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DIOPHANTINE APPROXIMATION AND CONTINUED FRACTION EXPANSIONS OF ALGEBRAIC POWER SERIES IN POSITIVE CHARACTERISTIC

Abstract. In a recent paper [2], M. Buck and D. Robbins have given the continued fraction expansion of an algebraic power series when the base field is $\mathbb{F}_3$. We study its rational approximation property in relation with Roth's theorem, and we show that this element has an analog for each power of an odd prime number. At last we give the explicit continued fraction expansion of another classical example.

§1. Introduction.

Let K be a field. We denote $K((T^{-1}))$ the set of formal Laurent series with coefficients in K. If $\alpha = \sum_{k \leq k_0} a_k T^k$ is an element of $K((T^{-1}))$, with $a_{k_0} \neq 0$, we introduce the absolute value $|\alpha| = |T|^{k_0}$ and $|0| = 0$, with $|T| > 1$. It is well known that Roth's theorem (if α is an element of $K((T^{-1}))$, irrational algebraic over $K(T)$, then for all real $\epsilon > 0$ we have $|\alpha - P/Q| > |Q|^{-(2+\epsilon)}$ for all $P/Q) \in K(T)$ with $|Q|$ large enough) fails if K has a positive characteristic p. In this case, which is the one we consider here, Liouville's theorem (there is a real positive constant C such that $|\alpha - P/Q| \geq C|Q|^{-n}$ for all $P/Q \in K(T)$, where n is the degree of α over $K(T)$) holds and is optimal.

Many examples can be studied. A special case is the one where α satisfies an equation of the form $\alpha = (A\alpha^{p^s} + B)/(C\alpha^{p^s} + D)$ where A, B, C, D belong to $K[T]$, with $AD - BC \neq 0$, and s is a positive integer. Those elements have been studied by Baum and Sweet, Mills and Robbins, Voloch, de Mathan ([1],[5],[6],[7]). To simplify we will say that such an irrational algebraic element is an element of class I. It is also possible to study some particular rational functions, with coefficients in $K[T]$, of an element of class I (This was done by Voloch in [8]). For such simple examples, if d is a real number such that, for every $\epsilon > 0$, we have $|\alpha - P/Q| > |Q|^{-(d+\epsilon)}$ for $|Q|$ large enough, then there is a real positive constant C such that $|\alpha - P/Q| \geq C|Q|^{-d}$, for all P/Q. But all these examples seem to be exceptions. It seems that , except for "particular" elements, Roth's theorem holds, and for an irrational algebraic element , for all $\epsilon > 0$, we have $|\alpha - P/Q| > |Q|^{-(2+\epsilon)}$, for $|Q|$ large enough *but not* $|\alpha - P/Q| \geq C|Q|^{-2}$ for all P/Q. Nevertheless, no algebraic element α, for which this result could be established, was known. It has only been proved that if α is an algebraic element of degree n, not of class I, then Thue's theorem holds, i.e. $|\alpha - P/Q| > |Q|^{-([n/2]+\epsilon)}$, for $|Q|$ large enough ([3]).

M. Buck and D. Robbins have given the continued fraction expansion of a particular algebraic element of $\mathbb{F}_3((T^{-1}))$ ([2]). What is very curious in this example is that it does not belong to the set of exceptions already known. Indeed this element satisfies, for $|Q|$ large enough, $|\alpha - P/Q| > |Q|^{-(2+\epsilon)}$ but not $|\alpha - P/Q| \geq C|Q|^{-2}$, for all P/Q. Actually there are two real positive constants λ_1 and λ_2 such that, for some rationals P/Q with $|Q|$ arbitrary large, we have $|\alpha - P/Q| \leq |Q|^{-(2+\lambda_1/\sqrt{\log|Q|})}$, and for all rationals P/Q with $|Q| > 1$, we have $|\alpha - P/Q| \geq |Q|^{-(2+\lambda_2/\sqrt{\log|Q|})}$.

We have observed that $\alpha(T) = \beta^2(\sqrt{T})$ where β satisfies $\beta = 1/(T + \beta^3)$, that is to say β^2 is a rational function of an element of class I, but not such that it can be studied by the method mentioned above. This new approach allows us to give another proof of the result due to M. Buck and D. Robbins. Let α be an irrational element of $K((T^{-1}))$. Then it may be expanded uniquely as a continued fraction. We write this continued fraction expansion as $\alpha = [a_0, a_1, a_2, ..., a_n, ...]$, where $a_k \in K[T]$ for $k \geq 0$ and deg $a_k > 0$ for $k > 0$. With these notations, we will prove that, in $\mathbb{F}_3((T^{-1}))$, we have

$$[T, T^3, ..., T^{3^n}, ...]^2 = [\lim_n \Omega_n]$$

where $(\Omega_n)_{n\geq 0}$ is a sequence of elements of $\mathbb{F}_3[T]$, defined inductively by

$$\Omega_0 = \emptyset, \qquad \Omega_1 = T^2, \qquad \Omega_n = \Omega_{n-1}, 2T^2, \Omega_{n-2}^{(3)}, 2T^2, \Omega_{n-1} \quad \text{for } n \geq 2$$

and $\lim_n \Omega_n$ denotes the sequence begining by Ω_n for all $n \geq 0$. This has been obtained by studying a general case. Let q be a power of an odd prime number p, then we have considered, in $\mathbb{F}_p((T^{-1}))$, the continued fraction expansion of $[T, T^q, ..., T^{q^n}, ...]^{(q+1)/2}$. We have not been able to describe it entirely for $q > 3$, but we show that it has an interesting structure which implies the above result, for $q = 3$. The possibility of describing completely the general case, or even of improving the description given in this paper, is an open question.

At last we give the continued fraction expansion of a classical example of algebraic element, first introduced by K. Mahler.

§2.A badly approximable element .

In [2], M. Buck et D. Robbins have given the continued fraction expansion of an element of $\mathbb{F}_3((T^{-1}))$. If $K = \mathbb{F}_3$, they show that the algebraic equation

$$(1) \qquad x^4 + x^2 - Tx + 1 = 0$$

has a unique solution in $K((T^{-1}))$, the continued fraction expansion of which can be totally described. Indeed , they define recursively the following polynomial sequences :

$$(2) \qquad \Omega_0 = \emptyset, \qquad \Omega_1 = T, \qquad \Omega_n = \Omega_{n-1}, -T, \Omega_{n-2}^{(3)}, -T, \Omega_{n-1} \quad \text{for } n \geq 2$$

(Here $\Omega_k^{(3)}$ denotes the sequence obtained by cubing each element of Ω_k and commas indicate juxtaposition of sequences), then they prove that $[0, \Omega_n]$ is the begining for all $n > 0$ of the continued fraction expansion of this solution. Using this result we can prove:

Theorem A. *Let* α *be the unique root of* (1) *in* $\mathbb{F}_3((T^{-1}))$. *Then there exist explicit positive real constants* λ_1 *and* λ_2 *such that for some rationals* P/Q *with* $|Q|$ *arbitrary large, we have*

$$(3) \quad |\alpha - P/Q| \le |Q|^{-(2+\lambda_1/\sqrt{\deg Q})}$$

and, for all rationals P/Q *with* $|Q|$ *sufficiently large, we have*

$$(4) \quad |\alpha - P/Q| \ge |Q|^{-(2+\lambda_2/\sqrt{\deg Q})}$$

(We can take $\lambda_1 = 2/\sqrt{3}$ *and* $\lambda_2 > 2/\sqrt{3}$.*)*

Proof: We write $\alpha = [a_0, a_1, a_2, ..., a_n, ...]$. For $k > 0$, we put $d_k = \deg a_k$ and $P_k/Q_k = [a_0, ..., a_k]$.
It results, from the inductive definition (2), that all partial quotients are monomials, and all have a power of 3 as degree.
For $i \ge 1$, we define $k_i = \inf\{k \ge 1/ \quad d_k = 3^i\}$. If $k_i \le k < k_{i+1}$, we have $d_k \le d_{k_i} = 3^i$. For each $n \ge 0$, let us define the sequence Ω_n^* of the degrees of the elements of Ω_n. We get :

$$\Omega_0^* = \emptyset, \quad \Omega_1^* = 1, \quad \Omega_2^* = 1111, \quad \Omega_3^* = 11111311111$$

From the recursive definition (2), we see, by induction on k, that

$$\sup \Omega_{2k+2}^* = \sup \Omega_{2k+1}^* = 3^k \quad \text{for } k \ge 0$$

therefore , for $k \ge 0$, $2k+1$ is the smallest integer n such that 3^k belongs to Ω_n^*. Again, from (2) and by induction on k, we see that Ω_{2k+1}^* has an odd number of terms, has 3^k as central term and is reversible. All of this leads to

$$(5) \qquad \sum_{a_k \in \Omega_{2i+1}} d_k = 3^i + 2 \sum_{k<k_i} d_k$$

Now we put $\omega_n = \sum_{a_k \in \Omega_n} d_k$. From (2), we obtain

$$(6) \quad \omega_0 = 0, \quad \omega_1 = 1, \quad \omega_n = 2\omega_{n-1} + 3\omega_{n-2} + 2 \quad \text{for } n \ge 2$$

It is easy to check that the sequence $((3^n - 1)/2)_{n \ge 0}$ is the one satisfying (6). Hence by (5), we have

$$(7) \quad \deg Q_{k_i - 1} = \sum_{k<k_i} d_k = (\omega_{2i+1} - 3^i)/2 = (3^{2i+1} - 2.3^i - 1)/4$$

Thus $3^i \ge (2/\sqrt{3})\sqrt{\deg Q_{k_i-1}}$, which gives $|T|^{-3^i} \le |Q_{k_i-1}|^{-2/\sqrt{3 \deg Q_{k_i-1}}}$.
Also we have, for $i \ge 1$

$$|\alpha - P_{k_i-1}/Q_{k_i-1}| = |T|^{-3^i} |Q_{k_i-1}|^{-2}$$

this shows that (3) holds for $P/Q = P_{k_i-1}/Q_{k_i-1}$ and for $i \geq 1$, with $\lambda_1 = 2/\sqrt{3}$.

On the other hand we see that deg $Q_{k_i-1} \leq$ deg $Q_k <$ deg $Q_{k_{i+1}-1}$ implies $|\alpha - P_k/Q_k| = |T|^{d_{k+1}}|Q_k|^{-2} \geq |T|^{-3^i}|Q_k|^{-2}$. As, by (7), the sequence $(3^i/\sqrt{\deg Q_{k_i-1}})_{i\geq 1}$ converges to $2/\sqrt{3}$, then, if $\lambda_2 > 2/\sqrt{3}$, we can write $3^i < \lambda_2\sqrt{\deg Q_{k_i-1}} \leq \lambda_2\sqrt{\deg Q_k}$, for i large enough. It follows that (4) holds for P_k/Q_k with k large enough. Since the convergents are the best rational approximations, this is also true for all P/Q with $|Q|$ large enough. So the theorem is proved.

Remark. The fact that for this element and for all $\epsilon > 0$, we have $|\alpha - P/Q| > |Q|^{-(2+\epsilon)}$, for $|Q|$ large enough but not $|\alpha - P/Q| \geq C|Q|^{-2}$ for all P/Q, implies that it is not of class I, according to the theorem proved in [5] or [7].
In the same paper [2], the authors have considered the unique solution, in $K((T^{-1}))$, of the algebraic equation (1), when the base field is $K = \mathbb{F}_{13}$. In that situation the solution is actually of class I. After some calculation, it can be seen that (1) implies $x = (Ax^{13} + B)/(Cx^{13} + D)$ with $A = T^2 + 1$, $B = T^5 + 2T^3 + 2T$, $C = 9T$ and $D = T^6 + T^4 + 11T^2 + 1$.
(We can observe that the conjecture made by the authors, ([6], p.404), implies $d_n = (13^{w_9(4n-1)}+2)/3$ where $w_9(k)$ is the greatest power of 9 dividing k. Using notations as above and as in [5], it is possible to compute the approximation exponent of this solution, called α. We have $\nu(\alpha) = 1 + \limsup_{k\geq 1} \deg a_{k+1}/\deg Q_k = 5/3$. It can be seen that $|\alpha - P/Q| \geq |T|^{-1}|Q|^{-8/3}$ for all $(P,Q) \in K[T] \times K[T]\backslash\{0\}$.)

§3. A power of a simple element of class I.

Here we come back to the element of $\mathbb{F}_3((T^{-1}))$, mentioned above, first introduced by W. Mills and D. Robbins in [6], satisfying

$$(1) \qquad x^4 + x^2 - Tx + 1 = 0$$

Let p be an odd prime number, q a power of p, and let $K = \mathbb{F}_p$. We consider the element α_q of $K((T^{-1}))$ defined by its continued fraction expansion:

$$(2) \qquad \alpha_q = [0, T, T^q, ..., T^{q^n}, ...]$$

This element is of class I, being the unique root, in $K((T^{-1}))$, of the algebraic equation

$$(3) \qquad x^{q+1} + Tx - 1 = 0$$

We put $r = (q+1)/2$ and we consider the element θ_q, of $K((T^{-1}))$, defined by $\theta_q = \alpha_q^r$. We observe that (3) implies $\alpha_q = (1/T)(1 - \alpha_q^{2r})$ which leads to $\theta_q = (1/T^r)(1 - \theta_q^2)^r$. So θ_q is a solution of the algebraic equation

$$(4) \qquad x = (1/T^r)(1 - x^2)^r$$

If x is a solution of (4), in $K((T^{-1}))$, we must have $|x| \leq 1$. Since otherwise $|x| > 1$ gives $|(1 - x^2)^r| = |x|^{2r}$, and by (4), $|T|^r = |x|^q$ which is impossible. We consider the set $E = \{x \in K((T^{-1}))/ \quad |x| \leq 1\}$, and the map f of E into itself defined by $f(x) = (1/T^r)(1-x^2)^r$. Then we can see that f is a contraction mapping, E is complete, and therefore $f(x) = x$ has a unique solution in E. So θ_q is the unique root of (4) in $K((T^{-1}))$. Also, the coefficients of this equation are elements of $K(T^r)$, thus its solution θ_q is an element of $K((T^{-r}))$. Then we can introduce the element θ_q^* of $K((T^{-1}))$, defined by $\theta_q(T) = \theta_q^*(T^r)$. So θ_q^* is the unique solution, in $K((T^{-1}))$, of the algebraic equation

$$(5) \qquad x = (1/T)(1-x^2)^r$$

Now we see that, if $q = 3$, we have $\theta_3^* = (1/T)(1-(\theta_3^*)^2)^2 = (1/T)(1+(\theta_3^*)^2+(\theta_3^*)^4)$, so that θ_3^* is the root of (1) in $\mathbb{F}_3((T^{-1}))$.

Here we shall see that the link between θ_q and α_q is simple enough to give a partial description of the continued fraction expansion of this element, this description being complete for $q = 3$. We start from the continued fraction expansion of α_q. Let us consider the usual two sequences of polynomials of $K[T]$, defined inductively by

$$P_0 = 0, P_1 = 1, Q_0 = 1, Q_1 = T, P_n = T^{q^{n-1}} P_{n-1} + P_{n-2} \quad Q_n = T^{q^{n-1}} Q_{n-1} + Q_{n-2}$$

for $n \geq 2$. So $(P_n/Q_n)_{n\geq 0}$ is the sequence of the convergents to α_q. By (2), for $n \geq 1$, we have

$$P_n/Q_n = [0, T, T^q, ..., T^{q^{n-1}}] = 1/(T+[0, T, T^q, ..., T^{q^{n-2}}]^q) = 1/(T+(P_{n-1}/Q_{n-1})^q)$$

Since P_n and Q_n are coprime and both unitary, we obtain

$$(6) \qquad \begin{cases} P_0 = 0 & P_n = Q_{n-1}^q \\ Q_0 = 1 & Q_n = TQ_{n-1}^q + P_{n-1}^q \quad \text{for } n \geq 1 \end{cases}$$

Now let us consider the continued fraction expansion of θ_q. We set $\theta_q = [a_0, a_1, ..., a_n, ...]$. We observe that $a_0 = 0$ from the definition of θ_q since $|\alpha_q| < 1$. Then we introduce the usual two sequences of polynomials of $K[T]$, defined inductively by

$$U_0 = 0, U_1 = 1, V_0 = 1, V_1 = a_1, \quad U_n = a_n U_{n-1} + U_{n-2} \quad V_n = a_n V_{n-1} + V_{n-2}$$

for $n \geq 2$. So $(U_n/V_n)_{n\geq 0}$ is the sequence of the convergents to θ_q.

First we are going to give some special sub-sequences of convergents to θ_q.

We use the following auxiliary results:

Lemma 1. *For $n \geq 0$, the polynomial a_n is an odd polynomial in the indeterminate T^r and the rational $(P_n/Q_n)^r$ is a convergent to θ_q.*

Proof: We know that equation (5) has θ_q^* as unique solution in $K((T^{-1}))$. From (5) we see that

$$\theta_q^*(-T) = (-1/T)(1-(\theta_q^*(-T))^2)^r \quad \text{thus} \quad -\theta_q^*(-T) = (1/T)(1-(-\theta_q^*(-T))^2)^r$$

Therefore $-\theta_q^*(-T)$ is also solution of (5), and we have $-\theta_q^*(-T) = \theta_q^*(T)$. That is to say θ_q^* is an odd element of $K((T^{-1}))$, and by induction we see that the partial quotients of the continued fraction expansion of θ_q^* are odd polynomials of $K[T]$. If we write $\theta_q^* = [a_0^*(T), a_1^*(T), ..., a_n^*(T), ...]$, then, because of the identity $\theta_q^*(T^r) = \theta_q(T)$, we have $a_n(T) = a_n^*(T^r)$.

Now we show that $(P_n/Q_n)^r$ is a convergent to θ_q. Indeed, for $n \geq 0$

$$|\alpha_q^r - (P_n/Q_n)^r| \ = \ |\alpha_q - P_n/Q_n| \ |\sum_{0\leq i\leq r-1} \alpha_q^i (P_n/Q_n)^{r-1-i}|$$

Since $|\alpha_q| \ = \ |P_n/Q_n| \ = \ |T|^{-1}$, we have r terms in the sum , each with absolute value $|T|^{-r+1}$ and dominant coefficient 1. Therefore, as r and p are coprime, this becomes

$$|\alpha_q^r - (P_n/Q_n)^r| \ = \ |\alpha_q - P_n/Q_n| \ |T|^{-r+1} = \ |Q_n Q_{n+1}|^{-1} |T|^{-r+1}$$

From (6) we get $|Q_{n+1}| \ = \ |Q_n|^q |T|$, which gives

$$(7) \qquad |\theta_q - (P_n/Q_n)^r| \ = \ |Q_n^r|^{-2} |T|^{-r}$$

This shows that $(P_n/Q_n)^r$ is a convergent to θ_q, and the Lemma is proved.

Lemma 2. *Let P and Q be two polynomials of $K[T]$, with $Q \neq 0$, and n a positive integer. If*

$$(8) \qquad |Q| < |Q_n|^r \ \textit{and} \ |PQ_n^r - QP_n^r| < \frac{|Q_n|^r}{|Q|}$$

then P/Q is a convergent to θ_q. Moreover, if P and Q are coprime and the convergent P/Q is U_k/V_k, then we have

$$(9) \quad |a_{k+1}| \ = \ |PQ_n^r - QP_n^r|^{-1} |Q|^{-1} |Q_n|^r$$

Proof: By (7) and (8), we have

$$|\theta_q - (P_n/Q_n)^r| \ = \frac{1}{|Q_n|^{q+1}|T|^r} < \frac{1}{|Q_n|^r|Q|} \leq \frac{|PQ_n^r - QP_n^r|}{|Q_n|^r|Q|}$$

since $|Q| < |Q_n|^r$ and $(P_n, Q_n) = 1$ implies $PQ_n^r - QP_n^r \neq 0$. Hence

$$|\theta_q - (P_n/Q_n)^r| < |P/Q - (P_n/Q_n)^r|$$

Therefore

$$|\theta_q - P/Q| \ = \ |\theta_q - (P_n/Q_n)^r + (P_n/Q_n)^r - P/Q| \ = \ |P/Q - (P_n/Q_n)^r|$$

and by (8)

$$|\theta_q - P/Q| < |Q|^{-2}$$

This shows that P/Q is a convergent to θ_q. Now if P and Q are coprime and $P/Q = U_k/V_k$, we have $|Q| = |V_k|$. Besides, we know that

$$|\theta_q - U_k/V_k| \ = \ |V_k|^{-2} |a_{k+1}|^{-1}$$

Since

$$|\theta_q - U_k/V_k| \ = \ |P/Q - (P_n/Q_n)^r|$$

it is clear that (9) holds. So Lemma 2 is proved.

Lemma 3. *Let us consider the elements of $K(T)$, defined by*

$$\Theta_q(T) = \frac{T^q}{(T^2+1)^r} \quad \textit{and} \quad \Theta'_q(T) = \frac{T^q}{(T^2-1)^r}$$

Then we have the following continued fraction expansions in $K(T)$:

$$(10) \quad \Theta_q(T) = [0, T, 2T, 2T, \ldots\ldots, 2T, T] \qquad (2T \textit{ is repeated } q-1 \textit{ times})$$

$$(11) \quad \Theta'_q(T) = [0, T, -2T, 2T, \ldots\ldots, -2T, 2T, -T] \qquad (-2T, 2T \textit{ is repeated } \frac{q-1}{2} \textit{ times})$$

Proof: Let $(R_k)_{0\leq k\leq q+1}$ be the sequence of elements of $K(T)$, defined inductively by:

$$(12) R_0 = 0,\ R_1 = 1,\ R_k = 2TR_{k-1} + R_{k-2} \text{ for } \quad 2 \leq k \leq q,\ R_{q+1} = TR_q + R_{q-1}$$

Then , by the usual property of a linear recurrent sequence, we have

$$(12)' \qquad R_k = \frac{1}{2\sqrt{T^2+1}}((T+\sqrt{T^2+1})^k - (T-\sqrt{T^2+1})^k) \quad \text{for} \quad 1 \leq k \leq q$$

Now we introduce the sequence $(S_k)_{0\leq k\leq q+1}$ of elements of $K[T]$, defined inductively by

$$(13) S_0 = 1,\ S_1 = T,\ S_k = 2TS_{k-1} + S_{k-2} \text{ for } \quad 2 \leq k \leq q,\ S_{q+1} = TS_q + S_{q-1}$$

So $(R_k/S_k)_{0\leq k\leq q+1}$ are the convergents to $[0, T, 2T....2T, T]$, and (10) will be proved if we show that: (14) $R_{q+1} = T^q$ and $S_{q+1} = (T^2+1)^r$
First we prove that (13)′ $S_k = TR_k + R_{k-1}$ holds for $1 \leq k \leq q$. By induction, since S_k and R_k satisfy the same recursive relation, it suffices to see that (13)′ is satisfied for $k = 1$ and $k = 2$.
Now we prove that: (15) $R_q = (T^2+1)^{r-1}$ and $S_q = T^q$
Indeed, by (12)′, we have

$$R_q = \frac{1}{2\sqrt{T^2+1}}((T^q + (\sqrt{T^2+1})^q) - (T^q - (\sqrt{T^2+1})^q)) = (T^2+1)^{r-1}$$

$$R_{q-1} = \frac{1}{2\sqrt{T^2+1}}(\frac{T^q + (\sqrt{T^2+1})^q}{T+\sqrt{T^2+1}} - \frac{T^q - (\sqrt{T^2+1})^q}{T-\sqrt{T^2+1}}) = T^q - T(T^2+1)^{r-1}$$

Then, by (13)′, we get $S_q = TR_q + R_{q-1} = T^q$.
By (12), we also get $R_{q+1} = TR_q + R_{q-1} = T^q$. Now we compute S_{q+1}. From the classical identity $R_{q+1}S_q - S_{q+1}R_q = -1$, we obtain, with (14) and (15), $S_{q+1}R_q = T^{2q} + 1 = (T^2+1)^q$, hence $S_{q+1} = (T^2+1)^r$. So (10) is proved.

Now we show that (11) is a consequence of (10). Let u be a square root of -1, eventually in an extension of K. We have

$$u\Theta_q(uT) = \frac{u^{2r}T^q}{(-T^2+1)^r} = \frac{T^q}{(T^2-1)^r} = \Theta'_q(T)$$

From this identity and (10), it follows that

$$\Theta'_q(T) = u[0, uT, 2uT, 2uT, ..., 2uT, uT]$$

Using the property of the multiplication of a continued fraction expansion by a scalar, we have

$$\Theta'_q(T) = [0, T, 2u^2T, 2T, ..., 2T, u^2T] = [0, T, -2T, 2T, ..., 2T, -T]$$

So (11) is proved.

We observe, from $(12)'$ and $(13)'$, that the polynomial R_i has the opposite parity to the integer i, and the polynomial S_i has the same parity as the integer i. For $0 \le i \le q+1$, we introduce the elements of $K[T]$, defined by

$$(16) \qquad \begin{cases} R'_i = R_i(uT) \quad \text{and} \quad S'_i(T) = -uS_i(uT) \qquad \text{for i odd} \\ R'_i = uR_i(uT) \quad \text{and} \quad S'_i(T) = S_i(uT) \qquad \text{for i even} \end{cases}$$

Since we have $u(R_i/S_i)(uT) = (R'_i/S'_i)(T)$, it is clear, by the same argument as above, that R'_i/S'_i are the convergents to $\Theta'_q(T)$.

Lemma 4. *For $1 \le i \le q$, let R_i, S_i, R'_i and S'_i be the elements of $K[T]$ introduced in Lemma 2. Notations being as above, for $n \ge 0$, we put*

$$R_{i,n} = P_n^r R_i(Q_n^r) \quad \textit{and} \quad S_{i,n} = S_i(Q_n^r) \qquad \textit{for n odd}$$

$$R_{i,n} = P_n^r R'_i(Q_n^r) \quad \textit{and} \quad S_{i,n} = S'_i(Q_n^r) \qquad \textit{for n even}$$

Then, for $n \ge 0$, $R_{i,n}/S_{i,n}$ is a convergent to θ_q. Further $R_{i,n}$ and $S_{i,n}$ are coprime, and if $m(i,n)$ is the integer such that $U_{m(i,n)}/V_{m(i,n)} = R_{i,n}/S_{i,n}$, then $a_{m(i,n)+1} = \lambda_{i,n}T^r$, where $\lambda_{i,n}$ is a non-zero element of K.

Moreover, for $n \ge 0$, we have:

$$(17) \quad R_{1,n}/S_{1,n} = P_n^r/Q_n^r \quad , \quad R_{q,n}/S_{q,n} = Q_{n+1}^{r-1}P_n^q/P_{n+1}^r$$

and the convergent preceding $R_{1,n}/S_{1,n}$ is $R_{q,n-1}/S_{q,n-1}$, i.e.

$$(18) \quad R_{q,n-1}/S_{q,n-1} = U_{m(1,n)-1}/V_{m(1,n)-1} \quad \textit{for all} \quad n \ge 1$$

Proof: Let n and i be integers such that $n \ge 0$ and $1 \le i \le q$. We shall apply Lemma 2 with $P = R_{i,n}$ and $Q = S_{i,n}$. First, by (13) and (16), we have $|S_i| = |S'_i| = |T|^i$ hence we have $|S_{i,n}| = |Q_n|^{ri}$. Then by (6), $|Q_n|^i \le |Q_n|^q < |Q_{n+1}|$. Thus we have $|S_{i,n}| < |Q_{n+1}|^r$, which is the first part of condition (8). We put $\delta_{i,n} = R_{i,n}Q_{n+1}^r - S_{i,n}P_{n+1}^r$. For n odd , we have

$\delta_{i,n} = P_n^r R_i(Q_n^r)Q_{n+1}^r - S_i(Q_n^r)P_{n+1}^r$

By (6), (14) and since we have $P_{n+1}Q_n - P_nQ_{n+1} = -1$, we get

$\delta_{i,n} = (Q_n^{2r}+1)^r R_i(Q_n^r) - S_i(Q_n^r)Q_n^{qr}$

$\delta_{i,n} = S_{q+1}(Q_n^r)R_i(Q_n^r) - S_i(Q_n^r)R_{q+1}(Q_n^r)$

$\delta_{i,n} = \Delta_i(Q_n^r)$ with $\Delta_i = S_{q+1}R_i - S_iR_{q+1}$

In the same way, for n even, by (6), (14) and since we have $P_{n+1}Q_n - P_nQ_{n+1} = 1$, we get

$\delta_{i,n} = (Q_n^{2r}-1)^r R'_i(Q_n^r) - S'_i(Q_n^r)Q_n^{qr}$

We observe, from (14) and (16), that $R'_{q+1} = (-1)^r T^q$ and $S'_{q+1} = (-T^2+1)^r$, so we obtain

$\delta_{i,n} = (-1)^r\Delta'_i(Q_n^r)$, with $\Delta'_i = S'_{q+1}R'_i - S'_iR'_{q+1}$.

Also we have $|R_{q+1}/S_{q+1} - R_i/S_i| = 1/|S_{i+1}S_i|$ and therefore

$$|\Delta_i| = |S_{q+1}S_i|\,|R_{q+1}/S_{q+1} - R_i/S_i| = |S_{q+1}|/|S_{i+1}|$$

By (12) and (13), we see that $|S_i| = |T|^i$ and $|R_i| = |T|^{i-1}$, then we get $|\Delta_i| = |T|^{q-i}$. In the same way, by (16) $|S_i| = |S'_i|$, $|R_i| = |R'_i|$, so we obtain $|\Delta'_i| = |T|^{q-i}$. Thus, as $|S_{i,n}| = |Q_n|^{ri}$, and by (6) $|Q_{n+1}| > |Q_n|^q$, we get

$$|\delta_{i,n}| = |Q_n|^{r(q-i)} < |Q_{n+1}|^r/|S_{i,n}|$$

which is the second part of condition (8), and so by Lemma 2, $R_{i,n}/S_{i,n}$ is a convergent to θ_q, for $n \geq 0$ and $1 \leq i \leq q$.

Now we prove that $R_{i,n}$ and $S_{i,n}$ are coprime. First we show that Δ_i and S_i are coprime (the same for Δ'_i and S'_i). We have $\Delta_i + S_iT^q = (T^2+1)^rR_i$ (or $\Delta'_i + (-1)^rS'_iT^q = (-T^2+1)^rR'_i$). Hence, since R_i and S_i are coprime (or R'_i and S'_i are coprime), we see that if A is a prime common divisor of Δ_i and S_i (or of Δ'_i and S'_i) , then it divides T^2+1 (or T^2-1). Now if S_i has such a divisor then we have $S_i(u) = 0$ or $S_i(-u) = 0$, where u is a square root of -1. From $(13)'$ we deduce

$$S_0(u) = 1 \qquad S_1(u) = u \qquad S_i(u) = 2uS_{i-1}(u) + S_{i-2}(u) \quad \text{for} 1 \leq i \leq q$$

and this implies $S_i(u) = u^i$ for $1 \leq i \leq q$. As S_i is alternatively an odd or even polynomial, we also have $S_i(-u) = (-1)^iS_i(u)$. Therefore $S_i(\pm u) \neq 0$, and consequently Δ_i and S_i are coprime. For Δ'_i and S'_i, the same proof holds. Here we have to prove that $S'_i(\pm 1) \neq 0$, and this is derived from (16), and the fact that $S_i(\pm u) \neq 0$. Hence there are polynomials E and F of $K[T]$ such that

$$E\Delta_i + FS_i = 1 \quad \text{wherefrom} \quad E(Q_n^r)\Delta_i(Q_n^r) + F(Q_n^r)S_i(Q_n^r) = 1$$

Thus $\Delta_i(Q_n^r)$ and $S_i(Q_n^r)$ are coprime (the same for $\Delta'_i(Q_n^r)$ and $S'_i(Q_n^r)$). Now we return to $R_{i,n}$ and $S_{i,n}$. If B is a common divisor of both of them, then B divides $R_{i,n}Q_{n+1}^r - S_{i,n}P_{n+1}^r = \Delta_i(Q_n^r)$ and $S_{i,n} = S_i(Q_n^r)$ (or $(-1)^r\Delta'_i(Q_n^r)$ and $S'_i(Q_n^r)$), and therefore divides 1. So we have the desired result.

Then Lemma 2 applies. By (9), we obtain

$$|a_{m(i,n)+1}| = |\delta_{i,n}|^{-1}|S_{i,n}|^{-1}|Q_{n+1}|^r = |Q_n|^{-r(q-i)}|Q_n|^{-ri}|Q_{n+1}|^r = |T|^r$$

since $|Q_{n+1}| = |T|\ |Q_n|^q$. So by Lemma 1, $a_{m(i,n)+1} = \lambda_{i,n}T^r$, where $\lambda_{i,n}$ is a non-zero element of K.

Now we explicit $R_{1,n}/S_{1,n}$ and $R_{q,n}/S_{q,n}$. Since $R_1 = R'_1 = 1$ and $S_1 = S'_1 = T$, the definition gives immediately the first part of (17) . By (15), we have $(R_q/S_q)(T) = (T^2+1)^{r-1}/T^q$. By (15) and (16), we obtain $(R'_q/S'_q)(T) = (T^2-1)^{r-1}/T^q$. Therefore $R_{q,n}/S_{q,n} = P_n^r(Q_n^{2r} + (-1)^{n-1})^{r-1}/Q_n^{rq}$. Moreover, by (6), we have $Q_n^q = P_{n+1}$ and then $Q_n^{q+1} - (-1)^n = P_nQ_{n+1}$. So $R_{q,n}/S_{q,n} = P_n^{2r-1}Q_{n+1}^{r-1}/P_{n+1}^r$, and (17) is proved. Finally, we have

$$|S_{q,n}| \ = \ |Q_n|^{qr} = (|Q_{n+1}|/|T|)^r = \ |S_{1,n+1}|/|T|^r$$

Since the denominators of the convergents are polynomials of $K[T^r]$, $R_{q,n}/S_{q,n}$ must be the convergent preceding $R_{1,n+1}/S_{1,n+1}$. This is (18), and so Lemma 4 is proved.

Now we can describe partially the continued fraction expansion of θ_q. With the notations of Lemma 4, we can write $R_{i,n}/S_{i,n} = [0, a_1, ..., a_{m(i,n)}]$, for $n \geq 0$ and for $1 \leq i \leq q$. We put $\Omega_{1,n} = a_1, a_2, ..., a_{m(1,n)}$, for all $n \geq 1$.
We can give explicitly $\Omega_{1,1}$ and $\Omega_{1,2}$. By (17), we have $R_{1,n}/S_{1,n} = [0, \Omega_{1,n}] = (P_n/Q_n)^r$. By (6), we get $R_{1,1}/S_{1,1} = (P_1/Q_1)^r = 1/T^r$, so $\Omega_{1,1} = a_1 = T^r$. Further, by (6) and with the notations of Lemma 3, we have

$$R_{1,2}/S_{1,2} = (P_2/Q_2)^r = T^{qr}/(T^{q+1}+1)^r = \Theta_q(T^r)$$

Therefore, by (10), we get

$$(19) \quad \Omega_{1,2} = T^r, 2T^r, 2T^r, ..., 2T^r, T^r \quad \text{(q+1 terms)}$$

We observe that, for $n \geq 1$, we have $m(1,n) < m(2,n) < ..... < m(q,n)$. Indeed $|S_{i+1,n}| > |S_{i,n}|$, since $|S_{i,n}| = |Q_n|^{ir}$ and $|Q_n| > 1$, for $n \geq 1$. Then we put $\Omega'_{i,n} = a_{m(i-1,n)+1}, ..., a_{m(i,n)}$, for $n \geq 1$ and $2 \leq i \leq q$. We define also $\Omega_{i,n}$ by $\Omega'_{i,n} = a_{m(i-1,n)+1}, \Omega_{i,n}$ and $\Omega'_{1,n}$ by $\Omega_{1,n} = T^r, \Omega'_{1,n}$.

If $\Omega = x_1, x_2, ..., x_k$ is a sequence of polynomials, we denote $\widetilde{\Omega}$ the sequence obtained by reversing the terms of Ω, i.e. $\widetilde{\Omega} = x_k, x_{k-1}, ..., x_1$. Also if ϵ is a non-zero element of K we write $\epsilon\Omega$ for $\epsilon x_1, \epsilon^{-1}x_2, ..., \epsilon^{(-1)^{k-1}}x_k$. Notice that if $[\Omega]$ denotes the element of $K(T)$ which has Ω as continued fraction expansion, we have $\epsilon[\Omega] = [\epsilon\Omega]$. Now we can prove the following result.

Lemma 5. *There exists a sequence $(\epsilon_n)_{n\geq 1}$ of non-zero elements of K, such that*

$$(20) \quad a_{m(1,n)-k} = \epsilon_n^{(-1)^k} a_{k+1} \quad \textit{for each } (k,n) \textit{ with } 0 \leq k \leq m(1,n)-1 \textit{ and } n \geq 1$$

Further we have for $n \geq 2$

$$(21) \quad \begin{cases} \Omega_{q,n} = \epsilon_{n+1}^{\pm 1}\widetilde{\Omega'}_{1,n} & \Omega_{q-i,n} = \epsilon_{n+1}^{\pm 1}\widetilde{\Omega}_{i+1,n} \quad \textit{for } 1 \leq i \leq r-2 \\ \lambda_{q,n} = \epsilon_{n+1}^{\pm 1} & \lambda_{q-i,n} = \epsilon_{n+1}^{\pm 1}\lambda_{i,n} \quad \textit{for } 1 \leq i \leq r-1 \end{cases}$$

Proof. By (17) and (18), we can write

$$(22) \qquad U_{m(1,n)} = \epsilon'_n P_n^r \qquad V_{m(1,n)} = \epsilon'_n Q_n^r$$

and

$$(23) \qquad U_{m(1,n)-1} = \epsilon''_n P_{n-1}^q Q_n^{r-1} \qquad V_{m(1,n)-1} = \epsilon''_n P_n^r$$

where ϵ'_n and ϵ''_n are non-zero elements of K. We write $\epsilon_n = \epsilon'_n/\epsilon''_n$.
From the definition of V_k, for each $k \geq 1$, we have $V_k/V_{k-1} = [a_k, a_{k-1}, ..., a_1]$, so we can write $V_{m(1,n)}/V_{m(1,n)-1} = [a_{m(1,n)}, a_{m(1,n)-1}, ..., a_1]$.
On the other hand, by (22) and (23), we have

$$\frac{V_{m(1,n)}}{V_{m(1,n)-1}} = \epsilon_n . \frac{V_{m(1,n)}}{U_{m(1,n)}} = \frac{\epsilon_n}{[0, a_1, ...a_{m(1,n)}]} = \epsilon_n [a_1, ..., a_{m(1,n)}]$$

therefore

$$[a_{m(1,n)}, ..., a_1] = \epsilon_n [a_1, ..., a_{m(1,n)}] = [\epsilon_n a_1, ..., \epsilon_n^{(-1)^{i-1}} a_i, ..., \epsilon_n^{(-1)^{m(1,n)-1}} a_{m(1,n)}]$$

This implies (20) and can be written $\widetilde{\Omega}_{1,n} = \epsilon_n \Omega_{1,n}$.

By Lemma 4 and (18), we have $a_{m(1,n+1)} = a_{m(q,n)+1} = \lambda_{q,n} T^r$, so we can write

$$\Omega_{1,n+1} = \Omega_{1,n}, \Omega'_{2,n}, ..., \Omega'_{q,n}, \lambda_{q,n} T^r$$

since, we also have $a_{m(i,n)+1} = \lambda_{i,n} T^r$, for $1 \leq i \leq q-1$, we obtain

$$(24) \quad \Omega_{1,n+1} = T^r, \Omega'_{1,n}, \lambda_{1,n} T^r, \Omega_{2,n}, \lambda_{2,n} T^r, ..., \Omega_{q,n}, \lambda_{q,n} T^r$$

For each finite sequence of non-zero polynomials, we define its degree as being the sum of the degrees of its terms. We have $\deg \Omega_{1,n} = \deg S_{1,n} = r \deg Q_n$ and, for $2 \leq i \leq q$, $\deg \Omega'_{i,n} = \deg S_{i,n} - \deg S_{i-1,n} = r \deg Q_n$. We put $\omega_n = rq \deg Q_{n-1}$. As $\deg Q_n = q \deg Q_{n-1} + 1$, we get $\deg \Omega_{1,n} = \omega_n + r$ and $\deg \Omega'_{1,n} = \omega_n$. Also, for $2 \leq i \leq q$, $\deg \Omega'_{i,n} = \omega_n + r$ and $\deg \Omega_{i,n} = \omega_n$. If we write the sequence of the degrees of the components in the right side of (24), we obtain the sequence, of $2q+1$ terms: $r, \omega_n, r, \omega_n, ..., r, \omega_n, r$. As this sequence is reversible and $\widetilde{\Omega}_{1,n+1} = \epsilon_{n+1} \Omega_{1,n+1}$, it is clear that $\Omega_{q,n} = \epsilon_{n+1}^{\pm 1} \widetilde{\Omega'}_{1,n}$, $\Omega_{q-1,n} = \epsilon_{n+1}^{\pm 1} \widetilde{\Omega}_{2,n}$,....,$\Omega_{r+1,n} = \epsilon_{n+1}^{\pm 1} \widetilde{\Omega}_{r-1,n}$, and also $\lambda_{q,n} T^r = \epsilon_{n+1}^{\pm 1} T^r$, $\lambda_{q-1,n} T^r = \epsilon_{n+1}^{\pm 1} \lambda_{1,n} T^r$,.......,$\lambda_{r,n} T^r = \epsilon_{n+1}^{\pm 1} \lambda_{r-1,n} T^r$. This is (21). So Lemma 5 is proved.
We can observe that if $\epsilon_n = 1$ then the sequence $\Omega_{1,n}$ is reversible, i.e. $\widetilde{\Omega}_{1,n} = \Omega_{1,n}$. This is actually the case if $m(1,n)$ is odd, say $m(1,n) = 2l+1$, then by (20) we have $a_{l+1} = \epsilon_n^{(-1)^l} a_{l+1}$ and therefore $\epsilon_n = 1$. Notice that we have $\epsilon_1 = 1$ and, since $\Omega_{1,2}$ is reversible by (19), we also have $\epsilon_2 = 1$.

Now we consider the case $q = 3$, $r = 2$. Since $K = \mathbb{F}_3$, we have $\epsilon_n^{\pm 1} = \epsilon_n$, and (20) becomes $(20)' \quad a_{m(1,n)-k} = \epsilon_n a_{k+1} \quad$ for $0 \leq k \leq m(1,n)-1$ and for $n \geq 1$. Using Lemma 5, (24) becomes

$$(24)' \quad \Omega_{1,n+1} = \Omega_{1,n}, \lambda_{1,n} T^2, \Omega_{2,n}, \epsilon_{n+1} \lambda_{1,n} T^2, \epsilon_{n+1} \widetilde{\Omega}_{1,n}$$

In this case the continued fraction expansion of θ_3 will be given explicitly below. We prove the result already obtained by M. Buck and D. Robbins in [2].

Theorem B. *If $q = 3$, we have*

$$(25) \qquad \Omega_{1,n+1} = \Omega_{1,n}, 2T^2, \Omega_{1,n-1}^{(3)}, 2T^2, \Omega_{1,n} \qquad \text{for } n \geq 2$$

Here $\Omega_{1,n-1}^{(3)}$ denotes the sequence obtained by cubing each element of $\Omega_{1,n-1}$.

Proof. Let n be an integer with $n \geq 2$. First we are going to describe $\Omega_{2,n}$. We have $U_{m(1,n)}/V_{m(1,n)} = [0, \Omega_{1,n}]$, $U_{m(1,n)+1}/V_{m(1,n)+1} = [0, \Omega_{1,n}, \lambda_{1,n}T^2]$ and $U_{m(2,n)}/V_{m(2,n)} = [0, \Omega_{1,n}, \lambda_{1,n}T^2, \Omega_{2,n}]$. If we denote $x_{2,n}$, the element of $K(T)$ defined by $[\Omega_{2,n}]$, then it is a classical fact that we have

$$(26) \quad \frac{U_{m(2,n)}}{V_{m(2,n)}} = \frac{x_{2,n}U_{m(1,n)+1} + U_{m(1,n)}}{x_{2,n}V_{m(1,n)+1} + V_{m(1,n)}}$$

We know that $U_{m(2,n)}/V_{m(2,n)} = R_{2,n}/S_{2,n}$. We have $R_2(T) = 2T$, $S_2(T) = 2T^2+1$ and also $R_2'(T) = uR_2(uT) = -2T$, $S_2'(T) = S_2(uT) = -2T^2 + 1$. It follows that $R_{2,n}/S_{2,n} = P_n^2Q_n^2/(Q_n^4 + (-1)^n)$. We put

$$(27) \quad P' = P_n^2Q_n^2 \quad \text{and} \quad Q' = Q_n^4 + (-1)^n$$

Then formula (26) can be solved for $x_{2,n}$, and by (22),we obtain

$$(26)' \quad x_{2,n} = \epsilon_n' \frac{P_n^2Q' - Q_n^2P'}{V_{m(1,n)+1}P' - U_{m(1,n)+1}Q'}$$

We have to determine $U_{m(1,n)+1}/V_{m(1,n)+1}$. We use Lemma 2, and the fact that $R_{3,n-1}/S_{3,n-1}$ and $R_{1,n}/S_{1,n}$ are, by Lemma 4, the two convergents preceding it.

So, we consider the polynomials P and Q of $K[T]$, defined by

$$(28) \qquad P = 2T^2P_n^2 + P_{n-1}^3Q_n \quad \text{and} \quad Q = 2T^2Q_n^2 + P_n^2$$

We apply Lemma 2, to show that P/Q is a convergent to θ_3. First we have deg $Q =$ 2deg Q_n+2 and thus $Q \neq 0$. By (28) and (6), we have $PQ_n^2 - QP_n^2 = P_{n-1}^3Q_n^3 - P_n^4 = P_{n-1}^3Q_n^3 - P_n^3Q_{n-1}^3 = (-1)^n$, so that $(P, Q) = 1$. Since 2deg $Q_n + 2 <$ 2deg Q_{n+1} for $n \geq 2$, the first part of condition (8), i.e. $|Q| < |Q_{n+1}|^2$, is satisfied. Let us show that $|PQ_{n+1}^2 - QP_{n+1}^2| < |Q_{n+1}|^2/|Q|$, is also satisfied. We put

$$X_1 = Q_{n+1}^2P_n^2 - Q_n^2P_{n+1}^2 \quad \text{and} \quad X_2 = P_{n-1}^3Q_nQ_{n+1}^2 - P_n^2P_{n+1}^2$$

By (28), we observe that $PQ_{n+1}^2 - QP_{n+1}^2 = 2T^2X_1 + X_2$. As $P_{n+1}Q_n - Q_{n+1}P_n = (-1)^n$, and using (6), we have

$$X_1 = (-1)^{n+1}(2Q_nP_{n+1} + (-1)^{n+1}) = (-1)^{n+1}(2Q_n^4 + (-1)^{n+1}) = (-1)^nQ_n^4 + 1$$

then

$$X_2 = Q_{n+1}^2P_{n-1}^3Q_n - P_{n+1}^2P_n^2 = (Q_{n+1}/Q_n)^2((-1)^n + P_n^4) - P_{n+1}^2P_n^2$$

$$X_2 = (Q_{n+1}/Q_n)^2(-1)^n + (P_n/Q_n)^2X_1$$

$X_2 = ((Q_{n+1}/Q_n)^2 + (P_nQ_n)^2)(-1)^n + (P_n/Q_n)^2$

We put $X = PQ_{n+1}^2 - QP_{n+1}^2$. As $X = 2T^2X_1 + X_2$, we have

$X = 2T^2 + (-1)^n(2T^2Q_n^4 + (Q_{n+1}/Q_n)^2 + (P_nQ_n)^2 + (-1)^n(P_n/Q_n)^2)$

$X = 2T^2 + (-1)^n(2T^2Q_n^4 + (TQ_n^2 + P_n^3/Q_n)^2 + (P_n/Q_n)^2(Q_n^4 + (-1)^n))$

As $Q_n^4 - TP_nQ_n^3 - P_n^4 = P_{n+1}Q_n - Q_{n+1}P_n = (-1)^n$, we get

$X = 2T^2 + (-1)^n(2TQ_nP_n^3 + P_n^6/Q_n^2 + (P_n/Q_n)^2(2Q_n^4 - TP_nQ_n^3 - P_n^4))$

$X - 2T^2 = (-1)^n(TQ_nP_n^3 + 2P_n^2Q_n^2) = (-1)^nP_n^2Q_n(TP_n - Q_n) = (-1)^{n+1}P_n^2Q_nP_{n-1}^3$

Since, for $n \geq 2$, $|P_{n-1}^3| < |Q_n|$ and $|P_n| < |Q_n|$, this equality implies

$$|X| < |Q_n|^4 = \frac{|Q_{n+1}|^2}{|Q|}$$

so (8) is satisfied. Hence P/Q is a convergent to θ_3, and , since deg Q =deg $V_{m(1,n)}$+ 2 and $\theta_3 \in \mathbb{F}_3((T^{-2}))$, it is the next after $U_{m(1,n)}/V_{m(1,n)}$. Therefore we can write

$$(29) \qquad U_{m(1,n)+1} = \eta_n P \quad \text{and} \quad V_{m(1,n)+1} = \eta_n Q$$

where η_n is an inversible element of $\mathbb{F}_3$. By (22), (23), (28), and $\epsilon^{-1} = \epsilon$ for $\epsilon \in \mathbb{F}_3^*$, the first equality of (29) can be written

$$a_{m(1,n)+1}U_{m(1,n)} + U_{m(1,n)-1} = \eta_n\epsilon_n' 2T^2U_{m(1,n)} + \eta_n\epsilon_n'' U_{m(1,n)-1}$$

Since we have deg $U_{m(1,n)}$ >deg $U_{m(1,n)-1}$, it follows that $a_{m(1,n)+1} = \eta_n\epsilon_n' 2T^2$ and $\eta_n\epsilon_n'' = 1$, i.e. $\eta_n = \epsilon_n''$. Thus, since $\epsilon_n'\epsilon_n'' = \epsilon_n$, we obtain

$$(30) \qquad a_{m(1,n)+1} = \epsilon_n 2T^2$$

Now we come back to $(26)'$. By (29), as $\eta_n = \epsilon_n''$ and $\epsilon_n'\epsilon_n'' = \epsilon_n$, $(26)'$ implies

$$(31) \qquad x_{2,n} = \epsilon_n \frac{P_n^2Q' - Q_n^2P'}{QP' - PQ'}$$

So we can compute $x_{2,n}$. By (27) and (6),

$P_n^2Q' - Q_n^2P' = P_n^2(Q_n^4 + (-1)^n) - Q_n^2P_n^2Q_n^2 = (-1)^nP_n^2 = (-1)^nQ_{n-1}^6$

By (27), (28), and (6),

$QP' - PQ' = P_n^2Q_n^2(2T^2Q_n^2 + P_n^2) - (Q_n^4 + (-1)^n)(2T^2P_n^2 + P_{n-1}^3Q_n)$

$QP' - PQ' = P_n^4Q_n^2 - Q_n^5P_{n-1}^3 - (-1)^n(2T^2P_n^2 + P_{n-1}^3Q_n)$

$QP' - PQ' = Q_n^2(P_nQ_{n-1} - Q_nP_{n-1})^3 + (-1)^n(T^2P_n^2 - Q_n^2 + TQ_nP_n)$

$QP' - PQ' = (-1)^n(T^2P_n^2 + Q_n^2 + TQ_nP_n)$

$QP' - PQ' = (-1)^n(Q_n - TP_n)^2 = (-1)^nP_{n-1}^6$

Hence, by (31), we obtain (32) $x_{2,n} = \epsilon_n (Q_{n-1}/P_{n-1})^6$. Now we observe that

$$[a_1, ..., a_{m(1,n-1)}] = 1/[0, a_1, ..., a_{m(1,n-1)}] = 1/(P_{n-1}/Q_{n-1})^2 = (Q_{n-1}/P_{n-1})^2$$

and , since $K = \mathbb{F}_3$, we have

$$\epsilon_n (Q_{n-1}/P_{n-1})^6 = [\epsilon_n a_1^3, ..., \epsilon_n a_{m(1,n-1)}^3]$$

So, by (32) and $x_{2,n} = [\Omega_{2,n}]$, we obtain

$$(33) \quad \Omega_{2,n} = \epsilon_n a_1^3, ..., \epsilon_n a_{m(1,n-1)}^3$$

According to (30) and (33), we can write $(24)'$ in the following way

$$(34) \quad \Omega_{1,n+1} = \Omega_{1,n}, \epsilon_n 2T^2, \epsilon_n a_1^3, ..., \epsilon_n a_{m(1,n-1)}^3, \epsilon_{n+1}\epsilon_n 2T^2, \epsilon_{n+1}\widetilde{\Omega}_{1,n}$$

So by Lemma 5 and $(20)'$ we have simultaneously $\epsilon_n a_{m(1,n-1)}^3 = \epsilon_{n+1}\epsilon_n a_1^3$, which implies $a_{m(1,n-1)} = \epsilon_{n+1} a_1$ and $a_{m(1,n-1)} = \epsilon_{n-1} a_1$. Therefore $\epsilon_{n+1} = \epsilon_{n-1}$ for all $n \geq 2$. Since $\epsilon_2 = \epsilon_1 = 1$, it follows that $\epsilon_n = 1$ for all $n \geq 1$. Finally, by $(20)'$, the sequence $\Omega_{1,n}$ is reversible for all $n \geq 1$, and so $\widetilde{\Omega}_{1,n} = \Omega_{1,n}$. So (34) becomes (25) for $n \geq 2$, and the theorem is proved.

Remark. We have observed the begining of the continued fraction expansion of θ_q by computer, for $q \leq 27$. In all cases and for the values of n that we could reach, we had

$$\epsilon_n = 1, \quad \lambda_{q,n} = 1, \quad \lambda_{i,n} = 2 \quad \text{for } 1 \leq i < q \quad \text{and} \quad \Omega_{r,n} = \Omega_{1,n-1}^{(q)}$$

as it does happen for $q = 3$. So, for $q > 3$, we can conjecture that (24) becomes

$$\Omega_{1,n+1} = \Omega_{1,n}, 2T^r, \Omega_{2,n}, ..., \Omega_{r-1,n}, 2T^r, \Omega_{1,n-1}^{(q)}, 2T^r, \widetilde{\Omega}_{r-1,n}, ..., \widetilde{\Omega}_{2,n}, 2T^r, \Omega_{1,n}$$

For $n \geq 2$, we denote $J_{n+1}(q) = 2T^r, \Omega_{2,n}, ..., \Omega_{r-1,n}, 2T^r$ and $j_n(q)$ the degree of $J_n(q)$, we have $j_{n+1}(q) = (r-2)\omega_n + (r-1)r = (r-2)rq\deg\, Q_{n-1} + (r-1)r$. We denote $j'_n(q)$ the highest degree in T^r of the terms in $J_n(q)$, then we have $j'_{n+1}(q) \leq \omega_n/r = q\deg\, Q_{n-1}$. Now we observe that if $j'_n(q)$ were not too large, then the number of terms in $J_n(q)$ would increase with n , because $j_n(q)$ does so. In that direction, we have observed the following data about $J_n(q)$:

Table giving the number of terms of $J_n(q)$ and (between brackets) the highest degree (in T^r)of those terms.

n:q	5	7	9	11	13
3	5(3)	13(3)	21(5)	35(5)	49(7)
4	22(3)	93(3)	154(9)	413(5)	754(7)
5	99(7)	599(7)	1239(15)		
n:q	17	19	23	25	27
3	85(9)	111(9)	167(11)	193(13)	231(13)
4	1844(9)	2677(9)			

Of course we expect the element θ_q to satisfy Roth's theorem for all power q of an odd prime number p, as it does for $q = 3$. Using the same arguments as the one developed in §**2.**, this would result from the conjecture $\Omega_{r,n} = \Omega^{(q)}_{1,n-1}$ and $j'_{2k+1}(q) < q^k$, $j'_{2k+2}(q) \leq q^k$ (cf. *Table*).
If we replace the element θ_q by the element α_q^k, for $1 \leq k \leq r$, we can see, as we did for θ_q, that $(P_n/Q_n)^k$ is a convergent to α_q^k , as soon as k and p are coprime. Therefore, in that situation, the approximation exponent of α_q^k is at least $(q+1)/k - 1$. We may suppose that this approximation exponent is indeed equal to $(q+1)/k - 1$ (i.e. there are no essentially better approximations to α_q^k than $(P_n/Q_n)^k$, consequently θ_q satisfies Roth's theorem). This is proved, in [8], for $(q+1)/k$ sufficiently large. If it were true for all k, with $(k,p) = 1$, we wonder whether it could be established without the help of the continued fraction expansion of α_q^k.

§4. The continued fraction expansion of a classical example .

In this last section we would like to give a result which is indirectly connected with the subject presented above. When we started our investigation from Buck and Robbins paper ([2]), we studied the method they have used to be able to describe the continued fraction expansion of θ_3^* . Their idea is to start from an algebraic element, to observe the begining of its continued fraction expansion by computer, to guess its pattern and then to show that the element defined by this expansion satisfies the desired equation. We have tried to apply this approach to the celebrated example given by Mahler in [4], and so we have succeeded in describing entirely the continued fraction expansion of this element. Curiously this result does not seem to be known, so we give it here. We will only give a brief survey of the proof.

We have the following result:

Theorem C. *Let p be a prime number, $q = p^s$ for $s \in \mathbb{N} - \{0\}$, $q > 2$, and $K = \mathbb{F}_p$. Let α be the element of $K((T^{-1}))$, defined by*

$$(1) \qquad \alpha = 1/T + \alpha^q \qquad \text{and} \qquad |\alpha| = |T|^{-1}$$

Let us define the sequence $(\Omega_n)_{n>0}$ of finite sequences of elements of $K[T]$, recursively by :

$$(R) \qquad \Omega_1 = T \qquad \Omega_n = \Omega_{n-1}, -T^{(q-2)q^{n-2}}, -\widetilde{\Omega}_{n-1} \qquad \text{for } n \geq 2$$

where $\widetilde{\Omega} = a_m, a_{m-1}, ..., a_1$ and $-\Omega = -a_1, -a_2, ..., -a_m$, if $\Omega = a_1, a_2, ..., a_m$. Let Ω_∞ be the infinite sequence begining by Ω_n for all $n \geq 1$. Then the continued fraction expansion of α is $[0; \Omega_\infty]$

To prove this, we start from the element $\alpha = [0; \Omega_\infty]$. For $n \geq 1$, we put

$$\Omega_n = a_1, a_2, ..., a_{m(n)} \quad r_n/s_n = [0, a_1, a_2, ..., a_{m(n)-1}] \quad t_n/u_n = [0, a_1, a_2, ..., a_{m(n)}]$$

Then we show, from the relation (R), that, for $n \geq 1$, we have

$$r_n = -u_n z_n^2 \qquad s_n = 1 - u_n z_n \qquad t_n = 1 + u_n z_n \qquad u_n = T^{q^{n-1}}$$

where $z_n = \sum_{0 \leq k \leq n-2} T^{-q^k}$, for $n \geq 2$, and $z_1 = 0$. Now we define $\delta_n = r_n/s_n - (r_n/s_n)^{q-1}(t_n/u_n) - T^{-1}$. It is clear that δ_n tends to $\alpha - \alpha^q - T^{-1}$. At last we show that $\lim_n \delta_n = 0$, and so the proof is complete.

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