Baily - Borel compactification

Goal: compactify the quotient X/Γ of a Hermitian symmetric space by an arithmetic subgroup Γ of $G = \mathrm{Hol}(X)^+$.

For instance: if $X = \mathbb{H}$ and $\Gamma = PSL_2(\mathbb{Z}) \subset G = PSL_2(\mathbb{R})$, then $X/\Gamma = \mathbb{P}^1_{\mathbb{C}}$.

Def: (Arithmetic subgroups)

1. Let G be a linear algebraic group defined over $\mathbb{Q}$. We say a subgroup $\Gamma \subset G(\mathbb{Q})$ is arithmetic if, for some $\mathbb{Q}$-embedding $G \hookrightarrow Gl_n$, Γ is commensurable with $G(\mathbb{Z}) \overset{def}{=} G(\mathbb{Q}) \cap Gl_n(\mathbb{Z})$

   (commensurable means that $\Gamma \cap G(\mathbb{Z})$ has finite index in both Γ and $G(\mathbb{Z})$).

2. (see Baily-Borel, p. 472)

   Let H be a connected real Lie group.

   A subgroup $\Gamma \subset H$ is arithmetic if there exist

   (a) an algebraic group G over $\mathbb{Q}$;

   (b) an arithmetic subgroup $\Gamma_0 \subset G(\mathbb{Q})$ (as in 1 above)

   (c) a surjective morphism $\varphi: G(\mathbb{R})^+ \longrightarrow H$ with compact kernel such that

   $$\varphi(\Gamma_0 \cap G(\mathbb{R})^+) = \Gamma.$$

Rmk: In 1, if Γ is commensurable with $G(\mathbb{Z})$ with respect to one $\mathbb{Q}$-embedding $G \hookrightarrow Gl_n$, then it is so for all $\mathbb{Q}$-embeddings (see Platonov & Rapinchuk, thm 4.1)

Eg.

$$\Gamma(N) \overset{def}{=} \left\{ g \in SL_n(\mathbb{Z}) \mid g \equiv \begin{pmatrix} 1 & & \\ & 1 & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \bmod N \right\}$$

is arithmetic in $G = SL_n$ (as an algebraic group or Lie group)

The main theorem is

(Baily-Borel* 1966) Let X/Γ be the quotient of a Hermitian symmetric space of non-compact type by a torsion-free arithmetic subgroup $\Gamma \subset \mathrm{Hol}(X)^+$. Then X/Γ has a canonical realisation as a Zariski open subset of a projective algebraic variety $\overline{X/\Gamma}^{BB}$ over $\mathbb{C}$.

Rmk: 1 $\overline{X/\Gamma}^{BB}$ will be normal, but in general very singular.

2 Canonical means: if Y is any quasi-projective variety over $\mathbb{C}$ such that $Y^{an} \cong X/\Gamma$ then Y is isomorphic to X/Γ viewed as a quasi-projective variety via $X/\Gamma \hookrightarrow \overline{X/\Gamma}^{BB}$.

This was actually proved later by Borel in 1972.

The proof of B-B follows the

Baby Baily-Borel (compactification of $\mathbb{H}/\Gamma$ where $\Gamma = PSL_2(\mathbb{Z}) \subset \mathrm{Hol}(X)^+ = PSL_2(\mathbb{R})$)

Here $PSL_2(\mathbb{R})$ acts on $X = \mathbb{H}$ by Möbius transformations.

In order to compactify X/Γ, we follow the steps:

I Compactify $X = \mathbb{H}$ in such a way that the $\mathrm{Hol}(X)^+ = PSL_2(\mathbb{R})$ action extends to a continuous action on $\overline{X}$:

$$\overline{X} = \mathbb{H} \cup \mathbb{P}^1_{\mathbb{R}} = \mathbb{H} \cup \mathbb{R} \cup \{\infty\}$$

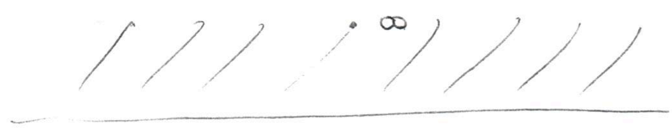
Observe that the $PSL_2(\mathbb{R})$-action on $\overline{X}$ has 2 orbits:

$$X = \mathbb{H} = PSL_2(\mathbb{R}) \cdot i \qquad \text{(original space)}$$
$$\overline{X} \setminus X = \mathbb{P}^1_{\mathbb{R}} = PSL_2(\mathbb{R}) \cdot \infty \qquad \text{(boundary)}$$

II Select the "rational" points of the boundary $\overline{X} \setminus X$ and define

$$X^* = X \cup \mathbb{P}^1_{\mathbb{Q}} \subset \overline{X}$$

where the topology now is defined by



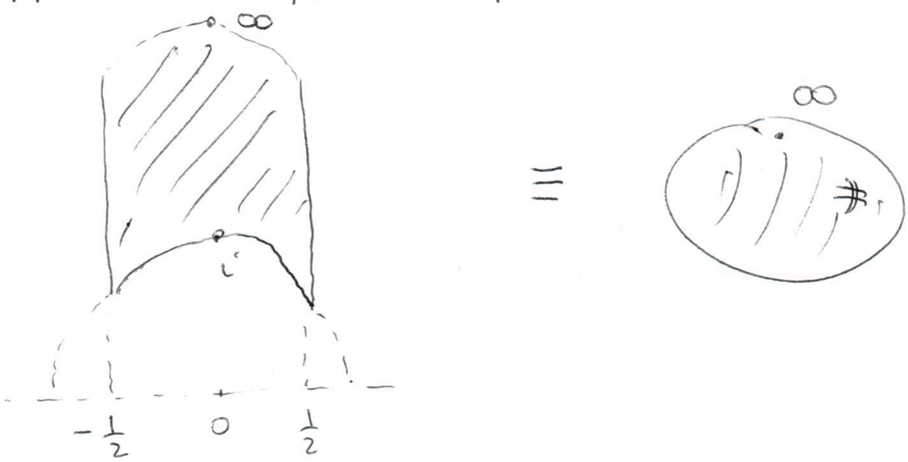
neighbourhoods of ∞ $(\mathrm{Im}(z) > c)$

$P \in \mathbb{Q}$

neighbourhoods of P = tangent circles at P

III Take the quotient X^*/Γ, yielding a compact Riemann surface

$$\overline{X/\Gamma}^{BB} \overset{def}{=} X^*/\Gamma = X/\Gamma \cup \{\infty\} = \mathbb{P}^1_{\mathbb{C}}$$



In general, we follow a similar approach, but with a few catches:

1. For an arbitrary symmetric domain X, how to obtain $\overline{X}$?

   Answer: Harish-Chandra embedding.

2. How to identify the "rational boundary components" and describe the X^*-topology?

   Answer: they are those whose normalisers are maximal parabolic subgroups defined over $\mathbb{Q}$.

Silly fact: if G is a group acting transitively on a set Y and $y_0 \in Y$ with $P = \mathrm{Stab}(y_0)$, and moreover $N_G(P) = P$, then we have bijections

$$\begin{array}{ccccc} G/P & \xrightarrow{\sim} & Y & \xrightarrow{\sim} & \{\text{conjugates of } P\} \\ gP & \longmapsto & g\cdot y_0 & \longmapsto & \mathrm{Stab}(g\cdot y_0) = gPg^{-1} \end{array}$$

In the Baby Baily-Borel, for $Y = \mathbb{P}^1_{\mathbb{R}}$ and $y_0 = \infty$, we have

$$P = \mathrm{Stab}(y_0) = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \in SL_2(\mathbb{R}) \right\} \Big/ \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

which is a parabolic subgroup of $PSL_2(\mathbb{R})$ (in particular, $N_{PSL_2(\mathbb{R})}(P) = P$). Hence there is a bijection

$$\left\{ \begin{array}{c} \text{rational points} \\ \text{in the boundary} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{(maximal) parabolic subgroups} \\ \text{of } PSL_2 \text{ defined over } \mathbb{Q} \end{array} \right\}$$

Moreover P has the following "Levi decomposition"

$$P = \underbrace{\left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\}}_{\substack{\text{unipotent} \\ \text{radical}}} \cdot \underbrace{\left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \right\}}_{\text{torus}}$$

and we can describe the neighbourhood of $y_0 = \infty$ solely in terms of P: for $t, r > 0$

$$N_\infty(t,r) \overset{\text{def}}{=} \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \cdot i \;\middle|\; \begin{array}{l} a > t \text{ and} \\ |b| < r \end{array} \right\}$$

$$= \left\{ a^2 i + b \;\middle|\; a > t \text{ and } |b| < r \right\}$$

form a system of neighbourhoods of $y_0 = \infty$. The unipotent part is "translation in the horizontal direction" and is limited, while the torus part is "translation in the vertical direction".

Borel embedding

Let X = irreducible Hermitian symmetric space of non-compact type.
(a bounded Hermitian symmetric domain)

$G = \mathrm{Hol}(X)^+$ (a simple real Lie group)

Then G acts transitively on X. Choosing a base point $x_0 \in X$ and defining

$K = \mathrm{Stab}(x_0)$ (a maximal compact subgroup of G)

we have a diffeomorphism

$$\begin{aligned} G/K &\xrightarrow{\sim} X \\ gK &\longmapsto g x_0 \end{aligned}$$

(see Helgason, thm 4.2, p. 113 for the description of the differential structure of G/K)

Write $s \in G$ for the symmetry at x_0 (so that $ds = -\mathrm{id}$ on $T_{x_0} X$). Then

$\theta \overset{\text{def}}{=} \mathrm{Ad}(s)$ is a Cartan involution on $\underline{g} \overset{\text{def}}{=} \mathrm{Lie}(G)$

[Recall: a Lie algebra involution θ is Cartan if $-B_\theta(X, Y) \overset{\text{def}}{=} -B(X, \theta Y)$ is positive definite on $\underline{g}$; here B denotes the Killing form]

We have a corresponding Cartan decomposition

$$\underline{g} = \underline{k} \oplus \underline{p} \quad \text{(orthogonal sum w.r.t. } B)$$

where
$$\begin{aligned} \underline{k} &= \mathrm{Lie}(K) = +1 \text{ eigenspace of } \theta \\ \underline{p} &= T_{x_0}(X) = -1 \text{ eigenspace of } \theta \end{aligned}$$

Denote the complexifications of $\underline{g}$ and G by $\underline{g}_{\mathbb{C}} = \underline{g} \otimes_{\mathbb{R}} \mathbb{C}$ and $G_{\mathbb{C}}$.

Def (Compact Dual) Let

$$\begin{cases} \underline{g}_c = \underline{k} \oplus i\underline{p} \subset \underline{g}_{\mathbb{C}} \\ G_c = \text{Corresponding analytic subgroup of } G_{\mathbb{C}} \end{cases}$$

Then $X_c \overset{\text{def}}{=} G_c/K$ is an irreducible Hermitian symmetric space of compact type, called the compact dual of X.

Thm (Borel embedding) There is a holomorphic embedding

$$X \hookrightarrow X_c$$

Moreover the G-action on X extends to a continuous action on the closure $\overline{X}$ of X in X_c.

E.g (Unit disk) Let

$$X = \mathbb{D} = \{ z \in \mathbb{C} \text{ s.t } |z| < 1 \}$$

with the usual Bergmann metric $\dfrac{dz \otimes d\bar{z}}{(1-|z|^2)^2}$. Then

$$G = \mathrm{Hol}(X)^+ = PSU(1,1) = SU(1,1)/\{\pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\}$$ (action by Möbius transf.)

Here we denote by $SU(m,n)$ the special indefinite unitary group, which is the subgroup of $SL(m+n, \mathbb{C})$ of all linear transformations preserving the indefinite Hermitian form

$$\psi(\underline{z}; \underline{w}) = -\sum_{1 \leq i \leq m} z_i \overline{w_i} + \sum_{m+1 \leq j \leq m+n} z_j \overline{w_j}$$

For $m = n = 1$, we have

$$SU(1,1) = \left\{ \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} : \begin{array}{l} \alpha, \beta \in \mathbb{C} \\ |\alpha|^2 - |\beta|^2 = 1 \end{array} \right\}$$

(As a side remark: we have an isomorphism

$$\varphi : \mathbb{H} \xrightarrow{\sim} \mathbb{D}$$
$$z \longmapsto \frac{z-i}{z+i}$$

and $SU(1,1) = \varphi \, SL_2(\mathbb{R}) \, \varphi^{-1}$, hence this is the ye old example of the complex upper half plane in disguise. But it turns out to be easier to work with the $\mathbb{D}$-model in this case)

We take as base point $x_0 = 0$. Then

$$K = \mathrm{Stab}(x_0) = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \bar{\alpha} \end{pmatrix} : \begin{matrix} \alpha \in \mathbb{C} \\ |\alpha| = 1 \end{matrix} \right\} \Big/ \{\pm 1\} \cong U(1).$$

The Lie algebra of G can be computed by

$$\mathfrak{g} = \mathrm{Lie}(G) = \mathfrak{su}(1,1) = \ker\big(G(\mathbb{C}[\varepsilon]) \to G(\mathbb{C}) \big)$$

where $\varepsilon^2 = 0$ as usual. Then

$$\mathfrak{g} = \mathfrak{su}(1,1) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{C}) : \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \varepsilon \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G(\mathbb{C}[\varepsilon]) \right\}$$

$$= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{C}) : \begin{matrix} \overline{1+\varepsilon a} = 1 + \varepsilon d, \quad \overline{\varepsilon b} = \varepsilon c, \text{ and} \\ (1+\varepsilon a)(\overline{1+\varepsilon a}) - (\varepsilon b)(\overline{\varepsilon b}) = 1 \end{matrix} \right\}$$

$$= \left\{ \begin{pmatrix} it & \beta \\ \bar{\beta} & -it \end{pmatrix} : \beta \in \mathbb{C},\ t \in \mathbb{R} \right\}$$

Similarly,

$$\mathfrak{k} = \mathrm{Lie}(K) = \left\{ \begin{pmatrix} it & 0 \\ 0 & -it \end{pmatrix} ; t \in \mathbb{R} \right\} \cong \mathfrak{u}(1).$$

The reflexion at $x_0 = 0$ is $s(z) = -z$, which is represented by the matrix $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \in SU(1,1)$. Then

$$\mathrm{Ad}(s) \begin{pmatrix} it & \beta \\ \bar{\beta} & -it \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} it & \beta \\ \bar{\beta} & -it \end{pmatrix} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} it & 0 \\ 0 & -it \end{pmatrix} + \begin{pmatrix} 0 & -\beta \\ -\bar{\beta} & 0 \end{pmatrix}$$

Hence the Cartan decomposition given by $\theta = \mathrm{Ad}(s)$ is

$$\mathfrak{su}(1,1) = \underbrace{\left\{ \begin{pmatrix} it & 0 \\ 0 & -it \end{pmatrix} ; t \in \mathbb{R} \right\}}_{\mathfrak{k} = \mathrm{Lie}(K)} \oplus \underbrace{\left\{ \begin{pmatrix} 0 & \beta \\ \bar{\beta} & 0 \end{pmatrix} ; \beta \in \mathbb{C} \right\}}_{\mathfrak{p} = T_{x_0} X}$$

The complexification of $\mathfrak{su}(1,1)$ is just $\mathfrak{sl}(2,\mathbb{C})$:

$$\mathfrak{su}(1,1) \otimes_{\mathbb{R}} \mathbb{C} \xrightarrow{\sim} \mathfrak{sl}(2,\mathbb{C})$$

$$A \otimes \alpha \longmapsto \alpha A$$

(whose inverse map is given by

$$\mathfrak{sl}_2(\mathbb{C}) \xrightarrow{\sim} \mathfrak{su}(1,1)\otimes_{\mathbb{R}}\mathbb{C}$$

$$\begin{pmatrix} a & b \\ c & -a\end{pmatrix} \longmapsto \frac{1}{2}\begin{pmatrix} a-\bar{a} & b+\bar{c} \\ \bar{b}+c & -a+\bar{a}\end{pmatrix}\otimes 1 + \frac{1}{2i}\begin{pmatrix} a+\bar{a} & b-\bar{c} \\ -\bar{b}+c & -a-\bar{a}\end{pmatrix}\otimes i$$

as can be easily checked).

Hence the compact dual is obtained by

$$\mathfrak{g}_c = \underline{k} \oplus i\underline{p} = \left\{ \begin{pmatrix} it & \gamma \\ -\bar{\gamma} & -it\end{pmatrix} : \begin{array}{c}\gamma\in\mathbb{C}\\ t\in\mathbb{R}\end{array}\right\} = \underline{\mathfrak{su}}(2)$$

(which is the Lie subalgebra of $\mathfrak{sl}_2(\mathbb{C})$ consisting of <u>skew-Hermitian</u> <u>matrices</u>, i.e., the analogue of purely imaginary numbers in the matrix world). The corresponding subgroup of $SL_2(\mathbb{C})$ is $SU(2)$, where $SU(n)$ denotes as usual the <u>special unitary group</u>, i.e., the subgroup of $SL(n,\mathbb{C})$ consisting of linear transformations preserving the Hermitian form

$$\psi(\underline{z},\underline{w}) = \sum_{1\le i\le n} z_i\overline{w_i}$$

In other words:

$$SU(n) = \{ A \in SL_n(\mathbb{C}) \mid A\cdot A^* = I \}$$

(here A^* = conjugate transpose of A)

Finally, we can explicitly describe the compact dual $X_c = SU(2)/U(1)$. It's just $\mathbb{P}^1_{\mathbb{C}}$! To see that, observe that both $Gl_2(\mathbb{C})$, $SL_2(\mathbb{C})$ and SU_2 act transitively on $\mathbb{P}^1_{\mathbb{C}}$. Writing $x_0=(0;1)$ we have

$$P \overset{\text{def}}{=} \operatorname{Stab}_{Gl_2(\mathbb{C})}(x_0) = \left\{ \begin{pmatrix} a & 0 \\ c & d\end{pmatrix} \in Gl_2(\mathbb{C})\right\}$$

and
$$K = \operatorname{Stab}_{SU(2)}(x_0) = \operatorname{Stab}_{Gl_2(\mathbb{C})}(x_0)\cap SU(2)$$
$$= \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1}\end{pmatrix} : \begin{array}{c}\alpha\in\mathbb{C}\\ |\alpha|=1\end{array}\right\} \cong U(1)$$

Hence the inclusions of $SU(1,1)$ and $SU(2)$ in their common

Complexification $SL(2;\mathbb{C})$ ~~[illegible]~~

$$\begin{tikzcd} SU(2) \arrow[r, hook] & SL(2,\mathbb{C}) \\ & SU(1,1) \arrow[u, hook] \end{tikzcd}$$

induces maps of symmetric spaces (by acting on x_0)

$$\begin{tikzcd} X_c = SU(2)/K \arrow[r, "\approx"] & SL(2,\mathbb{C})/P = \mathbb{P}^1_{\mathbb{C}} \\ & SU(1,1)/K = X \arrow[u, hook, "\text{Borel embedding.}"'] \end{tikzcd}$$

(compact dual) (original symmetric space)

The image of the Borel embedding in $X_c = \mathbb{P}^1_{\mathbb{C}}$ is

$$G \cdot x_0 = \left\{ \begin{pmatrix} \alpha & \beta \\ \bar\beta & \bar\alpha \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} : \begin{array}{l} \alpha, \beta \in \mathbb{C} \\ |\alpha|^2 - |\beta|^2 = 1 \end{array} \right\}$$

$$= \left\{ (\beta : \bar\alpha) \in \mathbb{P}^1_{\mathbb{C}} \quad \text{where} \quad \begin{array}{l} \alpha, \beta \in \mathbb{C}, \\ |\alpha|^2 - |\beta|^2 = 1 \end{array} \right\}$$

But $(\beta : \bar\alpha) = \left(\frac{\beta}{\bar\alpha} : 1\right)$ and $|\alpha|^2 - |\beta|^2 = 1 \Rightarrow 1 - \left|\frac{\beta}{\alpha}\right|^2 = \frac{1}{|\alpha|}$

$\Leftrightarrow \left|\frac{\beta}{\bar\alpha}\right|^2 = 1 - \frac{1}{|\alpha|^2} < 1$, and conversely given any point $z \in \mathbb{D}$, we may choose $\alpha = \frac{1}{\sqrt{1-|z|^2}}$ and $\beta = \frac{z}{\sqrt{1-|z|^2}}$ so that $\beta/\bar\alpha = z$. with $|\alpha|^2 - |\beta|^2 = 1$. To sum up, the Borel embedding in this case is just the embedding of $\mathbb{D}$ as the ~~lower~~ south hemisphere of $\mathbb{P}^1_{\mathbb{C}}$, viewed as the Riemann sphere.

Next, we describe another important type of 'canonical' embedding.

Harish-Chandra embedding

We now realise X as a bounded symmetric domain inside the tangent ~~plane~~ space at x_0; first we need some notations.

First, choose

$$\underline{h} \subset \underline{k} \quad \text{a Cartan subalgebra}$$

so that $\underline{h} \otimes \mathbb{C} \subset \underline{k} \otimes \mathbb{C}$ is also a Cartan subalgebra; set

$$\Delta = \Delta(\underline{g} \otimes \mathbb{C} ; \underline{h} \otimes \mathbb{C}) \quad \text{the corresponding root system.}$$

In the (real) Cartan decomposition

$$\underline{g}_c = \underline{k} \oplus i \underline{p}$$

we identified $i\underline{p} = T_{x_0} X_c$, hence $\underline{p} \otimes \mathbb{C}$ is the complexified tangent space of X_c at x_0. It is possible to choose an ordering of Δ so that

$$p^+ \overset{def}{=} \bigoplus_{\alpha \in \Delta_+} (\underline{g}_{\mathbb{C}})_\alpha = \text{holomorphic tangent space of } X_c \text{ at } x_0$$

($+i$ eigenspace of the almost complex structure J_{x_0})

<u>Thm</u> (Harish-Chandra) The exponential map induces a complex analytic diffeomorphism

$$\xi : p^+ \xrightarrow{\approx} U \subset X_c$$
$$v \longmapsto \exp(v) \cdot x_0$$

between p^+ and an open dense subset U of X_c such that $U \supset X$. (This realises X as a bounded domain $\xi^{-1}(X) \subset p^+$)

<u>E.g.</u> Let X_c = Grassmanian of n-planes inside $\mathbb{C}^{m+n}$

Then Gl_{m+n} and Sl_{m+n} both act transitively on X_c.

But so does $SU(m+n,\mathbb{C})$ since the unitary group permutes the ~~set~~ orthonormal bases of the n-dimensional subspaces of $\mathbb{C}^{m+n}$. Take the base point $x_0 \in X_c$ to be

$$x_0 = \text{span of } e_{m+1}, e_{m+2}, \dots, e_{m+n}$$

where the $e_i \in \mathbb{C}^{m+n}$ are the standard basis vectors. We shall write an element of $GL(m+n,\mathbb{C})$ in block form

$$\begin{array}{c c} & \begin{array}{cc} m & n \end{array} \\ \begin{array}{c} m \\ n \end{array} & \begin{pmatrix} A & B \\ C & D \end{pmatrix} \end{array}$$

Then $$P \overset{\text{def}}{=} \mathrm{Stab}_{SL(m+n;\mathbb{C})}(x_0) = \left\{ \begin{pmatrix} A & 0 \\ C & D \end{pmatrix} : \det A \cdot \det D = 1 \right\}$$

$$K \overset{\text{def}}{=} \mathrm{Stab}_{SU(m+n)}(x_0) = P \cap SU(m+n)$$

$$= \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} : \begin{array}{l} A \in U(m), \quad D \in U(n) \\ \det A \cdot \det D = 1 \end{array} \right\}$$

$$\cong S(U(m) \times U(n))$$

Then $X_c = SL_{m+n}(\mathbb{C}) \cdot x_0 = SU_{m+n} \cdot x_0$ and the Hermitian symmetric space of compact type X_c has presentations

$$X_c = SL_{m+n}(\mathbb{C})/P = SU_{m+n}/K$$

with $G_c \overset{\text{def}}{=} \mathrm{Hol}(X_c)^+ = SU(m+n)$.

The symmetry at x_0 is given as a matrix

$$s = \begin{pmatrix} -I_m & 0 \\ 0 & I_n \end{pmatrix} \cdot a$$ where $a \in \mathbb{C}^\times$ is any number s.t. $a^{m+n} \cdot (-1)^m = 1$

The Lie algebra of $SU(m+n)$ is the set of traceless skew-Hermitian matrices. Now $\theta = \mathrm{Ad}(s)$ gives the following Cartan decomposition:

$$\underline{su}(m+n) = \underbrace{\left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \right\}}_{\underline{k} = \mathrm{Lie}(K)} \oplus \underbrace{\left\{ \begin{pmatrix} 0 & B \\ -B^* & 0 \end{pmatrix} \right\}}_{\underline{p} = T_{x_0} X_c}$$

(with $\mathrm{Tr}\, A + \mathrm{Tr}\, D = 0$) Hence the non-compact dual is

$$\underline{su}(m,n) = \underbrace{\left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \right\}}_{\underline{k}} \oplus \underbrace{\left\{ \begin{pmatrix} 0 & C \\ C^* & 0 \end{pmatrix} \right\}}_{i\underline{p}}$$

and $G = SU(m,n)$ with $X = G/K$. The common complexification of $SU(m+n)$ and $SU(m,n)$ is $SL(m+n, \mathbb{C})$. Now we may choose

$$\underline{h} = \left\{ \begin{pmatrix} it_1 & & & \\ & it_2 & & \\ & & \ddots & \\ & & & it_{m+n} \end{pmatrix} : \begin{array}{c} t_1 + t_2 + \cdots + t_{m+n} = 0 \\ t_i \in \mathbb{R} \end{array} \right\}$$

as Cartan subalgebra of $\underline{k}$. Then $\underline{h} \otimes \mathbb{C}$ is a Cartan subalgebra of $\underline{sl}(m+n) = \underline{su}(m,n) \otimes \mathbb{C}$; choosing the ordering ~~case~~ of $\Delta(\underline{sl}(m+n); \underline{h} \otimes \mathbb{C})$ so that the positive root space consists of strictly upper triangular matrices, we can now identify the holomorphic tangent space of X_c at x_0 with

$$p^+ = \bigoplus_{\alpha \in \Delta_+} (\underline{g} \otimes \mathbb{C})_\alpha = \left\{ \begin{pmatrix} 0 & C \\ 0 & 0 \end{pmatrix} : C \in M_{m \times n}(\mathbb{C}) \right\}$$

The Borel embedding is given by

$$\begin{array}{ccc} X_c = SU(m+n)/K & \xrightarrow{\sim} & SL(m+n)/P \\ & & \big\uparrow \text{Borel embedding.} \\ & & X = SU(m,n)/K \end{array}$$

while the Harish-Chandra embedding is obtained by taking the pre-image of X under

$$\xi : \mathfrak{p}^+ \longrightarrow X_c$$
$$\begin{pmatrix} 0 & C \\ 0 & 0 \end{pmatrix} \longmapsto \exp\begin{pmatrix} 0 & C \\ 0 & 0 \end{pmatrix} \cdot x_0 = \begin{pmatrix} I_m & C \\ 0 & I_n \end{pmatrix} \cdot x_0$$
$$= \text{span of columns of } \begin{pmatrix} C \\ I_n \end{pmatrix}$$

But the image of X in X_c under the Borel embedding is the set

$$X \cong \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot x_0 \;\middle|\; \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SU(m,n) \right\}$$
$$= \left\{ \text{span of columns of } \begin{pmatrix} B \\ D \end{pmatrix} \;\middle|\; \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SU(m,n) \right\}$$

Since we know that the image of ξ contains X, i.e., that any span of the columns of $\begin{pmatrix} B \\ D \end{pmatrix}$ above is of the form span of the columns of $\begin{pmatrix} C \\ I_n \end{pmatrix}$, we conclude that D is invertible and that we may identify

$$X \cong \left\{ \begin{pmatrix} BD^{-1} \\ I_n \end{pmatrix} : \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SU(m,n) \right\}$$

But by Witt's theorem, ~~since~~ the span of the columns of $\begin{pmatrix} B \\ D \end{pmatrix}$ will be in the $SU(m,n)$-orbit of x_0 if and only if these two spaces are isometric with respect to the Hermitian form

$$\psi(z,w) = -\sum_{1 \le i \le m} z_i \overline{w_i} + \sum_{m+1 \le i \le m+n} z_i \overline{w_i}.$$

In other words, the columns of $\begin{pmatrix} B \\ D \end{pmatrix}$ should generate a positive definite space with respect to $\sum_{m+1 \le i \le m+n} z_i \overline{w_i}$. Now the Gram matrix of the basis given by the columns of

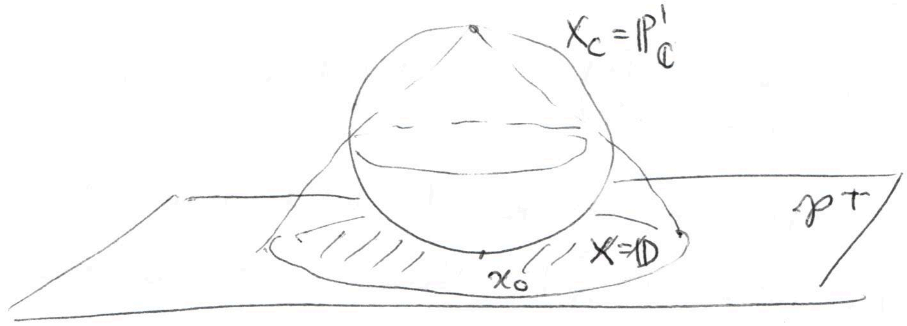
$\begin{pmatrix} Z \\ I_n \end{pmatrix}$ is $\begin{pmatrix} Z \\ I_n \end{pmatrix}^t \begin{pmatrix} -I_m & 0 \\ 0 & I_n \end{pmatrix} \begin{pmatrix} Z \\ I_n \end{pmatrix}$, i.e.,

$$-Z^t \bar{Z} + I_n$$

This is a positive definite matrix if and only if $I_n - Z^* Z$ is positive definite. We obtain the identification.

$$\xi^{-1}(X) = \left\{ \begin{pmatrix} 0 & Z \\ 0 & 0 \end{pmatrix} \in \mathfrak{p}^+ : I_n - Z^* Z \gg 0 \right\}$$

Observe that for $m = n = 1$, the Harish-Chandra embedding given above is just the stereographic projection of the disk $1 - \bar{z} z > 0$ in the holomorphic tangent plane of $X_c = \mathbb{P}^1_{\mathbb{C}}$:



We now define $\overline{\xi^{-1}(X)} \subset \mathfrak{p}^+$ to be the Baily-Borel compactification of the bounded symmetric domain X. Next we look closer at the boundary components $\overline{\xi^{-1}(X)} \setminus \xi^{-1}(X)$.

Boundary components (Korányi - Wolf theory)

To describe the boundary components, we need to describe a canonical way (up to conjugacy) of choosing the Cartan subalgebra of $\mathfrak{g} = \mathrm{Lie}(G)$ in a way compatible with the Cartan decomposition (some books call it a maximal non-compac Cartan subalgebra). We keep the notation of the previous section, namely

- $\underline{\mathfrak{h}} \subset \underline{\mathfrak{k}}$ a Cartan subalgebra (a maximally compact Cartan subalgebra)
- $\Delta = \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$

It is possible to choose root vectors E_μ ($\mu \in \Delta$) such that

$$[E_\mu, E_{-\mu}] = H_\mu \quad \text{with} \quad \nu(H_\mu) = \frac{2\,B(\nu,\mu)}{B(\mu,\mu)} \qquad (\mu, \nu \in \Delta)$$

and the complex conjugation of $\mathfrak{g}_{\mathbb{C}}$ with respect to $\mathfrak{g}$ swaps E_μ and $E_{-\mu}$ if they belong to $\mathfrak{p} \otimes \mathbb{C}$. Letting

$$X_\mu \overset{\text{def}}{=} E_\mu + E_{-\mu} \qquad \text{and} \qquad Y_\mu \overset{\text{def}}{=} i(E_\mu - E_{-\mu})$$

they form a real basis of $\mathfrak{p}$.

We say that two roots $\alpha, \beta \in \Delta$ are strongly orthogonal if both $\alpha \pm \beta$ are not roots (strong orthogonality implies orthogonality in the usual sense). Let Ψ be a maximal set of strongly orthogonal roots of non-compact type (i.e. the corresponding root spaces are contained in $\mathfrak{p} \otimes \mathbb{C}$), say

$$\Psi = \{\alpha_1, \alpha_2, \dots, \alpha_r\}$$

Write $X_i = E_{\alpha_i} + E_{-\alpha_i}$ (see above). Then

$$\underline{\mathfrak{a}} \overset{\text{def}}{=} \mathbb{R} X_1 + \dots + \mathbb{R} X_r$$

defines a maximal abelian subspace of $\mathfrak{p}$. Now we can define a relative root system (not reduced in general) $\Delta_{\mathbb{R}} = \Delta(\mathfrak{g}, \mathfrak{a})$. They have been classified, and are known to be of two types: either BC_r or C_r.

E.g. Let $\mathfrak{g} = \mathfrak{su}(2,2)$ so that $\mathfrak{g}\otimes\mathbb{C} = \mathfrak{sl}(4;\mathbb{C})$. We have a Cartan decomposition.

$$\mathfrak{su}(2,2) = \underline{k} \oplus \underline{p} \quad \text{where}$$

$$\underline{k} = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} : \begin{array}{l} A, D \in M_2(\mathbb{C}), \quad A + A^* = D + D^* = 0, \\ tr(A) + tr(D) = 0 \end{array} \right\}$$

and

$$\underline{p} = \left\{ \begin{pmatrix} 0 & B \\ B^* & 0 \end{pmatrix} : B \in M_2(\mathbb{C}) \right\}$$

We may choose as maximal compact Cartan subalgebra

$$\underline{h} = \left\{ \begin{pmatrix} it_1 & & 0 \\ & \ddots & \\ 0 & & it_4 \end{pmatrix} : t_j \in \mathbb{R} \text{ and } t_1 + \dots + t_4 = 0 \right\} \subseteq \underline{k}$$

Notice that complex conjugation of $\mathfrak{sl}(4;\mathbb{C})$ with respect to $\mathfrak{su}(2,2)$ is given by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \longmapsto \begin{pmatrix} -A^* & C^* \\ B^* & -D^* \end{pmatrix}$$

Hence we may choose $\Psi = \{\alpha, \beta\}$ as a maximal set of strongly orthogonal roots of non-compact type, where $\alpha, \beta \in \Delta(\mathfrak{g}\otimes\mathbb{C}, \underline{h})$ are given by

$$\alpha \begin{pmatrix} z_1 & & \\ & \ddots & \\ & & z_4 \end{pmatrix} = z_1 - z_4 \quad \text{and} \quad \beta \begin{pmatrix} z_1 & & \\ & \ddots & \\ & & z_4 \end{pmatrix} = z_2 - z_3 .$$

The normalised root vectors are.

$$E_\alpha = \begin{pmatrix} 0&0&0&1 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix}, \quad E_{-\alpha} = E_\alpha^t, \quad H_\alpha = [E_\alpha, E_{-\alpha}] = \begin{pmatrix} 1 &&& \\ & 0 && \\ && 0 & \\ &&& -1 \end{pmatrix}$$

and

$$E_\beta = \begin{pmatrix} 0&0&0&0 \\ 0&0&1&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix}, \quad E_{-\beta} = E_\beta^t, \quad H_\beta = [E_\beta, E_{-\beta}] = \begin{pmatrix} 0 &&& \\ & 1 && \\ && -1 & \\ &&& 0 \end{pmatrix}$$

The corresponding maximal abelian subspace of $\underline{p}$ is

$$\underline{a} = \mathbb{R}\cdot X_\alpha + \mathbb{R}\cdot X_\beta = \left\{ \begin{pmatrix} & & & a \\ & & b & \\ & b & & \\ a & & & \end{pmatrix} : a, b \in \mathbb{R} \right\} \subset \underline{p}$$

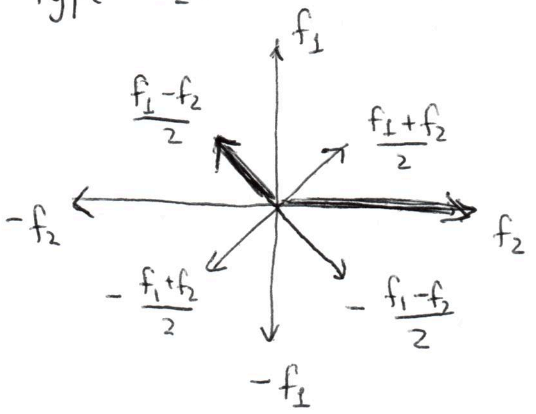
Consider the linear functionals. $f_i : \underline{a} \longrightarrow \mathbb{R}$

$$f_1 \begin{pmatrix} & & & a \\ & & b & \\ & b & & \\ a & & & \end{pmatrix} = 2a \qquad \text{and} \qquad f_2 \begin{pmatrix} & & & a \\ & & b & \\ & b & & \\ a & & & \end{pmatrix} = 2b$$

The relative root system is given by

$$\Delta_{\mathbb{R}} = \Delta_{\mathbb{R}}(\underline{g}, \underline{a}) = \left\{ \pm f_1, \pm f_2, \pm \frac{f_1+f_2}{2}, \pm \frac{f_1-f_2}{2} \right\}$$

which is of type C_2:



We may choose an ordering such that the positive roots are

$$\Delta^+_{\mathbb{R}} = \left\{ f_1, f_2, \frac{f_1+f_2}{2}, \frac{f_1-f_2}{2} \right\}$$

with basis $\left\{ \frac{f_1-f_2}{2}, f_2 \right\}$.

Back to the general case, denoting $f_i : \underline{a} \longrightarrow \mathbb{R}$ the coordinates with respect to the basis $X_i/2$, it is known by the classification theorem that $\Delta_{\mathbb{R}}$ has a basis

$$\left\{ \gamma_1 = \frac{f_1-f_2}{2}, \gamma_2 = \frac{f_2-f_3}{2}, \dots, \gamma_{r-1} = \frac{f_{r-1}-f_r}{2}, \gamma_r = \frac{f_r}{2} \right\} \qquad (BC_r)$$

or

$$\left\{ \gamma_1 = \frac{f_1-f_2}{2}, \gamma_2 = \frac{f_2-f_3}{2}, \dots, \gamma_{r-1} = \frac{f_{r-1}-f_r}{2}, \gamma_r = f_r \right\} \qquad (C_r)$$

Let $\Delta^+_{\mathbb{R}}$ be the set of positive roots with respect to these bases.

For $b = 1, 2, \dots, r$, set

$$\underline{a}_b = \mathbb{R}(X_1 + X_2 + \cdots + X_b) \qquad \text{(a 1-dim subspace of } \underline{a}\text{)}$$

$$A_b = \exp \underline{a}_b \subset G$$

$$\underline{n} = \bigoplus_{\gamma \in \Delta^+_{\mathbb{R}}} \mathfrak{g}_\gamma \qquad \text{(a nil. potent Lie algebra)}$$

$$N = \exp \underline{n} \qquad \text{(a unipotent subgroup of } G\text{)}$$

$$P_b = Z(A_b) \cdot N \qquad \text{(a maximal parabolic subgroup of } G\text{)}$$

$$x_b = E_1 + \cdots + E_b \in X_c \qquad \text{(via the Harish-Chandra embedding)}$$

$$\mathcal{L}_b = \sum_{\substack{\gamma \text{ is in the span} \\ \text{of } \gamma_{b+1}, \dots, \gamma_r}} \mathfrak{g}_\gamma \oplus [\mathfrak{g}_\gamma, \mathfrak{g}_{-\gamma}]$$

L_b = corresponding analytic subgroup of G

$$F_b = L_b \cdot x_b \subset X_c$$

<u>Thm</u> (Korányi - Wolf & Baily - Borel) Let $\overline{X}$ denote the closure of X in $\mathfrak{p}^+$ (via the Harish-Chandra embedding). Then

$$\overline{X} = \bigcup_{1 \leq b \leq r} G \cdot x_b$$

Moreover, the boundary is given by

$$\overline{X} \setminus X = \bigcup_{k \in K} \bigcup_{1 \leq b \leq r} k \cdot F_b$$

and the <u>boundary components</u> $k \cdot F_b$ $(k \in K, 1 \leq b \leq r)$ are also Hermitian symmetric spaces of non-compact type of lower dimension. Its <u>normaliser</u> is

$$N_G(F_b) \overset{\text{def}}{=} \{ g \in G \mid g(F_b) = F_b \} = P_b,$$

a maximal parabolic subgroup of G. We have a bijection

$$\left\{ \begin{array}{l} \text{boundary} \\ \text{components} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{maximal parabolic} \\ \text{subgroups of } G \end{array} \right\}$$

$$k.F_b \longmapsto N_G(kF_b) = k\,P_b\,.k^{-1} \qquad (k \in K)$$

E.g. Back to our example of $G = SU(2,2)$, we look at the case $b = 1$. We have

$$\underline{\mathfrak{a}_1} = \mathbb{R} X_1 = \left\{ \begin{pmatrix} & & & a \\ & & 0 & \\ & 0 & & \\ a & & & \end{pmatrix} : a \in \mathbb{R} \right\}$$

$$A_1 = \exp \underline{\mathfrak{a}_1} = \left\{ \begin{pmatrix} x & 0 & 0 & y \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ y & 0 & 0 & x \end{pmatrix} : \begin{array}{l} x, y \in \mathbb{R} \\ x^2 - y^2 = 1 \end{array} \right\} \cong \mathbb{G}_m(\mathbb{R}).$$

$$Z(A_1) = \left\{ \begin{pmatrix} x & 0 & 0 & y \\ 0 & \alpha & \beta & 0 \\ 0 & \bar{\beta} & \bar{\alpha} & 0 \\ y & 0 & 0 & x \end{pmatrix} \begin{array}{l} x, y \in \mathbb{R}, \quad x^2 - y^2 = 1 \\ \alpha, \beta \in \mathbb{C}, \quad |\alpha|^2 - |\beta|^2 = 1 \end{array} \right\} \subseteq SU(2,2).$$

Next, we compute N. First, the root spaces of $\Delta^+_{\mathbb{R}} = \{ f_1, f_2, \frac{f_1 \pm f_2}{2} \}$ are

$$\mathfrak{g}_{f_1} = \mathbb{R} \cdot \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & i & -i & 0 \\ 0 & i & -i & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} ; \qquad \mathfrak{g}_{f_2} = \mathbb{R} \cdot \begin{pmatrix} i & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & -i \end{pmatrix}$$

$$\mathfrak{g}_{\frac{f_1+f_2}{2}} = \left\{ \begin{pmatrix} 0 & z & -z & 0 \\ -\bar{z} & 0 & 0 & \bar{z} \\ -\bar{z} & 0 & 0 & \bar{z} \\ 0 & z & -z & 0 \end{pmatrix} : z \in \mathbb{C} \right\} ; \quad \mathfrak{g}_{\frac{f_1-f_2}{2}} = \left\{ \begin{pmatrix} 0 & z & z & 0 \\ -\bar{z} & 0 & 0 & \bar{z} \\ \bar{z} & 0 & 0 & -\bar{z} \\ 0 & z & z & 0 \end{pmatrix} : z \in \mathbb{C} \right\}$$

Hence the analytic subgroup corresponding to $\underline{\mathfrak{n}} = \mathfrak{g}_{f_1} \oplus \mathfrak{g}_{f_2} \oplus \mathfrak{g}_{\frac{f_1+f_2}{2}} \oplus \mathfrak{g}_{\frac{f_1-f_2}{2}}$ is

$$N = \left\{ \begin{pmatrix} 1-c & \alpha & \beta & c \\ -\gamma & 1-d & d & \gamma \\ -\delta & -d & 1+d & \delta \\ -c & \alpha & \beta & 1+c \end{pmatrix} : \begin{array}{l} c, \alpha, \beta, \gamma, \delta \in \mathbb{C}, \quad d \in i\mathbb{R}, \\ c + \bar{c} = |\gamma|^2 - |\delta|^2, \\ \alpha = \bar{\gamma}(1-d) + \bar{\delta} d, \\ \beta = \bar{\gamma} d - \bar{\delta}(1+d) \end{array} \right\}$$

(These computations are more easily performed using a maximal compact abelian subspace instead of $\underline{a}$; one conjugates everything by the "inverse Cayley transform"

$$D = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

so that $D \cdot \underline{a} D^{-1}$ is contained in the main diagonal. For more about Cayley transforms and relative root systems in the real case, see Knapp's "Lie groups beyond an introduction")

Finally,

$$\mathfrak{l}_1 = \mathfrak{g}_{f_2} \oplus \mathfrak{g}_{-f_2} \oplus [\mathfrak{g}_{f_2} ; \mathfrak{g}_{-f_2}] = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & it & \beta & 0 \\ 0 & \bar{\beta} & -it & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} : \begin{array}{l} t \in \mathbb{R} \\ \beta \in \mathbb{C} \end{array} \right\}$$

with corresponding analytic subgroup

$$L_1 = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \alpha & \beta & 0 \\ 0 & \bar{\beta} & \bar{\alpha} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} : \begin{array}{l} \alpha, \beta \in \mathbb{C} \\ |\alpha|^2 - |\beta|^2 = 1 \end{array} \right\} \cong SU(1,1).$$

Hence the boundary component F_1 is given by

$$F_1 = L_1 \cdot x_1$$ (action with respect to the Harish-Chandra embedding)

and the normaliser of F_1 is the maximal parabolic subgroup

$$P_1 = Z(A_1) \cdot N$$

To describe these more explicitly, we recall that we may identify $\mathfrak{p}^+$ with the set $M_2(\mathbb{C})$:

$$\mathfrak{p}^+ = \left\{ \begin{pmatrix} 0 & Z \\ 0 & 0 \end{pmatrix} \right\} \longleftrightarrow Z \in M_2(\mathbb{C})$$

The action of G on $\mathfrak{p}^+ = M_2(\mathbb{C})$ is given by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot \exp\begin{pmatrix} 0 & Z \\ 0 & 0 \end{pmatrix} \cdot x_0$$

(x_0 = base point of X_c = span of e_3, e_4 in Grass $(4,2)$)

$$= \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} I & Z \\ 0 & I \end{pmatrix} \cdot x_0 = \text{span of columns of } \begin{pmatrix} AZ+B \\ CZ+D \end{pmatrix}$$

$$= \text{span of columns of } \begin{pmatrix} (AZ+B)(CZ+D)^{-1} \\ I \end{pmatrix} \quad \text{if } Z \in \overline{X}$$

$$= \exp\begin{pmatrix} 0 & (AZ+B)(CZ+D)^{-1} \\ 0 & 0 \end{pmatrix} \cdot x_0 \quad \text{if } Z \in \overline{X}$$

That is, $Z \mapsto (AZ+B)(CZ+D)^{-1}$ with the identification $\mathfrak{p}^+ = M_2(\mathbb{C})$.

Now F_1 corresponds to the set of matrices

$$\xi^{-1}\left(L_1 \cdot \exp\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cdot x_0 \right) = \xi^{-1}\left(\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & \alpha & \beta & 0 \\ 0 & \bar\beta & \bar\alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot x_0 \right)$$

$$= \left\{ \left(\begin{array}{c|c} 0 & \begin{matrix} 0 & 1 \\ \beta/\bar\alpha & 0 \end{matrix} \\ \hline 0 & 0 \end{array} \right) : |\alpha|^2 - |\beta|^2 = 1 \right\} \cong \left\{ \begin{pmatrix} 0 & 1 \\ z & 0 \end{pmatrix} : |z| < 1 \right\}$$

which is isomorphic to $\mathbb{D}$, a lower dimensional symmetric space. Let us check that P_1 normalises F_1. First N acts as

$$\begin{pmatrix} 1-c & \alpha & \beta & c \\ -\gamma & 1-d & d & \gamma \\ -\delta & -d & 1+d & \delta \\ -c & \alpha & \beta & 1+c \end{pmatrix} \cdot_{HC} \begin{pmatrix} 0 & 1 \\ z & 0 \end{pmatrix} = \left(\begin{pmatrix} 1-c & \alpha \\ -\gamma & 1-d \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ z & 0 \end{pmatrix} + \begin{pmatrix} \beta & c \\ d & \gamma \end{pmatrix} \right)$$

$$\times \left(\begin{pmatrix} -\delta & -d \\ -c & \alpha \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ z & 0 \end{pmatrix} + \begin{pmatrix} 1+d & \delta \\ \beta & 1+c \end{pmatrix} \right)^{-1}$$

$$= \begin{pmatrix} 0 & 1 \\ \frac{(1-d)z+d}{-dz+1+d} & 0 \end{pmatrix}$$ and $z \mapsto \dfrac{(1-d)z+d}{-dz+1+d}$ is the Möbius action of $\begin{pmatrix} 1-d & d \\ -d & 1+d \end{pmatrix} \in SU(1,1)$.

On the other hand, $Z(A_1)$ acts as

$$\begin{pmatrix} x & 0 & 0 & y \\ 0 & \alpha & \beta & 0 \\ 0 & \bar\beta & \bar\alpha & 0 \\ y & 0 & 0 & x \end{pmatrix} \cdot_{HC} \begin{pmatrix} 0 & 1 \\ z & 0 \end{pmatrix} = \left(\begin{pmatrix} x & 0 \\ 0 & \alpha \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ z & 0 \end{pmatrix} + \begin{pmatrix} 0 & y \\ \beta & 0 \end{pmatrix} \right) \times \left(\begin{pmatrix} 0 & \bar\beta \\ y & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ z & 0 \end{pmatrix} + \begin{pmatrix} \bar\alpha & 0 \\ 0 & x \end{pmatrix} \right)^{-1}$$

$$= \begin{pmatrix} 0 & 1 \\ \frac{\alpha z + \beta}{\bar\beta z + \bar\alpha} & 0 \end{pmatrix} \quad \text{and} \quad z \mapsto \frac{\alpha z+\beta}{\bar\beta z + \bar\alpha} \text{ belongs to } SU(1,1).$$

So indeed F_1 is normalised by P_1.

A much easier computation shows that $F_2 = \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$ is just a "point". Therefore the boundary of X in X_c is the union of K-orbits of F_1 and F_2, namely

$$K \cdot F_1 = \left\{ A \begin{pmatrix} 0 & 1 \\ z & 0 \end{pmatrix} D^{-1} : \begin{array}{l} |z| < 1 \text{ and} \\ A, D \in U(2) \text{ with } \det A \cdot \det D = 1 \end{array} \right\}$$

$$K \cdot F_2 = \left\{ A \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} D^{-1} : A, D \in U(2) \text{ with } \det A \cdot \det D = 1 \right\}$$

Recall that X has Harish-Chandra image given by

$$X = \left\{ Z = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{C}) \;\middle|\; I - Z^* Z \gg 0 \right\}$$

positive definite.

Hence

$$\bar X = \left\{ Z = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{C}) \;\middle|\; I - Z^* Z \geq 0 \right\} \text{ (positive semi-definite)}$$

The K-orbit $K \cdot F_1$ corresponds to the elements Z of the boundary such that the Hermitian matrix $I - Z^* Z$ has exactly 1 eigenvalue 1, while $K \cdot F_2$ covers the matrices Z such that $I - Z^* Z$ has both eigenvalues equal to 1. The set $\bar X$ itself is a "ball" of matrices $Z \in M_2(\mathbb{C})$ such that the maximum eigenvalue of $I - Z^* Z$ is at most 1.

Topological compactification of locally symmetric spaces

From now on, we let G denote a simple algebraic group over $\mathbb{Q}$ ~~where~~ such that $G(\mathbb{R})^+ = \mathrm{Hol}(X)^+$ for an irreducible Hermitian symmetric space of non-compact type X. Let $\Gamma \subset G(\mathbb{R})$ be a torsion-free arithmetic subgroup. Our goal is to compactify the locally symmetric space X/Γ topologically first, then later as an analytic space and projective variety.

Write

$$X^* = X \cup \left\{ \begin{array}{l} \text{points of boundary components } k.F_b \\ \text{whose normiliser } N_G(kF_b) \text{ is defined over } \mathbb{Q} \end{array} \right\}$$

Let P a minimal parabolic $\mathbb{Q}$-subgroup of G. If G has a maximal $\mathbb{Q}$-split torus S, we may write $P = M \cdot S \cdot U$ where

① U is the unipotent subgroup normalised by S whose Lie algebra is the sum of positive root spaces (for some choice of ordering of $\Delta_{\mathbb{Q}}(G, S)$, the relative root system);

② M is the unique normal $\mathbb{Q}$-subgroup s.t. $M \cap S$ is finite and $Z(S) = M \cdot S$ with M ~~also~~ anisotropic over $\mathbb{Q}$.

Def: A Siegel domain $\mathfrak{S}$ in $\mathcal{P}^+$ (notation as in the section of the H-C embedding) is a set of the form

$$\mathfrak{S} = \omega \cdot A_t \cdot x_0$$

with $\omega \subseteq (M \cdot U)_{\mathbb{R}}$ compact and

$$A_t \stackrel{\text{def}}{=} \left\{ a \in S(\mathbb{R})^+ \mid \gamma(a) \geq t \text{ for all } \underset{\gamma \text{ simple}}{\gamma \in \Delta_{\mathbb{Q}}(G, S)} \right\}$$

for some $t > 0$.

E.g. For $G = SL_2$ and $S = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \right\}$, we have $M = 1$ is trivial and $U = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \right\}$ for the positive root $\gamma \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = a^2$; we obtain the decomposition

$$P = \left\{ \begin{pmatrix} a & c \\ 0 & a^{-1} \end{pmatrix} \right\} = \underbrace{\left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \right\}}_{S} \cdot \underbrace{\left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \right\}}_{U}$$

Letting G act on $\mathbb{H}$ by Möbius transformations, and choosing $x_0 = i$ as base point, we have that a Siegel domain is of the form $(t, T > 0)$

$$\mathfrak{S} = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \cdot i \ : \ \begin{matrix} |b| \leq T \\ a^2 \geq t \end{matrix} \right\}$$

$$= \left\{ a^2 i + b \ : \ a \geq \sqrt{t}, \ |b| \leq T \right\} \subseteq \mathbb{H}$$

(in fact, we may replace $|b| \leq T$ by $b \in C$ for some compact set $C \subseteq \mathbb{R}$, but obviously any Siegel set will be contained in one of the above)

Thm (Godement) There exists a Siegel domain $\mathfrak{S}$ and a finite subset $C \subset G(\mathbb{Q})$ such that $\Omega = C \cdot \mathfrak{S}$ is a fundamental set for Γ in X, i.e., it verifies

(i) $\Gamma(\Omega) = X$, and

(ii) For any $g \in G(\mathbb{Q})$,

$$\{ \gamma \in \Gamma \mid g(\Omega) \cap \gamma(\Omega) \neq \emptyset \}$$

is finite.

In Baily-Borel, they choose Ω above so that $\overline{\Omega}$ (the closure of Ω in $\mathcal{D}^+$) is contained in the union of finitely many boundary components, and $\Gamma(\overline{\Omega}) = X^*$.

Next, they apply the following theorem, which essentially says that you may "translate" the neighbourhood of a boundary component to another, defining a new topology on X^* (for $\mathbb{H}$, you get the topology discussed in the Baby Baily Borel).

<u>Thm</u> (Satake) Consider the following topology on X^*: a system of neighbourhoods of $x \in X^*$ is given by all subsets $U \subset X$, $U \ni x$, such that

(i) $\gamma(x) = x \Rightarrow \gamma(U) = U \quad (\gamma \in \Gamma)$

(ii) $\gamma(x) \in \bar{\Omega} \Rightarrow \gamma(U) \cap \bar{\Omega}$ is a neighbourhood of $\gamma(x)$ in the natural topology of $\bar{\Omega}$.

Then the above topology is the unique one satisfying.

(a) it induces the natural topology on X, and on the closure of any fundamental set Ω' for any $\Gamma' \subset G$ arithmetic;

(b) G operates continuously on X^*;

(c) if $x \not\sim_\Gamma x'$, there exist neighbourhoods $U \ni x$, $U' \ni x'$ such that $\Gamma(U) \cap U' = \emptyset$;

(d) for each $x \in X^*$, there is a system of neighbourhoods of x such that $\gamma(U) = U$ if $\gamma(x) = x$, and $\gamma(U) \cap U = \emptyset$ otherwise.

Now we define

$$\overline{X/\Gamma}^{BB} = X^*/\Gamma$$

where X^* is given the Satake topology above, and X^*/Γ is given the quotient topology. Then $\overline{X/\Gamma}^{BB}$ will be a compact Hausdorff space (property (c) accounts for the Hausdorff property), and X/Γ will be open everywhere dense in X^*/Γ. Moreover, the boundary $\overline{X/\Gamma}^{BB} \setminus X/\Gamma$ will be the finite union of subspaces F_i/Γ_i where F_i is a Hermitian symmetric space of non-compact type, and Γ_i is an arithmetic subgroup of $\mathrm{Hol}(F_i)^+$.

Analytic Structure

To define the analytic structure, B-B prove a slightly more general result: let

V = compact Hausdorff space with countable basis

Suppose that

$$V = V_0 \sqcup V_1 \sqcup \cdots \sqcup V_m$$

where V_i = irreducible normal analytic space. (for instance, $V = \overline{X/\Gamma}^{BB}$, $V_0 = X/\Gamma$, V_i = boundary components for $i>0$).

Define a sheaf on V by

$$\mathcal{A}(U) = \left\{ f: U \to \mathbb{C} \;\middle|\; \begin{array}{l} f \text{ is continuous and } f|_{U\cap V_i} \text{ is} \\ \text{analytic for } i=0,1,\dots,m \end{array} \right\}$$

Thm (BB) Suppose:

1. For each $d \in \mathbb{N}$, the stratum
$$V_{(d)} \overset{\text{def}}{=} \bigcup_{\dim_{\mathbb{C}} V_i \le d} V_i$$
is closed. For $i>0$, $\dim_{\mathbb{C}} V_i < \dim_{\mathbb{C}} V_0$, and V_0 is open and dense in V;
2. Each $v \in V$ has a fundamental set of open neighbourhoods $\{U_j\}$ such that $U_j \cap V_0$ is connected for all j;
3. The restrictions to V_i of local sections of $\mathcal{A}$ define the structure sheaf of V_i;
4. Each $v \in V$ has a neighbourhood U_v whose points are separated by functions on $\mathcal{A}(U_v)$.

Then $(V, \mathcal{O})$ is an irreducible normal analytic space, and for each $d \leq \dim V$, $V_{(d)}$ is an analytic subspace of V.

In order to apply the theorem, the difficult part is to show that condition (4) holds. This is done by explicitly constructions; more precisely, B-B consider sections of a locally free rank 1 $\mathcal{O}$-module extending the m-th tensor power $\omega_{X/\Gamma}^{\otimes m}$ of the canonical sheaf on X/Γ for $m \gg 0$.

These sections (which they call Poincaré - Eisenstein series) are all of the form (automorphic forms of weight m):

$$w \longmapsto P_{f,m}(w) \stackrel{\text{def}}{=} \sum_{\gamma \in \Gamma} \mu_{X^b}(\gamma, w)^m f(\gamma(w))$$

where X^b = Harish-Chandra realisation of X

f = polynomial in X^b

$\mu_{X^b}(g, w)$ = determinant of the Jacobian, at w, of the translation action by $g \in G$, acting on X^b.

Quotients of PE-series of same weight m give the desired functions, for they show

<u>Thm</u> (BB) For any Γ-inequivalent points $a_1, \ldots, a_n \in X^b$ and $(b_1, \ldots, b_n) \in \mathbb{C}^n$, there exist $m \gg 0$ and a polynomial f in X^b such that $P_{f,m}(a_i) = b_i$ for $i = 1, \ldots, n$.

Finally, they show $\overline{X/\Gamma}^{BB}$, now a compact normal irreducible analytic space, is also a projective variety over $\mathbb{C}$, by

Constructing an embedding

$$\overline{X/\Gamma}^{BB} \xhookrightarrow{(E_0 : E_1 : \cdots : E_N)} \mathbb{P}^N$$

with E_i P.E series as above (they need to show the analytic structure first since they use normality and separation ~~of~~ of points — but not tangents — to prove that the above map is an embedding). This completes the BB-compactification.

We still mention $\quad (\mathbb{D}^\times = \mathbb{D} \setminus \{0\})$

<u>Thm</u> (Borel, Kwack, Kobayashi) Let $f : (\mathbb{D}^\times)^a \times \mathbb{D}^b \longrightarrow X/\Gamma$ be any holomorphic map. Then f extends to a holomorphic map $f : \mathbb{D}^{a+b} \longrightarrow \overline{X/\Gamma}^{BB}$.

Using the theorem, it is easy to prove the uniqueness of the algebraic structure of X/Γ (as a quasi-projective variety): if $Y = X/\Gamma$ with algebraic structure given by the B-B embedding into $\overline{X/\Gamma}^{BB}$, and Y' is any other algebraic quasi-projective variety with $(Y')^{an} = X/\Gamma$, then by Hironaka we may embed $Y' \hookrightarrow \overline{Y'}$ with $\overline{Y'}$ projective and such that $\overline{Y'} \setminus Y'$ has normal crossings, i.e., ~~...~~ each $y \in \overline{Y'}^{an}$ has a complex neighbourhood N s.t. $N \cap Y'^{an} = (\mathbb{D}^\times)^a \times \mathbb{D}^b$. By the theorem, we may extend $f : (Y')^{an} \xrightarrow{\approx} Y^{an}$ to a holomorphic map $\tilde{f} : \overline{Y'}^{an} \longrightarrow \overline{Y}^{an} = \overline{X/\Gamma}^{BB}$, and now it follows by Chow/GAGA that there is an algebraic ~~...~~ ~~...~~ $g : \overline{Y'} \longrightarrow \overline{X/\Gamma}^{BB}$ restricting to an algebraic isomorphism $Y' \xrightarrow{\approx} X/\Gamma = Y$.