CONTINUED FRACTIONS FOR ALGEBRAIC FORMAL POWER SERIES OVER A FINITE BASE FIELD

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ABSTRACT

We consider the continued fraction expansion of certain algebraic formal power series when the base field is finite. We are concerned by the property of the sequence of partial quotients being bounded or unbounded. We formalize the approach introduced by L. Baum and M. Sweet in [1], which applies to the elements of a particular subset of algebraic power series. We illustrate this method with a result when the base field is $\mathbb{F}_2$.

§1. Introduction.

Let K be a field. We consider the field $K((T^{-1}))$ of formal Laurent series in T^{-1}. If $\alpha \in K((T^{-1}))$, and $\alpha \neq 0$, we have $\alpha = \sum_{k \leq k_0} a_k T^k$, with $k_0 \in \mathbb{Z}$, $a_k \in K$ and $a_{k_0} \neq 0$. We define the degree of α, by deg $\alpha = k_0$, and deg $0 = -\infty$. Then we define the ultrametric absolute value by $|\alpha| = |T|^{\deg \alpha}$ and $|0| = 0$, where $|T|$ is a fixed real number greater than 1. This field $K((T^{-1}))$ can be identified with the completion of $K(T)$ for this absolute value. Like in the classical case of the real numbers, we have a continued fraction theory, the partial quotients being elements of $K[T]$. Here we are concerned with the case when the base field K is finite.

In 1976 L. Baum and M. Sweet [1] showed that the unique solution in $K((T^{-1}))$ of the irreducible equation

$$Tx^3 + x + T = 0, \tag{1}$$

when the base field is $K = \mathbb{F}_2$, has a continued fraction expansion with partial quotients of bounded degree. They observed that no real algebraic number of degree ≥ 3 has yet been shown to have bounded or unbounded partial quotients.

In 1986, W. Mills and D. Robbins [8] observed that equation (1) should be looked at in a special way, that is as $x = T/(Tx^2 + 1)$. Indeed they suggested considering the set, which we will call $\mathcal{H}$, of the irrational elements in $K((T^{-1}))$, satisfying an algebraic equation of the following form

$$x = (Ax^q + B)/(Cx^q + D), \tag{2}$$

where A, B, C, and D are elements of $K[T]$, and q is a power of the characteristic p of the field K. In their paper they described an algorithm to give the explicit continued fraction expansion of an element satisfying (2). Hence they could give the explicit continued fraction of the solution of (1). Also, using this algorithm, they gave an example of a non-quadratic element in $\mathcal{H}$ with bounded partial quotients, when $K = \mathbb{F}_p$, and for all p greater than 2.

Later, the rational approximation properties of the elements in $\mathcal{H}$, were studied independently by J. Voloch [9], and B. de Mathan [7]. They showed that, for $\alpha \in \mathcal{H}$, if $\liminf_{|Q| \to \infty} |Q|^2 |\alpha - P/Q| = 0$, then there exist a real number

$\mu > 2$ and a real number $\delta > 0$ such that $\liminf_{|Q|\to\infty} |Q|^{\mu}|\alpha - P/Q| = \delta$, where $P, Q \in K[T]$ and $Q \neq 0$.

The set $\mathcal{H}$ contains elements which are very well approximated by rationals. A famous example in $\mathbb{F}_p((T^{-1}))$, which was given by K. Mahler in 1949 [5], satisfies the algebraic equation $x = 1/T + x^p$. For this element α, algebraic of degree p, we have rationals P/Q, with $|Q|$ arbitrarily large, and $|\alpha - P/Q| = |Q|^{-p}$. With respect to this, B. de Mathan and the author [3], have recently shown that if an algebraic element does not belong to $\mathcal{H}$, then it cannot be too well aproximated by rationals : if $\alpha \notin \mathcal{H}$ and is algebraic of degree $n > 1$ over $K(T)$, then, for all $\epsilon > 0$, we have $|Q\alpha - P| > |Q|^{-([n/2]+\epsilon)}$, for all $P/Q \in K(T)$ with $|Q|$ large enough. This last property highlights the peculiarity of this set $\mathcal{H}$.

In this paper, as observed by L. Baum and M. Sweet and later by W. Mills and D. Robbins, we want to stress the fact that this set $\mathcal{H}$ also contains non-quadratic elements which are badly approximable by rationals. By this we mean that, for such an element α, we have $|\alpha - P/Q| \geq C|Q|^{-2}$ for all $P/Q \in K(T)$, where C is a fixed positive real number. In other words, these elements have bounded partial quotients in their continued fraction expansion.

§2. The main result.

Let p be a prime number, $K = \mathbb{F}_p$, the field with p elements, and $q = p^s$ where s is a positive integer. Let A, B, C and $D \in K[T]$, coprime, such that $\Delta = AD - BC \neq 0$. Let us suppose that there is an irrational $\alpha \in K((T^{-1}))$, such that

$$\alpha = (A\alpha^q + B)/(C\alpha^q + D). \tag{1}$$

Let f be the linear fractional transformation defined on $K((T^{-1}))\backslash\{-D/C\}$, or $K((T^{-1}))$ if $C = 0$, by $f(x) = (Ax + B)/(Cx + D)$. We observe that f is invertible and for $x \neq A/C$ we have $f^{-1}(x) = (Dx - B)/(-Cx + A)$.

We are interested in the continued fraction expansion for α. The formalisation of the continued fraction algorithm in $K((T^{-1}))$ apparently goes back to E. Artin's work in his thesis; for general references on this subject see for instance [1,6]. Since α is not rational, this expansion is infinite and will be denoted by $\alpha = [a_0, a_1, a_2, ..., a_n, ...]$. The $a_i \in K[T]$ are called the *partial quotients*, and we have $\deg a_i > 0$ for $i > 0$. We consider the sequence $(U_n/V_n)_{n\geq 0}$ of the convergents to α, which is defined for $n \geq 2$, by

$$U_n = a_n U_{n-1} + U_{n-2} \quad \text{and} \quad V_n = a_n V_{n-1} + V_{n-2},$$

with the initial conditions $U_0 = a_0$, $U_1 = a_0 a_1 + 1$, $V_0 = 1$ and $V_1 = a_1$. We introduce the set E of all the convergents to α. When we write $U/V \in E$, we suppose that U and V are coprime polynomials, so that U and V are defined up to a multiplicative factor of K^*. Let $P, Q \in K[T]$, with $Q \neq 0$ and $\gcd(P, Q) = 1$, such that P/Q is a convergent to α. If $P/Q = [a_0, a_1, ..., a_n]$, we will denote $a(P, Q) = a_{n+1}$. We recall that, for $n \geq 0$, we have $|\alpha - U_n/V_n| = |V_n V_{n+1}|^{-1}$ (see for instance [6]). Thus for $P/Q \in E$, we obtain the following equation :

$$|Q\alpha - P| = |Q|^{-1}\ |a(P, Q)|^{-1}. \tag{2}$$

Now we put

$$\begin{cases} P_1 = DP - BQ \\ Q_1 = -CP + AQ \end{cases} \quad \text{and} \quad \begin{cases} P_2 = AP^q + BQ^q \\ Q_2 = CP^q + DQ^q \end{cases} \tag{3}$$

so that we have

$$P_1/Q_1 = f^{-1}(P/Q) \quad \text{and} \quad P_2/Q_2 = f((P/Q)^q). \tag{4}$$

We observe that, for $x, y \in K((T^{-1})) \setminus \{-D/C, A/C\}$, we have

$$f(x) - f(y) = \frac{\Delta(x-y)}{(Cx+D)(Cy+D)} \quad \text{and} \quad f^{-1}(x) - f^{-1}(y) = \frac{\Delta(x-y)}{(A-Cx)(A-Cy)}. \tag{5}$$

Since $\Delta \neq 0$, we have $\alpha \neq A/C$. Therefore, by (4) and (5), we get, if $Q_1 \neq 0$,

$$\alpha^q - P_1/Q_1 = f^{-1}(\alpha) - f^{-1}(P/Q) = \frac{\Delta(Q\alpha - P)}{(A - C\alpha)Q_1}, \tag{6}$$

and also, if $Q_2 \neq 0$,

$$\alpha - P_2/Q_2 = f(\alpha^q) - f((P/Q)^q) = \frac{\Delta(Q\alpha - P)^q}{(C\alpha^q + D)Q_2}. \tag{7}$$

Now, let us introduce two subsets, E_1 and E_2, of E. If $C = 0$, then we put $E_1 = E_2 = E$. If $C \neq 0$, then

$$E_1 = \{P/Q \in E \ : \ |\alpha - P/Q| < |\alpha - A/C| \ \},$$

and

$$E_2 = \{P/Q \in E \ : \ |\alpha - P/Q| < |\alpha^q + D/C|^{1/q} \ \}.$$

It is clear that if $P/Q \in E_1$, we have $|C(P/Q) - A| = |C\alpha - A|$, and this implies, by (3), $Q_1 \neq 0$ and

$$|Q_1| = |Q| \ |C\alpha - A|. \tag{8}$$

In the same way, if $P/Q \in E_2$, then we have $|C(P/Q)^q + D| = |C\alpha^q + D|$, and this implies, by (3), $Q_2 \neq 0$ and

$$|Q_2| = |Q|^q \ |C\alpha^q + D|. \tag{9}$$

Then, if $P/Q \in E_1$, the three equations (2), (6) and (8) lead to

$$|Q_1\alpha^q - P_1| = |\Delta| \ |a(P,Q)|^{-1}|Q_1|^{-1}. \tag{10}$$

In the same way, if $P/Q \in E_2$, the three equations (2), (7) and (9) lead to

$$|Q_2\alpha - P_2| = |\Delta| \ |a(P,Q)|^{-q}|Q_2|^{-1}. \tag{11}$$

Now we put, for $i = 1$ or 2, $\delta_i(P,Q) = \gcd(P_i, Q_i)$. The systems (3) can be solved and we get, respectively,

$$\begin{cases} CP_1 + DQ_1 = \Delta Q \\ AP_1 + BQ_1 = \Delta P \end{cases} \quad \text{and} \quad \begin{cases} -CP_2 + AQ_2 = \Delta Q^q \\ DP_2 - BQ_2 = \Delta P^q \end{cases}$$

This shows that, for $i = 1$ or 2, $\delta_i(P,Q)$ divides Δ. Let us put, for $i = 1$ or 2, $P_i = P_i'\delta_i(P,Q)$ and $Q_i = Q_i'\delta_i(P,Q)$. Then the equations (10) and (11) can be respectively written :

$$|Q_1'\alpha^q - P_1'| = |\Delta|\ |\delta_1(P,Q)|^{-2}\ |a(P,Q)|^{-1}|Q_1'|^{-1} \qquad (10')$$

and

$$|Q_2'\alpha - P_2'| = |\Delta|\ |\delta_2(P,Q)|^{-2}\ |a(P,Q)|^{-q}|Q_2'|^{-1}. \qquad (11')$$

After these preliminaries, we can now establish the following three lemmas. The first is a consequence of some basic properties of the continued fraction algorithm for formal power series.

LEMMA 1. *Let α be an irrational element in $K((T^{-1}))$, satisfying (1). If $P/Q \in E_1$ and if P_1' or Q_1' is not a q-th power of an element of $K[T]$, then we have*

$$|a(P,Q)| < |\Delta|\ |\delta_1(P,Q)|^{-2}.$$

PROOF: The proof is based upon two general properties of the continued fraction expansion of an element in $K((T^{-1}))$. Let P and $Q \in K[T]$ be such that $Q \neq 0$ and $\gcd(P,Q) = 1$. The first classical result is the following : if $|Q\alpha - P| < |Q|^{-1}$ then P/Q is a convergent to α (see for instance [6]). The second one is : if $|Q\alpha - P| = |Q|^{-1}$ then there are two consecutive convergents to α, U/V and U'/V', and two non-zero elements in K, λ and μ such that $P = \lambda U + \mu U'$ and $Q = \lambda V + \mu V'$. Both results are established in [1], for $K = \mathbb{F}_2$, and can be transposed without difficulty to the general case.
We are now going to show that if $(*)$ $|a(P,Q)| \geq |\Delta|\ |\delta_1(P,Q)|^{-2}$, then P_1' and Q_1' are q-th powers in $K[T]$, which will prove this lemma. We observe that if we have a strict inequality in $(*)$, then, by $(10')$ we obtain

$$|Q_1'\alpha^q - P_1'| < |Q_1'|^{-1}.$$

According to the previous remark, this shows that P_1'/Q_1' is a convergent to α^q. Now, due to the Frobenius homomorphism, if $\alpha = [a_0, a_1, ..., a_n, ..]$ then $\alpha^q = [a_0^q, a_1^q, ..., a_n^q, ..]$, and therefore the convergents to α^q are the q-th powers of the convergents to α. Thus P_1' and Q_1' are q-th powers in $K[T]$. Now if we have equality in $(*)$, then, by $(10')$, we get

$$|Q_1'\alpha^q - P_1'| = |Q_1'|^{-1}.$$

According to the second result, mentioned above, P_1', and Q_1', will be a linear combination, with coefficients in K, of two q-th powers in $K[T]$, and therefore also a q-th power in $K[T]$. So the lemma is proved.

The second lemma is a formalisation and a generalisation of an idea found in [1].

LEMMA 2. *Let α be an irrational element in $K((T^{-1}))$, satisfying (1). Let us suppose that we have $|a(P,Q)| \leq |\Delta|^{1/(q-1)}$, assuming that either $P/Q \in E$ and $P/Q \notin E_1$, or $P/Q \in E_1$ and $|\delta_1(P,Q)| < |\Delta|$. Then, for all $P/Q \in E$, we have $|a(P,Q)| \leq |\Delta|^{1/(q-1)}$.*

PROOF: We recall that if $P/Q \in E$, then $\delta_1(P,Q)$ divides Δ, therefore we have $|\delta_1(P,Q)| \leq |\Delta|$. Let us suppose that there exists $P/Q \in E$ such that $|a(P,Q)| > |\Delta|^{1/(q-1)}$, and moreover that P/Q is chosen so that $|a(P,Q)|$ is minimal. Then we must have $P/Q \in E_1$ and $|\delta_1(P,Q)| = |\Delta|$. So $(10')$ becomes

$$|Q_1'\alpha^q - P_1'| = |\Delta|^{-1}\ |a(P,Q)|^{-1}\ |Q_1'|^{-1}$$

This shows that P_1'/Q_1' is a convergent to α^q, and therefore there exist U and $V \in K[T]$, coprime, such that $P_1' = U^q$ and $Q_1' = V^q$. Then the above equation implies

$$|V\alpha - U| = |\Delta|^{-1/q}\ |a(P,Q)|^{-1/q}\ |V|^{-1}.$$

This shows that U/V is a convergent to α and that we have

$$|a(U,V)| = |\Delta|^{1/q}\ |a(P,Q)|^{1/q}.$$

Since $|a(P,Q)| > |\Delta|^{1/(q-1)}$, we obtain

$$|\Delta|^{1/(q-1)} < |a(U,V)| < |a(P,Q)|$$

which contradicts our assumption that $|a(P,Q)|$ is minimal. So the proof is complete.

The last lemma is due to Mills and Robbins and first appeared in [8]. The ideas involved have been developed independently by Voloch [9], and de Mathan [7]. We will see in this lemma that the bound $|\Delta|^{1/(q-1)}$, introduced in Lemma 2, is a critical value.

LEMMA 3. *Let α be an irrational element in $K((T^{-1}))$, satisfying (1). Let us suppose that there exists $P/Q \in E_2$ such that*

$$i)|\alpha - P/Q| < |\Delta|^{-1/(q-1)}\ |C\alpha^q + D|^{2/(1-q)} \quad \text{and} \quad ii)|a(P,Q)| > |\Delta|^{1/(q-1)}.$$

Then the sequence of the partial quotients of the continued fraction expansion for α is unbounded.

PROOF: We first show that if $P/Q \in E_2$ is such that *ii*) is satisfied, then there is $U/V \in E$, such that $|a(U,V)| > |a(P,Q)|$. We use equation $(11')$, which was stated above. Consequently, we have

$$|\alpha - P_2'/Q_2'| \leq |\Delta|\ |a(P,Q)|^{-q}\ |Q_2'|^{-2}.$$

Now the hypothesis *ii*) implies that $|\Delta|\ |a(P,Q)|^{-q} < 1$, therefore P_2'/Q_2' is a convergent to α and we have

$$|a(P_2', Q_2')|^{-1} \leq |\Delta|\ |a(P,Q)|^{-q}.$$

Thus we obtain $|a(P_2', Q_2')| \geq |\Delta|^{-1}\ |a(P,Q)|^q > |a(P,Q)|$. Now let us prove that $P_2'/Q_2' \in E_2$. By (7) and (9), we have

$$|\alpha - P_2'/Q_2'| = |\alpha - P/Q|^q\ |\Delta|\ |C\alpha^q + D|^{-2}.$$

On the other hand, the hypothesis *i)* implies that

$$|\alpha - P/Q|^{q-1} < |\Delta|^{-1}\ |\ |C\alpha^q + D|^2.$$

Combining those two relations, we get

$$|\alpha - P_2'/Q_2'| < |\alpha - P/Q|.$$

So we see that the hypotheses in the lemma hold for the convergent P_2'/Q_2', and step by step we obtain a strictly increasing sub-sequence for the absolute values of the partial quotients. This completes the proof of the lemma.

REMARK. This last lemma shows that a great number of irrational elements in $K((T^{-1}))$, satisfying (1), will have an unbounded sequence of partial quotients. This is certainly the case if the critical bound $|\Delta|^{1/(q-1)}$ is less than $|T|$, and it is necessarily so if $|\Delta|$ is fixed and if q is large enough (i.e. $q > 1 + \deg\Delta$). In order to get examples with a bounded sequence of partial quotients, we will use Lemmas 1 and 2. The basic idea is that the linear fractional transformation which is involved in equation (1), has to be be chosen such that the polynomials P_1' and Q_1' cannot be both a q-th power in $K[T]$.

To illustrate the possible use of what has just been discussed, we give an example below. We are aware that this example remains very specific, and close to the example introduced by L. Baum and M. Sweet in [1]. It would be particularly interesting, if possible, to extend this type of result to characteristic other than 2. We prove the following theorem :

THEOREM. *Let l be a positive integer. Let $D \in \mathbb{F}_2[T]$ be such that $D(0) = 1$. We consider the algebraic equation*

$$(E) \qquad Tx^3 + Dx + T^l = 0.$$

Let α be an irrational solution of (E) in $\mathbb{F}_2((T^{-1}))$. Then

i) if $|\alpha| \geq |T|^{-(l+1)}$, the sequence of the partial quotients of the continued fraction expansion for α is bounded by $|T|^{l+1}$.

ii) if $|\alpha| < |T|^{-(l+1)}$, the sequence of the partial quotients of the continued fraction expansion for α is unbounded.

REMARK. The existence of an irrational solution of (E) depends on the choice of D and of l. We can indicate some cases where this solution exists and is unique in $\mathbb{F}_2((T^{-1}))$. So for $l = 1$ and $D = 1$, the solution of (E) is the cubic example given by L. Baum and M. Sweet, we have $|\alpha| = 1$ and the partial quotients of its continued fraction expansion are bounded by $|T|^2$. Also if $m = \deg D$, and if $1 \leq l \leq m$, with $(l,m) \neq (1,1)$, equation (E) has a unique solution α, with $|\alpha| = |T|^{l-m}$. In this last situation, if this solution is irrational, the theorem implies that the partial quotients of its continued fraction expansion are bounded by $|T|^{l+1}$, if and only if $\lfloor m/2 \rfloor \leq l \leq m$.

PROOF: Equation (E) can be written $x = T^l/(Tx^2+D)$. We are therefore in the above situation, with $A=0$, $B=T^l$, $C=T$ and $p=q=2$. We have $\Delta = T^{l+1}$.

To prove the first part $i)$ of the theorem, we shall apply Lemma 2. Let us first show that if $P/Q \notin E_1$, then we have $|a(P,Q)| \leq |\Delta| = |T|^{l+1}$. Here we have $P/Q \in E_1$ if and only if $|\alpha - P/Q| < |\alpha|$. So we see that, if $|\alpha| \geq 1$, then $E_1 = E$. If $|T|^{-(l+1)} \leq |\alpha| < 1$, then the first convergent is 0 and $E_1 = E \setminus \{0\}$. In the first case, as $E_1 = E$, there is nothing to prove. In the second one, we have to estimate $|a(0,1)|$. But then we have

$$|a(0,1)| = |\alpha|^{-1} \leq |T|^{l+1} = |\Delta|.$$

The hypothesis of Lemma 2 will be satisfied if, for $P/Q \in E_1$, $|\delta_1(P,Q)| < |T|^{l+1}$ implies that $|a(P,Q)| \leq |T|^{l+1}$.
Here we have, by (3),

$$P_1 = DP + T^l Q \quad \text{and} \quad Q_1 = TP$$

We know that $\delta_1(P,Q)$ divides T^{l+1}. Let us consider the different possible values of $\delta_1(P,Q)$. At each step, we will use the fact that $D(0)=1$, that is to say that T does not divide D.
It is clear that $\delta_1(P,Q)=1$ if and only if T does not divide P.
Furthermore, for $1 \leq i \leq l-1$, we see that $\delta_1(P,Q) = T^i$ if and only if T^i divides P and T^{i+1} does not divide P.
If T^l divides P then $\delta_1(P,Q) = T^l$ or T^{l+1}. But then $P_1 = T^l(D(P/T^l)+Q)$ and T does not divide Q. Thus T divides $D(P/T^l)+Q$ if and only if T does not divide P/T^l. Consequently, we have $\delta_1(P,Q) = T^l$ if and only if T^{l+1} divides P.
Now we can show that, for $0 \leq i \leq l-1$, if $\delta_1(P,Q) = T^i$, then Q_1' is not a square. Indeed we have $Q_1' = T(P/T^i)$ and the factor P/T^i is not divisible by T, thus Q_1' cannot be a square. We can apply Lemma 1 to this situation, and we get $|a(P,Q)| < |\Delta|$.
It remains to study the case when $\delta_1(P,Q) = T^l$. Equation $(10')$ becomes

$$|Q_1'\alpha^2 - P_1'| = |T|^{-l+1}\ |a(P,Q)|^{-1}|Q_1'|^{-1}.$$

Thus P_1'/Q_1' is a convergent to α^2 and there exists $U/V \in E$ such that $P_1' = U^2$ and $Q_1' = V^2$. If we report this in the last equation, we obtain

$$|V\alpha - U| = |T|^{(-l+1)/2}\ |a(P,Q)|^{-1/2}|V|^{-1}.$$

This shows that $|a(U,V)| = |a(P,Q)|^{1/2}\ |T|^{(l-1)/2}$. On the other hand, T does not divide $P_1' = D(P/T^l)+Q$, since T divides P/T^l, and T does not divide Q. Thus T does not divide U, and this implies that $\delta_1(U,V) = 1$, as we have seen above. According to the previous result, we have $|a(U,V)| < |\Delta|$. Therefore we can write

$$|a(U,V)| = |a(P,Q)|^{1/2}\ |T|^{(l-1)/2} \leq |T|^l.$$

Hence again, we have $|a(P,Q)| \leq |T|^{l+1} = |\Delta|$.
Thus we can apply Lemma 2 and it follows that the sequence of partial quotients is bounded by $|T|^{l+1}$, which completes the proof of *i*).

Finally, we prove part *ii*) of the theorem. To do so we will apply Lemma 3. Here $|\alpha| < |T|^{-(l+1)}$, hence the first convergent is 0 and we have

$$|a(0,1)| = |\alpha|^{-1} > |T|^{l+1} = |\Delta|.$$

We have to see that the convergent 0 is in E_2 and satisfies the condition *i*) of this lemma. Hence it is necessary to have

$$|\alpha| < |\alpha^2 + D/T|^{1/2} \quad \text{and} \quad |\alpha| < |\Delta|^{-1}\ |T\alpha^2 + D|^2.$$

This is certainly true, since we have $|\alpha| < |T|^{-(l+1)}$ and $|T\alpha^2 + D| = |D| \geq 1$. Lemma 3 applies, this completes the proof of *ii*) and of the theorem.

§3. Conclusion.

The possibility of describing the two subsets of $\mathcal{H}$, formed on the one hand by the elements with bounded partial quotients, and on the other hand by the elements with unbounded partial quotients, remains an open question. We have to add that those subsets are both stable by a linear fractional transformation with polynomial coefficients, as well as by the Frobenius homomorphism, and also by changing T into a polynomial in T.

In the case when the base field is $\mathbb{F}_2$, L. Baum and M. Sweet [2] have obtained algebraic elements with bounded partial quotients of a different type, which do not belong to $\mathcal{H}$. B. de Mathan and the author have shown that this phenomenon has an explanation using methods of differential algebra. We also proved that this special phenomenon cannot happen if the characteristic is 3 [4]. Those algebraic elements are possibly exceptions and perhaps the unique algebraic ones, with bounded partial quotients, outside the set $\mathcal{H}$.

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