphi_{p^-}(z,t);$$

if $x \in [yz]$ and $y \in [xt]$,

$$s_{xy}^{<}(\mathbf{1}_z, \mathbf{1}_t) = \sum_{\substack{u \sim z \\ u \notin [xz]}} s_{xy}(\mathbf{1}_u, \mathbf{1}_t) = -\frac{1}{2} \sum_{\substack{u \sim z \\ u \notin [xz]}} \varphi_p(u,t) = -\frac{1}{2}\varphi_{p^-}(z,t);$$

and, if $x \in [yz] \cap [yt]$,

$$s_{xy}^{<}(\mathbf{1}_z, \mathbf{1}_t) = \sum_{\substack{u \sim z \\ u \notin [xz]}} \sum_{\substack{v \sim t \\ v \notin [xt]}} s_{xy}(\mathbf{1}_u, \mathbf{1}_v) = 0.$$

Comparing the formulae for $s_{yx}^{\vee<}$ and $s_{xy}^{<}$ shows that $s^{<} = s^{\vee<\vee}$, that is, s is radical.

Assume now k is odd, $k = 2\ell + 1$, $\ell \geq 0$. We get, for $z \neq t$ in $S^{\ell}(xy)$,

$$\begin{aligned} s_{xy}^{\vee}(\mathbf{1}_z, \mathbf{1}_t) &= -\varphi_p(z,t) \quad \text{if } y \in [xz] \cap [xt] \\ &= -\frac{1}{2}\varphi_p(z,t) \quad \text{if } [xy] \subset [zt] \\ &= 0 \quad \text{if } y \notin [xz] \cup [xt], \end{aligned}$$

so that in particular $s_{xy} + s_{xy}^{\vee} = p_{xy}$. Also, if $\ell \geq 1$, for $z \neq t$ in $S^{\ell}(x)$, if $y \in [xz] \cap [xt]$,

$$s_{xy}^{\vee<}(\mathbf{1}_z, \mathbf{1}_t) = s_{xy}^{\vee}(\mathbf{1}_z, \mathbf{1}_t) = -\varphi_p(z,t);$$

if $y \in [xt]$ and $y \notin [xz]$,

$$s_{xy}^{\vee<}(\mathbf{1}_z, \mathbf{1}_t) = \sum_{\substack{u \sim z \\ u \notin [xz]}} s_{xy}^{\vee}(\mathbf{1}_u, \mathbf{1}_t) = -\frac{1}{2}\varphi_{p^-}(z,t);$$

and, if $y \notin [xz] \cup [xt]$,

$$s_{xy}^{\vee<}(\mathbf{1}_z, \mathbf{1}_t) = \sum_{\substack{u \sim z \\ u \notin [xz]}} \sum_{\substack{v \sim t \\ v \notin [xt]}} s_{xy}^{\vee}(\mathbf{1}_u, \mathbf{1}_v) = 0.$$

Thus, $s^{\vee<}$ is the root of p^-. In particular, $s^{\vee<\vee}$ is given by (3.5) applied to p^-. Finally, again for $z \neq t$ in $S^\ell(xy)$, if $y \in [xz] \cap [zt]$, we have

$$s^<_{xy}(\mathbf{1}_z, \mathbf{1}_t) = s_{xy}(\mathbf{1}_z, \mathbf{1}_t) = 0;$$

if $y \in [xt]$ and $y \notin [xz]$,

$$s^<_{xy}(\mathbf{1}_z, \mathbf{1}_t) = \sum_{\substack{u\sim z\\ u\notin[xz]}} s_{xy}(\mathbf{1}_u, \mathbf{1}_t) = -\frac{1}{2}\varphi_{p^-}(z,t);$$

and, if $y \notin [xz] \cup [xt]$,

$$\begin{aligned} s^<_{xy}(\mathbf{1}_z, \mathbf{1}_t) = \sum_{\substack{u\sim z\\ u\notin[xz]}} \sum_{\substack{v\sim t\\ v\notin[xt]}} s_{xy}(\mathbf{1}_u, \mathbf{1}_v) = - \sum_{\substack{u\sim z\\ u\notin[xz]}} \sum_{\substack{v\sim t\\ v\notin[xt]}} \varphi_p(u,v) \\ = -\varphi_{p^-}(u,v). \end{aligned}$$

Comparing the latter three formulae with (3.5) applied to p^-, we get $s^< = s^{\vee<\vee}$, that is, s is radical. □

3.7. **Roots and pseudoweights.** Recall that in Subsection 2.4, we have drawn a link between the canonical map, the weight map and the pseudoweight construction for pseudokernels. In particular, there is a natural linear map that sends a pseudokernel in $\mathcal{L}_k$ to the cohomology class of the symmetrization of its pseudoweight, which is an element of $\mathcal{W}_k$. The next Lemma says that the root map R_k may be seen as the adjoint map of this latter map, when the space $\mathcal{W}_k$ is identified with the dual space of $\mathcal{F}_k$ through the weight pairing.

Proposition 3.22. *Let $k \geq 1$, p be in $\mathcal{F}_k$ and $s = R_k p$ be the root of p. Let L be in $\mathcal{L}_k$ and v be the pseudoweight of L. Then we have*

$$\langle s, L\rangle = \frac{1}{2} \sum_{(x,y)\in\Gamma\backslash X_k} \frac{1}{|\Gamma_x \cap \Gamma_y|}\varphi_p(x,y)v(x,y).$$

We start with a version of Proposition 3.22 for non-necessarily Γ-invariant objects.

Lemma 3.23. *Let $k \geq 1$, p be a k-quadratic field and s be the root of p. Let L be a finitely supported k-pseudokernel and v be the pseudoweight of L. Then v is finitely supported and we have*

$$\sum_{(x,y)\in X_1} \langle s_{xy}, L_{xy}\rangle = \frac{1}{2} \sum_{(x,y)\in X_k} \varphi_p(x,y)v(x,y).$$

Proof. We will check that the formula holds when L varies in a generating subset of the vector space of finitely supported k-pseudokernels.

Assume k is even, $k = 2\ell$, $\ell \geq 1$. We fix $x \sim y$ and $z \neq t$ in $S^\ell(x)$. Then, we define a $(k+1)$-pseudokernel L by setting

$$L_{ab}(c,d) = \mathbf{1}_{a=x}\mathbf{1}_{b=y}\mathbf{1}_{\{c,d\}=\{z,t\}}, \quad a \sim b \in X, \quad c \neq d \in S^\ell(a).$$

In other words, the symmetric bilinear form associated with L_{ab} on $V_0^\ell(a)$ is 0 if $(a,b) \neq (x,y)$, and the symmetric bilinear form associated with L_{xy} on $V_0^\ell(x)$ is

$$(\varphi,\psi) \mapsto -\frac{1}{2}\varphi(z)\psi(t) - \frac{1}{2}\varphi(t)\psi(z).$$

In particular, by Lemma I.C.5, we have

$$\sum_{(a,b)\in X_1} \langle s_{ab}, L_{ab}\rangle = -s_{xy}(\mathbf{1}_z, \mathbf{1}_t) \tag{3.6}$$

To conclude, we now need to compute the pseudoweight v of L. Corollary I.8.19 and Definition I.8.19 give

$$v(a,b) = \sum_{\substack{c\in S^\ell(a)\\ a\notin[bc]}} L_{cc_-}(a,b), \quad (a,b) \in X_k,$$

where as usual, for c as above, c_- is the neighbour of c on $[ac]$. By using the precise definition of L, this gives

$$v(a,b) = \mathbf{1}_{a=z}\mathbf{1}_{y\in[xz]}\mathbf{1}_{t\in[xb]} + \mathbf{1}_{a=t}\mathbf{1}_{y\in[xt]}\mathbf{1}_{z\in[xb]}, \quad (a,b) \in X_k. \tag{3.7}$$

Now we apply (3.6) and (3.7) to the three cases in Definition 3.19. If $y \notin [xz] \cup [xt]$, we get $v = 0$ and $\sum_{(a,b)\in X_1} \langle s_{ab}, L_{ab}\rangle = 0$. If $y \in [xz]$ and $y \notin [xt]$, we get $v = \mathbf{1}(z,t)$, hence

$$\sum_{(a,b)\in X_k} \varphi_p(a,b)v(a,b) = \varphi_p(z,t) = -2s_{xy}(\mathbf{1}_z,\mathbf{1}_t) = 2\sum_{(a,b)\in X_1}\langle s_{ab}, L_{ab}\rangle.$$

If $y \in [xz] \cap [xt]$, set $h = k - d(z,t) \geq 2$. As φ_p is a quadratic type function, we have

$$\begin{aligned}\sum_{(a,b)\in X_k} \varphi_p(a,b)v(a,b) &= \sum_{\substack{b\in S^h(t)\\ [tb]\cap[xt]=\{t\}}} \varphi_p(z,b) + \sum_{\substack{b\in S^h(z)\\ [zb]\cap[xz]=\{z\}}} \varphi_p(t,b)\\ &= 2\varphi_p(z,t) = -2s_{xy}(\mathbf{1}_z,\mathbf{1}_t) = 2\sum_{(a,b)\in X_1}\langle s_{ab}, L_{ab}\rangle.\end{aligned}$$

The case where k is even follows.

Assume k is odd, $k = 2\ell+1$, $\ell \geq 0$. We now fix $x \sim y$ and $z \neq t$ in $S^\ell(xy)$. We define a k-pseudokernel L by setting

$$L_{ab}(c,d) = \mathbf{1}_{a=x}\mathbf{1}_{b=y}\mathbf{1}_{\{c,d\}=\{z,t\}}, \quad a \sim b \in X, \quad c \neq d \in S^\ell(ab).$$

Still by Lemma I.C.5, we have

$$\sum_{(a,b)\in X_1} \langle s_{ab}, L_{ab}\rangle = -s_{xy}(\mathbf{1}_z, \mathbf{1}_t) \tag{3.8}$$

Again, we compute the pseudoweight v of L. Corollary I.8.19 and Definition I.8.19 give

$$v(a,b) = \sum_{\substack{c\in S^{\ell+1}(a)\\ a\notin[bc]}} L_{c_-c}(a,b), \quad (a,b)\in X_k.$$

This gives

$$v(a,b) = \mathbf{1}_{a=z}\mathbf{1}_{y\notin[xz]}\mathbf{1}_{t\in[xb]} + \mathbf{1}_{a=t}\mathbf{1}_{y\notin[xt]}\mathbf{1}_{z\in[xb]}, \quad (a,b)\in X_k. \tag{3.9}$$

We now apply (3.8) and (3.9) to the three cases in Definition 3.20. If $y\in[xz]\cap[xt]$, we get $v=0$ and $\sum_{(a,b)\in X_1}\langle s_{ab}, L_{ab}\rangle = 0$. If $y\notin[xz]$ and $y\in[xt]$, we get $v=\mathbf{1}(z,t)$, hence

$$\sum_{(a,b)\in X_k} \varphi_p(a,b)v(a,b) = \varphi_p(z,t) = -2s_{xy}(\mathbf{1}_z,\mathbf{1}_t) = 2\sum_{(a,b)\in X_1}\langle s_{ab}, L_{ab}\rangle.$$

If $y\notin[xz]\cup[xt]$, set $h=k-d(z,t)\geq 1$. As φ_p is a quadratic type function, we have

$$\begin{aligned}\sum_{(a,b)\in X_k} \varphi_p(a,b)v(a,b) &= \sum_{\substack{b\in S^h(t)\\ [tb]\cap[xt]=\{t\}}} \varphi_p(z,b) + \sum_{\substack{b\in S^h(z)\\ [zb]\cap[xz]=\{z\}}} \varphi_p(t,b)\\ &= 2\varphi_p(z,t) = -2s_{xy}(\mathbf{1}_z,\mathbf{1}_t) = 2\sum_{(a,b)\in X_1}\langle s_{ab}, L_{ab}\rangle\end{aligned}$$

and the case where k is odd follows. □

We will deduce Proposition 3.22 from Lemma 3.23. To this aim, we need to show that every Γ-invariant pseudokernel can be obtained from a finitely supported one.

Lemma 3.24. *Let $k\geq 1$ and L be a finitely supported k-pseudokernel. Set $\overline{L} = \sum_{\gamma\in\Gamma}\gamma L$. Then the map $L\mapsto\overline{L}$ maps the space of finitely supported k-pseudokernels onto $\mathcal{L}_k$.*

Proof. Indeed, fix $S\subset X_1$ a section of the quotient map $X_1\to\Gamma\backslash X_1$, that is, $X_1=\Gamma S$ and $\Gamma s\cap S=\{s\}$ for any s in S. If M is in $\mathcal{L}_k$, for $(x,y)\in X_1$, set $L_{xy}=\frac{1}{|\Gamma_x\cap\Gamma_y|}M_{xy}$ if (x,y) is in S and $L_{xy}=0$ else. Then L is a finitely supported k-pseudokernel and $\overline{L}=M$. □

To relate summation formulae on X and on $\Gamma\backslash X$, we shall use the standard

Lemma 3.25. *Let A be a set and G be a group acting on A such that, for every a in A, its stabilizer G_a in G is finite. If φ is a finitely supported function on A, set*

$$\overline{\varphi}(a) = \sum_{g \in G} \varphi(ga), \quad a \in A.$$

Then, we have

$$\sum_{a \in A} \varphi(a) = \sum_{a \in G \backslash A} \frac{1}{|G_a|} \overline{\varphi}(a).$$

Proof of Proposition 3.22. By Lemma 3.24, it suffices to prove the result for Γ-invariant k-pseudokernels which are of the form $\overline{L}$ where L is a finitely supported k-pseudokernel. For such a L, let v be its pseudoweight. If p is in $\mathcal{F}_k$ and s is the root of p, Lemma 3.23 gives

$$\sum_{(x,y) \in X_1} \langle s_{xy}, L_{xy} \rangle = \frac{1}{2} \sum_{(x,y) \in X_k} \varphi_p(x,y) v(x,y).$$

Now, the pseudoweight of $\overline{L}$ is $\overline{v} = \sum_{\gamma \in \Gamma} \gamma v$, and Lemma 3.25 gives

$$\sum_{(x,y) \in X_1} \langle s_{xy}, L_{xy} \rangle = \sum_{(x,y) \in \Gamma \backslash X_1} \frac{1}{|\Gamma_x \cap \Gamma_y|} \langle s_{xy}, \overline{L}_{xy} \rangle = \langle s, \overline{L} \rangle$$

and

$$\sum_{(x,y) \in X_k} \varphi_p(x,y) v(x,y) = \sum_{(x,y) \in \Gamma \backslash X_k} \frac{1}{|\Gamma_x \cap \Gamma_y|} \varphi_p(x,y) \overline{v}(x,y).$$

The result follows. $\square$

4. Tight quadratic fields

For $k \geq 2$, recall that $\mathcal{W}_k$ stands for the vector space of cohomology classes of Γ-invariant symmetric functions on X_k. We say that the cohomology class of such a function w is non-negative if the associated bilinear form Φ_w on H_0^ω is non-negative and we let $\mathcal{W}_k^+$ be the closed convex cone of non-negative cohomology classes. By Theorem I.7.17, the convex cone $\mathcal{W}_k^+$ is the image of the convex cone $\mathcal{K}_k^+$ of non-negative Γ-invariant k-dual kernels by the weight map $W_k : \mathcal{K}_k \to \mathcal{W}_k$.

Now Propositions I.4.11 and I.11.2 say that the space $\mathcal{F}_k$ of Γ-invariant k-quadratic fields may be considered as the dual space of $\mathcal{W}_k$ through the weight pairing. We say that a p in $\mathcal{F}_k$ is tight if, for any non-negative (K, K^-) in $\mathcal{K}_k$, one has $[p, (K, K^-)] \geq 0$. We denote by $\mathcal{F}_k^{\mathrm{tgt}} \subset \mathcal{F}_k$ the convex cone of tight Γ-invariant k-quadratic fields. It can be seen as the dual cone of $\mathcal{W}_k^+$ (see Subsection 4.2 below).

We say that a k-quadratic pseudofield s is non-negative if all the symmetric bilinear forms s_{xy}, $x \sim y \in X$, are non-negative. We denote by $\mathcal{M}_k^+ \subset \mathcal{M}_k$ the closed convex cone of Γ-invariant non-negative k-quadratic pseudofields and we set $(\mathcal{M}_k^1)^+ = \mathcal{M}_k^1 \cap \mathcal{M}_k^+$ to be the closed convex cone of Γ-invariant non-negative radical k-quadratic pseudofields.

Theorem 4.1. *Let $k \geq 2$. Then a Γ-invariant k-quadratic field is tight if and only if it is the shoot of a Γ-invariant non-negative radical k-quadratic pseudofield. In other words, we have $\mathcal{F}_k^{\mathrm{tgt}} = P_k((\mathcal{M}_k^1)^+)$.*

The proof of this result is the objective of this section.

4.1. **Non-negative extensions.** In this subsection, we prove a technical result that will allow us to show that certain pseudokernels are non-negative by knowing that some extensions of them are.

Proposition 4.2. *Let $k \geq 2$ and L, M be in $\mathcal{L}_k$. The $(k+1)$-pseudokernel $L^> + M^{>\vee}$ is non-negative if and only if there exists A, B in $\mathcal{L}_{k-1}$ and C, D in $\mathcal{L}_k^+$ with $L = A^> + C$, $M = B^> + D$ and $A + B^\vee$ non-negative.*

The proof relies on an adaptation of the argument in Lemma I.8.12.

Lemma 4.3. *Let $W_0, W_1, \ldots, W_d$ ($d \geq 2$) be finite-dimensional real vector spaces and, for $1 \leq i \leq d$, let $\varpi_i : W_i \to W_0$ be a surjective linear map. We set W to be the fibered product*

$$\{w = (w_1, \cdots, w_d) \in W_1 \times \cdots \times W_d | \forall 1 \leq i, j \leq d \quad \varpi_i(w_i) = \varpi_j(w_j)\}$$

and $\pi_i : W \to W_i$, $0 \leq i \leq d$, to be the natural surjective linear map. Assume $q_1, \ldots q_d$ to be symmetric bilinear forms on $W_1, \ldots, W_d$ and set $q = \pi_1^\star q_1 + \cdots + \pi_d^\star q_d$. Assume q is positive definite. Then the following holds.

For $1 \leq i \leq d$, the restriction r_i of q_i to $U_i = \ker \varpi_i$ is positive definite and we have $W_i = U_i \oplus V_i$ where V_i is the q_i-orthogonal subspace to U_i, that is,

$$V_i = \{v \in W_i | \forall u \in U_i \quad q_i(u, v) = 0\}.$$

Let $\alpha_i : W_i \to U_i$ be the projection in this decomposition. Then, there exists a unique symmetric bilinear form p_i on W_0 with $q_i = \alpha_i^\star r_i + \varpi_i^\star p_i$.

The symmetric bilinear form $p = p_1 + \cdots + p_d$ on W_0 is positive definite.

Proof. Fix $1 \leq i \leq d$ and pick $v_i \neq 0$ in V_i with $\varpi_i(v_i) = 0$. For $j \neq i$, we set $v_j = 0$, so that the vector $v = (v_j)_{1 \leq j \leq d}$ belongs to W and we have $\pi_i(v) = v_i$. By definition, we have $r_i(v_i, v_i) = q_i(v_i, v_i) = q(v, v) >$

0 and r_i is positive definite. In particular, it is non-degenerate, so that $U_i \cap V_i = \{0\}$. Now, the natural map $w_i \mapsto q_i(w_i, .)$ from W_i to the dual space W_i^* induces an injective linear map $W_i/U_i \hookrightarrow V_i^*$, so that we have $\dim W_i - \dim U_i \leq \dim V_i$, that is, $\dim W_i \leq \dim(U_i \oplus V_i)$ and we get $W_i = U_i \oplus V_i$.

Let $\alpha_i : W_i \to U_i$ and $\beta_i : W_i \to V_i$ be the projections in this decomposition. As U_i and V_i are q_i-orthogonal, we have $q_i = \alpha_i^\star r_i + \beta_i^\star s_i$, where s_i is the restriction of q_i to V_i. Now, ϖ_i induces an isomorphism from U_i onto W_0, so that there exists a unique symmetric bilinear form p_i on W_0 with $\beta_i^\star s_i = \varpi_i^\star p_i$.

Finally, we set $V \subset W$ to be the subspace defined by

$$V = \{w \in W | \forall 1 \leq i \leq d \quad \pi_i(w) \in V_i\}.$$

By construction, the restriction of q to V is equal to $\pi_0^\star(p_1 + \cdots + p_d)$, hence $p_1 + \cdots + p_d$ is positive definite on W_0. □

The proof of Proposition 4.2 also uses an elementary geometric property of convex cones. Recall that a closed convex cone in a finite-dimensional vector space is said to be proper if it contains no vector line.

Lemma 4.4. *Let V be a finite-dimensional vector space and $\mathcal{C} \subset V$ be a proper closed convex cone. Then, for every u in V, the set $\mathcal{C} \cap (u - \mathcal{C})$ is compact in V.*

Proof. We prove the contraposition. Assume that there exists u in V such that $\mathcal{C} \cap (u - \mathcal{C})$ is not compact and let us show that $\mathcal{C}$ is not proper. Equip V with a norm $\|.\|$. Let (v_n) be a sequence of elements of $\mathcal{C}$ with $u - v_n \in \mathcal{C}$ for any n and $\|v_n\| \xrightarrow[n\to\infty]{} \infty$. Then, after extracting, we can assume that there exists w in V with $\|v_n\|^{-1} v_n \xrightarrow[n\to\infty]{} w$. As v_n is in $\mathcal{C}$, we have $w \in \mathcal{C}$. As $u - v_n$ is in $\mathcal{C}$ and $\|v_n\| \xrightarrow[n\to\infty]{} \infty$, we also have $-w \in \mathcal{C}$. The Lemma follows. □

Proof of Proposition 4.2. Note that one direction of the equivalence is obvious. We prove the other one. Let L, M be in $\mathcal{L}_k$ with $L^> + M^{>\vee}$ non-negative.

We will first establish the case where $L^> + M^{>\vee}$ belongs to the interior of the cone $\mathcal{L}_k^+$, and then use an approximation argument to deal with the general case. Thus, we first assume that, for any $x \sim y$ in X, the symmetric bilinear form associated to $(L + M^\vee)_{xy}$ by Lemma I.5.1 is positive definite. In that case, Proposition I.4.5, Proposition I.4.6 and Lemma 4.3 tell us that we may write $L = A^> + C$ and $M = B^> + D$

where C and D are in $\mathcal{L}_k^+$ and A and B are in $\mathcal{L}_{k-1}$ and that the $(k-1)$-pseudokernel $A + B^\vee$ is non-negative, which should be proved.

Now, we will use an approximation argument to conclude. We pick a Γ-invariant exact k-dual kernel (K, K^-) (see Definition I.5.12 and Definition I.5.13 for this notion). For example, (K, K^-) can be the k-dual kernel obtained by orthogonal extension from the harmonic kernel (see Subsection I.5.5, Subsection I.9.6 and Subsection I.10.5). We consider the k-dual prekernel K as a k-pseudokernel and the $(k-1)$-dual prekernel K^- as a $(k-1)$-pseudokernel. We note that the orthogonal extension (K^+, K) of (K, K^-) may be defined by $K^+ = K^> + (K - K^{->})^{>\vee}$. For $0 < \varepsilon \leq 1$, we set $L_\varepsilon = L + \varepsilon K$ and $M_\varepsilon = M + \varepsilon(K - K^{->})$, so that $L_\varepsilon^> + M_\varepsilon^{>\vee} = L^> + M^{>\vee} + \varepsilon K^+$ and the symmetric bilinear forms associated to this $(k+1)$-pseudokernel are positive definite. Therefore, there exists C_ε and D_ε in $\mathcal{L}_k^+$ and A_ε and B_ε in $\mathcal{L}_{k-1}$ such that $A_\varepsilon + B_\varepsilon^\vee$ is non-negative and $L_\varepsilon = A_\varepsilon^> + C_\varepsilon$ and $M_\varepsilon = B_\varepsilon^> + D_\varepsilon$. In particular, we have

$$\begin{aligned} L^> + M^{>\vee} + \varepsilon K^+ = L_\varepsilon^> + M_\varepsilon^{>\vee} &= A_\varepsilon^{>>} + B_\varepsilon^{>>\vee} + C_\varepsilon^> + D_\varepsilon^{>\vee} \\ &= (A_\varepsilon + B_\varepsilon^\vee)^{>>} + C_\varepsilon^> + D_\varepsilon^{>\vee} \end{aligned}$$

(where we have used Lemma 2.4). We get, for $0 < \varepsilon \leq 1$,

$$C_\varepsilon^> \in \mathcal{L}_{k+1}^+ \cap (L^> + M^{>\vee} + K^+ - \mathcal{L}_{k+1}^+).$$

As $\mathcal{L}_{k+1}^+$ clearly is a proper closed convex cone in the finite-dimensional vector space $\mathcal{L}_{k+1}$, by Lemma 4.4, the set $\mathcal{L}_{k+1}^+ \cap (L^> + M^{>\vee} + K^+ - \mathcal{L}_{k+1}^+)$ is compact. As the extension map $L \mapsto L^>$ is injective, the set $\{C_\varepsilon | 0 < \varepsilon \leq 1\}$ is bounded in $\mathcal{L}_k$. In the same way, for any $0 < \varepsilon \leq 1$, we have

$$D_\varepsilon^{>\vee} \in \mathcal{L}_{k+1}^+ \cap (L^> + M^{>\vee} + K^+ - \mathcal{L}_{k+1}^+),$$

hence the set $\{D_\varepsilon | 0 < \varepsilon \leq 1\}$ is bounded in $\mathcal{L}_k$. Therefore, we can find a sequence $\varepsilon_n \xrightarrow[n\to\infty]{} 0$ and C, D in $\mathcal{L}_k^+$ with $C_{\varepsilon_n} \xrightarrow[n\to\infty]{} C$ and $D_{\varepsilon_n} \xrightarrow[n\to\infty]{} D$. For any n, we have $L_{\varepsilon_n} = A_{\varepsilon_n}^> + C_{\varepsilon_n}$. Hence, A_{ε_n} has a limit A in $\mathcal{L}_{k-1}$ and $L = A^> + C$. In the same way, B_{ε_n} has a limit B in $\mathcal{L}_{k-1}$ and $M = B^> + D$. As for any n, $A_{\varepsilon_n} + B_{\varepsilon_n}^\vee$ is non-negative, so is $A + B^\vee$ and we are done. □

Let us draw a first consequence of this Lemma that will be useful shortly.

Corollary 4.5. *Let $k \geq 1$ and L be in $\mathcal{L}_k$. If $L^{\vee>\vee} - L^>$ is non-negative, then $L = 0$.*

Proof. We prove this result by induction on $k \geq 1$.

If $k = 1$, for $x \sim y$ in X, we set $u(x,y) = L_{xy}(x,y)$. By the assumption, for any x in X and f in $V_0^1(x)$, we have

$$\sum_{\substack{z\sim x\\ z\neq y}} u(z,x)f(z)^2 \geq u(x,y)f(y)^2. \tag{4.1}$$

For $y \sim x$, applying this property to the function

$$f = \mathbf{1}_y - \frac{1}{d(x)-1}\sum_{\substack{z\sim x\\ z\neq y}} \mathbf{1}_z,$$

we get

$$\frac{1}{(d(x)-1)^2}\sum_{\substack{z\sim x\\ z\neq y}} u(z,x) \geq u(x,y). \tag{4.2}$$

Define a linear operator T on the space F_1 of Γ-invariant functions on X_1 by setting

$$Tv(x,y) = \frac{1}{(d(x)-1)^2}\sum_{\substack{z\sim x\\ z\neq y}} v(z,x), \quad v \in F_1, \quad x \sim y \in X.$$

As in the proof of Proposition I.10.13, since $d(x) \geq 3$ for any x in X, the operator T has operator norm $\leq \frac{1}{2}$ when F_1 is equipped with the supremum norm. In particular, $T^n \xrightarrow[n\to\infty]{} 0$. Now, (4.2) reads as $Tu \geq u$. As T maps non-negative functions to non-negative functions, we get $T^n u \geq u$ for any integer $n \geq 0$, which gives $u \leq 0$. Fix x in X and choose y, z and t to be three pairwise different neighbours of x: this is possible as $d(x) \geq 3$. By applying (4.1) to the function $f = \mathbf{1}_z - \mathbf{1}_t$, we obtain

$$u(z,x) + u(t,x) \geq 0.$$

As $u \leq 0$, we get $u = 0$ as required.

Assume $k \geq 2$ and the result holds for $k-1$. By Proposition 4.2, we can find A, B in $\mathcal{L}_{k-1}$ and C, D in $\mathcal{L}_k^+$ with $-L = A^{>} + C$ and $L^{\vee} = B^{>} + D$ and $A + B^{\vee}$ is non-negative. We get

$$B^{\vee>\vee} - B^{>} \geq -A^{>\vee} - B^{>} = (L+C)^{\vee} + (-L^{\vee} + D) = C^{\vee} + D \geq 0.$$

By the induction assumption, we get $B = 0$. In particular A is non-negative. As $-L = A^{>} + C$, L is non-positive, hence so is $L^{\vee}$. As $L^{\vee} = D$, $L^{\vee}$ is non-negative. We get $L = 0$ as required. □

4.2. **Convex cones.** We recall some notions of convex geometry and we show that there exists a duality between the notion of a non-negative quadratic pseudofield and the one of a non-negative pseudokernel.

Let V be a finite dimensional vector space with dual space V^*. Assume $\mathcal{C} \subset V$ is a closed convex cone. The dual cone of $\mathcal{C}$ is the set of φ in V^* with $\langle \varphi, x \rangle \geq 0$ for any x in $\mathcal{C}$. This is a closed convex cone and an application of the geometric form of Hahn-Banach theorem gives $\mathcal{C}^{**} = \mathcal{C}$.

We know how to transport these objects under linear maps.

Lemma 4.6. *Let V, W be finite dimensional vector spaces and $T : V \to W$ be a linear map. Let $\mathcal{C}$ be a closed convex cone in V with $\mathcal{C} \cap \ker T = \{0\}$. Then $T\mathcal{C}$ is closed in W.*

Proof. As every linear map is the composition of an injective linear map with a surjective one, and as the result is obvious in the injective case, we can assume that T is surjective. Note also that it suffices to prove the result when the null space of T has dimension 1, the general case following from a direct induction argument.

In other words, we are reduced to prove that, for any v in $V \smallsetminus (\mathcal{C} \cup (-\mathcal{C}))$, the set $\mathbb{R}v + \mathcal{C}$ is closed in V. Let (t_n) be a sequence in $\mathbb{R}$ and (u_n) be a sequence in $\mathcal{C}$ with $u_n - t_n v \xrightarrow[n\to\infty]{} w$ for w in V. If (t_n) is bounded, we may assume that (t_n) is convergent and the conclusion follows. If not, after extracting a subsequence, and up to replacing v by $-v$, we may assume that $t_n \xrightarrow[n\to\infty]{} +\infty$. In particular, we have $t_n^{-1} u_n \xrightarrow[n\to\infty]{} v$, hence v belongs to $\mathcal{C}$, a contradiction. $\square$

Remark 4.7. The assumption in Lemma 4.6 is necessary. Let $\mathcal{C} \subset \mathbb{R}^3$ be the cone

$$\mathcal{C} = \{(x, y, z) \in \mathbb{R}^3 | x^2 \geq y^2 + z^2 \text{ and } x \geq 0\}$$

and T be the linear map from $\mathbb{R}^3$ to $\mathbb{R}^2$, $(x, y, z) \mapsto (x - y, z)$. A direct computation shows that we have

$$T\mathcal{C} = \{(u, v) \in \mathbb{R}^2 | u > 0\} \cup \{(0, 0)\}.$$

Thus, the convex cone $\mathcal{C}$ is closed, but $T\mathcal{C}$ is not.

Corollary 4.8. *Let V, W be finite dimensional vector spaces and $T : V \to W$ be a linear map. Let $\mathcal{C}$ be a closed convex cone in V with $\mathcal{C} \cap \ker T = \{0\}$, then we have $(T\mathcal{C})^* = (T^*)^{-1}(\mathcal{C}^*)$.*

Here comes a first case where we can determine a dual cone.

Lemma 4.9. *Let V be a finite-dimensional vector space. Then, with respect to the quadratic duality, the dual cone of the convex cone $\mathcal{Q}_+(V)$ of non-negative symmetric bilinear forms on V is the cone $\mathcal{Q}_+(V^*)$.*

See Appendix I.C for the definition of the quadratic duality.

Proof. Let q in $\mathcal{Q}(V^*)$. If q belongs to the dual convex cone of $\mathcal{Q}_+(V)$, then for every φ in V^*, as φ^2 belongs to $\mathcal{Q}_+(V)$, we have $q(\varphi,\varphi) = \langle q, \varphi^2\rangle \geq 0$. Conversely, every non-negative symmetric bilinear form p on V may be written as $p = \varphi_1^2 + \cdots + \varphi_r^2$ for some $\varphi_1, \ldots, \varphi_r$ in V^*. Thus, if q is in $\mathcal{Q}_+(V^*)$, we have

$$\langle q, p\rangle = \langle q, \varphi_1^2\rangle + \cdots + \langle q, \varphi_r^2\rangle = q(\varphi_1,\varphi_1) + \cdots + q(\varphi_r,\varphi_r) \geq 0,$$

hence q belongs to the dual cone of $\mathcal{Q}_+(V)$. □

From this and the definition of the duality between quadratic pseudofields and pseudokernels in Subsection I.10.2, we get

Corollary 4.10. *Let $k \geq 1$. Then the convex cone $\mathcal{M}_k^+$ of non-negative Γ-invariant k-quadratic pseudofields is the dual cone of the convex cone $\mathcal{L}_k^+$ of non-negative Γ-invariant k-pseudokernels.*

4.3. **Non-negative quadratic pseudofields, non-negative pseudokernels.** We will now finish the proof of Theorem 4.1. This relies on the notions introduced above and on Proposition 4.2.

To check the assumption of Lemma 4.6, we will use the easy

Lemma 4.11. *Let $k \geq 2$ and s be in $(\mathcal{M}_k^1)^+$ a non-negative Γ-invariant radical $(k+1)$-quadratic pseudofield. If $P_k s = 0$, then $s = 0$.*

Proof. Indeed, if a sum of a non-negative symmetric bilinear forms is zero, each of them is zero. □

Proof of Theorem 4.1. Let us denote by $I_k : \mathcal{M}_k^1 \hookrightarrow \mathcal{M}_k$ the inclusion map. Then the null space of the adjoint map $I_k^* : \mathcal{L}_k \to (\mathcal{M}_k^1)^*$ is the space $\{M^{\vee>\vee} - M^{>} | M \in \mathcal{L}_{k-1}\}$. Corollary 4.5 and Lemma 4.6 imply that $I_k^*\mathcal{L}_k^+$ is closed in $(\mathcal{M}_k^1)^*$ and Corollary 4.8 implies that the dual cone of $I_k^*\mathcal{L}_k^+$ is $(\mathcal{M}_k^1)^+$.

Now, Theorem I.8.32 and Corollary 2.12 say that the map $I_k^* C_k$ gives rise to a linear embedding $\overline{C}_k : \mathcal{W}_k \to (\mathcal{M}_k^1)^*$. Corollary 3.14 says that, when $\mathcal{F}_k$ is identified with the dual space of $\mathcal{W}_k$ through the weight pairing, the adjoint operator of $\overline{C}_k$ is the shoot map $P_k : \mathcal{M}_k^1 \to \mathcal{F}_k$. By Lemma 4.6 and Lemma 4.11, the convex cone $P_k((\mathcal{M}_k^1)^+)$ is closed in $\mathcal{F}_k$. By Corollary 4.8, its dual cone in $\mathcal{W}_k$ is $\overline{C}_k^{-1} I_k^*\mathcal{L}_k^+$. The result follows since Proposition 2.13 implies that $\overline{C}_k^{-1} I_k^*\mathcal{L}_k^+ = \mathcal{W}_k^+$ □

4.4. **Reduction of nonnegative radical quadratic pseudofields.** To conclude this Section, we will now prove a refinement of Proposition 3.12, namely that reduction is surjective over non-negative radical quadratic pseudofields.

Proposition 4.12. *Let $k \geq 2$. Then the reduction map $s \mapsto s^-$ maps the convex cone $(\mathcal{M}^1_{k+1})^+$ of non-negative Γ-invariant radical $(k+1)$-quadratic pseudofields onto the convex cone space $(\mathcal{M}^1_k)^+$ of non-negative Γ-invariant radical k-quadratic pseudofields.*

Again, the proof will follow from a duality argument. To check the assumption in Lemma 4.6, we will need

Lemma 4.13. *Let $k \geq 1$ and s be a nonnegative radical $(k+1)$-quadratic pseudofield. If $s^- = 0$, we have $s = 0$.*

Proof. Note that, if V is a real vector space and $V_1, \ldots, V_d$ are subspaces which span V, then a nonnegative symmetric bilinear form on V is zero if and only if its restriction to V_i is zero for any $1 \leq i \leq d$.

Now, assume k is odd, $k = 2\ell+1$, $\ell \geq 0$. In that case, by Proposition I.4.5, for x in X, the space $\overline{V}^{\ell+1}(x)$ is spanned by the spaces $I^\ell_{xy}\overline{V}^\ell(xy)$, $y \sim x$. Fix $y \sim x$. As $s^{\vee<} = s^- = 0$, we have

$$\sum_{\substack{z\sim x\\ z\neq y}} (I^\ell_{xy})^\star s_{xz} = 0,$$

hence, as all these symmetric bilinear forms on $\overline{V}^\ell(xy)$ are non-negative, for all $z \sim x$, $z \neq y$, $(I^\ell_{xy})^\star s_{xz} = 0$. As s is radical, we have $s^< = s^{\vee<\vee} = 0$, that is $(I^\ell_{xy})^\star s_{xy} = 0$. We have proved that, for all x in X, for any $y, z \sim x$, $(I^\ell_{xy})^\star s_{xz} = 0$. By the remark above, we get $s = 0$ as required. The proof in the odd case is analoguous. $\square$

Here comes the main argument of the proof.

Lemma 4.14. *Let $k \geq 2$ and L be in $\mathcal{L}_k$. Assume that there exists M in $\mathcal{L}_k$ such that $L^{>\vee} + M^{\vee>\vee} - M^>$ is non-negative. Then, there exists N in $\mathcal{L}_{k-1}$ such that $L + N^{\vee>\vee} - N^{>\vee}$ is non-negative.*

Proof. Indeed, Proposition 4.2 implies that there exists A, B in $\mathcal{L}_{k-1}$ and C, D in $\mathcal{L}^+_k$ such that $A + B^\vee$ is non-negative and $-M = A^> + C$ and $L + M^\vee = B^> + D$. We get

$$L = B^> + D - M^\vee \geq B^> - M^\vee = B^> + A^{>\vee} + C^\vee \geq B^> - B^{\vee>\vee}$$

and the result follows. $\square$

Example 4.15. In general, the statement of Lemma 4.14 is false for $k = 1$. Indeed, let A be a finite set with at least three elements. There exists a function $u : A \to \mathbb{R}$ which takes a negative value but which is such that $\sum_{a\in A} u(a)f(a)^2 \geq 0$ for any function f on A with $\sum_{a\in A} f(a) = 0$. Let Γ be the group of Example I.2.1 associated with A. Then, the data of u determines the data of L in $\mathcal{L}_1 \smallsetminus \mathcal{L}_1^+$ such that $L^\vee = L$ and $K = L^> + L^{>\vee}$ is nonnegative. Let w be a weight function for (K, L) and Φ_w be the associated quadratic form on H_0^ω. For (x, y) in X_1, denote by $v(x, y) = v(y, x)$ the value of u on the element of A associated with the edge (x, y). From Lemma I.5.9 and Corollary I.7.9, a direct computation gives, for any θ in H_0^ω,

$$\Phi_w(\theta, \theta) = \frac{1}{2} \sum_{(x,y)\in X_1} v(x, y)\theta(U_{xy})^2 = \frac{1}{2} \sum_{x\in X} \sum_{y\sim x} v(x, y)\theta(U_{xy})^2.$$

In particular, Φ_w is non-negative, so that, by Theorem I.7.17 and Theorem I.8.32, there exists a 2-pseudokernel M such that $(K, L) + M$ is a non-negative 2-dual kernel, and hence, the canonical 2-pseudokernel N of $(K, L) + M$ is non-negative. By Lemma 2.10, since $L^\vee = L$, we have

$$N = L^> + M^{>\vee} - M^{\vee>} = L^{>\vee} + L^> - L^{\vee>\vee} + M^{>\vee} - M^{\vee>}.$$

In other words, setting $A = M^\vee - L$, we get that $L^> + A^{\vee>\vee} - A^>$ is non-negative although L is not.

Proof of Proposition 4.12. Note that we have $((\mathcal{M}_{k+1}^1)^+)^{\vee<} \subset (\mathcal{M}_k^1)^+$ by definition and we only need to prove the reverse inclusion. By Lemma 4.6 and Lemma 4.13, $((\mathcal{M}_{k+1}^1)^+)^{\vee<}$ is closed in $\mathcal{M}_k^1$, so that we can prove the result by looking at the dual cones. By Corollary 4.5, Corollary 4.8 and Corollary 4.10, the dual cone of $(\mathcal{M}_k^1)^+$ is $I_k^*(\mathcal{M}_k^+)$. Again by Corollary 4.8, the dual cone of $((\mathcal{M}_{k+1}^1)^+)^{\vee<}$ is the set

$$(((\mathcal{M}_{k+1}^1)^+)^{\vee<})^* = I_k^*\{L \in \mathcal{L}_k | I_{k+1}^*(L^{>\vee}) \in I_{k+1}^*(\mathcal{M}_{k+1}^+)\}.$$

Thus, Lemma 4.14 says that we have $(((\mathcal{M}_{k+1}^1)^+)^{\vee<})^* = ((\mathcal{M}_k^1)^+)^*$ and the Proposition follows. $\square$

5. Non-negative bilinear forms and non-negative functions

In this Section, as an application of Theorem 4.1, we show that, for w a symmetric Γ-invariant function on X_k, $k \geq 2$, if the bilinear form Φ_w is non-negative, then w can be assumed to have non-negative values.

Theorem 5.1. *Let $k \geq 2$ and w be a symmetric Γ-invariant function on X_k. Assume the symmetric bilinear form Φ_w associated to w*

on H_0^ω is non-negative. Then, w is cohomologous to a function with non-negative values, that is, there exists a skew-symmetric Γ-invariant function v on X_{k-1} such that, for any (x, y) in X_k, one has

$$w(x, y) \geq v(x, y_1) + v(x_1, y).$$

See Section I.3 for the definition of H_0^ω and Φ_w.

The proof of this result will rely on a duality argument. Theorem 4.1 gives a dual characterization of the functions w such that Φ_w is non-negative. We will establish a dual characterization of the functions w such that w is cohomologous to a function with non-negative values and compare the two characterizations in order to conclude. The latter one will be formulated in analogy with Livšic Theorem in hyperbolic dynamics (see [1, Theorem 19.2.1]), as in [4]. In particular, its proof will require us to give a more powerful version of the closing Lemma than Lemma I.2.6. To show this enhanced closing Lemma, we will begin by establishing an equidistribution result in $\Gamma \backslash X_1$ to ensure that Γ contains sufficiently many hyperbolic elements.

5.1. **Patterson-Sullivan functions.** In this Subsection, we adapt Patterson-Sullivan theory (see for example [7]) to our language. The following results will be no surprise for the experts of this domain. In the context of the proof of Theorem 5.1, we will use them to get a proof of the strong closing Lemma 5.8 below.

Proposition 5.2. *There exists a positive Γ-invariant function u on X_1 and a number $\rho \geq 2$ such that, for every $x \sim y$ in X, one has*

$$\sum_{\substack{z \sim y \\ z \neq x}} u(y, z) = \rho u(x, y).$$

The number ρ is unique and the function u is unique up to multiplication by a positive number.

Let F_1 be the space of functions on X_1 that are Γ-invariant. We define a linear endomorphism R of F_1 by setting

$$Rv(x, y) = \sum_{\substack{z \sim y \\ z \neq x}} v(y, z), \quad v \in F_1, \quad x \sim y \in X. \tag{5.1}$$

The number ρ is the spectral radius of R.

Remark 5.3. This number only depends on X, not on Γ. Indeed, as u is positive by Proposition 5.2, there exists $C > 1$ such that, for any

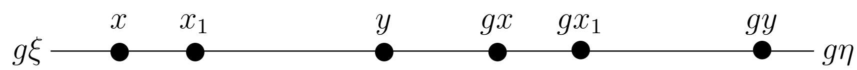

FIGURE 1. The first case in the proof of Lemma 5.4

$n \geq 1$ and $x \sim y$ in X, one has

$$\frac{1}{C}\rho^n u(xy) = \frac{1}{C}R^n u(xy) \leq |\{z \in X | d(x,z) = n+1, y \in [xz]\}| \\ \leq CR^n u(xy) = C\rho^n u(xy).$$

For this reason, we call ρ the growth rate of X.

The function u is said to be a Patterson-Sullivan function of Γ. If X is homogeneous, the function u is constant and ρ is $d-1$ where d is the degree of X.

We will need to state new results in the language of Subsection I.2.2.

Lemma 5.4. *Let $x \neq y$ be in X. There exists a hyperbolic element of Γ whose axis contains the segment $[xy]$. Equivalently, the set of pairs of fixed points of hyperbolic elements of Γ is dense in $\partial^2 X$.*

Proof. Recall that U_{xy} is the set of ξ in ∂X such that y belongs to $[x\xi)$. By Proposition I.2.3, there exists ξ in U_{yx} and η in U_{xy} such that the orbit $\Gamma(\xi, \eta)$ is dense in $\partial^2 X$. Set $k = d(x,y)$ and

$$\Gamma_k = \{g \in \Gamma | d(x, gx) \geq k \text{ and } d(y, gy) \geq k\}.$$

By assumption, $\Gamma \smallsetminus \Gamma_k$ is finite. As every element in X has at least three neighbours, ∂X has no isolated points and $\Gamma_k(\xi, \eta)$ is still dense in $\partial^2 X$. Therefore, we can find g in Γ such that $d(x, gx) \geq k$, $d(y, gy) \geq k$, $g\xi \in U_{yx}$ and $g\eta \in U_{xy}$. Let us show that g is hyperbolic and that $[xy]$ is contained in the axis of g.

Indeed, note that the assumption imply the inclusion

$$[xy] \subset ((g\xi)(g\eta)).$$

Now, let x_1 and y_1 be the neighbours of x and y on $[xy]$. Consider the points gx and gy on the geodesic line $((g\xi)(g\eta))$.

If gy belongs to the geodesic ray $[y(g\eta))$, as gx belongs to the geodesic ray $[(gy)(g\xi))$ and $d(gx, gy) = d(x, y)$, gx belongs to the geodesic ray $[x(g\eta))$ (see Figure 1). As $d(x, gx) \geq k$, we have $[xy] \subset [x(gx)]$ and in particular, x_1 is the neighbour of x on $[x(gx)]$. Since gx and gy both belong to $[y(g\eta))$, we have $gx_1 \notin [x(gx)]$. Therefore, by Lemma I.2.5, g is hyperbolic and $[x(gx)]$ is contained in the axis of g. All the more so is $[xy]$.

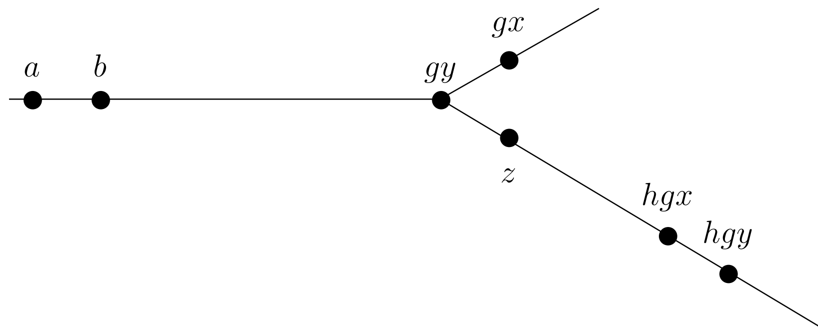

FIGURE 2. The second case in the proof of Lemma 5.5

If gy does not belong to the geodesic ray $[y(g\eta))$, as $d(gy, y) \geq k$, we must have $gy \in [x(g\xi))$. Since gx belongs to $[(gy)(g\xi))$, we get that gx belongs to $[x(g\xi))$ and, as above, we conclude that g is hyperbolic and that $[xy]$ is contained in the axis of g. $\square$

Given two edges (a, b) and (x, y) in X_1, say that the edge (x, y) is visible from the edge (a, b) if b and x belong to the segment $[ay]$.

Corollary 5.5. *Let $a \sim b$ and $x \sim y$ be in X. There exists γ in Γ such $(\gamma x, \gamma y)$ is visible from (a, b).*

Proof. By Lemma I.2.4, there exists g in Γ such that a does not belong to the segments $[b(gx)]$ and $[b(gy)]$. If (gx, gy) is visible from (a, b), we can set $\gamma = g$ and we are done.

If no, that is, if gy belongs to the segment $[b(gx)]$, the situation is as in Figure 2. Then, we chose a neighbour $z \neq gx$ of gy that does not belong to $[b(gy)]$ (which exists as $d(gy) \geq 3$). By Lemma 5.4, we can find a hyperbolic element h of Γ whose axis contains $[z(gx)]$. As h acts as a non trivial translation on its axis, up to replacing h by its inverse, hgx is the neighbour of hgy on the segment $[(hgy)z]$. Thus, (hgx, hgy) is visible from (a, b) and we can set $\gamma = hg$. $\square$

After these geometric preliminaries, we can prove the Proposition. This relies on classical arguments from the theory of Markov chains.

Proof of Proposition 5.2. Let R be as in (5.1). Let $\rho \geq 0$ be the spectral radius of R. As any x in X admits at least 3 neighbours, we have $R^n\mathbf{1} \geq 2^n\mathbf{1}$, $n \geq 0$, hence $\rho \geq 2$. As R maps non-negative functions to non-negative functions, by Perron-Frobenius Theorem (see for example Lemma I.10.18), there exists a non zero non-negative element u in F_1 which is an eigenvector of R with eigenvalue ρ, that is, $Ru = \rho u$.

Let v be any non-zero non-negative function on X_1 and λ be a real number with $Rv = \lambda v$. In particular, we have $\lambda \geq 0$. We claim that λ is positive and that v only takes positive values. Indeed, let $x \sim y$ be such that $v(x, y) > 0$. Fix $a \sim b$ in X and let γ be as in Corollary 5.5. Set $n = d(a, \gamma x)$. We have

$$\lambda^n v(a, b) = R^n v(a, b) \geq v(\gamma x, \gamma y) = v(x, y) > 0,$$

hence $\lambda \neq 0$ and v is positive. In particular, u is positive.

Besides, as v is positive, we can find $\varepsilon > 0$ with $v \geq \varepsilon u$. Fix $x \sim y$ in X. For any $n \geq 0$, we get

$$\lambda^n v(x, y) = R^n v(x, y) \geq \varepsilon R^n u(x, y) = \rho^n \varepsilon u(x, y),$$

hence $\lambda = \rho$. Let us show that v is a multiple of u, which finishes the proof. This is an application of the maximum principle. Set $\alpha = \sup_{X_1} \frac{v}{u}$ and let $Y \subset X_1$ be the set of (x, y) in X_1 with $v(x, y) = \alpha u(x, y)$. This is a Γ-invariant subset of X_1.

Fix (x, y) in Y. We claim that we have $(y, z) \in Y$ for all $z \sim y$, $z \neq x$. Indeed, we have

$$\alpha = \frac{v(x, y)}{u(x, y)} = \sum_{\substack{z \sim y \\ z \neq x}} \frac{v(y, z)}{\rho u(x, y)} = \sum_{\substack{z \sim y \\ z \neq x}} \frac{v(y, z)}{u(y, z)} \frac{u(y, z)}{\rho u(x, y)}$$

and $\sum_{\substack{z \sim y \\ z \neq x}} \frac{u(y,z)}{\rho u(x,y)} = 1$. As, for all z as above $v(y, z) \leq \alpha u(y, z)$, we get $v(y, z) = \alpha u(y, z)$.

Since Y is Γ-invariant, Corollary 5.5 shows that $Y = X_1$ and we are done. $\square$

For x, y in X, say that x and y are evenly related if the distance $d(x, y)$ is an even integer. This relation is an equivalence relation with two equivalence classes. Denote by Y this set of equivalence classes, so that Y has two elements. We shall say that the action of Γ on X is bipartite if the induced action on Y is trivial. Equivalently, the action of Γ on X is bipartite if, for any x in X and $\gamma \in \Gamma$, the distance $d(x, \gamma x)$ is even.

For u in F_1, let $u^\vee$ be the function $(x, y) \mapsto u(y, x)$ on X_1. We equip F_1 with the usual scalar product

$$(u, v) \mapsto \langle u, v \rangle = \sum_{(x,y) \in \Gamma \backslash X_1} \frac{1}{|\Gamma_x \cap \Gamma_y|} u(x, y) v(x, y). \tag{5.2}$$

The proof of Proposition 5.2 yields a description of the other eigenvalues of R with modulus ρ. We split the statement according to whether the action of Γ on X is bipartite or not.

Corollary 5.6. *Let ρ be the growth rate of X and u be a Patterson-Sullivan function on X_1.*

Assume that the action of Γ on X is not bipartite. Then, ρ is the unique complex eigenvalue of R with modulus ρ. In particular, for any v in F_1, we have

$$\rho^{-n}R^n v \xrightarrow[n\to\infty]{} \frac{\langle v, u^\vee\rangle}{\langle u, u^\vee\rangle}u. \tag{5.3}$$

Assume that the action of Γ on X is bipartite. Then, the complex eigenvalues of R with modulus ρ are ρ and $-\rho$. Let χ be a function on X that is constant with value 1 on one of the classes of the even distance equivalence relation, and -1 on the other. Then, the function

$$\chi u : (x, y) \mapsto \chi(x)u(x, y), X_1 \to \mathbb{R}$$

is, up to a scalar multiple, the unique eigenvector of R with eigenvalue $-\rho$. In particular, for any v in F_1, we have

$$\rho^{-n}R^n v - \frac{1}{\langle u, u^\vee\rangle}(\langle v, u^\vee\rangle u - (-1)^n\langle v, (\chi u)^\vee\rangle \chi u) \xrightarrow[n\to\infty]{} 0. \tag{5.4}$$

Proof. We start by studying the complex eigenvectors of R associated with eigenvalues of modulus ρ. Let v be a Γ-invariant complex valued function on X_1 such that $Rv = \rho e^{i\theta}v$ for some real number θ. For $x \sim y$ in X, we have

$$\frac{v(x, y)}{u(x, y)} = e^{-i\theta}\sum_{\substack{z\sim y\\ z\neq x}}\frac{v(y, z)}{\rho u(x, y)} = e^{-i\theta}\sum_{\substack{z\sim y\\ z\neq x}}\frac{v(y, z)}{u(y, z)}\frac{u(y, z)}{\rho u(x, y)}. \tag{5.5}$$

As $Ru = \rho u$, by convexity, we have

$$\frac{|v(x, y)|}{u(x, y)} \leq \sum_{\substack{z\sim y\\ z\neq x}}\frac{|v(y, z)|}{u(y, z)}.$$

By using the maximum principle as in the proof of Proposition 5.2, we get that $|v|$ is a multiple of u, and we can assume $|v| = u$. Set $\varphi = \frac{v}{u}$, so that φ has constant modulus 1. From (5.5) and the fact that $Ru = \rho u$ we get that, for any x, y, z in X with $x \sim y$, $y \sim z$ and $x \neq z$, we have

$$\varphi(x, y) = e^{-i\theta}\varphi(y, z). \tag{5.6}$$

In particular, if t is a neighbour of y with $t \neq x$ and $t \neq z$ (which exists as $d(y) \geq 3$), we have

$$\varphi(t, y) = e^{-i\theta}\varphi(y, z) = e^{-i\theta}\varphi(y, x).$$

Therefore, we have shown that $\varphi(y,z)=\varphi(y,x)$ for any y in X and any two neighbours x,z of y. We write $\psi(y)$ for this value. Then, (5.6) implies that, for any $x\sim y$ in X, we have

$$\psi(x)=e^{-i\theta}\psi(y).$$

By reversing the roles of x and y, we also have $\psi(y)=e^{-i\theta}\psi(x)$, which gives $e^{2i\theta}=1$, that is, $e^{i\theta}\in\{-1,1\}$. If $e^{i\theta}=1$, then ψ is constant. If $e^{i\theta}=-1$, then ψ is a multiple of the function χ of the statement.

Assume the action of Γ is not bipartite. Equivalently, the function χ is not Γ-invariant. In this case, we have just proved that any complex eigenvalue of R other than ρ has modulus $<\rho$ and that the eigenspace associated to ρ is $\mathbb{R}u$. Set L to be the endomorphism of F_1 defined by

$$Lv(x,y)=\sum_{\substack{z\sim x\\ z\neq y}}v(z,x)=(Rv^\vee)^\vee(x,y),\quad v\in F_1,\quad x\sim y\in X. \tag{5.7}$$

Then, the eigenspace of L associated to ρ is $\mathbb{R}u^\vee$. By using Lemma I.9.11, one can show that L and R are adjoint to each other with respect to the scalar product $\langle .,.\rangle$. Therefore, the orthogonal subspace of $u^\vee$ is the unique R-invariant complementary subspace to $\mathbb{R}u$. We have just shown that the restriction of R to that space has spectral radius $<\rho$. Thus (5.3) follows by elementary linear algebra.

Assume the action of Γ is bipartite. Then, the function χ is Γ-invariant and R admits two eigenvalues with modulus ρ, which are ρ itself, with eigenspace $\mathbb{R}u$, and $-\rho$, with eigenspace $\mathbb{R}\chi u$. Let still L be as in (5.7). Then $\mathbb{R}u^\vee$ is the eigenline of L associated to the eigenvalue ρ and $\mathbb{R}(\chi u)^\vee$ is the eigenline of L associated to the eigenvalue $-\rho$. Therefore, R has spectral radius $<\rho$ in the invariant subspace

$$\{v\in F_1|\langle v,u\rangle=\langle v,(\chi u)^\vee\rangle=0\}.$$

Again, (5.4) follows by elementary linear algebra and by noticing that $\langle\chi u,(\chi u)^\vee\rangle=-\langle u,u^\vee\rangle$. □

5.2. **Cohomology classes with non-negative values.** In order to prove Theorem 5.1, we give a characterization of functions that are cohomologous to functions with non-negative values in the spirit of Livšic Theorem in hyperbolic dynamics (see [1, Theorem 19.2.1]).

Let us use the language of Subsection I.2.2. Let γ be a hyperbolic element in Γ and (ξ,η) be the invariant geodesic line of γ. Choose a parametrization $(x_h)_{h\in\mathbb{Z}}$ of $(\xi\eta)$ and denote by $\ell\geq 1$ the translation length of γ, so that $\gamma x_h=x_{h+\ell}$, $h\in\mathbb{Z}$. For $k\geq 1$ and w a Γ-invariant function on X_k, we set

$$\sum_{[\gamma]} w = \sum_{h=0}^{\ell-1} w(x_h, x_{h+k}).$$

Indeed, this number only depends on w and the conjugacy class $[\gamma]$ of γ in Γ.

Recall that $\mathscr{S}$ stands for the space of parametrized geodesic lines of X and $T : \mathscr{S} \to \mathscr{S}$ for the time shift. In Subsection I.2.5, we have identified the space of Γ-invariant functions on X_k, $k \geq 1$, with a subspace of the space of smooth functions on $\Gamma\backslash\mathscr{S}$. If w is a Γ-invariant function on X_k, we associate to w the function

$$(x_h)_{h\in\mathbb{Z}} \mapsto w(x_0, x_k), \mathscr{S} \to \mathbb{R}.$$

As in Subsection I.2.3, two smooth functions f and g on $\Gamma\backslash\mathscr{S}$ are said to be cohomologous if $f - g = h - h \circ T$ for some smooth function h on $\Gamma\backslash\mathscr{S}$. The following statement is a direct analogue of the main result of [4]. The proof will follow the same lines.

Proposition 5.7. *Let $k \geq 2$ and w be a Γ-invariant function on X_k. The following are equivalent:*
(i) *For any T-invariant Borel probability measure μ on $\Gamma\backslash\mathscr{S}$, one has*

$$\int_{\Gamma\backslash\mathscr{S}} f \mathrm{d}\mu \geq 0,$$

where f is a smooth function on $\Gamma\backslash\mathscr{S}$ that is cohomologous to the function associated to w.
(ii) *For any hyperbolic element γ of Γ, one has*

$$\sum_{[\gamma]} w \geq 0.$$

(iii) *The function w is cohomologous to a non-negative function, that is, there exists a Γ-invariant function v on X_{k-1} such that, for any (x, y) in X_k, one has*

$$w(x, y) \geq v(x, y_1) - v(x_1, y).$$

If w is symmetric on X_k, the function v can be assumed to be skew symmetric.

The proof will use the following version of the closing Lemma, which is a reinforcement of Lemma I.2.6:

Lemma 5.8. *There exists $c \geq 0$ with the following property: for any $x \neq y$ in X, there exists a hyperbolic element γ in Γ with invariant geodesic line $(\xi\eta)$ such that*

$$[xy] \subset [x(\gamma x)] \subset (\xi\eta) \text{ and } d(y, \gamma x) \leq c.$$

Proof. We will deduce this fact from the equidistribution statements in Corollary 5.6. We keep the notation of this result: F_1 is the space of Γ-invariant functions on X_1, equipped with the usual scalar product from (5.2), R is the operator defined in (5.1), ρ is the growth rate of X and u is a Patterson-Sullivan function as in Proposition 5.2. By Corollary 5.6, for v in F_1 with $\langle v, u^\vee \rangle = 1$, we have

$$\rho^{-2n} R^{2n} v + \rho^{-2n-1} R^{2n+1} v \xrightarrow[n \to \infty]{} \frac{2}{\langle u, u^\vee \rangle} u. \tag{5.8}$$

As $u^\vee$ takes only positive values, the set

$$K = \{v \in F_1 | v \geq 0 \text{ and } \langle v, u^\vee \rangle = 1\}$$

is compact. Therefore, the convergene of linear operators in (5.8) is uniform on K. Thus, as u is positive, there exists $n \geq 1$ such that, for any non zero $v \geq 0$ in F_1, the function $\rho^{-2n} R^{2n} v + \rho^{-2n-1} R^{2n+1} v$ is positive everywhere. Let $x \neq y$ be in X and x_1 and y_1 be the neighbours of x and y on $[xy]$. Applying the previous property to the function $v = \sum_{\gamma \in \Gamma} \mathbf{1}_{\gamma(x,x_1)}$, tells us that there exists γ in Γ such that $d(\gamma x_1, y) \leq 2n+1$ and $(\gamma x, \gamma x_1)$ is visible from (y_1, y) (that is, we have $[y(\gamma x)] \subset [y_1(\gamma x_1)]$). In particular, we have $[xy] \subset [x(\gamma x)]$, $x_1 \in [x(\gamma x)]$ and $\gamma x_1 \notin [x(\gamma x)]$, so that, by Lemma I.2.5, γ is hyperbolic and its axis contains $[x(\gamma x)]$, hence $[xy]$. The Lemma follows by taking $c = 2n$. $\square$

Now that we have an adapted closing Lemma, we can borrow the main idea of the proof of Proposition 5.7 from [4].

Proof of Proposition 5.7. (iii)⇒(i). Assume v is as in the statement. Let f and h be the smooth functions on $\mathscr{S}$ associated to w and v. For $s = (x_h)_{h \in \mathbb{Z}}$ in $\mathscr{S}$, we have $f(s) = w(x_0, x_k)$ and $h(s) = v(x_0, x_{k-1})$ so that, by assumption, $f \geq h - h \circ T$. The conclusion follows.

(i)⇒(ii). Let γ be a hyperbolic element with translation length $\ell \geq 1$ and $s = (x_h)_{h \in \mathbb{Z}}$ be a parametrization of the invariant geodesic line of γ. We have $T^\ell s = \gamma s$, hence the probability measure on $\Gamma \backslash \mathscr{S}$,

$$\mu = \frac{1}{\ell} \sum_{h=0}^{\ell-1} \delta_{T^h \Gamma s}$$

is T-invariant. By construction, we have

$$\int_{\Gamma\backslash\mathscr{S}} f \mathrm{d}\mu = \frac{1}{\ell}\sum_{[\gamma]} w,$$

where f is the smooth function associated with w. The conclusion follows.

(ii)⇒(iii). As in Subsection I.8.1, for $j \geq k$ and (x,y) in X_j, we write

$$\sum_{[xy]} w = \sum_{h=0}^{j-k} w(x_h, x_{h+k}),$$

where $x_0 = x, x_1, \dots, x_{j-1}, x_j = y$ is the geodesic path from x to y. Let c be as in Lemma 5.8. By this Lemma, for x, y in X with $d(x,y) \geq k$, we have

$$\sum_{[xy]} w \geq \sum_{[\gamma]} w - c \sup |w|,$$

where γ is a certain hyperbolic element. By assumption, we get

$$\inf_{\substack{x,y\in X\\ d(x,y)\geq k}} \sum_{[xy]} w > -\infty.$$

We use this property to define the function v. For (x,y) in X_{k-1}, we set

$$v(x,y) = \inf_{\substack{z\in X\\ y\in[xz]\\ d(x,z)\geq k}} \sum_{[xz]} w.$$

We claim that v satifies the required property. Indeed, let (x,y) be in X_k and x_1 and y_1 be the neighbours of x and y on $[xy]$. Note that we have

$$\{z \in X | y \in [x_1 z] \text{ and } d(x_1, z) \geq k\} \\ \subset \{z \in X | y_1 \in [xz] \text{ and } d(x,z) \geq k+1\}$$

and that, for such a z, we have

$$w(x,y) + \sum_{[x_1 z]} w = \sum_{[xz]} w.$$

We get

$$w(x,y) + v(x_1, y) \geq v(x, y_1)$$

as required. If w is symmetric, we also have

$$w(x,y) + v(y_1, x) \geq v(y, x_1),$$

hence

$$2w(x,y) + v(x_1,y) - v(y,x_1) \geq v(x,y_1) - v(y_1,x),$$

that is, we may replace v by a skew-symmetric function. □

5.3. **Bilinear forms and invariant distributions.** The proof of Theorem 5.1 will use a duality property. Thus, we will now study distributions on $\Gamma\backslash\mathscr{S}$.

We start by describing general phenomena. Let U be a locally compact totally discontinuous topological space, endowed with a proper action of a discrete group G. If φ is a smooth function with compact support on U, the function $\overline{\varphi}$ defined by

$$\overline{\varphi}(u) = \sum_{g\in G} \varphi(gu), \quad u \in U,$$

is smooth with compact support on $G\backslash U$. The map $\varphi \mapsto \overline{\varphi}$ sends the space $\mathcal{D}(U)$ of compactly supported smooth functions on U onto $\mathcal{D}(G\backslash U)$. The adjoint map $\mathcal{D}^*(G\backslash U) \to \mathcal{D}^*(U)$ is injective and its range is exactly the space $\mathcal{D}^*(U)^G$ of G-invariant distributions on U.

In our concrete situation, the space $\mathscr{S}$ of parametrized geodesic lines is equipped with two commuting group actions. One is the natural action of Γ. The other is the action of $\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$ defined by the time shift T and the time reversal $\iota : (x_h)_{h\in\mathbb{Z}} \mapsto (x_{-h})_{h\in\mathbb{Z}}$. The quotient of $\mathscr{S}$ by the $\mathbb{Z}$-action generated by T may be identified with $\partial^2 X$ by the map that forgets the parametrization of a geodesic line.

Say that a distribution on $\partial^2 X$ is symmetric if it is invariant under the map $(\xi, \eta) \mapsto (\eta, \xi)$. We have just defined a natural identification of the space of (ι, T)-invariant distributions in $\mathcal{D}^*(\Gamma\backslash\mathscr{S})$ with the space of Γ-invariant symmetric distributions in $\mathcal{D}^*(\partial^2 X)$. We claim that the latter space can be identified with the space $\mathcal{Q}(\overline{\mathcal{D}}(\partial X))$ of symmetric bilinear forms on $\overline{\mathcal{D}}(\partial X)$. Indeed, this comes from the following extension of Lemma 4.14:

Lemma 5.9. *Assume U is a compact totally discontinuous topological space. Set $U_2 = \{(x, y) \in U^2 | x \neq y\}$ and $\iota : U_2 \to U_2$ to be the natural involution $(x, y) \mapsto (y, x)$. Write $\overline{\mathcal{D}}(U)$ for the quotient of the space $\mathcal{D}(U)$ by the constant functions. For φ, ψ in $\mathcal{D}(U)$ and (x, y) in U_2, set*

$$\Xi(\varphi, \psi) = \frac{1}{2}(\varphi(x) - \varphi(y))(\psi(x) - \psi(y)).$$

Then $\Xi(\varphi, \psi)$ is a smooth compactly supported function on U_2, that is, it belongs to $\mathcal{D}(U_2)$. If θ is a distribution on U_2, let q_θ be the symmetric bilinear form on $\overline{\mathcal{D}}(U)$ defined by

$$q_\theta(\varphi, \psi) = \theta(\Xi(\varphi, \psi)), \quad \varphi, \psi \in \overline{\mathcal{D}}(U).$$

Then, the map $\theta \mapsto q_\theta$ is a linear isomorphism from the space $\mathcal{D}^*(U_2)^\iota$ of ι-invariant distributions on U_2 onto the space $\mathcal{Q}(\overline{\mathcal{D}}(U))$ of symmetric bilinear forms on $\overline{\mathcal{D}}(U)$.

Proof. Note that, as φ and ψ are smooth, that is, locally constant, the function $\Xi(\varphi, \psi)$ has compact support in U_2 and the bilinear form q_θ is well-defined.

Let $U = \bigsqcup_{a \in A} U_a$ be a partition of U into finitely many closed open subsets. Then, the space $V \subset \mathcal{D}(U)$ of functions that are constant on each of the U_a may be identified with the space of functions on the finite set A. The space $\mathcal{D}(U)$ is the union of all such subspaces V. The Lemma follows by applying Lemma 4.14 to each of them. $\square$

Thus, in view of Proposition I.4.1, we can identify the space of symmetric distributions in $\mathcal{D}^*(\partial^2 X)$ with the space of quadratic type functions on X_*. This identification is given by the following formula.

Corollary 5.10. *Let θ be a symmetric distribution on $\partial^2 X$ and φ be the associated quadratic type function on X_*. For (x, y) in X_*, we have*

$$\varphi(x, y) = \frac{1}{2}\theta(\mathbf{1}_{V_{xy}}),$$

where

$$V_{xy} = \{(\xi, \eta) \in \partial^2 X | [xy] \subset (\xi\eta)\}. \tag{5.9}$$

Remark 5.11. With the notation of the Corollary, one easily checks that the function $(x, y) \mapsto \frac{1}{2}\theta(\mathbf{1}_{V_{xy}})$ is of quadratic type.

Proof. Let q be the symmetric bilinear form on $\overline{\mathcal{D}}(\partial X)$ defined in Lemma 5.9. Recall from Subsection I.4.1, that we have set

$$U_{xy} = \{\xi \in \partial X | y \in [x\xi)\}$$

and

$$\varphi(x, y) = -q(\mathbf{1}_{U_{xy}}, \mathbf{1}_{U_{yx}}).$$

For $\xi \neq \eta$ in ∂X, we have

$$(\mathbf{1}_{U_{xy}}(\xi) - \mathbf{1}_{U_{xy}}(\eta))(\mathbf{1}_{U_{yx}}(\xi) - \mathbf{1}_{U_{yx}}(\eta)) = -\mathbf{1}_{V_{xy}}(\xi, \eta).$$

The conclusion follows by Lemma 5.9. $\square$

We conclude this discussion by giving a formula for computing the value of a distribution on a given function thanks to these identifications.

Lemma 5.12. *Let θ be a (ι, T)-invariant distribution on $\mathscr{S}$ with associated quadratic type function φ on X_*. Let w be a finitely supported function on X_k and f be the smooth compactly supported function on $\mathscr{S}$*

$$f : (x_h)_{h\in\mathbb{Z}} \mapsto w(x_0, x_k), \mathscr{S} \to \mathbb{R}.$$

Then we have

$$\langle \theta, f\rangle = \frac{1}{2} \sum_{(x,y)\in X_k} w(x,y)\varphi(x,y).$$

Let θ be a (ι, T)-invariant distribution on $\Gamma\backslash\mathscr{S}$ with associated quadratic type function φ on X_*. Fix $k \geq 1$ and let w be a Γ-invariant function on X_k and f be a smooth function on $\Gamma\backslash\mathscr{S}$ that belongs to the cohomology class associated to w. We have

$$\langle \theta, f\rangle = \frac{1}{2} \sum_{(x,y)\in \Gamma\backslash X_k} \frac{1}{|\Gamma_x \cap \Gamma_y|} w(x,y)\varphi(x,y).$$

Proof. We establish the first formula. It suffices to prove the claim when w is the indicator function $\mathbf{1}_{\{(a,b),(b,a)\}} = \mathbf{1}_{(a,b)} + \mathbf{1}_{(b,a)}$ for some (a, b) in X_k. As in the statement, let f be the associated function on $\mathscr{S}$ and identify the quotient of $\mathscr{S}$ by $T^{\mathbb{Z}}$ with $\partial^2 X$. Then the function $\overline{f} = \sum_{n\in\mathbb{Z}} f \circ T^n$ is the indicator function of the set V_{ab} defined in (5.9) and the formula follows by Corollary 5.10.

As in the proof of Proposition 3.22, the second formula follows from the first by using Lemma 3.25. □

Let us describe the behaviour of Radon measures under our chain of identifications. Recall that, if U is a second countable totally discontinous locally compact space, a distribution θ on U is a Radon measure if and only if, for any non-negative function φ in $\mathcal{D}(U)$, one has $\theta(\varphi) \geq 0$. If $\mathcal{V}$ is a collection of compact open subsets of U which form a basis of the topology of U, then θ is a Radon measure if and only if one has $\theta(\mathbf{1}_V) \geq 0$ for any V in $\mathcal{V}$. If G is a discrete group acting properly on U, the natural identification between $\mathcal{D}^*(G\backslash U)$ and $\mathcal{D}^*(U)^G$ induces an identification of the space of Radon measures on $G\backslash U$ with the space of G-invariant Radon measures on U.

In our case, the sets $(V_{xy})_{(x,y)\in X_1}$ of (5.9) form a basis of the topology of ∂X. Thus, by using Corollary 5.10, we get

Lemma 5.13. *Let φ be a quadratic type function on X_*. Then the associated symmetric distribution on $\partial^2 X$ is a Radon measure if and only if φ takes non-negative values. If φ is Γ-invariant, the associated (ι, T)-invariant distribution on $\Gamma\backslash\mathscr{S}$ is a Radon measure if and only φ takes non-negative values.*

Note that, as $\Gamma\backslash\mathscr{S}$ is compact, a Radon measure on this space is the same as a finite Borel measure.

5.4. **Radon measures and tight quadratic fields.** Theorem 5.1 will be a consequence of the following dual statement which uses the vocabulary that we have just introduced.

Proposition 5.14. *Let μ be a (ι, T)-invariant Borel probability measure on $\Gamma\backslash\mathscr{S}$ and φ be the associated quadratic type function on X_*. Fix $k \geq 2$ and let p be the k-quadratic field that is the natural image of φ in $\mathcal{F}_k$. Then p is tight.*

See Section I.4 for the relations between quadratic type functions and quadratic fields. See Section 4 for the definition of tight quadratic fields.

The proof relies on the construction of the root map in Subsection 3.6 and the following elementary remark.

Lemma 5.15. *Let A be a finite set. Let V be the space of real-valued functions on A and $\overline{V} = V/\mathbb{R}$ be its quotient by the space of constant functions. Let p be a symmetric bilinear form on $\overline{V}$. For $a \neq b$ in A, set*

$$\varphi(a, b) = -p(\mathbf{1}_a, \mathbf{1}_b).$$

Assume that φ takes non-negative values on $A_2 = \{(a, b) \in A^2 | a \neq b\}$. Then, p is non-negative as a bilinear form.

Proof. Indeed, by a direct computation (or by Lemma I.C.5), for f in $\overline{V}$, one has

$$p(f, f) = \frac{1}{2} \sum_{(a,b)\in A_2} \varphi(a, b)(f(a) - f(b))^2.$$

□

Proof of Proposition 5.14. By Lemma 5.13, as μ is a positive measure, the function φ has non-negative values. Let s be the root of p. By Definition 3.19, Definition 3.20 and Lemma 5.15, the quadratic pseudofield s is non-negative. By Proposition 3.21, s is radical and the shoot of s is p. Therefore p is tight by Theorem 4.1. □

As announced, Theorem 5.1 directly follows from Proposition 5.14.

Proof of Theorem 5.1. As in the statement, let $k \geq 2$ and w be a symmetric Γ-invariant function on X_k such that the associated bilinear form Φ_w is non-negative on H_0^ω. We will show that w is cohomologous to a non-negative function by applying Proposition 5.7. Thus, we take a T-invariant Borel probability measure μ on $\Gamma\backslash\mathscr{S}$ and we will prove that $\int_{\Gamma\backslash\mathscr{S}} f\mathrm{d}\mu \geq 0$, where f is the smooth function on $\Gamma\backslash\mathscr{S}$ associated with w. As w is symmetric and μ is T-invariant, we have $\int_{\Gamma\backslash\mathscr{S}} f\mathrm{d}\mu = \int_{\Gamma\backslash\mathscr{S}} f\circ\iota\mathrm{d}\mu$, so that we may replace μ by $\frac{1}{2}(\mu+\iota_*\mu)$ and assume that μ is (ι, T)-invariant. Let φ be the quadratic type function associated to μ as in Subsection 5.3. By Proposition 5.14, the restriction of φ to X_k is the quadratic type function associated to a tight k-quadratic field p in $\mathcal{F}_k$. By the definition of the weight pairing in Theorem I.11.4 and by Lemma 5.12, if (K, K^-) is a k-dual kernel with w as a weight function, we have

$$\int_{\Gamma\backslash\mathscr{S}} f\mathrm{d}\mu = [p, (K,K)^-].$$

As p is tight and Φ_w is non-negative, we get $\int_{\Gamma\backslash\mathscr{S}} f\mathrm{d}\mu \geq 0$. As this is true for any T-invariant measure μ, w is cohomologous to a non-negative function by Proposition 5.7. □

6. Skew quadratic fields

We have shown in Proposition 3.21 that the root map is a section of the shoot map. Thus, the space of radical quadratic pseudofields splits in a natural way as the direct sum of the image of the root map and of the null space of the shoot map. We will now develop a study of the elements of this null space, which we will call skew quadratic fields. This theory will play the role of a skew-symmetric counterpart to the theory of quadratic fields.

6.1. **Skew quadratic fields and the shoot map.**

We begin by defining formally skew quadratic fields.

Definition 6.1. Let $k \geq 1$. If $k = 1$, a 1-skew quadratic field is a 1-quadratic pseudofield s with $s^\vee = -s$. If $k \geq 2$, a k-skew quadratic field is a k-quadratic pseudofield s with $s^\vee = -s$ and $s^{<\vee} = -s^<$.

We can relate skew quadratic fields with the null space of the shoot map.

Lemma 6.2. *Let $k \geq 1$ and s be a k-quadratic pseudofield. Then s is a k-skew quadratic field if and only if s is radical and the shoot of s is 0.*

The proof is immediate.

Remark 6.3. From now on, we adopt the following notational convention. If u is a function on X_1, for $k \geq 1$, L a k-pseudokernel and s a k-quadratic pseudofield, we let uL and us be the pseudokernel and the quadratic pseudofield defined by $(x, y) \mapsto u(x, y)L_{xy}$ and

$(x,y) \mapsto u(x,y)s_{xy}$. We get $(uL)^> = uL^>$ and $(us)^< = us^<$ as well as $\langle us, L\rangle = \langle s, uL\rangle$ when u, L and s are Γ-invariant. As in Subsection 5.1, we also write $u^\vee$ for the function $(x,y) \mapsto u(y,x)$ on X_1, so that, if k is odd, we have $(uL)^\vee = u^\vee L^\vee$ and $(us)^\vee = u^\vee s^\vee$. Finally, if v is a function on X, we use the same letter v to denote the function $(x,y) \mapsto v(x)$ on X_1. Then, $v^\vee$ stands for the function $(x,y) \mapsto v(y)$.

Remark 6.4. For $k \geq 2$, in Remark 2.8, we have identified the space of k-dual prekernels with a subspace of the space of k-pseudokernels. In the same way, we can identify the space of k-quadratic fields with a subspace of the space of k-quadratic pseudofields. Indeed, if p is a k-quadratic field, we can define a k-quadratic pseudofield s by setting, for $x \sim y$ in X, $s_{xy} = p_x$ if k is even and $s_{xy} = p_{xy}$ if k is odd. Then, with this identification, if k is even (resp. odd), a k-quadratic pseudofield s is the one associated with a k-quadratic field if and only if $s^\vee = (d-1)s$ and $s^{<\vee} = s^<$ (resp. $s^\vee = s$ and $s^{<\vee} = (d-1)s^<$). Thus, the notion of a skew quadratic field may be seen as the skew-symmetric counterpart of the notion of a quadratic field. We will develop the study of skew quadratic fields by keeping this idea in mind.

For $k \geq 1$, we let $\mathcal{G}_k$ denote the space of Γ-invariant k-skew quadratic fields. By Proposition 3.21 and Lemma 6.17, we have $\mathcal{M}^1_k = \mathcal{G}_k \oplus R_k\mathcal{F}_k$.

The notion of a skew quadratic field behaves well under direct restriction. Lemma 3.4 gives

Lemma 6.5. *Let $k \geq 2$ and s be a k-skew quadratic field. Then $s^<$ is a $(k-1)$-skew quadratic field.*

The orthogonal subspace of the space of k-skew quadratic fields in the space of k-pseudokernels is the image of the canonical map.

Lemma 6.6. *Let $k \geq 1$ and s be k-quadratic pseudofield.*

If $k = 1$, then s is a 1-skew quadratic field if and only if, for every L in $\mathcal{L}_1$, one has $\langle s, L + L^\vee\rangle = 0$.

If $k \geq 2$, then s is a k-skew quadratic field if and only if, for every (K, K^-) in $\mathcal{K}_k$, one has $\langle s, C_k(K, K^-)\rangle = 0$.

Proof. If $k = 1$, this directly follows from Definition 6.1, the inversion map $L \mapsto L^\vee$ being an involution of the space of 1-pseudokernels.

If $k \geq 2$, this is a direct consequence of Lemma 3.13 and Lemma 6.2. □

We get an analogue for skew quadratic fields of Lemma I.11.1 for quadratic fields.

Corollary 6.7. *Let $k \geq 2$. Then direct restriction maps $\mathcal{G}_k$ onto $\mathcal{G}_{k-1}$.*

Proof of Corollary 6.7. As usual, this follows from the dual injectivity statement.

If $k = 2$, we must show that, if L is a 1-pseudokernel and $L^{>}$ is the canoncial 2-pseudokernel of a 2-dual kernel (K, K^{-}), then $L^{\vee} = L$. Indeed, with the convention of Remark 2.8, we have $L^{>} = K - K^{->}$, hence

$$(L + K^{-})^{>\vee} = K^{\vee} = (d-1)K = (d-1)(L + K^{-}).$$

Lemma 2.5 gives $L = -K^{-}$ and we are done.

If $k \geq 3$, we must show that, if L is a $(k-1)$-pseudokernel and $L^{>}$ is a canonical pseudokernel, then L already is a canonical pseudokernel. Indeed, assume (K, K^{-}) is a k-dual kernel and $L^{>} = K - K^{->}$.

If k is even, we again have

$$(L + K^{-})^{>\vee} = K^{\vee} = (d-1)K = (d-1)(L + K^{-})$$

and Lemma 2.5 now says that there exists a $(k-2)$-pseudokernel J with

$$L + K^{-} = J^{>} \text{ and } (d-1)(L + K^{-}) = J^{\vee>}.$$

In particular, we have $J^{\vee} = (d-1)J$, that is, J is a $(k-2)$-dual prekernel. We get $L + K^{-} = J^{>}$, hence L is the canonical $(k-1)$-pseudokernel of $-(K^{-}, J)$.

If k is odd, we have

$$(L + K^{-})^{>\vee} = K^{\vee} = K = L + K^{-}$$

and Lemma 2.5 says that there exists a $(k-2)$-pseudokernel J with

$$L + K^{-} = J^{>} = J^{\vee>}.$$

Thus J is a $(k-2)$-dual prekernel and L is the canonical $(k-1)$-pseudokernel of $-(K^{-}, J)$. □

6.2. **Orthogonal extension of skew quadratic fields.** Still in analogy with the case of quadratic fields, we now show how the choice of a Euclidean field allows to define a section of direct restriction of skew quadratic fields.

We first introduce a new notation. Let $k \geq 2$ and p be a k-Euclidean field. If s is a $(k-1)$-quadratic pseudofield, we let $s^{>_p}$ denote the k-quadratic pseudofield defined by, for $x \sim y$ in X,

$$\begin{aligned} s_{xy}^{>_p} &= (I_{xy}^{\ell-1,\dagger p})^{\star} s_{xy} \quad k = 2\ell, \ell \geq 1. \\ &= (J_{xy}^{\ell,\dagger p})^{\star} s_{xy} \quad k = 2\ell+1, \ell \geq 1. \end{aligned}$$

We can relate this notation with previously studied operations. First, by the very definition of the objects, we have

Lemma 6.8. *Let $k \geq 2$, p be a k-Euclidean field and s be a $(k-1)$-quadratic pseudofield. Then we have $s^{>_p<} = s$.*

As in Lemma 3.4, we have

Lemma 6.9. *Let $k \geq 3$, p be a k-Euclidean field and s be a $(k-2)$-quadratic pseudofield. Then we have*

$$s^{\vee >_{p-} >_p} = s^{>_{p-} >_p \vee}.$$

We also have a direct consequence of Lemma I.10.8:

Lemma 6.10. *Let $k \geq 2$, p be a k-Euclidean field and s be a k-quadratic pseudofield. We have $s^{>_{p^+} \vee <} = s^{<\vee >_p}$.*

Let p be a Γ-invariant k-Euclidean field. Recall from Definition I.10.4 and Definition I.10.5 that we may associate to p its transfer operator $T_p : \mathcal{M}_{k-1} \to \mathcal{M}_{k-1}$.

Corollary 6.11. *Let $k \geq 2$, p be a Γ-invariant k-Euclidean field and s be a Γ-invariant $(k-1)$-quadratic pseudofield. We have $T_p s = s^{\vee >_p \vee <}$.*

Remark 6.12. Let $k \geq 2$. As in Remark 6.4, let us identify k-quadratic fields with a subspace of k-quadratic pseudofields. The computation of the derivative of the orthogonal extension map $\eta_k : \mathcal{P}_k \to P_{k+1}$ in Proposition I.11.9 can be stated as, for $x \sim y$ in X, p in $\mathcal{P}_k$ and q in $\mathcal{F}_k$,

$$\begin{aligned} \mathrm{d}_p \eta_k(q) &= q^{>_{p^+}} + q^{>_{p^+}\vee} - q^{->_p>_{p^+}} \quad \text{if } k \text{ is even.} \\ &= q^{>_{p^+}} + q^{>_{p^+}\vee} - (d-1) q^{->_p>_{p^+}} \quad \text{if } k \text{ is odd.} \end{aligned}$$

We will define orthogonal extension of skew quadratic fields in analogy with this formula.

Definition 6.13. Let $k \geq 2$, p be a k-Euclidean field and s be a k-skew quadratic field. If k is even, we set

$$s^{+_p} = -s^{>_{p^+}} + s^{>_{p^+}\vee} + s^{<>_p>_{p^+}}. \tag{6.1}$$

If k is odd, we set

$$s^{+_p} = -s^{>_{p^+}} + \frac{1}{d-1} s^{>_{p^+}\vee} + \frac{1}{d-1} s^{<>_p>_{p^+}}. \tag{6.2}$$

We call s^{+_p} the p-orthogonal extension of s.

Proposition 6.14. *Let $k \geq 2$, p be a k-Euclidean field and s be a k-skew quadratic field. Then s^{+_p} is a $(k+1)$-skew quadratic field and $s^{+_p<} = -s$, that is, $s^{+_p-} = s^{+_p\vee <} = s$.*

To prove this, we will need to compute the square of the reversal map.

Lemma 6.15. *Let $k \geq 2$ be even and s be a k-quadratic pseudofield and L be a k-pseudokernel. We have $s^{\vee\vee} = (d-1)s + (d-2)s^{\vee}$ and $L^{\vee\vee} = (d-1)L + (d-2)L^{\vee}$.*

In case the tree X is not homogeneous, this Lemma uses the notation introduced in Remark 6.3.

Proof. Indeed, for $x \sim y$ in X, we have

$$s^{\vee\vee}_{xy} = \sum_{\substack{z\sim x\\ z\neq y}} s^{\vee}_{xz} = \sum_{\substack{z\sim x\\ z\neq y}} \sum_{\substack{t\sim x\\ t\neq z}} s_{xt} = \sum_{t\sim x} |\{z \in S^1(x) | z \notin \{y,t\}\}| s_{xt}$$
$$= (d(x)-1)s_{xy} + (d(x)-2)\sum_{\substack{t\sim x\\ t\neq y}} s_{xt}.$$

The proof for pseudokernels is the same. □

Proof of Proposition 6.14. If k is even, we have

$$s^{+_p\vee} = -s^{>_p+\vee} + s^{>_p+} + s^{<>_p>_p+\vee} = -s^{>_p+\vee} + s^{>_p+} + s^{<\vee>_p>_p+}$$
$$= -s^{>_p+\vee} + s^{>_p+} - s^{<>_p>_p+} = -s^{+_p\vee},$$

where we have used Corollary 6.9 and the fact that $s^{<\vee} = -s^{<}$. Besides, we have

$$s^{+_p<} = -s^{>_p+<} + s^{>_p+\vee<} + s^{<>_p>_p+<}.$$

Lemma 6.8 gives $s^{>_p+<} = s$ and $s^{<>_p>_p+<} = s^{<>_p}$ and Corollary 6.10 gives $s^{>_p+\vee<} = s^{<\vee>_p} = -s^{<>_p}$. We get $s^{+_p<} = -s$ as required. In particular, $s^{+_p<\vee} = -s^{+_p<}$ and s^{+_p} is a $(k+1)$-skew quadratic field.

If k is odd, we have

$$s^{+_p\vee} = -s^{>_p+\vee} + \frac{1}{d-1}s^{>_p+\vee\vee} + \frac{1}{d-1}s^{<>_p>_p+\vee}$$
$$= -s^{>_p+\vee} + s^{>_p+} + \frac{d-2}{d-1}s^{>_p+\vee} + \frac{1}{d-1}s^{<\vee>_p>_p+}$$
$$= s^{>_p+} - \frac{1}{d-1}s^{>_p+\vee} - \frac{1}{d-1}s^{<>_p>_p+},$$

where we have used Corollary 6.9, Lemma 6.15 and the fact that $s^{<\vee} = -s^{<}$. Now, we have

$$s^{+_p<} = -s^{>_p+<} + \frac{1}{d-1}s^{>_p+\vee<} + \frac{1}{d-1}s^{<>_p>_p+<}.$$

Lemma 6.8 still gives $s^{>_p+<} = s$ and $s^{<>_p>_p+<} = s^{<>_p}$ and Corollary 6.10 gives $s^{>_p+\vee<} = s^{<\vee>_p} = -s^{<>_p}$. We get $s^{+_p<} = -s$ as required and again, $s^{+_p<\vee} = -s^{+_p<}$, so that s^{+_p} is a $(k+1)$-skew quadratic field. □

6.3. **Skew quadratic fields and harmonic maps.** For any $k \geq 2$, Lemma I.11.1 shows that reduction of quadratic fields maps $\mathcal{F}_k$ onto $\mathcal{F}_{k-1}$ and the constructions in Section I.4.1 show that the projective limit of the projective system $(\mathcal{F}_k)_{k\geq 1}$ may be identified with the space of Γ-invariant symmetric bilinear forms on $\overline{\mathcal{D}}(\partial X)$. We now develop an analogue interpretation of the projective limit of the projective system $(\mathcal{G}_k)_{k\geq 1}$.

If V is a vector space, say that a skew-symmetric map $\varphi : X_1 \to V$ is harmonic if, for any x in X, one has $\sum_{y\sim x} \varphi(x, y) = 0$. Recall that $\mathcal{Q}(V)$ stands for the space of symmetric bilinear forms on V.

Proposition 6.16. *Let $\varphi : X_1 \to \mathcal{Q}(\overline{\mathcal{D}}(\partial X))$ be a harmonic skew-symmetric map. For any $k \geq 1$ and $x \sim y$ in X, define*

$$s_{k,xy} = (N_x^\ell)^\star \varphi(x, y) \quad k = 2\ell, \ell \geq 1.$$
$$s_{k,xy} = (N_{xy}^\ell)^\star \varphi(x, y) \quad k = 2\ell + 1, \ell \geq 0.$$

Then s_k is a k-skew quadratic field and $s_{k+1}^< = s_k$. One has $\varphi = 0$ if and only if $s_k = 0$ for any $k \geq 1$. Conversely, if, for any $k \geq 1$, s_k is a k-skew quadratic field and $s_{k+1}^< = s_k$, then $(s_k)_{k\geq 1}$ can be obtained in this way. In particular, the projective limit of the projective system $(\mathcal{G}_k)_{k\geq 1}$ may be identified with the space of Γ-equivariant harmonic skew-symmetric maps $X_1 \to \mathcal{Q}(\overline{\mathcal{D}}(\partial X))$.

The natural linear map $N_x^\ell : \overline{V}^\ell(x) \to \overline{\mathcal{D}}(\partial X)$, $x \in X$, $\ell \geq 1$, is defined in Subsection I.4.4. The analogue linear map $N_{xy}^\ell : \overline{V}^\ell(xy) \to \overline{\mathcal{D}}(\partial X)$, $x \sim y \in X$, $\ell \geq 0$, is defined in Subsection I.7.3.

Proof. The proof is a direct consequence of the relation (I.7.7). □

6.4. $(d-1)$**-radical quadratic pseudofields.** We are still developing the theory of skew quadratic fields in analogy with the theory of quadratic fields. It turns out that, in this context, we can define a skew analogue of radical quadratic pseudofields and the shoot map. This is the purpose of the following two subsections.

Definition 6.17. Let $k \geq 1$ and s be a k-pseudofield. If $k \geq 3$ is odd, we say that s is $(d-1)$-radical if one has $s^{\vee<\vee} = (d-1)s^<$. If $k \geq 2$ is even, we say that s is $(d-1)$-radical if one has $s^{\vee<\vee} = (d^\vee - 1)s^<$. If $k = 1$, by convention, every 1-pseudofield is said to be $(d-1)$-radical.

For $k \geq 1$, the space of Γ-invariant $(d-1)$-radical k-pseudofields is denoted by $\mathcal{M}_k^{(d-1)}$. We get an analogue of Lemma 3.8 and Proposition 3.12.

Proposition 6.18. *Let $k \geq 2$ and s be a $(d-1)$-radical k-pseudofield. If k is odd, then $s^{\vee<}$ is $(d-1)$-radical. If k is even, then $\frac{1}{d-1}s^{\vee<}$ is $(d-1)$-radical. In both cases, the associated linear map $\mathcal{M}_k^{(d-1)} \to \mathcal{M}_{k-1}^{(d-1)}$ is onto.*

Proof. Assume k is odd. By assumption, we have $s^{\vee<\vee} = (d-1)s^{<}$. We get

$$s^{\vee<\vee<\vee} = ((d-1)s^{<})^{<\vee} = ((d-1)s^{<<})^{\vee} = (d^{\vee}-1)s^{\vee<<},$$

where we have used Lemma 3.4. This tells us exactly that $s^{\vee<}$ is $(d-1)$-radical.

Assume k is even (and $k \geq 4$, else there is nothing to prove). We now have $s^{\vee<\vee} = (d^{\vee}-1)s^{<}$, hence, as $k-1$ is odd,

$$\left(\frac{1}{d-1}s^{\vee<}\right)^{\vee<\vee} = \left(\frac{1}{d^{\vee}-1}s^{\vee<\vee}\right)^{<\vee} = s^{<<\vee} = s^{\vee<<} = (d-1)\left(\frac{1}{d-1}s^{\vee<}\right)^{<},$$

the latter again following from Lemma 3.4. Thus, the $(k-1)$-pseudofield $\frac{1}{d-1}s^{\vee<}$ is $(d-1)$-radical.

Now, we must prove the surjectivity property. As usual, this will follow from a dual injectivity property.

First, assume $k=1$. In that case, we must show that, if L and M are in $\mathcal{L}_1$ and

$$\frac{1}{d-1}L^{>\vee} = (d^{\vee}-1)M^{>} - M^{\vee>\vee},$$

then $L=0$. Indeed, in this case, Lemma 2.5 says that $M=0$ and $\frac{1}{d-1}L = -M^{\vee}$, hence $L=0$.

Assume $k \geq 2$ is even. In that case, let L and M be in $\mathcal{L}_k$ and

$$L^{>\vee} = (d-1)M^{>} - M^{\vee>\vee}.$$

Lemma 2.5 says that there exists N in $\mathcal{L}_{k-1}$ with

$$L + M^{\vee} = N^{>} \text{ and } (d-1)M = N^{\vee>}.$$

As k is even, we have $(d-1)M^{\vee} = N^{\vee>\vee}$, hence

$$L = N^{>} - \frac{1}{d-1}N^{\vee>\vee} = (d^{\vee}-1)\left(\frac{1}{d^{\vee}-1}N\right)^{>} - \left(\frac{1}{d^{\vee}-1}N\right)^{\vee<\vee},$$

and the injectivity property follows.

Assume $k \geq 3$ is odd and let now L and M be in $\mathcal{L}_k$ with

$$\frac{1}{d-1}L^{>\vee} = (d^{\vee}-1)M^{>} - M^{\vee>\vee},$$

so that, still by Lemma 2.5, there exists N in $\mathcal{L}_{k-1}$ with

$$\frac{1}{d-1}L + M^\vee = N^> \text{ and } (d^\vee - 1)M = N^{\vee >}.$$

As k is odd, we get $(d-1)M^\vee = N^{\vee > \vee}$, hence $L = (d-1)N^> - N^{\vee > \vee}$, which should be proved. $\square$

6.5. **The skew shoot map.** The shoot map was defined in Definition 3.10. It maps radical pseudofields to quadratic fields. Skew quadratic fields appeared naturally as radical pseudofields in the null space of the shoot map. Thus, thanks to Proposition 3.15, we have an exact sequence

$$0 \to \mathcal{G}_k \to \mathcal{M}_k^1 \xrightarrow{P_k} \mathcal{F}_k \to 0.$$

We now introduce the skew shoot map which sends $(d-1)$-radical pseudofields to skew quadratic fields. The null space of the skew shoot map will be the space of quadratic fields and, with the convention of Remark 6.4, we will have an exact sequence

$$0 \to \mathcal{F}_k \to \mathcal{M}_k^{(d-1)} \xrightarrow{Q_k} \mathcal{G}_k \to 0.$$

Lemma 6.19. *Let $k \geq 1$ and t be a $(d-1)$-radical k-pseudofield.*

If k is odd, set $s = t^\vee - t$.

If k is even, set $s = \frac{1}{d-1}t^\vee - t$.

Then, s is a k-skew quadratic field.

Definition 6.20. Let $k \geq 1$, t be a $(d-1)$-radical k-pseudofield and s be as in Lemma 6.19. Then the skew quadratic field s is called the skew shoot of t. The linear map $Q_k : \mathcal{M}_k^{(d-1)} \to \mathcal{G}_k$ that sends a Γ-invariant $(d-1)$-radical k-pseudofield to its skew shoot is called the skew shoot map.

Proof of Lemma 6.19. By construction, we have $s^\vee = -s$. If $k = 1$, we are done.

If $k \geq 2$ and k is even, as t is $(d-1)$-radical, we have

$$s^< = \frac{1}{d-1}t^{\vee <} - t^< = \frac{1}{d-1}t^{\vee <} - \left(\frac{1}{d-1}t^{\vee <}\right)^\vee,$$

hence $s^{<\vee} = -s^<$ and s is a k-skew quadratic field.

If $k \geq 3$ and k is odd, we have

$$s^< = t^{\vee <} - t^< = t^{\vee <} - \frac{1}{d-1}t^{\vee < \vee},$$

hence again $s^{<\vee} = -s^<$ and s is a k-skew quadratic field. $\square$

The proof also yields

Corollary 6.21. *Let $k \geq 2$, t be a $(d-1)$-radical k-pseudofield and s be the skew shoot of t.*

If k is even, the skew shoot of $\frac{1}{d-1} t^{\vee <}$ is $-s^<$.

If k is odd, the skew shoot of $t^{\vee <}$ is $-s^<$.

Finally, we show that the null space of the skew shoot map is the space of quadratic fields, with the convention of Remark 6.4 for embedding quadratic fields inside pseudofields. We have the following counterpart to Lemma 6.2.

Lemma 6.22. *Let $k \geq 1$ and s be a k-pseudofield. Then s is a k-quadratic field if and only if s is $(d-1)$-radical and the skew shoot of s is 0.*

The proof is straightforward.

7. Skew dual kernels

We continue to develop the analogy between the theory of skew quadratic fields and the theory of quadratic fields by introducing skew dual kernels which are related to skew quadratic fields as dual kernels are related to quadratic fields. We then define a pairing between skew dual kernels and skew quadratic fields. When given in addition a Euclidean field, this allows us to introduce a natural scalar product on the space of skew quadratic fields, which we call the skew weight metric.

7.1. **Skew dual kernels and their orthogonal extensions.** We define skew dual kernels in analogy with dual kernels. Their orthogonal extensions are defined by copying the formula for the orthogonal extension of skew quadratic fields.

Definition 7.1. Let $k \geq 2$. A k-skew dual kernel is a pair (H, H^-) where H is a k-pseudokernel, H^- is a $(k-1)$-pseudokernel and $H^\vee = -H$ and $(H^-)^\vee = -H$.

The space of Γ-invariant k-skew dual kernels is denoted by $\mathcal{J}_k$.

By mimicking (6.1) and (6.2), we define the orthogonal extension of skew dual kernels.

Definition 7.2. (k even) Let $k \geq 2$ be an even integer and (H, H^-) be a k-skew dual kernel. We set

$$H^+ = -H^> + H^{>\vee} - H^{->>}.$$

Then the $(k+1)$-skew dual kernel (H^+, H) is called the orthogonal extension of (H, H^-).

Definition 7.3. (k odd) Let $k \geq 2$ be an odd integer and (H, H^-) be a k-skew dual kernel. We set

$$H^+ = -H^> + \frac{1}{d-1} H^{>\vee} - \frac{1}{d-1} H^{->>}.$$

Then the $(k+1)$-skew dual kernel (H^+, H) is called the orthogonal extension of (H, H^-).

As for dual kernels, there is a natural way for associating a skew dual kernel to a pseudokernel.

Definition 7.4. (k even) Let $k \geq 2$ be an even integer and L be $(k-1)$-pseudokernel. We set

$$H = L^> - \frac{1}{d-1} L^{>\vee} \text{ and } H^- = L^\vee - L.$$

We call (H, H^-) the k-skew dual kernel associated to L.

Definition 7.5. (k odd) Let $k \geq 2$ be an odd integer and L be $(k-1)$-pseudokernel. We set

$$H = L^> - L^{>\vee} \text{ and } H^- = L^\vee - (d-1)L.$$

We call (H, H^-) the k-skew dual kernel associated to L.

This defines an injective map from $\mathcal{L}_{k-1}$ to $\mathcal{J}_k$.

Lemma 7.6. *Let $k \geq 1$ and L be a k-pseudokernel. If the $(k+1)$-skew dual kernel associated to L is zero, then L is zero.*

Proof. We prove this by induction on k.

If $k = 1$, we have $L^> = \frac{1}{d-1} L^{>\vee}$ and Lemma 2.5 gives $L = 0$.

Assume $k \geq 2$ and the result holds for $k-1$. If k is even, we have $L^> = L^{>\vee}$, so that Lemma 2.5 says that there exists a $(k-1)$-pseudokernel M with $L = M^> = M^{\vee>}$, hence $M = M^\vee$. As we have $L^\vee = (d-1)L$, we get $M^{>\vee} = (d-1)M^>$, that is, the $(k-1)$-skew dual kernel associated to M is 0. By induction, we have $M = 0$, hence $L = 0$. The proof in the odd case is analogous. □

We get a compatibility relation with orthogonal extension.

Lemma 7.7. *Let $k \geq 2$ and L be a $(k-1)$-pseudokernel. Let (H, H^-) be the k-skew dual kernel associated to L and (H^+, H) be its orthogonal extension.*

If k is even, (H^+, H) is the $(k+1)$-skew dual kernel associated to $\frac{1}{d-1} L^{>\vee}$.

If k is odd, (H^+, H) is the $(k+1)$-skew dual kernel associated to $L^{>\vee}$.

Proof. Assume k is even. Let (J, J^-) be the $(k+1)$-skew dual kernel associated with $\frac{1}{d-1}L^{>\vee}$. By Lemma 6.15, we get

$$J^- = \frac{1}{d-1}L^{>\vee\vee} - L^{>\vee} = L^> + \frac{d-2}{d-1}L^{>\vee} - L^{>\vee} = L^> - \frac{1}{d-1}L^{>\vee} = H.$$

Besides, by definition of the orthogonal extension, we have

$$\begin{aligned} H^+ &= -\left(L^> - \frac{1}{d-1}L^{>\vee}\right)^> + \left(L^> - \frac{1}{d-1}L^{>\vee}\right)^{>\vee} - (L^\vee - L)^{>>} \\ &= -L^{>>} + \frac{1}{d-1}L^{>\vee>} + L^{>>\vee} - \frac{1}{d^\vee - 1}L^{>\vee>\vee} - L^{\vee>>} + L^{>>} \\ &= \left(\frac{1}{d-1}L^{>\vee}\right)^> - \left(\frac{1}{d-1}L^{>\vee}\right)^{>\vee} = J, \end{aligned}$$

where we have used Lemma 2.4.

Assume now k is odd and let (J, J^-) be the $(k+1)$-skew dual kernel associated with $L^{>\vee}$. We have $J^- = L^> - L^{>\vee} = H$ and

$$\begin{aligned} H^+ = -\left(L^> - L^{>\vee}\right)^> &+ \frac{1}{d-1}\left(L^> - L^{>\vee}\right)^{>\vee} \\ &- \frac{1}{d-1}\left(L^\vee - (d-1)L\right)^{>>}. \end{aligned}$$

By again using Lemma 2.4, we get $H^+ = L^{>\vee>} - \frac{1}{d-1}L^{\vee>>} = J$ and we are done. □

7.2. **The skew weight pairing.** We now introduce a pairing between Γ-invariant skew quadratic fields and Γ-invariant skew dual kernels which is an analogue of the weight pairing introduced in Definition I.11.5.

Definition 7.8. Let $k \geq 2$, s be a Γ-invariant k-skew quadratic field and (H, H^-) be a Γ-invariant k-skew dual kernel.

If k is even, we set

$$[s, (H, H^-)] = \left\langle \frac{d-1}{d}s, H \right\rangle + \left\langle \frac{1}{2}s^<, H^- \right\rangle.$$

If k is odd, we set

$$[s, (H, H^-)] = \left\langle \frac{1}{2}s, H \right\rangle + \left\langle \frac{1}{d}s^<, H^- \right\rangle.$$

We call the pairing $[.,.]$ the skew weight pairing.

Remark 7.9. Note that there is no analogue of the weight map in the theory of skew quadratic fields and skew dual kernels.

The skew weight pairing is degenerated on k-skew dual kernels associated to $(k-1)$-pseudokernels.

Lemma 7.10. *Let $k \geq 2$, s be a Γ-invariant k-skew quadratic field, L be a Γ-invariant $(k-1)$-pseudokernel and (H, H^-) be the k-skew dual kernel associated to L. We have $[s, (H, H^-)] = 0$.*

We shall see later that the converse is true: if a k-skew dual kernel has zero weight pairing with every k-skew quadratic field, then it is the skew dual kernel associated to a $(k-1)$-pseudokernel.

Proof. Assume k is even. Definition 7.4 and Definition 7.8 give

$$[s, (H, H^-)] = \left\langle \frac{d-1}{d} s, L^{>} - \frac{1}{d-1} L^{>\vee} \right\rangle + \left\langle \frac{1}{2} s^{<}, L^{\vee} - L \right\rangle.$$

We have

$$\left\langle \frac{d-1}{d} s, \frac{1}{d-1} L^{>\vee} \right\rangle = \left\langle \frac{1}{d} s, L^{>\vee} \right\rangle = \left\langle -\frac{1}{d} s^{<}, L \right\rangle,$$

since $s^{\vee} = -s$. As $s^{<\vee} = -s^{<}$, we get

$$[s, (H, H^-)] = \left\langle \frac{d-1}{d} s^{<} + \frac{1}{d} s^{<}, L \right\rangle - \langle s^{<}, L \rangle = 0.$$

Assume k is odd. Definition 7.4 and Definition 7.8 now give

$$[s, (H, H^-)] = \left\langle \frac{1}{2} s, L^{>} - L^{>\vee} \right\rangle + \left\langle \frac{1}{d} s^{<}, L^{\vee} - (d-1) L \right\rangle.$$

As above, we have

$$\left\langle \frac{1}{2} s, L^{>} - L^{>\vee} \right\rangle = \langle s^{<}, L \rangle$$

and

$$\left\langle \frac{1}{d} s^{<}, L^{\vee} - (d-1) L \right\rangle = \left\langle \frac{1}{d} s^{<\vee} - \frac{d-1}{d} s^{<}, L \right\rangle = -\langle s^{<}, L \rangle$$

and the result follows. □

We have an analogue of Corollary I.11.6.

Proposition 7.11. *Let $k \geq 2$, s be in $\mathcal{G}_{k+1}$ and (H, H^-) be in $\mathcal{J}_k$. Set (H^+, H) to be the orthogonal extension of (H, H^-). We have*

$$[s, (H^+, H)] = -[s^{<}, (H, H^-)].$$

Proof. Assume k is even. Definition 7.2 and Definition 7.8 give

$$\begin{aligned}
[s,(H^+,H)] &= \left\langle \frac{1}{2}s, -H^{>} + H^{>\vee} - H^{->>} \right\rangle + \left\langle \frac{1}{d}s^{<}, H \right\rangle \\
&= -\frac{1}{2}\left\langle s^{<}, H\right\rangle + \frac{1}{2}\left\langle s^{\vee<}, H\right\rangle - \frac{1}{2}\left\langle s^{<<}, H^{-}\right\rangle + \left\langle \frac{1}{d}s^{<}, H \right\rangle \\
&= -\left\langle s^{<}, H\right\rangle - \frac{1}{2}\left\langle s^{<<}, H^{-}\right\rangle + \left\langle \frac{1}{d}s^{<}, H \right\rangle \\
&= -[s,(H,H^-)],
\end{aligned}$$

where we have used that $s^\vee = -s$.

Assume k is odd. Definition 7.3 and Definition 7.8 now give

$$\begin{aligned}
[s,(H^+,H)] &= \left\langle \frac{1}{d}s, -(d-1)H^{>} + H^{>\vee} - H^{->>} \right\rangle + \left\langle \frac{1}{2}s^{<}, H \right\rangle \\
= -\left\langle \frac{d-1}{d}s^{<}, H \right\rangle &+ \left\langle \frac{1}{d}s^{\vee<}, H \right\rangle - \left\langle \frac{1}{d}s^{<<}, H^{-} \right\rangle + \left\langle \frac{1}{2}s^{<}, H \right\rangle \\
&= -[s,(H,H^-)],
\end{aligned}$$

where we have used that $s^\vee = -s$ and $\frac{d-1}{d} + \frac{1}{d} - \frac{1}{2} = \frac{1}{2}$. □

7.3. **The skew canonical pseudokernel.** We now introduce the analogue of the canonical map of dual kernels. This will allow us in particular to determine the null space of the weight pairing.

Definition 7.12. Let $k \geq 2$ and (H,H^-) be a k-skew dual kernel. The skew canonical k-pseudokernel L of (H,H^-) is defined by

$$L = H + H^{->}.$$

We write $D_k : \mathcal{J}_k \to \mathcal{L}_k$ for the linear map that sends a Γ-invariant k-skew dual kernel to its skew canonical k-pseudokernel. We call this map the skew canonical map. As in the case of dual kernels, we have a compatibility property with orthogonal extension.

Lemma 7.13. *Let $k \geq 2$ and (H,H^-) be a k-skew dual kernel with skew canonical k-pseudokernel L and orthogonal extension (H^+,H).*

If k is even, the skew canonical $(k+1)$-pseudokernel of (H^+,H) is $L^{>\vee}$.

If k is odd, the skew canonical $(k+1)$-pseudokernel of (H^+,H) is $\frac{1}{d-1}L^{>\vee}$.

Proof. Let M be the skew canonical $(k+1)$-pseudokernel of (H^+,H).

Assume k is even. In that case, Definition 7.2 gives

$$M = (-H^{>} + H^{>\vee} - H^{->>}) + H^{>} = H^{>\vee} + H^{-\vee>>} = (H + H^{->})^{>\vee} \\ = L^{>\vee},$$

where we have used that $H^{-\vee} = -H^{-}$ and Lemma 2.4.

Assume k is odd. Now, Definition 7.3 gives

$$M = (-H^{>} + \frac{1}{d-1} H^{>\vee} - \frac{1}{d-1} H^{->>}) + H^{>} = \frac{1}{d-1}(H^{>\vee} + H^{-\vee>>}) \\ = \frac{1}{d-1} L^{>\vee}.$$

□

As for dual kernels (see Lemma 2.10), the skew canonical k-pseudokernel of the k-skew dual kernel associated to a $(k-1)$-pseudokernel may be computed directly.

Lemma 7.14. *Let $k \geq 2$, M be a $(k-1)$-pseudokernel and (H, H^{-}) be the k-skew dual kernel associated to $M^{\vee}$. Let L be the skew canonical k-pseudokernel of (H, H^{-}).*

If k is even, we have $L = M^{>} - \frac{1}{d-1} M^{\vee>\vee}$.

If k is odd, we have $L = (d-1)M^{>} - M^{\vee>\vee}$.

Proof. Assume k is even. Then Definition 7.4 and Definition 7.12 give

$$L = M^{\vee>} - \frac{1}{d-1} M^{\vee>\vee} + (M - M^{\vee})^{>} = M^{>} - \frac{1}{d-1} M^{\vee>\vee}.$$

Assume k is odd. Definition 7.5 and Definition 7.12 now give

$$L = M^{\vee>} - M^{\vee>\vee} + (M^{\vee\vee} - (d-1)M^{\vee})^{>} = (d-1)M^{>} - M^{\vee>\vee},$$

where we have used Lemma 6.15. □

Again as for dual kernels, the skew canonical map is injective.

Lemma 7.15. *Let $k \geq 2$ and (H, H^{-}) be a k-skew dual kernel. If the skew canonical k-pseudokernel of (H, H^{-}) is 0, then $(H, H^{-}) = 0$.*

Proof. We will prove this by induction on k.

Assume $k = 2$. Then, we have $H^{->} = -H = H^{\vee} = -H^{->\vee}$. Lemma 2.5 gives $H^{-} = 0$, and hence $H = 0$.

Assume $k \geq 3$ and the results holds for $k-1$. Again, we have $H^{->} = -H = H^{\vee} = -H^{->\vee}$. Lemma 2.5 now says that there exists a $(k-1)$-pseudokernel J with $H^{-} = -J^{>} = J^{\vee>}$, so that $J^{\vee} = -J$ and (H^{-}, J) is a $(k-1)$-skew dual kernel with zero skew canonical k-pseudokernel. The induction assumption says that $H^{-} = 0$, hence also $H = 0$, which should be proved. □

Lemma 7.13 and Lemma 7.15 give directly

Corollary 7.16. *Let $k \geq 2$, (H, H^-) be a k-skew dual kernel, L be its skew canonical k-pseudokernel and M be a $(k-1)$-pseudokernel.*

If k is even, we have $L = M^> - \frac{1}{d-1} M^{\vee > \vee}$ if and only if (H, H^-) is the k-skew dual kernel associated to $M^\vee$.

If k is odd, we have $L = (d-1)M^> - M^{\vee > \vee}$ if and only if (H, H^-) is the k-skew dual kernel associated to $M^\vee$.

7.4. **Crossed duality.** We will show that the skew canonical map may be seen as the adjoint of the skew shoot map with respect to the skew weight pairing. This will allow us to describe the null space of this pairing in the space of skew dual kernels.

Lemma 7.17. *Let $k \geq 2$. Let t in $\mathcal{M}_k^{(d-1)}$ be a Γ-invariant $(d-1)$-radical k-pseudofield and set $s = Q_k t$ to be the skew shoot of t. Let (H, H^-) in $\mathcal{J}_k$ be a Γ-invariant k-skew dual kernel and set $L = D_k(K, K^-)$ to be the skew canonical k-pseudokernel of (K, K^-). We have*

$$[s, (H, H^-)] = -\langle t, L \rangle.$$

Proof. Again, this is a straightforward computation. If k is even, Definition 6.20 and Definition 7.8 give

$$[s, (H, H^-)] = \left\langle \frac{1}{d} t^\vee - \frac{d-1}{d} t, H \right\rangle + \frac{1}{2} \left\langle \frac{1}{d-1} t^{\vee <} - t^<, H^- \right\rangle.$$

As $H^\vee = -H$, we have $\left\langle \frac{1}{d} t^\vee - \frac{d-1}{d} t, H \right\rangle = -\langle t, H \rangle$. As $H^{-\vee} = -H^-$ and t is $(d-1)$-radical, we have

$$\begin{aligned}\left\langle \frac{1}{d-1} t^{\vee <}, H^- \right\rangle = - \left\langle \frac{1}{d^\vee - 1} t^{\vee < \vee} - t^<, H^- \right\rangle \\ = -\langle t^<, H^- \rangle = -\langle t, H^{->} \rangle.\end{aligned}$$

We get $[s, (H, H^-)] = -\langle t, H \rangle - \langle t, H^{->} \rangle = -\langle t, L \rangle$, as required.

If k is odd, still by Definition 6.20 and Definition 7.8, we have

$$[s, (H, H^-)] = \left\langle \frac{1}{2} t^\vee - \frac{1}{2} t, H \right\rangle + \left\langle \frac{1}{d} t^{\vee <} - \frac{1}{d} t^<, H^- \right\rangle.$$

As $H^\vee = -H$, we have $\left\langle \frac{1}{2} t^\vee - \frac{1}{2} t, H \right\rangle = -\langle t, H \rangle$. As $H^{-\vee} = -H^-$ and t is $(d-1)$-radical, we have

$$\begin{aligned}\left\langle \frac{1}{d} t^{\vee <}, H^- \right\rangle = - \left\langle \frac{1}{d} t^{\vee < \vee}, H^- \right\rangle = - \left\langle \frac{d-1}{d} t^<, H^- \right\rangle \\ = - \left\langle \frac{d-1}{d} t, H^{->} \right\rangle.\end{aligned}$$

Again, we get $[s,(H,H^-)] = -\langle t, H+H^{->}\rangle = -\langle t, L\rangle$, as required. $\square$

This allows us to determine the null space of the skew weight pairing in skew dual kernels.

Corollary 7.18. *Let $k \geq 2$ and (H,H^-) be in $\mathcal{J}_k$. Then one has $[s,(H,H^-)] = 0$ for every s in $\mathcal{G}_k$ if and only if (H,H^-) is the k-skew dual kernel associated to a $(k-1)$-pseudokernel.*

Proof. The *if* part was established in Lemma 7.10. Conversely, by Lemma 7.17, if $[s,(H,H^-)] = 0$ for any s in $\mathcal{G}_k$, then $D_k(H,H^-)$ belongs to the orthogonal subspace to $\mathcal{M}_k^{(d-1)}$ in $\mathcal{L}_k$. The conclusion follows from Definition 7.1 and Corollary 7.16. $\square$

By reasoning as for Proposition 3.15, from Proposition 6.18 and Lemma 7.17, one can show

Proposition 7.19. *Let $k \geq 1$. Then Q_k maps $\mathcal{M}_k^{(d-1)}$ onto $\mathcal{G}_k$. If $k \geq 2$ and k is even (resp. odd), then the map $t \mapsto (Q_k t, \frac{1}{d-1} t^{\vee<})$ (resp. $t \mapsto (Q_k t, t^{\vee<})$) sends $\mathcal{M}_k^{(d-1)}$ onto the space*

$$\{(s,u) \in \mathcal{G}_k \times \mathcal{M}_{k-1}^{(d-1)} | -s^< = Q_{k-1} u\}.$$

7.5. **The skew weight metric.** We conclude this Section by using the above constructed objects to equip the space $\mathcal{G}_k$ of Γ-invariant k-skew fields with a scalar product that is constructed in analogy with the weight metric of Section I.11.

Let $k \geq 2$ and p be a k-Euclidiean field. Let us introduce the analogue operation to that of Subsection 6.2 for pseudokernels. We use the notation of Subsection I.10.6 for the adjoint operators $I_{xy}^{\ell-1,*\dagger p}$ and $J_{xy}^{\ell,*\dagger p}$, $x \sim y \in X$, $\ell \geq 1$.

Let L be a k-pseudokernel and, for any $x \sim y$, let r_{xy}^L be the symmetric bilinear form associated with L_{xy}. If k is even, $k = 2\ell$, $\ell \geq 1$, we set $L_{xy}^{<_p}$ to be the function on $S^{\ell-1}(xy)^2$ associated with the bilinear form $(I_{xy}^{\ell-1,*\dagger p})^\star r_{xy}^L$ by Lemma I.5.1. If k is odd, $k = 2\ell+1$, $\ell \geq 1$, we set $L_{xy}^{<_p}$ to be the function on $S^\ell(x)^2$ associated with the bilinear form $(J_{xy}^{\ell,*\dagger p})^\star r_{xy}^L$ by Lemma I.5.1. Then $L^{<_p}$ is a $(k-1)$-pseudokernel.

We directly get:

Lemma 7.20. *Let $k \geq 2$ and p be a k-Euclidiean field. If L is a $(k-1)$-pseudokernel, then $L^{><_p} = L$.*

Lemma 7.21. *Let $k \geq 2$ and p be a Γ-invariant k-Euclidiean field. For L in $\mathcal{L}_k$ and s in $\mathcal{M}_{k-1}$, we have $\langle s^{>_p}, M\rangle = \langle s, M^{<_p}\rangle$.*

Let still p be in $\mathcal{P}_k$. If s is a Γ-invariant k-skew quadratic field, we let H be the k-pseudokernel obtained in the following way.

If k is even, $k = 2\ell$, $\ell \geq 1$, for $x \sim y$ in X, then H_{xy} is the function on $S^\ell(x)^2$ associated by Lemma I.5.1 to the bilinear form on $V_0^\ell(x)$ obtained from s_{xy} by the identification between $V_0^\ell(x)$ and $\overline{V}^\ell(x)$ through the scalar product p_x.

If k is odd, $k = 2\ell + 1$, $\ell \geq 1$, for $x \sim y$ in X, then H_{xy} is the function on $S^\ell(xy)^2$ associated by Lemma I.5.1 to the bilinear form on $V_0^\ell(xy)$ obtained from s_{xy} by the identification between $V_0^\ell(xy)$ and $\overline{V}^\ell(xy)$ through the scalar product p_{xy}.

In both cases, we have $H^\vee = -H$. We set $H^- = -H^{<_p}$. As $s^{<\vee} = -s^<$, one easily checks that $H^{<_p\vee} = -H^{<_p}$. Therefore, the pair (H, H^-) is a k-skew dual kernel. If r is an other element of $\mathcal{G}_k$, we set

$$h_p(r, s) = [r, (H, H^-)].$$

Proposition 7.22. *Let $k \geq 2$ and p be a Γ-invariant k-Euclidiean field. The bilinear form h_p on $\mathcal{G}_k$ is symmetric and positive definite. If r is in $\mathcal{G}_{k+1}$ and s is in $\mathcal{G}_k$, we have*

$$h_{p^+}(r, s^{+_p}) = -h_p(r^<, s).$$

We will call h_p the skew weight metric on $\mathcal{G}_k$.

Proof. This is a consequence of the previous constructions of the orthogonal extension and the skew weight pairing.

Indeed, let r, s be in $\mathcal{G}_k$.

If k is even, $k = 2\ell$, $\ell \geq 1$, for $x \sim y$ in X, let A_{xy} and B_{xy} be the endomorphisms that represent the bilinear forms s_{xy} and r_{xy} with respect to the scalar product p_x on $\overline{V}^\ell(x)$ and let A^-_{xy} and B^-_{xy} be the endomorphisms that represent the bilinear forms $s^<_{xy}$ and $r^<_{xy}$ with respect to the scalar product p^-_{xy} on $\overline{V}^{\ell-1}(xy)$. Then the Definition 7.8 of the weight pairing and that of h_p give

(7.1)

$$h_p(r, s) = \sum_{(x,y)\in\Gamma\backslash X_1} \frac{1}{|\Gamma_x \cap \Gamma_y|} \left(\frac{d(x) - 1}{d(x)} \operatorname{tr}(A_{xy} B_{xy}) - \frac{1}{2} \operatorname{tr}(A^-_{xy} B^-_{xy}) \right).$$

Symmetry follows. Besides, as $d(x) \geq 3$ for any x in X, we get $\frac{d(x)-1}{d(x)} > \frac{1}{2}$ and therefore, if $r = s \neq 0$, the latter is positive.

In the same way, if k is odd, $k = 2\ell + 1$, $\ell \geq 1$, for $x \sim y$ in X, let A_{xy} and B_{xy} be the endomorphisms that represent the bilinear forms s_{xy} and r_{xy} with respect to the scalar product p_{xy} on $\overline{V}^\ell(xy)$ and let

A^-_{xy} and B^-_{xy} be the endomorphisms that represent the bilinear forms $s^<_{xy}$ and $r^<_{xy}$ with respect to the scalar product p^-_x on $\overline{V}^\ell(x)$. Still by Definition 7.8, we may write

(7.2)

$$h_p(r,s) = \sum_{(x,y)\in\Gamma\backslash X_1} \frac{1}{|\Gamma_x \cap \Gamma_y|}\left(\frac{1}{2}\operatorname{tr}(A_{xy}B_{xy}) - \frac{1}{d(x)}\operatorname{tr}(A^-_{xy}B^-_{xy})\right).$$

Symmetry and positivity follow.

It remains to prove the adjointness property of orthogonal extension. We take r in $\mathcal{G}_{k+1}$ and s is in $\mathcal{G}_k$ and, as in the construction of h_p, we let H be the k-pseudokernel with $H^\vee = -H$ that is associated with s by the duality coming from p. By this construction, we have

$$h_p(r^<, s) = [r^<, (H, H^-)],$$

where $H^- = -H^{<_p}$. By comparing the Definition 6.13 of the orthogonal extension of s and Definitions 7.2 and 7.3 of the orthogonal extension (H^+, H) of (H, H^-), we get

$$h_p(r, s^{+_p}) = [r, (H^+, H)].$$

The conclusion now follows from Proposition 6.14 and Proposition 7.11. □

Recall from Corollary 6.7 that direct restriction is a surjective linear map $\mathcal{G}_{k+1} \to \mathcal{G}_k$.

Corollary 7.23. *Let $k \geq 2$ and p be a Γ-invariant k-Euclidiean field. The scalar product h_p on $\mathcal{G}_k$ is the Euclidean image of h_{p^+} on $\mathcal{G}_{k+1}$ by direct restriction.*

Euclidean images of nonnegative symmetric bilinear forms under surjective maps were defined in Appendix I.A.

Proof. We need to show that, with respect to the Euclidean structures h_{p^+} and h_p, the adjoint map of direct restriction $\mathcal{G}_{k+1} \to \mathcal{G}_k$ is an isometric embedding $\mathcal{G}_k \to \mathcal{G}_{k+1}$. Now, Proposition 7.22 precisely says that this adjoint map is the map $s \mapsto -s^{+_p}, \mathcal{G}_k \to \mathcal{G}_{k+1}$ and Proposition 6.14 ensures that, for s in $\mathcal{G}_k$, one has $-s^{+_p<} = s$. The conclusion follows. □

8. Projective limits of the weight metric and the skew weight metric

Let still $k \geq 2$ and p be a Γ-invariant k-Euclidean field. With the language of Appendix I.B, it follows from Corollary 7.23 above that the skew weight metrics associated to p and its orthogonal extensions allow to equip the projective system $(\mathcal{G}_j)_{j\geq k}$ with the structure of a straight nonnegative projective system. In the same way, if p is admissible, Proposition I.11.12 and Theorem I.11.16 ensure that the weight metrics associated to p and its orthogonal extensions allow to equip the projective system $(\mathcal{F}_j)_{j\geq k}$ with the structure of a straight nonnegative projective system. The purpose of this final Section is to give a concrete description of the Euclidean projective limits of both these systems.

8.1. **The projective system of skew quadratic fields.** We start with the case of skew quadratic fields. This case is easier and will motivate the formulation of the case of quadratic fields.

We recall some facts about Hilbert-Schmidt quadratic forms which can be deduced from the analagous properties of Hilbert-Schmidt operators (see for example [8]). Let H be a real Hilbert space with scalar product p and (v_i) be a Hilbert basis of H. A symmetric bilinear form q on H is said to be a Hilbert-Schmidt form if one has

$$\sum_{i,j} q(v_i, v_j)^2 < \infty.$$

This property is independent on the choice of the Hilbert basis. It implies q to be bounded with respect to the Hilbert norm on H. The space of Hilbert-Schmidt symmetric bilinear forms q is denoted by $\mathcal{Q}^2(H)$. The bilinear form $p_2^* : (q, r) \mapsto \sum_{i,j} q(v_i, v_j)r(v_i, v_j)$ is positive definite on $\mathcal{Q}^2(H)$ and it also does not depend of the basis. The space $\mathcal{Q}^2(H)$ is complete with respect to the associated Euclidean norm. If H has finite dimension, this scalar product may be written as $(q, r) \mapsto \mathrm{tr}(AB)$ where A and B are the endomorphisms which represent q and r with respect to p.

Now, let $k \geq 2$ and p be a Γ-invariant k-Euclidean field. As in Subsection I.4.6 and I.4.5, the successive orthogonal extensions of p define a scalar product on the space $\overline{\mathcal{D}}(\partial X)$ of classes of smooth functions on ∂X modulo the constant functions. The completion of $\overline{\mathcal{D}}(\partial X)$ with respect to this scalar product is denoted by H^p. By abuse of notation, we still denote by p the scalar product of H^p. As $\overline{\mathcal{D}}(\partial X)$ is dense in H^p, we may consider the space $\mathcal{Q}^2(H^p)$ of Hilbert-Schmidt symmetric bilinear forms on H^p as a subspace of the space $\mathcal{Q}(\overline{\mathcal{D}}(\partial X))$ of symmetric bilinear forms on $\overline{\mathcal{D}}(\partial X)$.

Recall from Proposition 6.16 that we may identify the projective limit of the system $(\mathcal{G}_k)_{k\geq 1}$ with the space of Γ-invariant harmonic skew symmetric maps $X_1 \to \mathcal{Q}(\overline{\mathcal{D}}(\partial X))$.

Proposition 8.1. *Let $k \geq 2$ and p be a Γ-invariant k-Euclidean field. Then the Euclidean projective limit of the projective system $(\mathcal{G}_j)_{j\geq k}$, equipped with the skew weight metric, is the space of Γ-invariant harmonic skew symmetric maps $X_1 \to \mathcal{Q}^2(H^p)$, equipped with the scalar product*

$$(r, s) \mapsto \sum_{(x,y)\in\Gamma\backslash X_1} \frac{1}{|\Gamma_x \cap \Gamma_y|} \frac{d(x) - 2}{2d(x)} p_2^*(r_{xy}, s_{xy}).$$

See Appendix I.B for the language of Euclidean projective limits.

The proof uses the following direct consequence of the definition of the Hilbert-Schmidt norm:

Lemma 8.2. *Let H be a Hilbert space and $(H_\ell)_{\ell\geq 0}$ be an increasing sequence of finite-dimensional subspaces of H such that $\bigcup_{\ell\geq 0} H_\ell$ is dense in H. Then, the Euclidean projective limit of the system $(\mathcal{Q}^2(H_\ell))_{\ell\geq 0}$ is the space $\mathcal{Q}^2(H)$.*

Proof of Proposition 8.1. Set $C = 6 \sup_{(x,y)\in X_1} |\Gamma_x \cap \Gamma_y|$.

If k is even, for r, s in $\mathcal{G}_k$, we can write (7.1) as

$$(8.1)\quad h_p(s, s) = \sum_{(x,y)\in\Gamma\backslash X_1} \frac{1}{|\Gamma_x \cap \Gamma_y|} \left(\frac{d(x) - 1}{d(x)} (p_x)_2^*(r_{xy}, s_{xy}) - \frac{1}{2} (p_{xy}^-)_2^*(r_{xy}^<, s_{xy}^<) \right).$$

As $d(x) \geq 3$ for any x in X, this gives

$$(8.2)\qquad h_p(s, s) \leq \sum_{(x,y)\in\Gamma\backslash X_1} (p_x)_2^*(r_{xy}, s_{xy}) \leq C h_p(s, s).$$

In the same way, if k is odd, for r, s in $\mathcal{G}_k$, (7.2) gives

$$(8.3)\quad h_p(s, s) = \sum_{(x,y)\in\Gamma\backslash X_1} \frac{1}{|\Gamma_x \cap \Gamma_y|} \left(\frac{1}{2} (p_{xy})_2^*(r_{xy}, s_{xy}) - \frac{1}{d(x)} (p_x^-)_2^*(r_{xy}^<, s_{xy}^<) \right).$$

As above, we get

$$(8.4)\qquad h_p(s, s) \leq \sum_{(x,y)\in\Gamma\backslash X_1} (p_{xy})_2^*(r_{xy}, s_{xy}) \leq C h_p(s, s).$$

Together with Lemma 8.2, (8.2) and (8.4) imply that a Γ-invariant harmonic skew symmetric map $s : X_1 \to \mathcal{Q}(\overline{\mathcal{D}}(\partial X))$ belongs to the Euclidean projective limit of $(\mathcal{G}_j)_{j\geq k}$ if and only if, for any (x, y) in X_1,

s_{xy} belongs to $\mathcal{Q}^2(H^p)$. For such an s and $j \geq k$, let s^j be the image of s in $\mathcal{G}_j$. Still by Lemma 8.2, we get, for (x, y) in X_1,

$$(p_x^j)_2^*(s_{xy}^j, s_{xy}^j) \xrightarrow[\substack{j\to\infty \\ j \text{ even}}]{} p_2^*(s_{xy}, s_{xy})$$
$$\text{and } (p_{xy}^j)_2^*(s_{xy}^j, s_{xy}^j) \xrightarrow[\substack{j\to\infty \\ j \text{ odd}}]{} p_2^*(s_{xy}, s_{xy}),$$

where p^j is the $(j-k)$-th orthogonal extension of p. By using the relation $\frac{d(x)-1}{d(x)} - \frac{1}{2} = \frac{d(x)-2}{2d(x)} = \frac{1}{2} - \frac{1}{d(x)}$, we get from (8.1) and (8.3),

$$h_{p^j}(s^j, s^j) \xrightarrow[j\to\infty]{} \sum_{(x,y)\in\Gamma\backslash X_1} \frac{1}{|\Gamma_x \cap \Gamma_y|} \frac{d(x)-2}{2d(x)} p_2^*(s_{xy}, s_{xy}).$$

The conclusion follows in view of Lemma I.B.3 and Definition I.B.4. $\square$

8.2. **The weight pairing and the standard pairing.** Our objective will now be to give a description of the Euclidean projective limit of the projective system $(\mathcal{F}_j)_{j\geq k}$, equipped with the weight metric associated to an admissible Γ-invariant k-Euclidean field, as in Section I.11. To do this, we will use the abstract formalism of Appendix A to relate the weight pairing and the weight metric to the language of traces on Von Neumann algebras.

Indeed, by definition, the Hilbert space H_0^ω of Subsection I.3.1 is Γ-isomorphic to a closed Γ-invariant subspace of the space $\ell^2(X_1)$. By assumption, Γ has finitely many orbits in X_1 and the stabilizers of the elements of X_1 in Γ are finite. Thus, with the language of Appendix A, the action of Γ on X_1 is standard. Therefore, we can apply to the representation of Γ in the Hilbert space H_0^ω the results of this Appendix. In particular, in the present Subsection, we relate the weight pairing of Definition I.11.5 with the standard quadratic pairing $\langle .,. \rangle_\Gamma$ of Subsection A.3.

Recall that by construction, we have a dense embedding of $\overline{\mathcal{D}}(\partial X)$ in the topological dual space $(H_0^\omega)'$ of H_0^ω. This gives a natural injection of $\mathcal{Q}^\infty((H_0^\omega)')^\Gamma$ inside $\mathcal{Q}(\overline{\mathcal{D}}(\partial X))^\Gamma$, where $\mathcal{Q}^\infty((H_0^\omega)')^\Gamma$ is the space of continuous Γ-invariant symmetric bilnear forms on $(H_0^\omega)'$. Besides, in view of the results of Section I.4, the latter may be identified with the projective limit of the projective system $(\mathcal{F}_k)_{k\geq 1}$ of Γ-invariant quadratic fields.

Proposition 8.3. *Let $k \geq 2$, (K, K^-) be a Γ-invariant k-dual kernel, $w : X_k \to \mathbb{R}$ be a Γ-invariant weight function for (K, K^-) and Φ_w be the associated symmetric bilinear form on H_0^ω. Let also Π be in*

$\mathcal{Q}^\infty((H_0^\omega)')^\Gamma$ and p be the image of Π by the natural map $\mathcal{Q}^\infty((H_0^\omega)')^\Gamma \to \mathcal{F}_k$. Then we have

$$\langle \Pi, \Phi_w \rangle_\Gamma = [p, (K, K^-)].$$

The bilinear form Φ_w is defined in Subsection I.3.2 and the notion of a k-dual kernel is introduced in Subsection I.5.1. The weight functions associated to a k-dual kernel are constructed in Subsection I.6.2. The weight pairing is defined in Subsection I.11.2.

Proof. Define a bilinear form Ψ_w on $\ell^2(X_1)$ by setting, for ρ, θ in $\ell^2(X_1)$,

$$\Psi_w(\rho, \theta) = -\frac{1}{2} \sum_{(x,y)\in X_k} w(x,y)\rho(x,x_1)\theta(y,y_1), \tag{8.5}$$

where, as usual, for (x,y) in X_k, x_1 and y_1 are the neighbours of x and y in the segment $[xy]$. By Lemma A.11, Ψ_w is a well-defined bounded bilinear form on $\ell^2(X_1)$. As w is a symmetric function, Ψ_w is symmetric. Let $I : H_0^\omega \to \ell^2(X_1)$ be the natural injection. Then (I.3.3) implies $I^\star \Psi_w = \Phi_w$.

On the other hand, identify $\ell^2(X_1)$ with its topological dual space by the usual scalar product and let $I^* : \ell^2(X_1) \to (H_0^\omega)'$ denote the adjoint map of I. Recall that, for (x,y) in X_1, U_{xy} denotes the closed open subset of ∂X,

$$U_{xy} = \{\xi \in \partial X | y \in [x\xi)\}.$$

By the construction of the embedding I in Subsection I.3.1, for θ in H_0^ω, when θ is viewed as a distribution on ∂X, we have

$$I\theta(x,y) = \theta(\mathbf{1}_{U_{xy}}).$$

This gives

$$I^*(\mathbf{1}_{(x,y)}) = \mathbf{1}_{U_{xy}},$$

where we have identified $\overline{\mathcal{D}}(\partial X)$ with a subspace of $(H_0^\omega)'$. Therefore, by (8.5), (A.3) and the definition of the standard pairing in Subsection A.3, we get

$$\begin{aligned} &\langle (I^*)^\star \Pi, \Psi_w \rangle_\Gamma = \\ &-\frac{1}{2} \sum_{((a,a'),(b,b'))\in\Gamma\backslash X_1^2} \frac{\Pi(\mathbf{1}_{U_{aa'}}, \mathbf{1}_{U_{bb'}})}{|\Gamma_a \cap \Gamma_{a'} \cap \Gamma_b \cap \Gamma_{b'}|} \sum_{(x,y)\in X_k} w(x,y)\mathbf{1}_{\substack{x=a,x_1=a',\\ y=b,y_1=b'}} \\ &\qquad = -\frac{1}{2} \sum_{(x,y)\in\Gamma\backslash X_k} \frac{1}{|\Gamma_x \cap \Gamma_y|} \Pi(\mathbf{1}_{U_{xx_1}}, \mathbf{1}_{U_{yy_1}}) w(x,y), \end{aligned} \tag{8.6}$$

where we also have used Lemma I.9.11. By the weight formula, that is, Theorem I.11.4, we have

$$-\frac{1}{2}\sum_{(x,y)\in\Gamma\backslash X_k}\frac{1}{|\Gamma_x\cap\Gamma_y|}\Pi(\mathbf{1}_{U_{xx_1}},\mathbf{1}_{U_{yy_1}})w(x,y)=[p,(K,K^-)].$$

The conclusion follows since, by Lemma A.9, we have

$$\langle (I^*)^\star\Pi,\Psi_w\rangle_\Gamma=\langle\Pi,I^\star\Psi_w\rangle_\Gamma=\langle\Pi,\Phi_w\rangle_\Gamma.$$

□

Proposition 8.3 yields the following corollary which we were not able to prove by a more direct method.

Corollary 8.4. *For $k\geq 2$, any admissible Γ-invariant k-Euclidean field is tight.*

Admissible Euclidean fields are defined in Section I.10; tight quadratic fields are defined in Section 4.

Proof. Let p be an admissible k-Euclidean field. Then the successive orthogonal extensions of p define a scalar product p^∞ on the dual space of H_0^ω. Let w be a Γ-invariant function on X_k such that Φ_w is non-negative and (K,K^-) be a non-negative k-dual kernel such that w is a weight function of (K,K^-). By Proposition 8.3, we have

$$[p,(K,K^-)]=\langle p^\infty,\Phi_w\rangle_\Gamma.$$

By Proposition A.10, we get $[p,(K,K^-)]\geq 0$. The conclusion follows. □

8.3. **The projective system of quadratic fields.** To conclude, for $k\geq 2$ and p an admissible Γ-invariant k-Euclidean field, we will use the previous constructions to describe the Euclidean projective limit of the projective system $(\mathcal{F}_j)_{j\geq k}$, equipped with the weight metric, by means of the standard scalar products of Subsection A.4.

As above, we identify the projective limit of the system $(\mathcal{F}_k)_{k\geq 1}$ with the space $\mathcal{Q}(\overline{\mathcal{D}}(\partial X))^\Gamma$ of Γ-invariant symmetric bilinear forms on $\overline{\mathcal{D}}(\partial X)$. Let $k\geq 2$ and p be a Γ-invariant admissible k-Euclidean field. We still set H^p to be the completion of $\overline{\mathcal{D}}(\partial X)$ with respect to the scalar product obtained from p by its successive orthogonal extensions as in Section I.4. Then we have a natural embedding of $\mathcal{Q}^\infty(H^p)^\Gamma$ inside $\mathcal{Q}(\overline{\mathcal{D}}(\partial X))^\Gamma$.

Assume now that p is admissible as in Definition I.10.1, so that, by Theorem I.7.6 and Theorem I.7.17, the space H^p may be considered as the topological dual space of the Hilbert space H_0^ω. As the representation of Γ in H_0^ω is standard, so is its representation on H^p and we may define the standard scalar product associated to p on $\mathcal{Q}^\infty(H^p)^\Gamma$, which we still write as p^*_Γ as in Definition A.13.

Theorem 8.5. *Let $k \geq 2$ and p be an admissible Γ-invariant k-Euclidean field. Then the Euclidean projective limit of the projective system $(\mathcal{F}_j)_{j\geq k}$, equipped with the weight metric g_p, is the completion of $\mathcal{Q}^\infty(H^p)^\Gamma$ with respect to the standard scalar product p^*_Γ. More precisely, when both of them are seen as subspaces of $\mathcal{Q}(\overline{\mathcal{D}}(\partial X))^\Gamma$, the Euclidean projective limit contains $\mathcal{Q}^\infty(H^p)^\Gamma$ as a dense subspace and the restriction of the scalar product of the Euclidean projective limit to $\mathcal{Q}^\infty(H^p)^\Gamma$ is p^*_Γ.*

We first prove a density result.

Lemma 8.6. *The space*

$$\bigcup_{k\geq 1}\{\Phi_w | w : X_k \to \mathbb{R}, w \textit{ is } \Gamma\textit{-invariant}\}$$

is dense in $\mathcal{Q}^\infty(H^\omega_0)^\Gamma$ with respect to the topology induced by a standard scalar product.

The fact that the statement does not depend on the choice of the standard scalar product is a consequence of Corollary A.16.

Proof. As in the proof of Proposition 8.3, we identify $\ell^2(X_1)$ and its dual space by the usual scalar product and we let $I : H^\omega_0 \to \ell^2(X_1)$ denote the natural inclusion. We equip H^ω_0 with the induced scalar product of the usual one on $\ell^2(X_1)$. Therefore, by Lemma A.14, the natural map $(I^*)^\star : \mathcal{Q}^\infty((H^\omega_0)')^\Gamma \to \mathcal{Q}^\infty(\ell^2(X_1))^\Gamma$. is an isometric embedding for the standard scalar products. Let Π be in $\mathcal{Q}^\infty((H^\omega_0)')^\Gamma$. In view of the proof of Proposition 8.3, for $x \neq y$ in X, if x_1 and y_1 are the neighbours of x and y in $[xy]$, we have

$$\begin{aligned}(8.7)\quad (I^*)^\star\Pi(\mathbf{1}_{(x,x_1)}, \mathbf{1}_{(y,y_1)}) &= (I^*)^\star\Pi(\mathbf{1}_{(x_1,x)}, \mathbf{1}_{(y_1,y)}) \\ &= -(I^*)^\star\Pi(\mathbf{1}_{(x_1,x)}, \mathbf{1}_{(y,y_1)}) = -(I^*)^\star\Pi(\mathbf{1}_{(x,x_1)}, \mathbf{1}_{(y_1,y)}) \\ &= \Pi(\mathbf{1}_{U_{x_1 x}}, \mathbf{1}_{U_{y_1 y}}) = -\psi(x,y),\end{aligned}$$

where, as in Subsection I.4.1, ψ is the quadratic type function associated to Π, when Π is viewed as a symmetric bilinear form on $\overline{\mathcal{D}}(\partial X)$.

We apply Lemma A.18 to the Γ-action on X_1 which is standard by assumption. This tells us that we may identify $\mathcal{Q}^\infty(\ell^2(X_1))^\Gamma$ with a space of Γ-invariant symmetric functions on X_1^2; then, every element φ in the completion of $\mathcal{Q}^\infty((H^\omega_0)')^\Gamma$ with respect to the standard scalar product may be seen as a Γ-invariant function on X_1^2 that is square-summable in $\Gamma\backslash X_1^2$. Due to (8.7), this function satisfies, for $x \neq y$ in

X,

$$\begin{aligned}(8.8)\quad \varphi((x,x_1),(y,y_1)) &= \varphi((x_1,x),(y_1,y)) \\ &= \varphi((x_1,x),(y,y_1)) = \varphi((x,x_1),(y_1,y)).\end{aligned}$$

Let φ be such a function and assume moreover that, for any $k \geq 1$ and any symmetric Γ-invariant function w on X_k, φ is orthogonal to Ψ_w with respect to the standard quadratic pairing, where Ψ_w is defined as in the proof of Proposition 8.3. Then, by reasoning as in (8.6), we get

$$\sum_{(x,y)\in\Gamma\backslash X_k} \frac{1}{|\Gamma_x \cap \Gamma_y|}\varphi((x,x_1),(y,y_1))w(x,y) = 0.$$

As this is true for any $k \geq 1$ and any function w, for any $x \neq y$ in X, we get $\varphi((x,x_1),(y,y_1)) = 0$. By (8.8), we get $\varphi = 0$.

By construction, for $k \geq 1$ and w a Γ-invariant function on X_k, we have $I^\star \Psi_w = \Phi_w$. Therefore, by Lemma A.9, we have just shown that, with respect to the standard pairing, in the completion of $\mathcal{Q}^\infty((H_0^\omega)')^\Gamma$, the orthogonal subspace of

$$\bigcup_{k\geq 1}\{\Phi_w | w : X_k \to \mathbb{R}, w \text{ is } \Gamma\text{-invariant}\}$$

is reduced to $\{0\}$. By Corollary A.17, the Lemma follows. $\square$

We will need the infinite dimensional version of Lemma I.C.3, whose proof is obtained in the same way. Recall that, if H is a Hilbert space, $\mathcal{Q}^\infty_{++}(H)$ is the set of coercive symmetric bilinear forms on H, that is, the set of scalar products which define the topology of H.

Lemma 8.7. *Let H be a Hilbert space with scalar product p and topological dual space H'. Write $\delta_H : \mathcal{Q}^\infty_{++}(H) \to \mathcal{Q}^\infty_{++}(H')$ for the natural bijection and $\theta_p : H \to H', v \mapsto p(v,.)$. Then δ_H is smooth and the differential of δ_H at p is the linear map*

$$\mathrm{d}(\delta_H)_p : \mathcal{Q}^\infty(H) \to \mathcal{Q}^\infty(H'), q \mapsto -(\theta_p^{-1})^\star q.$$

The proof of Theorem 8.5 will use an elementary argument of Hilbert spaces geometry that we formulate in the language of Appendix I.B.

Lemma 8.8. *Let $(X_\ell, q_\ell, \pi_\ell)_{\ell\geq 0}$ be a straight Euclidean projective system and $H \subset \varprojlim_{\ell\geq 0} X_\ell$ be its Euclidean projective limit, equipped with its natural scalar product q. Let L be another Hilbert space and $\varphi : L \to H$ be a continuous linear map. For $\ell \geq 0$, set $\varphi_\ell = \pi_\ell\varphi : L \to X_\ell$ and assume that the adjoint map of φ_ℓ is an isometry onto its image in L. Then, the adjoint of φ is also an isometry onto its image.*

Proof. Let $\ell \geq 0$. As $(\pi_\ell)_* q = q_\ell$, the adjoint $\pi_\ell^\dagger : X_\ell \to H$ of π_ℓ is an isometry onto its image in H. As $\varphi^\dagger \pi_\ell^\dagger = \varphi_\ell^\dagger$, $\varphi^\dagger$ induces an isometry from $\pi_\ell^\dagger X_\ell$ onto its image in L. The conclusion follows as $\bigcup_\ell \pi_\ell^\dagger X_\ell$ is dense in H. □

Proof of Theorem 8.5. The strategy of the proof is to aim at applying Lemma 8.8 to the projective system $(\mathcal{F}_j)_{j\geq k}$, equipped with the weight metric, and to the Hilbert space L which is the completion of $\mathcal{Q}^\infty(H^p)^\Gamma$ with respect to the standard metric. Therefore, we will construct linear maps $\mathcal{F}_j \to \mathcal{Q}^\infty(H^p)^\Gamma$ which will be adjoint to the natural maps $\mathcal{Q}^\infty(H^p)^\Gamma \to \mathcal{F}_j$. We now begin this construction.

Write $\delta : \mathcal{Q}^\infty_{++}(H_0^\omega) \to \mathcal{Q}^\infty_{++}((H_0^\omega)')$ for the natural bijection. Recall that $\mathcal{P}_k$ stands for the set of Γ-invariant k-Euclidean fields and $\mathcal{K}_k$ for the vector space of Γ-invariant k-dual kernels. By Proposition I.10.14 the natural map $\iota_k : \mathcal{P}_k \hookrightarrow \mathcal{K}_k$ is a smooth immersion.

Let first p be any element of $\mathcal{P}_k$ and p^∞ be the scalar product on $\overline{\mathcal{D}}(\partial X)$ obtained from p by successive orthogonal extenions. The construction of the k-dual kernel $(K, K^-) = \iota_k(p)$ in Subsection 5.1 and the definition of the Hilbert space H^{K,K^-} in Subsection 5.4 ensure that H^{K,K^-} is the space of distributions which are bounded with respect to p and that the natural bilinear form q^{K,K^-} on H^{K,K^-} is dual to p^∞. If p is admissible, then by definition, $H^{K,K^-} = H_0^\omega$ and the previous can be written as

$$p^\infty = \delta(q^{K,K^-}) = \delta(q^{\iota_k(p)}). \tag{8.9}$$

Recall that $\mathcal{W}_k$ is the space of cohomology classes of Γ-invariant symmetric functions on X_k and that $W_k : \mathcal{K}_k \to \mathcal{W}_k$ is the weight map. As in the proof of Theorem I.10.17, let $F_k : \mathcal{W}_k \to \mathcal{Q}^\infty_{++}(H_0^\omega)$ denote the map that sends the cohomology class of a function w to Φ_w. Let still p be admissible and $(K, K^-) = \iota_k(p)$. By Theorem 7.6, we have $q^{K,K^-} = \Phi_w$ where w is a Γ-invariant weight function of $(K, K)^-$. Thus, by using (8.9), we get

$$p^\infty = \delta F_k W_k \iota_k(p).$$

Finally, denote by $\pi_k : \mathcal{Q}(\overline{\mathcal{D}}(\partial X))^\Gamma \to \mathcal{F}_k$ the natural surjective linear map. Since by construction $\pi_k(p^\infty) = p$, we have established that, for p in $\mathcal{P}_k^{\mathrm{ad}}$,

$$p = \pi_k \delta F_k W_k \iota_k(p). \tag{8.10}$$

In the above relations, the linear maps π_k, F_k and W_k are linear and continuous. Indeed, W_k and F_k are defined on finite-dimensional vector spaces, whereas π_k is defined on $\mathcal{Q}^\infty((H_0^\omega)')^\Gamma$ by evaluating bilinear forms on elements of H_0^ω. Besides, the map ι_k is smooth by Proposition I.10.14 and the map δ is smooth by Lemma 8.7. Write $U_p : H^p = (H_0^\omega)' \to H_0^\omega$ for the isomorphism associated to p. By Lemma 8.7, we get, after differentiating in (8.10), for p in $\mathcal{P}_k^{\mathrm{ad}}$ and q in $\mathcal{F}_k$,

$$q = -\pi_k U_p^\star F_k W_k \mathrm{d}_p \iota_k(q). \tag{8.11}$$

In view of this, we define a linear map $\theta_k : \mathcal{F}_k \to \mathcal{Q}^\infty(H^p)^\Gamma$ by setting, for q in $\mathcal{F}_k$,

$$\theta_k q = -U_p^\star F_k W_k \mathrm{d}_p \iota_k(q), \tag{8.12}$$

so that (8.11) reads as

$$\pi_k \theta_k q = q. \tag{8.13}$$

We claim that, for Π in $\mathcal{Q}^\infty(H^p)^\Gamma$, we have the adjointness property

$$p_\Gamma^*(\Pi, \theta_k q) = g_p(\pi_k \Pi, q), \tag{8.14}$$

where g_p is the weight metric. Indeed, in view of the definition of the standard scalar product p_Γ^* in Subsection A.4 and by (8.12), we have,

$$p_\Gamma^*(\Pi, \theta_k q) = -\langle \Pi, F_k W_k \mathrm{d}_p \iota_k(q)\rangle_\Gamma = -[\pi_k \Pi, \mathrm{d}_p \iota_k(q)],$$

where the second equality follows from Proposition 8.3. By the definition of g_p in Subsection I.11.3, (8.14) follows.

Let us use these constructions to show that, when viewed as a subspace of $\mathcal{Q}(\overline{\mathcal{D}}(\partial X))^\Gamma$, the space $\mathcal{Q}^\infty(H^p)^\Gamma$ is contained in the Euclidean projective limit of the projective system $(\mathcal{F}_j)_{j\geq k}$, equipped with the weight metric. Indeed, for Π in $\mathcal{Q}^\infty(H^p)^\Gamma$, we have, by (8.14),

$$g_p(\pi_k \Pi, \pi_k \Pi) = p_\Gamma^*(\theta_k \pi_k \Pi, \Pi).$$

By Cauchy Schwartz inequality, we get

$$g_p(\pi_k \Pi, \pi_k \Pi)^2 \leq p_\Gamma^*(\theta_k \pi_k \Pi, \theta_k \pi_k \Pi) p_\Gamma^*(\Pi, \Pi).$$

Still by (8.14), we have

$$p_\Gamma^*(\theta_k \pi_k \Pi, \theta_k \pi_k \Pi) = g_p(\pi_k \theta_k \pi_k \Pi, \pi_k \Pi) = g_p(\pi_k \Pi, \pi_k \Pi),$$

where we have used (8.13). Thus, $g_p(\pi_k \Pi, \pi_k \Pi) \leq p_\Gamma^*(\Pi, \Pi)$. By applying this property to the successive orthogonal extensions of p, we get that $\mathcal{Q}^\infty(H^p)^\Gamma$ is contained in the Euclidean projective limit H of the projective system $(\mathcal{F}_j)_{j\geq k}$, and that this inclusion has norm ≤ 1.

Therefore, this inclusion induces a bounded linear map $\varphi : L \to H$, where L is the completion of $\mathcal{Q}^\infty(H^p)^\Gamma$ with respect to the standard scalar product. By (8.14), for $j \geq k$, the adjoint map of $\pi_j \varphi$ is θ_j and by (8.13), θ_j is an isometry onto its image. By Lemma 8.8, to conclude it suffices to prove that the null space of φ is reduced to $\{0\}$. This will follow from Lemma 8.6. Indeed, recall that by Theorem 8.32, the weight map $W_k : \mathcal{K}_k \to \mathcal{W}_k$ is surjective. Hence, by Proposition 8.3, for Π in L, if $\varphi\Pi = 0$, then, for any $j \geq 1$ and any w in $\mathcal{W}_j$, we have $\langle \Pi, \Phi_w \rangle_\Gamma = 0$. By Lemma 8.6, this implies $\Pi = 0$ as required. □

Theorem 8.5 yields the following corollary which we were not able to prove in a more direct way.

Corollary 8.9. *Let $k \geq 2$ and p, q be admissible Γ invariant Euclidean fields. There exists $C \geq 1$ such that, for any $j \geq k$, one has*

$$\frac{1}{C} g_{p^j} \leq g_{q^j} \leq C g_{p^j}$$

where g_{p^j} and g_{q^j} are the weight metrics associated with the $(j-k)$-th orthogonal extensions of p and q on $\mathcal{F}_j$.

Proof. This follows directly from Corollary A.16 and Theorem 8.5. □

Appendix A. Algebraic properties of standard representations

In this appendix, we achieve certain abstract constructions that were used in the proof of Proposition 8.3 and Theorem 8.5.

Let G be a discrete group. If A is a set, we shall say that an action of G on A is standard if G has finitely many orbits in A and the stabilizers in G of the elements of A are finite. If H is a (real) Hilbert space, equipped with a unitary action of G, we shall say that the representation of G on H is standard if there exists a standard action of G on a set A and a closed non necessarily unitary G-equivariant embedding of H inside the space $\ell^2(A)$ of square-summable functions on A.

It is a widely used fact in the theory of Von Neumann algebras (see [3, Chapter 8]) that the algebra $\mathcal{B}(H)^G$ of bounded G-equivariant endomorphisms of H can then be equipped with a natural trace. This functional allows to introduce algebraic structures on spaces related to H that are analogues of the algebraic structures studied in Appendix I.C. These constructions are classical, but we will give a presentation of them that is adapted to our purpose.

A.1. **Standard trace.** We begin by defining the above mentioned trace functional. If A is a set, we write $\langle ., . \rangle$ for the usual scalar product on $\ell^2(A)$. If H is a Hilbert space, we denote by $\mathcal{B}(H)$ the algebra of bounded endomorphisms of H. The following is a classical result from the theory of Von Neumann algebras:

Proposition A.1. *Let G be a discrete group and H be a Hilbert space equipped with a standard unitary representation of G. Then there exists a unique linear functional tr_G on $\mathcal{B}(H)^G$ with the following property:*

let A be a set with a standard G-action and $I : H \to \ell^2(A)$ be a closed G-equivariant embedding. Then, for every T in $\mathcal{B}(H)^G$ and S in $\mathcal{B}(\ell^2(A))^G$ with $SI = IT$ and $S\ell^2(A) \subset IH$, we have

$$\operatorname{tr}_G T = \sum_{a \in G\backslash A} \frac{1}{|G_a|} \langle \mathbf{1}_a, S\mathbf{1}_a \rangle.$$

The functional tr_G is a trace, meaning that, if H, L are Hilbert spaces equipped with standard unitary representations of G, for every bounded G-equivariant linear maps $U : H \to L$ and $V : L \to H$, we have

$$\operatorname{tr}_G(UV) = \operatorname{tr}_G(VU). \tag{A.1}$$

Remark A.2. The assumption on S and T in the statement means that, for a direct decompostion of $\ell^2(A)$ as $H \oplus L$, the operator S has a matrix of the form

$$\begin{pmatrix} T & * \\ 0 & 0 \end{pmatrix}.$$

Definition A.3. Let G be a discrete group and H be a standard unitary representation of G. Then the functional tr_G is called the standard trace on $\mathcal{B}(H)^G$.

Assume the set A is equipped with a standard G-action. We start building the standard trace in case $H = \ell^2(A)$. Let T be in $\mathcal{B}(\ell^2(A))^G$. We set

$$\operatorname{tr}_G(T) = \sum_{a \in G\backslash A} \frac{1}{|G_a|} \langle \mathbf{1}_a, T\mathbf{1}_a \rangle. \tag{A.2}$$

Lemma A.4. *Let G be a discrete group, A be a set equipped with a standard action of G and S, T be in $\mathcal{B}(\ell^2(A))^G$. For a, b in A, set $\varphi(a, b) = \langle \mathbf{1}_a, S\mathbf{1}_b \rangle$ and $\psi(a, b) = \langle \mathbf{1}_a, T\mathbf{1}_b \rangle$. Then we have*

$$\operatorname{tr}_G(ST) = \sum_{(a,b) \in G\backslash A^2} \frac{1}{|G_a \cap G_b|} \varphi(a, b)\psi(b, a). \tag{A.3}$$

In particular, we have $\operatorname{tr}_G(ST) = \operatorname{tr}_G(TS)$, that is, the functional tr_G is a trace on $\mathcal{B}(\ell^2(A))^G$.

Proof. Indeed, for a, b in A, we have

$$\langle \mathbf{1}_a, ST\mathbf{1}_a \rangle = \sum_{b \in A} \varphi(a, b)\psi(b, a),$$

meaning that the series is absolutely convergent. By the definition of the trace in (A.2), we get

$$\operatorname{tr}_G(ST) = \sum_{a \in G\backslash A} \frac{1}{|G_a|} \sum_{b \in A} \varphi(a, b)\psi(b, a).$$

By using Lemma I.9.11, (A.3) follows. □

Let $H \subset \ell^2(A)$ be a closed G-invariant subspace $I : H \to \ell^2(A)$ be the natural injection and $P : \ell^2(A) \to H$ be the orthogonal projection. For T in $\mathcal{B}(H)^G$, set

$$\operatorname{tr}_G(T) = \operatorname{tr}_G(ITP). \tag{A.4}$$

Lemma A.5. *Let G be a group and A be a set equipped with a standard G-action. Let H be a closed G-invariant subspace of $\ell^2(A)$. Then, for any S in $\mathcal{B}(H)^G$ with $IT = SI$ and $S\ell^2(A) \subset IH$, we have*

$$\operatorname{tr}_G(T) = \operatorname{tr}_G(S).$$

In particular, if $Q : \ell^2(A) \to H$ is a non necessarily orthogonal projection, we have

$$\operatorname{tr}_G(T) = \operatorname{tr}_G(ITQ).$$

Proof. As the range of S is contained in IH, we have $S = IPS$. Therefore, by Lemma A.4,

$$\operatorname{tr}_G(S) = \operatorname{tr}_G(IPS) = \operatorname{tr}_G(SIP) = \operatorname{tr}_G(ITP) = \operatorname{tr}_G(T).$$

If Q is as in the statement, we can set $S = ITQ$ and the conclusion follows. □

Proof of Proposition A.1. Let us prove the uniqueness property of the trace. We will show that the trace tr_G defined on $\mathcal{B}(H)^G$ in (A.4) does not depend on the embedding of H as a subspace of the space of square-integrable functions on some standard action of G.

Let A and B be two sets, both equipped with standard G-actions. Assume that we are given two G-equivariant closed embeddings $I : H \to \ell^2(A)$ and $J : H \to \ell^2(B)$. Set $C = A \sqcup B$ and equip it with the natural action. Identify $\ell^2(C)$ with $\ell^2(A) \oplus \ell^2(B)$. Then, a direct computation shows that the traces on $\mathcal{B}(\ell^2(A))^G$ and $\mathcal{B}(\ell^2(B))^G$ are the same as the ones induced by the closed embeddings of $\ell^2(A)$ and $\ell^2(B)$ inside $\ell^2(C)$. Denote by $P : \ell^2(C) \to \ell^2(A)$ and $Q : \ell^2(C) \to \ell^2(B)$ the orthogonal projections which we consider as endomorphisms of $\ell^2(C)$. Write $K = I \oplus J$ for the diagonal embedding of H in $\ell^2(C)$. We get

$$PK = I \text{ and } QK = J. \tag{A.5}$$

Temporarily write tr_G^A, tr_G^B and tr_G^C for the traces on $\mathcal{B}(H)^G$ associated with the embeddings I, J and K. Let $P' : \ell^2(A) \to H$ and $Q' : \ell^2(B) \to H$ be the orthogonal projections. Note that $P'P$ and $Q'Q$ are projections $\ell^2(C) \to H$. Therefore, for T in $\mathcal{B}(H)^G$, by Lemma A.5, we have

$$\operatorname{tr}_G^C(T) = \operatorname{tr}_G(KTP'P) = \operatorname{tr}_G(KTQ'Q). \tag{A.6}$$

By using (A.5), we get

$$\mathrm{tr}_G(KTP'P) = \mathrm{tr}_G(KTP'P^2) = \mathrm{tr}_G(PKTP'P) = \mathrm{tr}_G(ITP) = \mathrm{tr}_G^A(T)$$

and in the same way, $\mathrm{tr}_G(KTQ'Q) = \mathrm{tr}_G^B(T)$. By (A.6), we get

$$\mathrm{tr}_G^A(T) = \mathrm{tr}_G^C(T) = \mathrm{tr}_G^B(T)$$

as required. By Lemma A.5, we have built a functional on $\mathcal{B}(H)^G$ with the required properties.

To conclude, it remains to prove (A.1). Thus, we let H and L be Hilbert spaces with standard representations of G, and $U : H \to L$ and $V : L \to H$ be bounded G-equivariant linear maps. We set M to be the Hilbert space $H \oplus L$, and S and T to be the endomorphisms defined by the matrices

$$S = \begin{pmatrix} 0 & 0 \\ U & 0 \end{pmatrix} \text{ and } T = \begin{pmatrix} 0 & V \\ 0 & 0 \end{pmatrix}.$$

By the uniqueness of the trace and by Lemma A.4, we have

$$\mathrm{tr}_G(UV) = \mathrm{tr}_G(ST) = \mathrm{tr}_G(TS) = \mathrm{tr}_G(VU).$$

□

A.2. **Trace inequalities.** We will use the standard trace to write inequalities in $\mathcal{B}(H)^G$ which are analogues of inequalities for endomorphisms of finite dimensional vector spaces.

We first state a consequence of (A.3).

Corollary A.6. *Let G be a group and H be a Hilbert space with scalar product p. Assume H is equipped with a standard unitary representation of G. Then the bilinear form*

$$(S, T) \mapsto \mathrm{tr}_G(S^{\dagger p} T)$$

is a scalar product on $\mathcal{B}(H)^G$.

Proof. Let A be a standard G-set. If $H = \ell^2(A)$, the result is clear from (A.3).

In general, let $I : H \to \ell^2(A)$ be a closed embedding and assume first that H is equipped with the restriction of the usual scalar product of $\ell^2(A)$. Then, the adjoint of I is the orthogonal projection $P : \ell^2(A) \to H$. Therefore, for S, T in $\mathcal{B}(H)^G$, we have

$$\mathrm{tr}_G(S^\dagger T) = \mathrm{tr}_G(IS^\dagger TP) = \mathrm{tr}_G(IS^\dagger PITP) = \mathrm{tr}_G((ISP)^\dagger (ITP)),$$

and the result follows from the previous case.

Finally, suppose p is any scalar product on H, which defines the topology of H. We let V be the endomorphism of H which represents p with respect to the scalar product induced by the usual scalar product

$\langle .,. \rangle$ of $\ell^2(A)$. As p is positive and defines the topology of H, V is invertible and positive. For T in $\mathcal{B}(H)$ and x, y in H, we have

$$p(Tx, y) = \langle VTx, y \rangle = \langle x, T^\dagger Vy \rangle = p(x, V^{-1}T^\dagger Vy),$$

hence $T^{\dagger p} = V^{-1}T^\dagger V$. Let W be the square-root of V, that is, the unique self-adjoint endomorphism with respect to the usual scalar product which is non-negative and such that $W^2 = V$. As V commutes with G, so does W by uniqueness. For S, T in $\mathcal{B}(H)^G$, we have

$$\begin{aligned} \operatorname{tr}_G(S^{\dagger p}T) &= \operatorname{tr}_G(V^{-1}S^\dagger VT) = \operatorname{tr}_G(W^{-2}S^\dagger W^2 T) \\ &= \operatorname{tr}_G(W^{-1}S^\dagger W^2 T W^{-1}) = \operatorname{tr}_G((WSW^{-1})^\dagger (WTW^{-1})). \end{aligned}$$

The result follows by the previous case. □

Corollary A.7. *Let G be a group and H be a Hilbert space with a standard unitary representation of G. Let T be a non-negative self-adjoint element of $\mathcal{B}(H)^G$. Then we have*

$$\operatorname{tr}_G(T) \geq 0.$$

Proof. Let S be the square-root of T. Then S is self-adjoint and belongs to $\mathcal{B}(H)^G$. By Corollary A.6, we have

$$\operatorname{tr}_G(T) = \operatorname{tr}_G(S^2) \geq 0.$$

□

If H is a standard unitary representation of G, the trace of the identity endomorphism of H is called the Von Neumann dimension of H and denoted by $\dim_G H$. We get a bound of the norm of the trace functional on $\mathcal{B}(H)^G$.

Corollary A.8. *Let G be a group and H be a Hilbert space with a standard unitary representation of G. For any T in $\mathcal{B}(H)^G$, we have*

$$\operatorname{tr}_G(T) \leq \|T\| \dim_G(H),$$

where $\|T\|$ is the operator norm of T.

Proof. By Corollary A.6 we get, thanks to Cauchy-Schwarz inequality,

$$\operatorname{tr}_G(T)^2 \leq \operatorname{tr}_G(T^\dagger T) \dim_G H.$$

Besides, $T^\dagger T$ is self-adjoint and has norm $\|T\|^2$. Therefore, the endomorphism $\|T\|^2 - T^\dagger T$ is self-adjoint and non-negative. Thus, by Corollary A.7, we get

$$\operatorname{tr}_G(T^\dagger T) \leq \|T\|^2 \dim_G H.$$

The conclusion follows. □

A.3. **Standard quadratic pairing.** Now that we have introduced the trace, we can use it to define an analogue of the quadratic duality of Appendix I.C, that is, the natural duality between $\mathcal{Q}(V)$ and $\mathcal{Q}(V^*)$ where V is a finite dimensional vector space.

Let G be a group and H be a Hilbert space, equipped with a standard unitary representation of G. We write $\mathcal{Q}^\infty(H)$ for the space of bounded symmetric bilinear forms on H and $\mathcal{Q}^\infty(H)^G$ for the set of G-invariant such forms. Let H' be the topological dual space of H. Let q be in $\mathcal{B}(H)^G$ and let r be in $\mathcal{B}(H')^G$. There exists G-equivariant bounded linear operators $S : H \to H'$ and $T : H' \to H$ such that, for any v, w in H and φ, ψ in H', one has

$$q(v, w) = (Sv)(w) \text{ and } r(\varphi, \psi) = \varphi(T\psi).$$

We set

$$\langle q, r\rangle_G = \mathrm{tr}_G(ST) = \mathrm{tr}_G(TS).$$

We call $\langle ., .\rangle_G$ the standard quadratic pairing between $\mathcal{Q}^\infty(H)^G$ and $\mathcal{Q}^\infty(H')^G$.

This pairing behaves well under linear maps.

Lemma A.9. *Let G be a group and H and L be standard unitary representations of G. Let $S : H \to L$ be a G-equivariant bounded linear map, q be in $\mathcal{Q}^\infty(L)^G$ and r be in $\mathcal{Q}^\infty(H')^G$. We have*

$$\langle S^\star q, r\rangle_G = \langle q, (S')^\star r\rangle_G$$

where $S' : L' \to H'$ is the adjoint linear map of S.

Proof. Let $T : L \to L'$ and $U : H' \to H$ be the bounded linear maps associated to q and r. We have

$$\begin{aligned}\langle S^\star q, r\rangle_G = \mathrm{tr}_G((S'TS)U) = \mathrm{tr}_G(S'TSU) &= \mathrm{tr}_G(TSUS')\\ &= \mathrm{tr}_G(T(SUS')) = \langle q, (S')^\star r\rangle_G.\end{aligned}$$

□

The standard pairing allows to state an analogue of Lemma 4.9

Proposition A.10. *Let G be a group, H be a Hilbert space with a standard unitary representation of G and p be in $\mathcal{Q}^\infty(H)^G$. Then p is non-negative if and only if, for every non-negative q in $\mathcal{Q}^\infty(H')^G$, one has $\langle p, q\rangle_G \geq 0$.*

The proof uses a construction of bounded G-invariant bilinear forms.

Lemma A.11. *Let A be a standard G-set and φ be a finitely supported function on $\Gamma\backslash A^2$. Then, the bilinear form defined on finitely supported functions by $(\rho, \theta) \mapsto \sum_{(a,b)\in A^2} \varphi(a, b)\rho(a)\theta(b)$ is bounded on $\ell^2(A)$.*

Proof. We first note that, for ρ, θ in $\ell^2(A)$ and (a,b) in A^2, we have, by Cauchy-Schwarz inequality,

$$\sum_{g\in G} |\rho(ga)\theta(gb)| \leq \left(\sum_{g\in G} \rho(ga)^2\right)^{\frac{1}{2}} \left(\sum_{g\in G} \theta(gb)^2\right)^{\frac{1}{2}} \leq |G_a|^{\frac{1}{2}} |G_b|^{\frac{1}{2}} \|\rho\|_2 \|\theta\|_2 .$$

Therefore, if $S \subset \Gamma\backslash A^2$ is the support of φ, by Lemma 3.25, we have

$$\sum_{(a,b)\in A^2} |\varphi(a,b)\rho(a)\theta(b)| = \sum_{(a,b)\in G\backslash A^2} \frac{1}{|G_a \cap G_b|} |\varphi(a,b)| \sum_{g\in G} |\rho(ga)\theta(gb)| \leq |S| \|\varphi\|_\infty \sup_{a\in A} |G_a| \|\rho\|_2 \|\theta\|_2 .$$

The conclusion follows. ☐

Proof of Proposition A.10. First assume that p is non-negative. Let q be a G-invariant coercive bilinear form on H', that is, a scalar product which defines the topology of H'. We write $\langle .,. \rangle$ for the duality between H and H' and $S : H \to H'$ and $T : H' \to H$ for the linear maps associated to p and q. By definition, we have $\langle p, q\rangle_G = \mathrm{tr}_G(ST)$. For x, y in H', we have $q(x,y) = \langle x, Ty\rangle$ and

$$p(Tx, Ty) = \langle STx, Ty\rangle = q(STx, y).$$

Therefore, the G-equivariant endomorphism ST is self-adjoint and non-negative with respect to q on H'. By Corollary A.7, we get $\langle p, q\rangle_G = \mathrm{tr}_G(ST) \geq 0$. Besides, by Corollary A.8, the quadratic pairing is continuous on $\mathcal{Q}^\infty(H)^G \times \mathcal{Q}^\infty(H')^G$ for the norm topology. As the set of coercive symmetric bilinear forms is dense in the set of non-negative bilinear forms, we still get $\langle p, q\rangle_G \geq 0$ for any non-negative q in $\mathcal{Q}^\infty(H')^G$.

Conversely, suppose now that, for any non-negative q in $\mathcal{Q}^\infty(H')^G$, we have $\langle p, q\rangle_G \geq 0$, and let us show that p is non-negative.

Assume first that H is $\ell^2(A)$, equipped with the usual scalar product, where A is a standard G-set. Let θ be a finitely supported function on A and let us show that $p(\theta,\theta) \geq 0$. For (a,b) in A^2, set

$$\varphi(a,b) = \sum_{g\in G} \theta(ga)\theta(gb). \tag{A.7}$$

By construction, the support of φ has finite image in $G\backslash A^2$. Hence, by Lemma A.11, there exists q in $\mathcal{Q}^\infty(H)^G$ with

$$\varphi(a,b) = q(\mathbf{1}_a, \mathbf{1}_b) \quad a, b \in A.$$

We claim that q is non-negative. Indeed, if ρ is another finitely supported function, we have

$$\begin{aligned} q(\rho,\rho) = \sum_{(a,b)\in A^2} \varphi(a,b)\rho(a)\rho(b) &= \sum_{g\in G}\sum_{(a,b)\in A^2} \theta(ga)\theta(gb)\rho(a)\rho(b) \\ &= \sum_{g\in G}\left(\sum_{a\in A}\theta(ga)\rho(a)\right)^2 \geq 0. \end{aligned}$$

Let $U : \ell^2(A)' \to \ell^2(A)$ be the isomorphism associated to the usual scalar product. By assumption, we have $\langle p, U^\star q\rangle_G \geq 0$. Now, by (A.3), we get

$$\langle p, U^\star q\rangle_G = \sum_{(a,b)\in G\backslash A^2} \frac{1}{|G_a \cap G_b|} p(\mathbf{1}_a, \mathbf{1}_b)\varphi(a,b),$$

hence, thanks to Lemma 3.25 and the definition of φ in (A.7),

$$\langle p, U^\star q\rangle_G = \sum_{(a,b)\in A^2} p(\mathbf{1}_a, \mathbf{1}_b)\theta(a)\theta(b) = p(\theta,\theta).$$

We get $p(\theta,\theta) \geq 0$ as required.

In general, if $I : H \to \ell^2(A)$ is a closed embedding, we let $P : \ell^2(A) \to H$ be the orthogonal projection. Then, the adjoint P' of P is a closed embedding $H' \to \ell^2(A)'$. If q is a non-negative element of $\mathcal{Q}^\infty(\ell^2(A)')^G$, then $(P')^\star q$ is non-negative on H' and, by Lemma A.9, we have

$$\langle P^\star p, q\rangle_G = \langle p, (P')^\star q\rangle_G \geq 0.$$

By the previous case, $P^\star p$ is non-negative. As P is surjective, p is non-negative. □

A.4. **Standard scalar product.** We finally define an analogue of the $\mathrm{GL}(V)$-invariant Riemannian metric on the space $\mathcal{Q}_{++}(V)$ of positive definite symmetric bilinear forms on V, where V is a finite dimensional real vector space (see [2, Chapter VI]).

Let H be a standard unitary representation of G and p be a G-invariant scalar product on H which defines the topology of H. Then p induces a G-equivariant isomorphism from H onto H'. For q in $\mathcal{Q}^\infty(H)^G$, let q' be the element of $\mathcal{Q}^\infty(H')^G$ defined through this isomorphism. For q, r in $\mathcal{Q}^\infty(H)^G$, we set

$$p_G^*(q,r) = \langle q', r\rangle_G.$$

We have formulated the definition in this way in order to emphasize the relation with the standard quadratic pairing. Here comes an equivalent definition.

Lemma A.12. *Let G be a group and H be a standard unitary representation of G with scalar product p. Let q and r in $\mathcal{Q}^\infty(H)^G$ be respectively represented by the self-adjoint endomorphisms S and T. Then we have*

$$p^*_G(q, r) = \mathrm{tr}_G(ST).$$

In particular, the bilinear form p^*_G is symmetric and positive definite on $\mathcal{Q}^\infty(H)^G$.

Definition A.13. Given a group G and a standard unitary representation H with scalar product p as above, the associated scalar product p^*_G on $\mathcal{Q}^\infty(H)^G$ is called the standard scalar product.

Proof of Lemma A.12. For v, w in H, we have

$$q(v, w) = p(Sv, w) \text{ and } r(v, w) = p(Tv, w).$$

Let $U : H \to H'$ be the G-equivariant isomorphism associated to p. The linear map $H' \to H$ associated to q' is SU^{-1} and the linear map $H \to H'$ associated to r is UT. By definition, we get

$$p^*_G(q, r) = \langle q', r\rangle_G = \mathrm{tr}_G(SU^{-1}UT) = \mathrm{tr}_G(ST). \tag{A.8}$$

The remainder of the proof follows from Corollary A.6. □

The construction of the standard scalar product is natural in the following sense:

Lemma A.14. *Let G be a group and H and L be standard unitary representations of G, with scalar products p and q. Let $T : H \to L$ be a surjective G-equivariant continuous linear map with $T_\star p = q$. Then, for every r, s in $\mathcal{Q}(L)$, we have*

$$p^*_G(T^\star r, T^\star S) = q^*_G(r, s).$$

As in Appendix I.A, the notation $T_\star p = q$ means that the adjoint operator $T^\dagger$ of T is an isometric embedding $L \to H$.

Proof. Let A and B be the continuous bounded endomorphisms of L which represent r and s with respect to q. Then, $T^\star r$ and $T^\star s$ are respectively represented by $T^\dagger AT$ and $T^\dagger BT$ with respect to p. By Lemma A.12, we have $q^*_G(r, s) = \mathrm{tr}_G(AB)$ and, as by assumption $TT^\dagger$ is the identity operator of L,

$$\begin{aligned} p^*_G(T^\star r, T^\star S) = \mathrm{tr}_G(T^\dagger ATT^\dagger BT) = \mathrm{tr}_G(T^\dagger ABT) = \mathrm{tr}_G(ABTT^\dagger) \\ = \mathrm{tr}_G(AB). \end{aligned}$$

□

Bounded G-equivariant endomorphisms act as bounded endomorphisms on $\mathcal{Q}^\infty(H)^G$ with respect to the standard scalar product.

Lemma A.15. *Let G be a group, H be a standard unitary representation of G with scalar product p and T be in $\mathcal{B}(H)^G$. Write $\|T\|$ for the operator norm of T with respect to p. Then, for q in $\mathcal{Q}^\infty(H)^G$, we have*
$$p_G^*(T^\star q, T^\star q) \leq \|T\|^4 \, p_G^*(q,q).$$

Proof. Let S be the element of $\mathcal{B}(H)^G$ that represents q with respect to p. For x, y in H, we have
$$T^\star q(x,y) = q(Tx,Ty) = p(STx,Ty) = p(T^\dagger STx, y),$$
hence, by Lemma A.12,
$$p_G^*(T^\star q, T^\star q) = \mathrm{tr}_G(T^\dagger STT^\dagger ST).$$
For x, y in H, we have
$$\begin{aligned} p(T^\dagger STT^\dagger STx, y) = p(T^\dagger STx, T^\dagger STy) &\leq \|T\|^2 \, p(STx, STy) \\ &= \|T\|^2 \, p(T^\dagger S^2 Tx, y), \end{aligned}$$
that is, the symmetric endomorphism $\|T\|^2 \, T^\dagger S^2 T - T^\dagger STT^\dagger ST$ is nonnegative. Therefore, by Lemma A.7 and the trace property, we get
$$\mathrm{tr}_G(T^\dagger STT^\dagger ST) \leq \|T\|^2 \, \mathrm{tr}_G(T^\dagger S^2 T) = \|T\|^2 \, \mathrm{tr}_G(STT^\dagger S).$$
Again, for x, y in H,
$$p(STT^\dagger Sx, y) = p(T^\dagger Sx, T^\dagger Sy) \leq \|T\|^2 \, p(Sx, Sy) = \|T\|^2 \, p(S^2 x, y),$$
hence, still by Lemma A.7,
$$\mathrm{tr}_G(STT^\dagger S) \leq \|T\|^2 \, \mathrm{tr}_G(S^2).$$
The result follows from this chain of inequalities. □

Corollary A.16. *Let G be a group and H be a standard unitary representation of G with scalar product p. If q is another G-invariant scalar product on H which defines the topology of H, then the norms associated to the standard scalar products p_G^* and q_G^* on $\mathcal{Q}^\infty(H)^G$ are equivalent.*

Proof. As usual, let V be the positive invertible element of $\mathcal{B}(H)^G$ which represents q with respect to p, and let W be the square-root of V. For r in $\mathcal{Q}^\infty(H)^G$, we have $q_G^\star(W^\star r, W^\star r) = p_G^\star(r,r)$. The conclusion follows from Lemma A.15. □

In particular, the completion of $\mathcal{Q}^\infty(H)^G$ does not depend on the choice of a G-invariant scalar product. The following is a direct consequence of the definition of the standard scalar product.

Corollary A.17. *Let G be a group and H be a standard unitary representation of G with scalar product p. Then the standard quadratic pairing is continuous with respect to the topologies on $\mathcal{Q}^\infty(H)^G$ and $\mathcal{Q}^\infty(H')^G$ associated to a standard scalar product. This pairing defines an identification of the completion of $\mathcal{Q}^\infty(H')^G$ with the topological dual space of the completion of $\mathcal{Q}^\infty(H)^G$.*

We will need the following description of the completion of $\mathcal{Q}(\ell^2(A))^G$ with respect to the standard scalar product.

Lemma A.18. *Let G be a group and A be a standard G-set. We equip $\ell^2(A)$ with the usual scalar product. For any q in $\mathcal{Q}^\infty(\ell^2(A))^G$, let φ_q be the function on A^2*

$$\varphi_q : A^2 \to \mathbb{R}, (a,b) \mapsto q(\mathbf{1}_a, \mathbf{1}_b).$$

We use the map $\Phi : q \mapsto \varphi_q$ to identify $\mathcal{Q}^\infty(\ell^2(A))^G$ with a subspace of the space of symmetric G-invariant functions on A^2. Then, Φ may be extended as an isometry from the completion of $\mathcal{Q}^\infty(\ell^2(A))^G$ with respect to the standard scalar product onto the space of square-summable symmetric functions φ on $G\backslash A^2$, equipped with the scalar product

$$(\varphi, \psi) \mapsto \sum_{(a,b)\in\Gamma\backslash A^2} \frac{1}{|G_a \cap G_b|} \varphi(a,b)\psi(a,b). \tag{A.9}$$

Proof. Equation (A.3) implies directly that Φ induces an isometric embedding of the completion of $\mathcal{Q}^\infty(\ell^2(A))^G$ with respect to the standard scalar product inside the space of square-summable symmetric functions φ on $G\backslash A^2$, equipped with the scalar product (A.9). By Lemma A.11, the range of this embedding contains all symmetric functions with finite support in $G\backslash A^2$. The conclusion follows. □

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