ALGEBRAIC CONTINUED FRACTIONS IN $\mathbb{F}_q((T^{-1}))$ AND RECURRENT SEQUENCES IN $\mathbb{F}_q$

Alain Lasjaunias [1]
C.N.R.S.-UMR 5251, Université Bordeaux I, Talence 33405, FRANCE

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Abstract. There exists a particular subset of algebraic power series over a finite field which, for different reasons, can be compared to the subset of quadratic real numbers. The continued fraction expansion for these elements, called hyperquadratic, can sometimes be fully explicited. In this work, which is a continuation of [L1] and [L2], we describe this expansion for a wide family of hyperquadratic power series in odd characteristic. This leads to consider interesting recurrent sequences in the finite base field when it is not a prime field.

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1. Introduction

Formal power series over a finite field are analogues of real numbers. Like quadratic real numbers, for which the continued fraction expansion is well known, certain algebraic power series have a continued fraction expansion which can be explicitly described. Most of these power series belong to a particular subset of algebraic elements related to the existence of the Frobenius isomorphism in these power series fields. The reader may consult [BL] for further information on these elements called hyperquadratic. In a recent work [L1] we have introduced a family of hyperquadratic elements having a continued fraction expansion with a regular pattern. This expansion is linked to particular sequences in a finite field. Here we complete the study of these sequences. It is also worth mentioning that, in an unexpected way, the present work sheds a new light on an older one [LR].

We are concerned with power series over a finite field $\mathbb{F}_q$ of odd characteristic p. Given a formal indeterminate T, we consider the ring of

[1] Alain.Lasjaunias@math.u-bordeaux1.fr

polynomials $\mathbb{F}_q[T]$ and the field of rational functions $\mathbb{F}_q(T)$. Then if $|T|$ is a fixed real number greater than one, we introduce the ultrametric absolute value defined on the field $\mathbb{F}_q(T)$ by $|P/Q| = |T|^{\deg(P)-\deg(Q)}$. The completion of this field for this absolute value is the field of power series in $1/T$ over $\mathbb{F}_q$, which is often denoted by $\mathbb{F}_q((T^{-1}))$ or here simply by $\mathbb{F}(q)$. If $\alpha \in \mathbb{F}(q)$ and $\alpha \neq 0$, we have

$$\alpha = \sum_{k \leq k_0} u_k T^k, \qquad \text{where } k_0 \in \mathbb{Z}, u_k \in \mathbb{F}_q, u_{k_0} \neq 0 \quad \text{and} \quad |\alpha| = |T|^{k_0}.$$

We know that each irrational element α of $\mathbb{F}(q)$ can be expanded as an infinite continued fraction. This will be denoted by $\alpha = [a_1, \dots, a_n, \dots]$, where the $a_i \in \mathbb{F}_q[T]$ are non constant polynomials (except possibly for the first one) and are called the partial quotients of α. As usual the tail of the expansion, $[a_n, a_{n+1}, \dots]$, called the complete quotient, is denoted by α_n where $\alpha_1 = \alpha$. The numerator and the denominator of the truncated expansion $[a_1, a_2, \dots, a_n]$, which is called a convergent, are denoted by x_n and y_n. These polynomials, called continuants, are both defined by the same recursive relation :$K_n = a_n K_{n-1} + K_{n-2}$ for $n \geq 2$, with the initial conditions $x_0 = 1$ and $x_1 = a_1$ for the the sequence of numerators, while the initial conditions are $y_0 = 0$ and $y_1 = 1$ for the sequence of denominators. For a general account on continued fractions in power series fields and also for numerous references the reader may consult W. Schmidt's article [S].

In this note we consider continued fraction expansions for algebraic power series over a finite field. We recall that the first works in this area are due to L. Baum and M. Sweet [BS] and later to W. Mills and D. Robbins [MR]. The problem discussed here has been introduced in [L1]. In the next section we present the background of this problem and we state a technical lemma to go from characteristic zero to positive characteristic. In the third section, with Proposition A, we define a large class of algebraic continued fractions in the fields $\mathbb{F}(q)$. In the fourth section we state the main result, Theorem B, which gives an explicit description of these continued fractions under certain conditions. We also present an illustration in the field of power series over $\mathbb{F}_{27}$ which is Corollary B. At the end of this section we state a conjecture concerning a family of irreducible polynomials over $\mathbb{F}_p$. The last section is dedicated to the proof of Theorem B and its corollary.

2. A special pair of polynomials

For each integer $k \geq 1$, we consider the following pair of polynomials in $\mathbb{Q}[T]$:

$$P_k(T) = (T^2-1)^k \quad \text{and} \quad Q_k(T) = \int_0^T (x^2-1)^{k-1}dx.$$

We have the following finite continued fraction expansions in $\mathbb{Q}(T)$:

$$P_1(T)/Q_1(T) = [T, -T], \quad P_2(T)/Q_2(T) = [3T, T/3, -3T/4, -4T/3]$$

and more generally

$$P_k/Q_k = [v_{1,k}T, \ldots, v_{i,k}T, \ldots, v_{2k,k}T], \tag{1}$$

where the rational numbers $v_{i,k}$ for $k \geq 1$ and $1 \leq i \leq 2k$ are defined by $v_{1,k} = 2k-1$ and recursively, for $1 \leq i \leq 2k-1$, by

$$v_{i+1,k}v_{i,k} = (2k-2i-1)(2k-2i+1)(i(2k-i))^{-1}. \tag{2}$$

This continued fraction expansion for P_k/Q_k has been established in [L1]. Moreover we consider the rational numbers

$$\theta_k = (-1)^k 2^{-2k}\binom{2k}{k} \quad \text{and} \quad \omega_k = -(2k\theta_k)^{-2} \quad \text{for} \quad k \geq 1. \tag{3}$$

These rational numbers were introduced in [L1] in connection with the pair (P_k, Q_k). Indeed we have $Q_k(1) = -(2k\theta_k)^{-1}$ and also

$$v_{2k+1-i,k} = v_{i,k}\omega_k^{(-1)^{i+1}}. \tag{4}$$

We recall that throughout this note p is an odd prime number. Our aim is to obtain, by reducing the identity (1) modulo p, a similar identity in $\mathbb{F}_p(T)$. Clearly the integer k must be well chosen. The easiest way to do so is to assume that $2k < p$ and this is what we did in [L1]. Here we shall extend this to other values of k. We set $r = p^t$ where t is a positive integer. Then we introduce the subset $E(r)$ of integers k such that

$$k = mp^l + (p^l-1)/2 \quad \text{for} \quad 1 \leq m \leq (p-1)/2 \quad \text{and} \quad 0 \leq l \leq t-1. \tag{5}$$

For instance $E(3) = \{1\}$, $E(5) = \{1, 2\}$ and $E(25) = \{1, 2, 7, 12\}$. Note that we have $E(r) \subset \{1, \ldots, (r-1)/2\}$ with equality if $r = p$. Also $(r-1)/2 \in E(r)$ in all cases. We have the following result where $\mathbf{v}_p(x)$ is used to denote the p-adic valuation of a rational number x.

Lemma 1. *Let p and r be as above. Let k be a positive integer with $k \in E(r)$.*
1) For $1 \leq i \leq 2k-1$ we have $\mathbf{v}_p(i) = \mathbf{v}_p(2k-2i+1)$ and $\mathbf{v}_p(2k-i) =$

$\mathbf{v}_p(2k-2i-1)$. *For* $1 \le i \le 2k$ *we have* $\mathbf{v}_p(v_{i,k}) = 0$. *Consequently,in the sequel, for* $1 \le i \le 2k$, $v_{i,k}$ *and for* $1 \le i \le 2k-1$, $i/(2k-2i+1)$ *and* $(2k-i)/(2k-2i-1)$ *will be considered as elements of* $\mathbb{F}_p^*$.
2) For $0 \le i \le 2k$ *we have* $\mathbf{v}_p(\binom{2k}{i}) = 0$. *Consequently* θ_k, $2k\theta_k$ *and* ω_k *as well as* $\binom{2k}{i}$ *for* $0 \le i \le 2k$ *will be considered in the sequel as elements of* $\mathbb{F}_p^*$.
3) We can define in $\mathbb{F}_p[T]$ *the pair of polynomials*

$$P_k(T) = (T^2-1)^k \quad \text{and} \quad Q_k(T) = \sum_{0 \le i \le k-1} b_i T^{2i+1}$$

where $b_i = (-1)^{k-1-i}\binom{k-1}{i}(2i+1)^{-1} \in \mathbb{F}_p$. *The identity (1) holds in* $\mathbb{F}_p(T)$ *for the rational function* P_k/Q_k, *with the* $v_{i,k}$ *defined in* $\mathbb{F}_p^*$ *as above.*
Proof : Let $k \in E(r)$. According to (5) we have $2k+1 = (2m+1)p^l$ with $3 \le 2m+1 \le p$ and $0 \le l \le t-1$. For $1 \le i \le 2k-1$ we have $i < p^t$ and therefore $\mathbf{v}_p(i) \le \mathbf{v}_p(2k+1)$. This implies clearly that $\mathbf{v}_p(i) = \mathbf{v}_p(2k-2i+1)$. We also have $2k-2i-1 = 2(2k-i)-(2k+1)$ and consequently, changing i into $2k-i$, the same arguments show that $\mathbf{v}_p(2k-i) = \mathbf{v}_p(2k-2i-1)$. By (2), it follows that $\mathbf{v}_p(v_{i,k}v_{i+1,k}) = 0$ for $1 \le i \le 2k-1$. Since $\mathbf{v}_p(v_{1,k}) = \mathbf{v}_p(2k-1) = 0$ we have $\mathbf{v}_p(v_{i,k}) = 0$ for $1 \le i \le 2k$. So we have proved the first point. For the second one we use a classical formula on the p-adic valuation of $n!$. Indeed for an integer $n \ge 1$ and a prime p we have $\mathbf{v}_p(n!) = (n - s_p(n))/(p-1)$, where $s_p(n)$ denotes the sum of the digits of n when it is written in basis p. Since $k \in E(r)$ we can write $2k = 2mp^l + (p-1)(p^{l-1} + \cdots + 1)$. For $0 \le i \le 2k$, this writing implies the equality $s_p(2k-i) + s_p(i) = s_p(2k)$. Consequently we have $\mathbf{v}_p((2k-i)!) + \mathbf{v}_p(i!) = \mathbf{v}_p((2k)!)$ and therefore $\mathbf{v}_p(\binom{2k}{i}) = 0$ for $0 \le i \le 2k$. Now we prove the last point. According to (1), by a trivial integration, we can write in $\mathbb{Q}(T)$

$$\sum_{0 \le i \le k-1} b_i T^{2i+1} = (T^2-1)^k [0, v_{1,k}T, \ldots, v_{2k,k}T]$$

Since $k \in E(r)$, the right hand side of this equality can be reduced modulo p in $\mathbb{F}_p(T)$. Thus the left hand side is well defined by reduction modulo p, i.e. $\mathbf{v}_p(b_i) \ge 0$ for $0 \le i \le k-1$. Consequently the pair (P_k, Q_k) is well defined in $(\mathbb{F}_p[T])^2$ and we have the desired continued fraction expansion for the rational function $P_k(T)/Q_k(T)$ in $\mathbb{F}_p(T)$. This completes the proof of the lemma.

Given r and $k \in E(r)$, we need now to introduce a pair of finite sequences $(g_i)_{0 \le i \le 2k}$ and $(h_i)_{0 \le i \le 2k}$ of functions in $\mathbb{F}_p(X)$ which will be used further on. We set

$$
(G) \qquad \begin{cases} g_0(X) = \theta_k + X, \qquad g_{2k}(X) = 1/(\theta_k - X) \\ and \quad for \quad 1 \leq i \leq 2k-1 \\ g_i(X) = 2k\theta_k v_{i,k}(i/(2k-2i+1))\dfrac{\theta_k + w_{i,k}X}{\theta_k + w_{i-1,k}X} \end{cases}
$$

also

$$
(H) \qquad \begin{cases} h_0(X) = X/(\theta_k + X), \qquad h_{2k}(X) = X/(\theta_k - X) \\ and \quad for \quad 1 \leq i \leq 2k-1 \\ h_i(X) = (-1)^i \dbinom{2k}{i} \dfrac{\theta_k X}{(\theta_k + w_{i,k}X)(\theta_k + w_{i-1,k}X)} \end{cases}
$$

where

$$
w_{i,k} = (-1)^i \binom{2k-1}{i} \in \mathbb{F}_p \quad \text{for} \quad 0 \leq i \leq 2k-1.
$$

Due to Lemma 1, the functions defined above are not zero. Moreover we may have $w_{i,k} = 0$ for some i but $w_{i,k} - w_{i-1,k} = (-1)^i \binom{2k}{i} \neq 0$ for $1 \leq i \leq 2k-1$.

3. Continued fractions of type (r, l, k) in $\mathbb{F}(q)$

In [L1] a process to generate in $\mathbb{F}(q)$ algebraic continued fractions from certain polynomials in $\mathbb{F}_q[T]$ is presented. The following proposition is a particular case of a more general theorem (see [L1] Theorem 1, p. 332-333).

Proposition A. *Let p be an odd prime number. Set $q = p^s$ and $r = p^t$ with $s, t \geq 1$. Let k be an integer with $k \in E(r)$. Let $(P_k, Q_k) \in (\mathbb{F}_p[T])^2$ be defined as in Lemma 1. Let $l \geq 1$ be an integer. Let $(\lambda_1, \lambda_2, \ldots, \lambda_l)$ be a l-tuple in $(\mathbb{F}_q^*)^l$. Let $(\epsilon_1, \epsilon_2) \in (\mathbb{F}_q^*)^2$. There exists a unique infinite continued fraction $\alpha = [\lambda_1 T, \ldots, \lambda_l T, \alpha_{l+1}] \in \mathbb{F}(q)$ defined by*

$$
\alpha^r = \epsilon_1 P_k \alpha_{l+1} + \epsilon_2 Q_k.
$$

This element α is the unique root in $\mathbb{F}(q)$ with $|\alpha| \geq |T|$ of the algebraic equation

$$
y_l X^{r+1} - x_l X^r + (\epsilon_1 P_k y_{l-1} - \epsilon_2 Q_k y_l)X - \epsilon_1 P_k x_{l-1} + \epsilon_2 Q_k x_l = 0
$$

where x_l,x_{l-1},y_l and y_{l-1} are the continuants defined in the introduction.

Note that these continued fractions satisfy an algebraic equation of a particular type. The reader may consult the introduction of [BL] for a presentation of these particular algebraic power series which are called hyperquadratic. A continued fraction defined as in Proposition A is generated by the pair (P_k, Q_k) for $k \in E(r)$. Such a continued fraction will be called an expansion of type (r, l, k). When the pair (P_k, Q_k) is fixed, this expansion depends on the l-tuple $(\lambda_1, \lambda_2, \ldots, \lambda_l)$ in $(\mathbb{F}_q^*)^l$ and on the pair (ϵ_1, ϵ_2) in $(\mathbb{F}_q^*)^2$. When these $l+2$ elements in $\mathbb{F}_q^*$ are taken arbitrarily then the expansion has a regular pattern only up to a certain point (see [L1] Proposition 4.6, p. 347). In the next section we are concerned with a particular subfamily of these continued fractions.

4. Perfect continued fractions of type (r, l, k) in $\mathbb{F}(q)$

In previous works we have seen that an expansion of type (r, l, k), under certain conditions on $(\lambda_1, \lambda_2, \ldots, \lambda_l)$ and (ϵ_1, ϵ_2), may be given explicitly. A first example was given in [L1] Theorem 3. In [L2] a more general case was treated, but there we restricted ourselves to the case of a prime base field $\mathbb{F}_p$ and we also only considered the case $r = p$. Note that in this way we could prove the conjecture for the expansion of a quartic power series over $\mathbb{F}_{13}$ made by Mills and Robbins in [MR] p. 403. Here our aim is to describe explicitely many expansions of type (r, l, k) having a very regular pattern as Mills and Robbins' example does. To do so we need first to introduce further notations. Given $l \geq 1$ and $k \geq 1$, we define the sequence of integers $(f(n))_{n\geq 1}$ where $f(n) = (2k+1)n + l - 2k$. We also define the sequence of integers $(i(n))_{n\geq 1}$ in the following way :

$$i(n) = 1 \quad \text{if} \quad n \notin f(\mathbb{N}^*) \quad \text{and} \quad i(f(n)) = i(n) + 1.$$

Finally we introduce the sequence $(A_i)_{i\geq 1}$ of polynomials in $\mathbb{F}_p[T]$ defined recursively by

$$A_1 = T \quad \text{and} \quad A_{i+1} = [A_i^r / P_k] \quad \text{for} \quad i \geq 1$$

(here the square brackets denote the integer part, i.e. the polynomial part). Note that the sequence $(A_i)_{i\geq 1}$ depends on the polynomial P_k chosen with $k \in E(r)$. It is remarkable that if $2k = r - 1$ then this sequence of polynomials is constant and we have $A_i = T$ for $i \geq 1$.

For an arbitrary continued fraction of type (r, l, k) the sequence of partial quotients is based on the above sequence $(A_i)_{i\geq 1}$ but only up to a certain rank (see the remark after Lemma 5.1 below). Nevertheless it may happen that this sequence of partial quotients is entirely described by means of this sequence $(A_i)_{i\geq 1}$. The aim of the following theorem is to give this description as well as the conditions of its existence. These particular expansions of type (r, l, k), which are defined in this theorem, will be called perfect (This term was introduced in [L1], p 348).

Theorem B. *Let p be an odd prime and $q = p^s$, $r = p^t$ with $s, t \geq 1$ be given. Let $k \in E(r)$. Let $(A_i)_{i\geq 1}$ in $\mathbb{F}_p[T]$, $(f(n))_{n\geq 1}$ and $(i(n))_{n\geq 1}$ in $\mathbb{N}^*$ be the sequences defined above. Let $\alpha \in \mathbb{F}(q)$ be a continued fraction of type (r, l, k) defined by the l-tuple $(\lambda_1, \ldots, \lambda_l)$ in $(\mathbb{F}_q^*)^l$ and by the pair (ϵ_1, ϵ_2) in $(\mathbb{F}_q^*)^2$. Then the partial quotients of this expansion satisfy*

$$(I) \quad a_n = \lambda_n A_{i(n)} \quad \textit{where} \quad \lambda_n \in \mathbb{F}_q^* \quad \textit{for} \quad n \geq 1$$

if and only if we can define in $\mathbb{F}_q^*$

$$(II) \qquad \delta_n = 2k\theta_k[\lambda_n^r, \ldots, \lambda_1^r, 2k\theta_k\epsilon_2^{-1}] \quad \textit{for} \quad 1 \leq n \leq l$$

and we have (III)

$$\textit{either case} \quad (III_1): \qquad \delta_l = 4k^2\theta_k(\epsilon_1/\epsilon_2)^r$$

$$\textit{or case} \quad (III_2): \qquad \delta_l \neq 4k^2\theta_k(\epsilon_1/\epsilon_2)^r$$

and there exists in $\mathbb{F}_q^*$ a sequence $(\gamma_n)_{n\geq 1}$ defined recursively by

$$(\Gamma) \qquad \begin{cases} \gamma_1^r = (4k^2\theta_k\epsilon_1^r\delta_l^{-1} - \epsilon_2^r)\theta_k\delta_1^{-r} \\ \gamma_n = \gamma_{n-1}(\delta_n\delta_{n-1}\omega_k)^{-1} \quad \textit{for} \quad 2 \leq n \leq l \\ \gamma_{f(n)+i} = C_0 h_i(\gamma_n^r) \quad \textit{for} \quad 0 \leq i \leq 2k \quad \textit{and} \quad n \geq 1 \end{cases}$$

where

$$C_0 = \gamma_l\epsilon_1^r(\delta_1\gamma_1)^{-r}(\delta_l\omega_k)^{-1} \in \mathbb{F}_q^*.$$

If (II) and (III) hold then we can define recursively a sequence $(\delta_n)_{n\geq 1}$ in $\mathbb{F}_q^*$ by the initial values $\delta_1, \ldots, \delta_l$ given by (II) and the formulas

$$\delta_{f(n)+i} = \epsilon_1^{r(-1)^{n+i}}\delta_n^{r(-1)^i}g_{i,n} \quad \textit{for} \quad n \geq 1 \quad \textit{and for} \quad 0 \leq i \leq 2k, \qquad (D)$$

where $g_{i,n} = g_i(0)$ in case (III_1) and $g_{i,n} = g_i(\gamma_n^r)$ in case (III_2). Then the sequence $(\lambda_n)_{n\geq 1}$ in $\mathbb{F}_q^*$, introduced in (I), is defined recursively by the first values $\lambda_1, \lambda_2, \ldots, \lambda_l$ and the formulas

$$\lambda_{f(n)} = \epsilon_1^{(-1)^n}\lambda_n^r, \quad \lambda_{f(n)+i} = -v_{i,k}\epsilon_1^{(-1)^{n+i}}\delta_n^{(-1)^i} \qquad (LD)$$

for $n \geq 1$ and for $1 \leq i \leq 2k$.

In this theorem we have two conditions (II) and (III) which are not at the same level. Condition (II) is primary and clearly necessary to define recursively the sequence $(\delta_n)_{n\geq 1}$ in $\mathbb{F}_q^*$ by (D). This condition has already been pointed out in [L2] even though there we had only considered the simplest case where the base field is prime, that is $q = p$. Here it is necessary to underline that there is a mistake in the formula given there for δ_n when $1 \leq n \leq l$. Indeed in [L2] Theorem 1 condition (H_1), instead of $\delta_i = [2k\theta_k\lambda_i, \ldots, 2k\theta_k\lambda_1, \epsilon_2^{-1}]$ one should read $\delta_i = 2k\theta_k[\lambda_i, \ldots, \lambda_1, 2k\theta_k\epsilon_2^{-1}]$. Note that this last formula is in agreement with (II) in Theorem B, if the base field is prime and consequently the Frobenius isomorphism is reduced to the identity in $\mathbb{F}_p$. We must also add that this mistake has no consequence on Theorem 2 of [L2] because there the values of δ_i were computed with the right formula. Condition (III) is of a different kind. It is split into two distinct cases. In fact case (III_1) has also already been considered in [L2] Theorem 1 condition (H_2), again when the base field is prime.

We want now to discuss about case (III_2). This one is more complex because of a possible obstruction in the recursive definition of the sequence $(\gamma_n)_{n\geq 1}$ in $\mathbb{F}_q^*$. Actually it is conjectured that this second case can only happen if the base field $\mathbb{F}_q$ is a particular algebraic extension of the prime field $\mathbb{F}_p$. Indeed the sequence $(\gamma_n)_{n\geq 1}$ is clearly well defined if γ_n does not belong to $\mathbb{F}_p$ for all n. There is a sufficient condition to obtain that. We recall that the functions h_i for $0 \leq i \leq 2k$ involved in the recursive definition of the sequence $(\gamma_n)_{n\geq 1}$ are of two types : $h(x) = ax/(x+u)$ or $h'(x) = ax/(x+u_1)(x+u_2)$ where $a \in \mathbb{F}_p^*$ and $u, u_1, u_2 \in \mathbb{F}_p$. Consequently if x is an algebraic element over $\mathbb{F}_p$ of degree $d > 2$ then $h_i(x)$ has degree d (for all h_i of type h) or $d/2$ (eventually for some h_i of type h'). This remark implies that if the first l terms of the sequence γ_n have each one a degree over $\mathbb{F}_p$ different from a power of two and the constant C_0 has a degree over $\mathbb{F}_p$ which is a power of two, then by induction the degrees over $\mathbb{F}_p$ of all the terms remain greater than one and thus none of these terms belongs to $\mathbb{F}_p$. This sufficient condition for the existence of the sequence $(\gamma_n)_{n\geq 1}$ may also be necessary but this remains a conjecture. Observe that if this conjecture is true and if the base field is prime then the continued fraction can only be perfect in case (III_1). We will make a more precise conjecture in that direction at the end of this section. Before going further on, we need to point out the similarity with the problem discussed in [LR], particularly on pages 562-565. In this older work we had investigated the existence of algebraic continued fractions having linear partial quotients, and this matches the case $2k = r - 1$ in the present work. The approach in [LR] was singular and completely different from here, this forced us to make the restriction $l \geq r$.

Now we want to illustrate the occurrence of case (III_2) if the base field is $\mathbb{F}_q$ where $q = p^m$ and m is not a power of two. We take $p = 3$, $q = 27$ and $r = 3$, with $l = 1$ and $k = 1$. Since $2k = r - 1$, if the expansion is perfect, then all partial quotients are linear. The elements of the finite field $\mathbb{F}_{27}$ will be represented by means of a root u of the irreducible polynomial over $\mathbb{F}_3$: $P(X) = X^3 + X^2 - X + 1$. Then we have $u^{13} = -1$ and

$$\mathbb{F}_{27} = \{0, \pm u^i, \qquad 0 \leq i \leq 12\}.$$

We have the following corollary.

Corollary C. *Define the sequences* $(\gamma_n)_{n\geq 1}$ *and* $(\delta_n)_{n\geq 1}$ *in* $\mathbb{F}_{27}^*$ *as follows. The first is defined recursively by* $\gamma_1 = u$ *and*

$$\gamma_{3n-1} = \frac{\gamma_n^3}{1+\gamma_n^3}, \quad \gamma_{3n} = \frac{\gamma_n^3}{1-\gamma_n^6}, \quad \gamma_{3n+1} = \frac{\gamma_n^3}{1-\gamma_n^3} \quad \textit{for} \quad n \geq 1.$$

The second one, based upon the first one, is defined recursively by $\delta_1 = u^4$ *and*

$$\delta_{3n-1} = u^{5(-1)^n}\delta_n^3(1+\gamma_n^3), \quad \delta_{3n} = \frac{\gamma_n^3 - 1}{\delta_{3n-1}}, \quad \delta_{3n+1} = \frac{\delta_{3n-1}}{1-\gamma_n^6} \quad \textit{for} \quad n \geq 1.$$

We consider the following algebraic equation with coefficients in $\mathbb{F}_{27}[T]$

$$(E) \qquad X^4 - TX^3 - u^3TX + uT^2 - u^6 = 0.$$

This equation has a unique root α *in* $\mathbb{F}(27)$ *which can be expanded as an infinite continued fraction*

$$\alpha = [T, u^7T, u^2T, u^{11}T, -uT, \dots] = [\lambda_1 T, \dots, \lambda_n T, \dots],$$

where the sequence $(\lambda_n)_{n\geq 1}$ *in* $\mathbb{F}_{27}^*$ *is defined recursively by* $\lambda_1 = 1$ *and*

$$\lambda_{3n-1} = -u^{6(-1)^n}\lambda_n^3, \quad \lambda_{3n} = u^{6(-1)^{n+1}}\delta_n^{-1}, \quad \lambda_{3n+1} = -\lambda_{3n}^{-1} \quad \textit{for} \quad n \geq 1.$$

Before concluding this section, we make a conjecture in connection with the sequence $(\gamma_n)_{n\geq 1}$ described in Theorem B. For the sake of shortness we take $k = 1$. Let p be an odd prime. Let us consider the three elements of $\mathbb{F}_p(x)$

$$h_0(x) = \frac{2x}{2x-1}, \quad h_1(x) = \frac{4x}{1-4x^2} \quad \text{and} \quad h_2(x) = \frac{-2x}{2x+1}.$$

We define recursively a sequence $(u_n)_{n\geq 1}$ of rational functions in $\mathbb{F}_p(x)$ by

$$u_1(x) = x \quad u_{3n+i-1}(x) = h_i(u_n(x)) \quad \text{for} \quad 0 \leq i \leq 2 \quad \text{and} \quad n \geq 1.$$

Let $\mathcal{P}(p) \subset \mathbb{F}_p[x]$ be the subset of all monic polynomials irreducible over $\mathbb{F}_p$ which appear as a prime factor of the numerator or denominator of $u_n(x)$

for all $n \geq 1$. Let $\mathcal{P}_2(p) \subset \mathbb{F}_p[x]$ be the subset of all monic polynomials irreducible over $\mathbb{F}_p$ of degree 2^k for $k \geq 0$. Then the arguments developed after Theorem B show that we have $\mathcal{P}(p) \subset \mathcal{P}_2(p)$. We conjecture that $\mathcal{P}(p) = \mathcal{P}_2(p)$ holds for all odd primes p.

5. Proofs of Theorem B and Corollary C

In this section p, q and r are given as above. We consider an integer k with $k \in E(r)$ and an integer $l \geq 1$. Moreover the numbers $\theta_k, \omega_k \in \mathbb{F}_p^*$ and the natural integers $f(n), i(n)$ for $n \geq 1$ are defined as above. The proof of Theorem B will be divided into several steps.

Lemma 5.1. *Let $\alpha = [\lambda_1 T, \ldots, \lambda_l T, \alpha_{l+1}] \in \mathbb{F}(q)$ be a continued fraction of type (r, l, k) for the pair $(\epsilon_1, \epsilon_2) \in (\mathbb{F}_q^*)^2$. Then there exists a sequence $(\lambda_n)_{n\geq 1}$ in $\mathbb{F}_q^*$ such that we have*

$$(I) \quad a_n = \lambda_n A_{i(n)} \quad \text{for} \quad n \geq 1$$

if and only if there exists a sequence $(\delta_n)_{n\geq 0}$ in $\mathbb{F}_q^*$ such that we have

$$\lambda_{f(n)} = \epsilon_1^{(-1)^n} \lambda_n^r, \quad \lambda_{f(n)+i} = -v_{i,k} \epsilon_1^{(-1)^{n+i}} \delta_n^{(-1)^i} \qquad (LD)$$

for $1 \leq i \leq 2k$ and $n \geq 1$, with

$$\delta_n = 2k\theta_k^{i(n)} \lambda_n^r - (\omega_k \delta_{n-1})^{-1} \quad \text{for} \quad n \geq 1, \qquad (D_1)$$

where $\delta_0 = -(\omega_k \epsilon_2)^{-1}$.

This lemma, which is the first and the main step in the proof of Theorem B, is a direct consequence of [L1] Prop 4.6, p 347. There we proved that an expansion of type (r, l, k) for an arbitrary pair $(\epsilon_1, \epsilon_2) \in (\mathbb{F}_q^*)^2$ has the pattern given by (I), where the sequence $(\lambda_n)_{n\geq 1}$ is described by (D_1) and (LD), but only up to a certain rank (if δ_n ever vanishes in (D_1)). Remark that in the proof of Proposition 4.6 we made the restriction $2k < p$. This condition was sufficient to have in $\mathbb{F}_p(T)$ the identity (1) of Section 2 which is the fundament of the proof. But, according to Lemma 1 of Section 2, we may replace this condition by $k \in E(r)$ and this has no consequences for the proof of the proposition. Now to separate the sequence $(\delta_n)_{n\geq 0}$ from the sequence $(\lambda_n)_{n\geq 1}$, we have the following lemma.

Lemma 5.2. *Let $(\lambda_n)_{n\geq 1}$ and $(\delta_n)_{n\geq 0}$ be two sequences in $\mathbb{F}_q^*$. We assume that they satisfy (LD). Then these sequences satisfy (D_1) if and only if*

$$(II_0) \qquad \delta_n = 2k\theta_k[\lambda_n^r, \ldots, \lambda_1^r, \delta_0/(2k\theta_k)] \quad \text{for} \quad 1 \leq n \leq l,$$

$$\delta_{f(n)} + (\omega_k \delta_{f(n)-1})^{-1} = \theta_k \epsilon_1^{r(-1)^n} (\delta_n^r + (\omega_k \delta_{n-1})^{-r}) \quad for \quad n \geq 1 \qquad (D_2)$$

and

$$\delta_{f(n)+i} + (\omega_k \delta_{f(n)+i-1})^{-1} = -2k\theta_k v_{i,k} \epsilon_1^{r(-1)^{n+i}} \delta_n^{r(-1)^i} \qquad (D_3)$$

for $1 \leq i \leq 2k$ *and* $n \geq 1$.

Proof: First we assume that (D_1) holds for $n \geq 1$. For $1 \leq n \leq l$ we have $i(n) = 1$. By (3) from Section 2, we have $\omega_k = -(2k\theta_k)^{-2}$, consequently (D_1) for $1 \leq n \leq l$ can be written as

$$\delta_n = 2k\theta_k \lambda_n^r + (2k\theta_k)^2 \delta_{n-1}^{-1}.$$

By induction, it is clear that (D_1) for $1 \leq n \leq l$ is equivalent to (II_0). Now for $1 \leq j \leq 2k$ and for $n \geq 1$ we have $i(f(n)+j) = 1$ consequently, taking into account (LD), we have the equivalence between (D_3) and (D_1) at the rank $f(n)+j$. Observing that for $n \geq 1$ we have $i(f(n)) = i(n)+1$ and taking into account (LD), we see that (D_1) at the rank $f(n)$ and n implies (D_2). Reciprocally suppose (II_0), (D_2) and (D_3) are satisfied. Then (D_1) holds for $1 \leq n \leq l$ and also at the rank $f(n)+i$, for $n \geq 1$ and $1 \leq i \leq 2k$. On the other hand, if (D_1) and (D_2) hold at the rank $n \geq 1$, taking into account (LD), then (D_1) holds at the rank $f(n)$. Hence, with the cases already established and using induction, we see that (D_1) holds for $n \geq 1$. This completes the proof of the lemma.

In the next lemma we introduce a new sequence $(\gamma_n)_{n\geq 1}$ in $\mathbb{F}_q$ which is linked to the sequence $(\delta_n)_{n\geq 0}$.

Lemma 5.3. *Let* $(g_i)_{0\leq i\leq 2k}$ *be the sequence of functions in* $\mathbb{F}_p(X)$ *defined by* (G) *in Section 2. Let* $(\delta_n)_{n\geq 0}$ *be a sequence in* $\mathbb{F}_q^*$ *with* $\delta_0, \delta_1, \ldots, \delta_l$ *given. Then* $(\delta_n)_{n\geq 0}$ *satisfies* (D_2) *and* (D_3) *if and only if there exists a sequence* $(\gamma_n)_{n\geq 1}$ *in* $\mathbb{F}_q$ *such that we have*

$$\delta_{f(n)+i} = \epsilon_1^{r(-1)^{n+i}} \delta_n^{r(-1)^i} g_i(\gamma_n^r) \quad for \quad 0 \leq i \leq 2k \quad and \quad n \geq 1, \qquad (D)$$

with

$$\gamma_1^r = (\theta_k \delta_0^{-r} - \epsilon_1^r \delta_l^{-1})(\omega_k \delta_1)^{-r} \qquad (\Gamma_1)$$

and

$$\gamma_n = \gamma_{n-1}(\delta_n \delta_{n-1} \omega_k)^{-1} \quad for \quad n \geq 2. \qquad (\Gamma_2)$$

Proof: First we prove that the sequence $(g_i)_{0\leq i\leq 2k}$ in $\mathbb{F}_p(X)$, described in (G), can also be defined recursively by $g_0(X) = \theta_k + X$ and

$$g_{i+1}(X) = 2k\theta_k(-v_{i+1,k} + 2k\theta_k/g_i(X)) \quad \text{for} \quad 0 \leq i \leq 2k-1. \qquad (6)$$

For $i = 0$, (6) becomes

$$g_1(X) = 2k\theta_k(-v_{1,k} + 2k\theta_k/(\theta_k + X)).$$

Since $v_{1,k} = 2k - 1$, this equality implies

$$g_1(X) = 2k\theta_k(\theta_k - (2k-1)X)/(\theta_k + X).$$

This is in agreement with (G) for $i = 1$. Now we use induction on i. Let $1 \le i < 2k - 1$ and assume that $g_i(X)$ is as stated in (G). Then, by (2) from Section 2, we have

$$2k\theta_k/g_i(X) = \frac{(2k-i)v_{i+1,k}}{2k-2i-1}\frac{\theta_k + w_{i-1,k}X}{\theta_k + w_{i,k}X}. \tag{7}$$

Besides, a direct computation shows that, for $1 \le i < 2k - 1$, we also have

$$(2k-i)w_{i-1,k} - (2k-2i-1)w_{i,k} = (i+1)w_{i+1,k}. \tag{8}$$

Combining (6), (7) and (8), we get $g_{i+1}(X)$ as stated in (G). It remains to compute $g_{2k}(X)$. From (6) and (7) for $i = 2k - 1$, we obtain

$$g_{2k}(X) = 2k\theta_k(-v_{2k,k} - \frac{v_{2k,k}}{2k-1}\frac{\theta_k + (2k-1)X}{\theta_k - X}). \tag{9}$$

Recalling (3) and (4) from Section 2, we also have $v_{2k,k} = v_{1,k}\omega_k$ and consequently

$$v_{2k,k} = -(2k-1)(2k\theta_k)^{-2}. \tag{10}$$

Finally, combining (9) and (10), we get $g_{2k}(X) = 1/(\theta_k - X)$ and this is in agreement with (G) for $i = 2k$.

We set now

$$g_{i,n} = \delta_{f(n)+i}/(\epsilon_1^{r(-1)^{n+i}}\delta_n^{r(-1)^i}) \quad \text{for} \quad 0 \le i \le 2k \quad \text{and for} \quad n \ge 1. \tag{11}$$

Then we define the sequence $(\gamma_n)_{n\ge1} \in \mathbb{F}_q$ from the sequence $(\delta_n)_{n\ge1} \in \mathbb{F}_q^*$ by

$$\gamma_n^r = g_{0,n} - \theta_k \quad \text{for} \quad n \ge 1. \tag{12}$$

By definition, (12) becomes $g_{0,n} = g_0(\gamma_n^r)$, thus (11) implies (D) for $i = 0$. Now we prove that (D_3) is equivalent to (D) for $1 \le i \le 2k$. According to (11), we need to prove that (D_3) is equivalent to $g_{i,n} = g_i(\gamma_n^r)$ for $1 \le i \le 2k$ and $n \ge 1$. Using (11) and again $\omega_k = -(2k\theta_k)^{-2}$, (D_3) can be written as

$$g_{i,n} = 2k\theta_k(-v_{i,k} + 2k\theta_k/g_{i-1,n}) \quad \text{for} \quad 1 \le i \le 2k.$$

Since $g_{0,n} = g_0(\gamma_n^r)$, with the recursive definition of the sequence $(g_i)_{1\le i\le 2k}$, we see that (D_3) is equivalent to $g_{i,n} = g_i(\gamma_n^r)$ for $1 \le i \le 2k$. Now we shall see that (D_2) and (D_3) imply (Γ_1) and (Γ_2). Hence, with $(\gamma_n)_{n\geq 1}$ defined by (12), we have (D). For $n=1$, (D_2) becomes

$$\delta_{l+1}+(\omega_k\delta_l)^{-1}=\theta_k\epsilon_1^{-r}(\delta_1^r+(\omega_k\delta_0)^{-r}). \tag{13}$$

But, using (D) for $n=1$ and for $i=0$, we also have

$$\delta_{l+1}=\epsilon_1^{-r}\delta_1^r(\theta_k+\gamma_1^r). \tag{14}$$

Combining (13) and (14), we obtain the value for γ_1^r stated in (Γ_1). Now we assume that $n\geq 2$ and we recall that we have $f(n)-1=f(n-1)+2k$. Consequently, using (D) for $i=0$ and for $i=2k$, by (11), (D_2) implies

$$g_0(\gamma_n^r)\delta_n^r+(\omega_k g_{2k}(\gamma_{n-1}^r)\delta_{n-1}^r)^{-1}=\theta_k(\delta_n^r+(\omega_k\delta_{n-1})^{-r}). \tag{15}$$

Since, by (G), $g_0(\gamma_n^r)=\theta_k+\gamma_n^r$ and $g_{2k}(\gamma_{n-1}^r)=1/(\theta_k-\gamma_{n-1}^r)$, (15) gives

$$(\gamma_n\delta_n-\omega_k^{-1}\gamma_{n-1}\delta_{n-1}^{-1})^r=0$$

which is (Γ_2). Reciprocally we assume that both sequences satisfy (D), (Γ_1) and (Γ_2). First, as we have seen above, (D_3) hold for $1\leq i\leq 2k$ and $n\geq 1$. Then (D) for $n=1$ and $i=0$ implies $\theta_k+\gamma_1^r=\delta_{l+1}\epsilon_1^r\delta_1^{-r}$. Taking (Γ_1) into account, this implies (D_2) for $n=1$. Finally, for $n\geq 2$, we have seen that (Γ_2) implies (15). Using (D) for $i=0$ and for $i=2k$, (15) is equivalent to (D_2). The proof of the lemma is complete.

In the last lemma we describe the sequence $(\gamma_n)_{n\geq 1}$, if it is not identically zero.

Lemma 5.4. *Let $(h_i)_{0\leq i\leq 2k}$ be the sequence of functions in $\mathbb{F}_p(X)$ defined by (H) in Section 2. Let $(\delta_n)_{n\geq 1}$ and $(\gamma_n)_{n\geq 1}$ be two sequences in $\mathbb{F}_q^*$ with $\delta_1,\ldots,\delta_l$ and γ_1 given. We assume that they satisfy (D). Then they satisfy (Γ_2) if and only if we have*

$$\gamma_n=\gamma_{n-1}(\delta_n\delta_{n-1}\omega_k)^{-1}\quad \mathit{for}\quad 2\leq n\leq l \tag{Γ_2'}$$

and

$$\gamma_{f(n)+i}=C_0h_i(\gamma_n^r)\quad \mathit{for}\quad 0\leq i\leq 2k\quad \mathit{and}\quad \mathit{for}\quad n\geq 1, \tag{Γ_3}$$

where

$$C_0=\gamma_l\epsilon_1^r(\delta_1\gamma_1)^{-r}(\delta_l\omega_k)^{-1}\in\mathbb{F}_q^*.$$

Proof: First we prove that (Γ_2) implies (Γ_3). We will use the connection between the two sequences $(g_i)_{0\leq i\leq 2k}$ and $(h_i)_{0\leq i\leq 2k}$ in $\mathbb{F}_p(X)$. Indeed, from (G) and (H), an elementary calculation shows that we have

$$g_i(X)g_{i-1}(X)\omega_k=h_{i-1}(X)/h_i(X)\quad \mathit{for}\quad 1\leq i\leq 2k. \tag{16}$$

We also have

$$g_0(X)h_0(X) = X \quad \text{and} \quad h_{2k}(X) = Xg_{2k}(X). \tag{17}$$

For $1 \leq i \leq 2k$ and $n \geq 1$, using (D) and (16), we have

$$\omega_k \delta_{f(n)+i}\delta_{f(n)+i-1} = h_{i-1}(\gamma_n^r)/h_i(\gamma_n^r). \tag{18}$$

Applying (Γ_2) at the rank $f(n)+i$, (18) implies

$$\gamma_{f(n)+i}/\gamma_{f(n)+i-1} = h_i(\gamma_n^r)/h_{i-1}(\gamma_n^r). \tag{19}$$

Clearly, for $0 \leq i \leq 2k$ and $n \geq 1$, from (19) we obtain

$$\gamma_{f(n)+i} = \gamma_{f(n)}h_i(\gamma_n^r)/h_0(\gamma_n^r). \tag{20}$$

We assume now that $n \geq 2$. Recalling that $f(n)-1 = f(n-1)+2k$, by (D) for $i=0$ and $i=2k$ and (17), we also have

$$\delta_{f(n)}\delta_{f(n)-1} = (\delta_n\delta_{n-1})^r(\gamma_n/\gamma_{n-1})^r h_{2k}(\gamma_{n-1}^r)/h_0(\gamma_n^r). \tag{21}$$

Applying (Γ_2) at the rank n, (21) becomes

$$\omega_k\delta_{f(n)}\delta_{f(n)-1} = h_{2k}(\gamma_{n-1}^r)/h_0(\gamma_n^r). \tag{22}$$

Applying (Γ_2) at the rank $f(n)$, (22) implies

$$\gamma_{f(n)} = \gamma_{f(n)-1}h_0(\gamma_n^r)/h_{2k}(\gamma_{n-1}^r). \tag{23}$$

By (20) we also have

$$\gamma_{f(n)-1} = \gamma_{f(n-1)+2k} = \gamma_{f(n-1)}h_{2k}(\gamma_{n-1}^r)/h_0(\gamma_{n-1}^r). \tag{24}$$

Combining (23) and (24), we obtain

$$\gamma_{f(n)} = \gamma_{f(n-1)}h_0(\gamma_n^r)/h_0(\gamma_{n-1}^r). \tag{25}$$

Consequently, by (25), for $n \geq 1$ we have

$$\gamma_{f(n)}/h_0(\gamma_n^r) = C_0 = \gamma_{f(1)}/h_0(\gamma_1^r). \tag{26}$$

To compute C_0, we apply (Γ_2) at the rank $f(1) = l+1$ and (D) for $n=1$ and for $i=0$. We obtain

$$C_0 = \gamma_l\epsilon_1^r(\delta_1\gamma_1)^{-r}(\delta_l\omega_k)^{-1}.$$

Finally, combining (20) and (26), we get (Γ_3). We now prove that (Γ_2') and (Γ_3) imply (Γ_2). Hence (Γ_2) holds for $2 \leq n \leq l$. Moreover we obtain (19) directly from (Γ_3). Together with (18), this proves that (Γ_2) holds at the rank $f(n)+i$ for $n \geq 1$ and $1 \leq i \leq 2k$. We observe that (Γ_2) also holds for $l+1$. Indeed applying (Γ_3) for $n=1$ and $i=0$, together with the value of C_0, we obtain (Γ_2) for $l+1$. Now we assume that $n \geq 2$ and we apply (Γ_3) for $i=0$ and for $i=2k$. We have

$$\gamma_{f(n)}/\gamma_{f(n)-1} = h_0(\gamma_n^r)/h_{2k}(\gamma_{n-1}^r). \tag{27}$$

Combining (21) and (27) we obtain

$$\gamma_{f(n)}/\gamma_{f(n)-1} = (\delta_n\delta_{n-1})^r(\gamma_n/\gamma_{n-1})^r(\delta_{f(n)}\delta_{f(n)-1})^{-1}.$$

This shows that if (Γ_2) holds at the rank $n \geq 2$ then it holds at the rank $f(n)$. Consequently, with the cases already established and using induction, we see that (Γ_2) holds for $n \geq 2$. The proof of the lemma is complete.

Proof of Theorem B:

Let $\alpha \in \mathbb{F}(q)$ be a continued fraction of type (r,l,k) defined by the l-tuple $\lambda_1,\ldots,\lambda_l \in (\mathbb{F}_q^*)^l$ and the pair $(\epsilon_1,\epsilon_2) \in (\mathbb{F}_q^*)^2$. According to Lemma 5.1 and Lemma 5.2, the sequence of partial quotients for α satisfies (I) if and only if there exists a sequence $(\delta_n)_{n\geq 0}$ in $\mathbb{F}_q^*$ satisfying $(II)_0$, (D_2) and (D_3), where the sequence $(\lambda_n)_{n\geq 1}$ is based on $(\delta_n)_{n\geq 0}$ by (LD). Given the value for δ_0 in Lemma 5.1, the existence of this sequence requires condition (II) of Theorem B. According to Lemma 5.3, this sequence does exist if and only if there exists a sequence $(\gamma_n)_{n\geq 1}$ satisfying (D), (Γ_1) and (Γ_2). Now distinguish two cases :either $\epsilon_2^r\delta_l - 4k^2\theta_k\epsilon_1^r = 0$ or $\epsilon_2^r\delta_l - 4k^2\theta_k\epsilon_1^r \neq 0$. In the first case, which is case $(III)_1$ of Theorem B, by (Γ_1) and according to the previous value for δ_0, we have $\gamma_1 = 0$ and also, by (Γ_2), $\gamma_n = 0$ for $n \geq 2$. In the second case, which is case $(III)_2$ of Theorem B, again by (Γ_1) and (Γ_2), the sequence $(\gamma_n)_{n\geq 1}$ is in $\mathbb{F}_q^*$ and consequently, using Lemma 5.4, this sequence can be described by the formulas (Γ) of Theorem B. In both cases, the sequence $(\delta_n)_{n\geq 0}$ is described recursively from $\delta_1,\ldots,\delta_l$ and by (D) from the sequence $(\gamma_n)_{n\geq 1}$, identically zero or not. So the proof of the theorem is complete.

Proof of Corollary C:

First, due to the degrees of the polynomial coefficients of equation (E), we observe that if this equation has a root α in $\mathbb{F}(27)$ then we must have $|\alpha| = |T|$. Now, with Proposition A, we consider the continued fraction of type $(3,1,1)$, in $\mathbb{F}(27)$ defined by $\lambda_1 = 1$ and the pair $(-u^6, u^3) \in (\mathbb{F}_{27}^*)^2$. So we have $\alpha^3 = -u^6(T^2-1)\alpha_2 + u^3T$, where $\alpha_2 = 1/(\alpha - T)$. Hence this continued fraction satisfies equation (E). This proves that (E) has no other root in $\mathbb{F}(27)$ (and consequently this root is algebraic over $\mathbb{F}_{27}(T)$ of degree four). Now we need to prove that the expansion for α is perfect. Here we have $k = 1$ and $l = 1$, consequently $\theta_1 = 1$ and $f(n) = 3n - 1$ for $n \geq 1$. From Lemma 5.1, we also have $\delta_0 = u^{-3}$ and $\delta_1 = -\lambda_1^3 + \epsilon_2 = -1 + u^3 = u^4 \in \mathbb{F}_{27}^*$. So (II) holds. We compute now γ_1. We have $\gamma_1^3 = (4\theta_1\epsilon_1^3\delta_1^{-3} - \epsilon_2^3)\theta_1\delta_1^{-3} = u^3 \neq 0$. We are in case (III_2) if the sequence $(\gamma_n)_{n\geq1}$ can be defined. First we compute C_0. We have $C_0 = -\delta_1^{-1}(\gamma_1\epsilon_1^3)(\delta_1\gamma_1)^{-3} = 1$. Applying the formulas in (Γ) with the triplet (h_0, h_1, h_2) in $(\mathbb{F}_3(X))^3$ stated in (H), we obtain the recursive definition given in the corollary for $(\gamma_n)_{n\geq1}$. As $\gamma_1 = u$ has degree 3 over $\mathbb{F}_3$, by induction all the terms have the same degree over $\mathbb{F}_3$ and therefore this sequence $(\gamma_n)_{n\geq1}$ is well defined. Applying the formulas (D) with the triplet (g_0, g_1, g_2) in $(\mathbb{F}_3(X))^3$ stated in (G), we obtain the recursive definition for the sequence $(\delta_n)_{n\geq1}$ from $(\gamma_n)_{n\geq1}$ as stated in the corollary. Finally, applying the formulas (LD), the sequence $(\lambda_n)_{n\geq1}$ satisfies the recursive definition indicated in the corollary. This completes the proof.

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