# ALGEBRAIC AND BADLY APPROXIMABLE POWER SERIES OVER A FINITE FIELD

By

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**Abstract.** We will exhibit certain continued fraction expansions for power series over a finite field, with all the partial quotients of degree one, which are non-quadratic algebraic elements over the field of rational functions.

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## §1. Introduction.

Let $p$ be a prime number and $q = p^s$ where $s$ is a positive integer. Let $\mathbb{F}_q$ be the finite field with $q$ elements. We consider the ring of polynomials $\mathbb{F}_q[T]$, and the field of rational functions $\mathbb{F}_q(T)$, in an indeterminate $T$ with coefficients in $\mathbb{F}_q$. There is an ultrametric absolute value defined on $\mathbb{F}_q(T)$ by $|0| = 0$ and $|P/Q| = |T|^{\deg P - \deg Q}$ where $|T|$ is a fixed real number greater than one. The field obtained by completion from $\mathbb{F}_q(T)$, for this absolute value, is usually denoted $\mathbb{F}_q((T^{-1}))$. A non-zero element of this field is represented by a power series in the following way

$$\Theta = \sum_{k \leq k_0} \theta_k T^k \text{ where } k_0 \in \mathbb{Z}, \theta_k \in \mathbb{F}_q \text{ and } \theta_{k_0} \neq 0.$$

The absolute value extended to this field is then defined by $|\Theta| = |T|^{k_0}$.

This construction, which can be done with an arbitrary base field instead of $\mathbb{F}_q$, is an analogue of the construction of the field of real numbers from the ring of the rational integers. The field of power series with a finite base field has many interesting properties which have analogues in the real number case. Because of this analogy and to simplify the terminology, this field will be shortly denoted $\mathbb{F}(q)$ and we call its elements "formal numbers" over $\mathbb{F}_q$.

We are concerned with the continued fraction algorithm in this field $\mathbb{F}(q)$. For a survey on this subject see [Sch]. It is known, by applying an analogue of Liouville's theorem in fields of power series, that the quadratic power series over the field of rational functions are badly approximable by rationals. This terminology (first introduced by L. Baum and M. Sweet, see [BS2]) means that if $\Theta \in \mathbb{F}(q)$ is quadratic over $\mathbb{F}_q(T)$ there is a positive real number $C$ such that

$$|\Theta - P/Q| \geq C|Q|^{-2} \text{ for all } P, Q \in \mathbb{F}_q[T] \text{ with } Q \neq 0.$$

It is equivalent to say that the partial quotients in the continued fraction expansion for $\Theta$ are polynomials of bounded degree. Indeed if $d$ is the upper bound for these degrees then in the above formula we have $C = |T|^{-d}$.

The existence of badly approximable non-quadratic algebraic elements in $\mathbb{F}(q)$ is known. This fact was observed first by L. Baum and M. Sweet ( see [BS1] and [BS2], where $q = 2$ ). Later W. Mills and D. Robbins [MR] have given an example for all $q = p \geq 3$ with $d = 1$. In a recent work [L2] the first named author has given a family of examples for $q = 3$ and $d = 1$. In this paper we develop a similar but deeper approach in the most general setting.

## §2. Flat power series over $\mathbb{F}_q$.

Let us consider the subset of $\mathbb{F}_q[T]^{\mathbb{N}}$ defined by

$$\mathcal{A}(q) = \{A = (a_i)_{i\geq 1} \ : \ a_i \in \mathbb{F}_q[T] \text{ and } \deg a_i = 1 \text{ for } i \geq 1\}.$$

Then we consider the map $\Phi$ from $\mathcal{A}(q)$ into $\mathbb{F}(q)$ such that if $A \in \mathcal{A}(q)$ then $\Phi(A) = \Theta$ where $\Theta$ is defined by its continued fraction expansion in $\mathbb{F}(q) : \Theta = [0, a_1, a_2, \ldots, a_n, \ldots]$. We denote $\mathcal{E}(q) = \Phi\big(\mathcal{A}(q)\big)$.

Now for $A \in \mathcal{A}(q)$ we define two sequences $X(A) = (x_i)_{i\geq -1}$ and $Y(A) = (y_i)_{i\geq -1}$ of elements of $\mathbb{F}_q[T]$ by the following recursion

$$\begin{cases} x_{-1} = 1, \quad x_0 = 0 \qquad \text{and} \qquad x_n = a_n x_{n-1} + x_{n-2} \quad n \geq 1 \\ y_{-1} = 0, \quad y_0 = 1 \qquad \text{and} \qquad y_n = a_n y_{n-1} + y_{n-2} \quad n \geq 1 \end{cases} \tag{1}$$

It follows from these definitions that $(x_i/y_i)_{i\geq 0}$ is the sequence of the convergents to $\Phi(A)$ and that we have $x_i/y_i = [0, a_1, a_2, \ldots, a_i]$ for $i \geq 1$.

We can now state the following proposition.

**Proposition A.** *Let $p$ be a prime number and let $s$ and $t$ be two positive integers. We put $q = p^s$ and $r = p^t$. Let $A \in \mathcal{A}(q)$, $X(A)$, $Y(A)$ and $\Phi(A)$ be defined as above. Let $k$ be a non-negative integer. The two following conditions are equivalent :*

*(I) There is $\epsilon \in \mathbb{F}_q^*$ such that $\Phi(A)$ is a root of the algebraic equation*

$$y_k X^{r+1} - x_k X^r + \epsilon y_{k+r} X - \epsilon x_{k+r} = 0.$$

*(II) There is a sequence $(\epsilon_n)_{n\geq 0}$ of elements of $\mathbb{F}_q^*$ with $\epsilon_0 = 1$ such that for $n \geq 1$ we have*

$$\begin{cases} \epsilon_{n+1} x_{(n+1)r+k} = a_n^r \epsilon_n x_{nr+k} + \epsilon_{n-1} x_{(n-1)r+k} \\ \epsilon_{n+1} y_{(n+1)r+k} = a_n^r \epsilon_n y_{nr+k} + \epsilon_{n-1} y_{(n-1)r+k} \end{cases}$$

Proof: We first show that (I) implies (II). We put $\Theta = \Phi(A)$ and we have

$$\Theta = \frac{x_k \Theta^r + \epsilon x_{k+r}}{y_k \Theta^r + \epsilon y_{k+r}} = f(\Theta^r) \tag{2}$$

According to a basic property of the continued fraction algorithm, we recall that we have

$$|\Theta - x_n/y_n| = |a_{n+1}|^{-1}|y_n|^{-2} = |T|^{-1}|y_n|^{-2} \qquad \text{for } n \geq 0. \tag{3}$$

For $n \geq 0$, we set

$$\begin{cases} u_n = x_k x_n^r + \epsilon x_{k+r} y_n^r \\ v_n = y_k x_n^r + \epsilon y_{k+r} y_n^r \end{cases} \tag{4}$$

Thus we have for $n \geq 0$

$$\frac{u_n}{v_n} = f\left(\left(\frac{x_n}{y_n}\right)^r\right). \tag{5}$$

Now if $a, b \in \mathbb{F}(q)$, by straightforward calculation and using the Frobenius homomorphism, we obtain

$$f(a^r) - f(b^r) = \frac{\epsilon(y_{k+r}x_k - x_{k+r}y_k)(a-b)^r}{(y_k a^r + \epsilon y_{k+r})(y_k b^r + \epsilon y_{k+r})}. \tag{6}$$

Assume that $|a| \leq |T|^{-1}$ and $|b| \leq |T|^{-1}$. Since $|y_{k+r}| > |y_k|$, we have $|y_k a^r + \epsilon y_{k+r}| = |y_k b^r + \epsilon y_{k+r}| = |y_{k+r}|$. The absolute value being ultrametric, we also have

$$|x_{k+r}/y_{k+r} - x_k/y_k| = |\Theta - x_k/y_k| = |T|^{-1}|y_k|^{-2}.$$

Thus $|y_{k+r}x_k - x_{k+r}y_k| = |T|^{-1}|y_k|^{-1}|y_{k+r}|$. Finally (6) implies

$$|f(a^r) - f(b^r)| = |T|^{-1}|y_k|^{-1}|y_{k+r}|^{-1}|a-b|^r. \tag{7}$$

As $|\Theta| = |x_n/y_n| = |T|^{-1}$ for all $n \geq 0$, from (2), (5) and (7) we can write

$$|\Theta - u_n/v_n| = |f(\Theta^r) - f\big((x_n/y_n)^r\big)| = |T|^{-1}|y_k|^{-1}|y_{k+r}|^{-1}|\Theta - x_n/y_n|^r$$

and using (3) we obtain

$$|\Theta - u_n/v_n| = |T|^{-(r+1)}|y_k|^{-1}|y_{k+r}|^{-1}|y_n|^{-2r}.$$

Since $|a_n| = |T|$ for $n \geq 1$, it is clear that $|y_n| = |T|^n$ for all $n \geq 1$. By (4) we have $|v_n| = |y_n|^r|y_{k+r}|$. Thus we get

$$|\Theta - u_n/v_n| = |T|^{-1}|v_n|^{-2}.$$

Consequently we have $|v_n|^2|\Theta - u_n/v_n| < 1$, and this proves that $u_n/v_n$ is a convergent to $\Theta$. Put $u_n/v_n = x_m/y_m$. As $|\Theta - x_m/y_m| = |T|^{-1}|y_m|^{-2}$, we obtain $|v_n| = |y_m|$, $|u_n| = |x_m|$ and so that $\gcd(u_n, v_n) \in \mathbb{F}_q^*$. Since $|y_m| = |T|^m$ and $|v_n| = |T|^{rn+k+r}$, we get $m = r(n+1)+k$ and thus

$u_n/v_n = x_{r(n+1)+k}/y_{r(n+1)+k}$. This proves that there exists $\epsilon_{n+1} \in \mathbb{F}_q^*$, for $n \geq 0$, such that

$$\begin{cases} \epsilon_{n+1} x_{(n+1)r+k} = \epsilon x_{k+r} y_n^r + x_k x_n^r \\ \epsilon_{n+1} y_{(n+1)r+k} = \epsilon y_{k+r} y_n^r + y_k x_n^r \end{cases} \tag{8}$$

Observe that (8) is also true for $n = -1$ setting $\epsilon_0 = 1$. Now we assume that $n \geq 1$, and using the property (1) of the sequence $(x_n)_{n \geq -1}$, we can write

$$\epsilon_{n+1} x_{(n+1)r+k} = \epsilon x_{k+r} (a_n y_{n-1} + y_{n-2})^r + x_k (a_n x_{n-1} + x_{n-2})^r$$

which becomes using the Frobenius homomorphism

$$\epsilon_{n+1} x_{(n+1)r+k} = a_n^r (\epsilon x_{k+r} y_{n-1}^r + x_k x_{n-1}^r) + \epsilon x_{k+r} y_{n-2}^r + x_k x_{n-2}^r.$$

Finally applying (8) for $n-1$ and $n-2$ we obtain the desired formula

$$\epsilon_{n+1} x_{(n+1)r+k} = a_n^r \epsilon_n x_{nr+k} + \epsilon_{n-1} x_{(n-1)r+k}.$$

It is clear, by the same arguments, that the same holds with $y$ in place of $x$. Thus condition (II) is fulfilled.

We now prove that (II) implies (I). Let $\Theta = \Phi(A) = [0, a_1, a_2, \dots]$. First we observe that

$$\epsilon_{n+1} x_{(n+1)r+k} = \epsilon_1 x_{k+r} y_n^r + x_k x_n^r \tag{9}$$

is true for $n = -1$ and $n = 0$. Now we use induction on $n$ and we assume it is true for all integers less than $n$. From (II), we have

$$\epsilon_{n+2} x_{(n+2)r+k} = a_{n+1}^r \epsilon_{n+1} x_{(n+1)r+k} + \epsilon_n x_{nr+k}$$

which gives, using (9) for $n$ and $n-1$

$$\epsilon_{n+2} x_{(n+2)r+k} = a_{n+1}^r (\epsilon_1 x_{k+r} y_n^r + x_k x_n^r) + \epsilon_1 x_{k+r} y_{n-1}^r + x_k x_{n-1}^r.$$

Finally, using the property (1) of the sequence $(x_n)_{n \geq -1}$ and the Frobenius homomorphism, we obtain

$$\epsilon_{n+2} x_{(n+2)r+k} = \epsilon_1 x_{k+r} y_{n+1}^r + x_k x_{n+1}^r.$$

Thus (9) holds for all $n \geq 0$. For the same reasons we also have for $n \geq 0$

$$\epsilon_{n+1} y_{(n+1)r+k} = \epsilon_1 y_{k+r} y_n^r + y_k x_n^r. \tag{10}$$

Now, from (9) and (10), we get by dividing

$$\frac{x_{(n+1)r+k}}{y_{(n+1)r+k}} = \frac{\epsilon_1 x_{k+r} + x_k (x_n/y_n)^r}{\epsilon_1 y_{k+r} + y_k (x_n/y_n)^r} \tag{11}$$

and letting $n$ go to infinity we obtain the desired equation in (I)

$$\Theta = \frac{\epsilon_1 x_{k+r} + x_k \Theta^r}{\epsilon_1 y_{k+r} + y_k \Theta^r}$$

with $\epsilon = \epsilon_1$. So the proof of the proposition is complete.

We will denote by $\mathcal{F}_k^t(q)$ the subset of elements in $\mathcal{E}(q)$ which satisfy the two equivalent conditions of the proposition. Further we define $\mathcal{F}^t(q) = \bigcup_{k\geq 0} \mathcal{F}_k^t(q)$ and $\mathcal{F}(q) = \bigcup_{t\geq 1} \mathcal{F}^t(q)$. We call $\mathcal{F}(q)$ the set of "flat formal numbers" over $\mathbb{F}_q$. We observe that $\mathcal{F}^t(q)$ is a set of algebraic elements over $\mathbb{F}_q(T)$ of degree less or equal to $r+1$. We will first show that $\mathcal{F}_k^t(q)$ is not empty for all $k \geq 0$, $q = p^s$ and $r = p^t$.

Let us consider the special element, in $\mathcal{E}(q)$ for all $q = p^s$, defined by

$$\mathbf{e} = [0, T, T, \dots, T, \dots]. \tag{12}$$

Here $A$ is a constant sequence with $a_i = T$ for $i \geq 1$. If $X(A)$ and $Y(A)$ are the sequences defined above, it is easy to see that $x_n = y_{n-1}$ for $n \geq 0$. This element $\mathbf{e}$ is quadratic over $\mathbb{F}_q(T)$ and satisfies, according to (12), $\mathbf{e} = 1/(T + \mathbf{e})$, i.e.

$$\mathbf{e}^2 + T\mathbf{e} - 1 = 0. \tag{13}$$

Let $k$ and $r$ be two integers with $k \geq 0$ and $r \geq 2$. We consider the polynomial

$$g(X) = (X^2 + TX - 1)\Big(\sum_{0\leq i\leq r-1} (-1)^i y_{k+i} X^{r-1-i}\Big). \tag{14}$$

Now we will show that $g(X)$ can be written in another way. Developing the product in (14), the right side is

$$\sum_{0\leq i\leq r-1} \Big((-1)^i y_{k+i} X^{r+1-i} + (-1)^i T y_{k+i} X^{r-i} - (-1)^i y_{k+i} X^{r-1-i}\Big)$$

and this becomes by ordering the powers of $X$

$$y_k X^{r+1} - \omega_{k+1} X^r + Y + (-1)^{r-1}(T y_{k+r-1} + y_{k+r-2}) X + (-1)^r y_{k+r-1} \tag{15}$$

where $\omega_{k+1} = (y_{k+1} - T y_k)$, $Y = 0$ if $r = 2$ and else

$$Y = \sum_{0\leq i\leq r-3} (-1)^i (y_{k+i+2} - T y_{k+i+1} - y_{k+i}) X^{r-1-i}.$$

From the definition of the sequences $X(A)$ and $Y(A)$, we have

$$\begin{aligned}
&y_{k+1} - T y_k = y_{k-1} = x_k,\\
&y_{k+i+2} - T y_{k+i+1} - y_{k+i} = 0 \quad \text{for} \quad 0 \leq i \leq r-3 \quad (\text{ if } \ r \geq 3),\\
&T y_{k+r-1} + y_{k+r-2} = y_{k+r} \qquad \text{and} \qquad y_{k+r-1} = x_{k+r}.
\end{aligned}$$

Consequently, for $k \geq 0$ and $r \geq 2$, we have $Y = 0$ and so (15) becomes

$$g(X) = y_k X^{r+1} - x_k X^r + (-1)^{r-1} y_{k+r} X - (-1)^{r-1} x_{k+r}. \tag{16}$$

By (13) and (14) we have $g(\mathbf{e}) = 0$, thus (16) implies for $k \geq 0$ and $r \geq 2$

$$y_k \mathbf{e}^{r+1} - x_k \mathbf{e}^r + (-1)^{r-1} y_{k+r} \mathbf{e} - (-1)^{r-1} x_{k+r} = 0.$$

This shows that $\mathbf{e} \in \mathcal{F}_k^t(q)$ for all $k \geq 0$, $q = p^s$ and $r = p^t$.

REMARKS:

• It is well known that there is an exceptional subset of algebraic elements in $\mathbb{F}(q)$. These elements have been studied by different authors and have important properties of rational approximation (see [L1] for full references). We call them algebraic elements of class I. An element in $\mathbb{F}(q)$ is algebraic of class I if it is irrational and satisfies an algebraic equation of the form $X = (AX^r + B)/(CX^r + D)$ where $A, B, C, D \in \mathbb{F}_q[T]$ and $r = p^t$. The set of algebraic irrationals satifying such an equation is denoted $\mathcal{H}^t(q)$. We define $\mathcal{H}(q) = \bigcup_{t \geq 1} \mathcal{H}^t(q)$. Considering the equation satisfied by an element in $\mathcal{F}^t(q)$, it is clear that $\mathcal{F}^t(q) \subset \mathcal{H}^t(q)$ and $\mathcal{F}(q) \subset \mathcal{H}(q)$. This subset $\mathcal{H}(q)$ contains among others all algebraic elements of degree less or equal to three. Moreover its elements are either badly approximable (the sequence of the degrees of the partial quotients is bounded) or well approximable (the sequence of the degrees of the partial quotients increases quickly). It has been proved that if $r > 1 + \deg(AD - BC)$ in the above equation then the sequence of the degrees of the partial quotients is unbounded. It is interesting to notice that for the equation in the first condition of Proposition A we have $r = 1 + \deg(AD - BC)$.

Here it is interesting to come back to the analogy between the real numbers and the formal numbers. If we think of an equation corresponding to the one which defines the formal numbers in $\mathcal{H}(q)$, replacing the Frobenius homomorphism by the identity, we obtain an algebraic equation defining the quadratic real numbers. Indeed it is this particular form of the equation, where a quadratic real number appears as a fix point of a Moebius transformation with integer coefficients, which allows to develop an algorithm giving the continued fraction expansion of such a quadratic real number. This expansion is of course known to be ultimately periodic. It is important to recall that the same property is true for quadratic formal numbers over a finite field ( see [Sch] ). Using the corresponding equation for formal numbers with the Frobenius, Mills and Robbins [MR] have shown that it is possible to develop another algorithm to obtain the continued fraction expansion of an element in $\mathcal{H}(q)$. Unluckily this algorithm is of difficult use and the expansion can be awfully complicated for some elements in $\mathcal{H}(q)$.

• Baum and Sweet [BS2] have studied and described the set $\mathcal{E}(2)$. If we denote $\mathcal{Q}(2)$ the set of quadratic formal numbers over $\mathbb{F}_2$, then it results from this work and using an argument of differential algebra that we have $\mathcal{E}(2) \cap \mathcal{H}(2) \subset \mathcal{Q}(2)$ (see [L1] p. 225). This implies that $\mathcal{F}(2)$ can only contain quadratic elements.

• Mills and Robbins [MR] have given a non quadratic example of an element in $\mathcal{F}^1(p)$ for all prime numbers $p \geq 3$. In [L2] we have an example of a quartic element in $\mathcal{F}^1_k(3)$ for all $k \geq 0$.

## §3. A special class of flat power series over $\mathbb{F}_q$ .

In this section we consider a simpler case where in the sequence $A \in \mathcal{A}(q)$ all the polynomials $a_i$ are constant multiples of $T$. First we establish the following proposition.

**Proposition B.** *Let $p$ be a prime number and let $s$ and $t$ be two positive integers. We put $q = p^s$ and $r = p^t$. Let $A \in \mathcal{A}(q)$, $X(A)$ and $Y(A)$ be defined as above. We assume that $a_i = \lambda_i T$ for $i \geq 1$. If there exists a sequence $(\epsilon_i)_{i\geq 0}$ of elements of $\mathbb{F}_q^*$ with $\epsilon_0 = 1$ such that the condition (II) in Proposition A is satisfied then we have*

$$\epsilon_1 = \lambda_1^{-r} \prod_{r+k+1\leq i\leq 2r+k} \lambda_i \quad \textit{and} \quad \epsilon_{2l} = 1,\ \epsilon_{2l+1} = \epsilon_1 \quad \textit{for} \quad l \geq 0.$$

Proof: According to what we have established during the proof of Proposition A, we know that the equalities in condition (II) imply

$$\begin{cases} \epsilon_{n+1} x_{(n+1)r+k} = \epsilon_1 x_{k+r} y_n^r + x_k x_n^r \\ \epsilon_{n+1} y_{(n+1)r+k} = \epsilon_1 y_{k+r} y_n^r + y_k x_n^r \end{cases} \tag{1}$$

for $n \geq 0$. We will now use the following notation: if $a \in \mathbb{F}_q[T]$ and $a = \sum_{0\leq i\leq m} u_i T^i$ then we set $\bar{\epsilon}(a) = u_m$ and $\underline{\epsilon}(a) = u_0$. Considering the formulas defining the sequences $X(A)$ and $Y(A)$ and since $\underline{\epsilon}(a_n) = 0$ for $n \geq 1$, we observe that we have $\underline{\epsilon}(x_n) = \underline{\epsilon}(x_{n-2})$ and $\underline{\epsilon}(y_n) = \underline{\epsilon}(y_{n-2})$ for $n \geq 1$. Thus using the initial conditions, we obtain for $l \geq 0$

$$\begin{cases} \underline{\epsilon}(x_{2l+1}) = 1 \quad \text{and} \quad \underline{\epsilon}(x_{2l}) = 0 \\ \underline{\epsilon}(y_{2l+1}) = 0 \quad \text{and} \quad \underline{\epsilon}(y_{2l}) = 1. \end{cases} \tag{2}$$

From (1), we can write for $n \geq 0$

$$\begin{cases} \epsilon_{n+1} \underline{\epsilon}(x_{(n+1)r+k}) = \epsilon_1 \underline{\epsilon}(x_{r+k}) \underline{\epsilon}(y_n)^r + \underline{\epsilon}(x_k) \underline{\epsilon}(x_n)^r \\ \epsilon_{n+1} \underline{\epsilon}(y_{(n+1)r+k}) = \epsilon_1 \underline{\epsilon}(y_{r+k}) \underline{\epsilon}(y_n)^r + \underline{\epsilon}(y_k) \underline{\epsilon}(x_n)^r. \end{cases} \tag{3}$$

Let $l \geq 1$ be an integer. If $2rl + k$ is odd, the first equation in (3) gives, replacing $n$ by $2l - 1$,

$$\epsilon_{2l} \underline{\epsilon}(x_{2rl+k}) = \epsilon_1 \underline{\epsilon}(x_{r+k}) \underline{\epsilon}(y_{2l-1})^r + \underline{\epsilon}(x_k) \underline{\epsilon}(x_{2l-1})^r.$$

By (2), $\underline{\epsilon}(x_{2rl+k}) = \underline{\epsilon}(x_{2l-1}) = 1$ and $\underline{\epsilon}(y_{2l-1}) = 0$, hence we get $\epsilon_{2l} = 1$. If $2rl + k$ is even, the second equation in (3) gives, replacing $n$ by $2l - 1$,

$$\epsilon_{2l} \underline{\epsilon}(y_{2rl+k}) = \epsilon_1 \underline{\epsilon}(y_{r+k}) \underline{\epsilon}(y_{2l-1})^r + \underline{\epsilon}(y_k) \underline{\epsilon}(x_{2l-1})^r.$$

By (2), $\underline{\epsilon}(y_{2rl+k}) = \underline{\epsilon}(x_{2l-1}) = 1$ and $\underline{\epsilon}(y_{2l-1}) = 0$, hence we get $\epsilon_{2l} = \underline{\epsilon}(y_k)$ and, since $k$ must be even, we have again $\epsilon_{2l} = 1$. Consequently for all $l \geq 1$ we have

$$\epsilon_{2l} = 1. \tag{4}$$

The same type of arguments, using the equations in (3) and replacing $n$ by $2l$, implies that for all $l \geq 1$ we have

$$\epsilon_{2l+1} = \epsilon_1. \tag{5}$$

On the other hand, we can also deduce from the formulas defining the sequences $X(A)$ and $Y(A)$ that for $n \geq 1$ we have

$$\bar{\epsilon}(x_n) = \prod_{2\leq i\leq n} \bar{\epsilon}(a_i) \quad \text{and} \quad \bar{\epsilon}(y_n) = \prod_{1\leq i\leq n} \bar{\epsilon}(a_i) \tag{6}$$

where as usual the empty product is equal to 1. Observe that the formulas (6) are true without any particular assumption on the sequence $A \in \mathcal{A}(q)$. Since $\epsilon_1 x_{k+r} y_n^r$ is the term of highest degree in the right hand side of the first equation in (1), we can write

$$\epsilon_{n+1}\bar{\epsilon}(x_{(n+1)r+k}) = \epsilon_1\bar{\epsilon}(x_{k+r})\bar{\epsilon}(y_n)^r.$$

Now, applying (6) with $\bar{\epsilon}(a_i) = \lambda_i$, we obtain for $n \geq 1$

$$\epsilon_{n+1} \prod_{r+k+1\leq i\leq (n+1)r+k} \lambda_i = \epsilon_1 \prod_{1\leq i\leq n} \lambda_i^r. \tag{7}$$

Replacing $n$ by 1 in this equality and recalling that $\epsilon_2 = 1$, we get

$$\epsilon_1 = \lambda_1^{-r} \prod_{r+k+1\leq i\leq 2r+k} \lambda_i.$$

This completes the proof of the proposition.

We will now use the following notations. If $b_1, b_2, \ldots, b_l$ is a finite sequence of elements in $\mathbb{F}_q[T]$ and $m \in \mathbb{N}$, we write $(b_1, b_2, \ldots, b_l)^{[m]}$ for the sequence obtained by repeating the sequence $b_1, b_2, \ldots, b_l$ $m$ times if $m \geq 1$ and the empty sequence if $m = 0$. Further if $b_1, b_2, \ldots, b_l$ and $c_1, c_2, \ldots, c_m$ are two such sequences we denote $b_1, b_2, \ldots, b_l \oplus c_1, c_2, \ldots, c_m$ the sequence obtained by juxtaposition. We are now able to give the example of a family of flat formal numbers over an arbitrary finite field. We prove the following proposition.

**Proposition C.** *Let $p$ be a prime number. Let $s$ and $t$ be two positive integers. We set $q = p^s$ and $r = q^t$. Let $\alpha \in \mathbb{F}_q^*$. If $p \neq 2$, we assume that $\alpha \neq 2$ and put $\beta = 2 - \alpha$. Let $k$ be a non-negative integer. Let $\Theta_k^t(\alpha) \in \mathbb{F}(q)$ be defined by its continued fraction expansion*

$$\Theta_k^t(\alpha) = [0, T^{[k]}, \oplus_{i\geq 1}(T, (\alpha T, \beta T)^{[(r^i-1)/2]})^{[k+1]}\ ] \quad \textit{if} \quad p \neq 2$$

*and*

$$\Theta_k^t(\alpha) = [0, T^{[k]}, \oplus_{i\geq 1}(T, \alpha T^{[r^i-1]})^{[k+1]}\ ] \qquad \textit{if} \quad p = 2.$$

*Then* $\Theta_k^t(\alpha)$ *satisfies the algebraic equation*

$$y_k X^{r+1} - x_k X^r + (\alpha\beta)^{(r-1)/2} y_{k+r} X - (\alpha\beta)^{(r-1)/2} x_{k+r} = 0 \qquad \textit{if} \quad p \neq 2$$

*and*

$$y_k X^{r+1} + x_k X^r + y_{k+r} X + x_{k+r} = 0 \qquad \textit{if} \quad p = 2.$$

REMARK :

When $s, t$ and $k$ are fixed and $\alpha$ varies we obtain in $\mathbb{F}(q)$ $q-2$ different elements $\Theta_k^t(\alpha)$ if $p \neq 2$ and $q-1$ if $p = 2$. When the base field is $\mathbb{F}_2$ or $\mathbb{F}_3$ we only have the case when $\alpha = 1$ and thus $\Theta_k^t(1) =\mathbf{e}$. If the base field is larger, we also have non-quadratic elements. For instance, if the base field is $\mathbb{F}_4 = \{0, 1, u, u^2\}$ with $u^2 + u + 1 = 0$, taking $k = 0$ and $r = 4$ we have the element

$$\Theta_0^1(u) = [0, \oplus_{i\geq 1}(T, uT^{[4^i-1]})] \in \mathbb{F}(4)$$

which satisfies the algebraic equation

$$X^5 + (T^4 + u^2T^2 + 1)X + T^3 = 0.$$

PROOF: First we observe that if $\alpha = 1$, in both cases $p = 2$ or $p \neq 2$ and hence $\beta = 1$ also, then we have $\Theta_k^t(1)$=$\mathbf{e}$. In this case the result has already been proved in §**2**. So we assume that $\alpha \neq 1$.
Let $A \in \mathcal{A}(q)$ be the sequence such that $\Theta_k^t(\alpha) = \Phi(A)$. We will apply Proposition A. It is enough to prove that there exists a sequence $(\epsilon_i)_{i\geq 0}$ of elements of $\mathbb{F}_q^*$ with $\epsilon_0 = 1$ such that condition (II) is satisfied. Here all the polynomials in $A$ are linear and we put $a_i = \lambda_i T$ for $i \geq 1$. Thus we know, by Proposition B, that we must have

$$\epsilon_1 = \lambda_1^{-r} \prod_{r+k+1\leq i\leq 2r+k} \lambda_i \quad \text{and} \quad \epsilon_{n+1}/\epsilon_n = \epsilon_1^{(-1)^n} \quad \text{for } n \geq 0. \tag{8}$$

Consequently, by Proposition A, $\Theta_k^t(\alpha)$ will satisfy the equation in (I) with $\epsilon = \epsilon_1$ if we have the two conditions

$$\begin{cases} x_{(n+1)r+k} - x_{(n-1)r+k} = \epsilon_1^{(-1)^{n+1}} a_n^r x_{nr+k} \\ y_{(n+1)r+k} - y_{(n-1)r+k} = \epsilon_1^{(-1)^{n+1}} a_n^r y_{nr+k} \end{cases} \tag{9}$$

for $n \geq 1$. We first compute $\epsilon_1$. If $p \neq 2$, from the continued fraction expansion defining $\Theta_k^t(\alpha)$, we see that $\lambda_{r+k+1} = 1$ and the following $r-1$ coefficients $\lambda_i$ are alternatively $\alpha$ and $\beta$. By (8) and since $\lambda_1 = 1$, we obtain

$$\epsilon_1 = (\alpha\beta)^{(r-1)/2}. \tag{10}$$

If $p=2$, from the second continued fraction expansion, we see again that $\lambda_{r+k+1}=1$ and the following $r-1$ coefficients $\lambda_i$ are constantly $\alpha$. Again by (8), we obtain

$$\epsilon_1=\alpha^{r-1}. \tag{11}$$

We observe now that, since $r=q^t$, we have $\omega^r=\omega$ for all $\omega\in\mathbb{F}_q$. Thus if $p=2$ we have $\epsilon_1=1$ and if $p\neq 2$ we have $\epsilon_1^2=1$. This shows that the algebraic equation in (I) of Proposition A, in both cases $p\neq 2$ and $p=2$, is the one stated in Proposition C. Thus we only need to prove that (9) holds for $n\geq 1$ with the corresponding $\epsilon_1$ in each case. It is indeed sufficient to prove the first equality in (9), the second one involving $y$ can be obtained in the same manner. Since $\epsilon_1^{-1}=\epsilon_1$ in both cases, and since $a_n^r=\lambda_n^rT^r=\lambda_nT^r$, this equality can be written

$$x_{(n+1)r+k}-x_{(n-1)r+k}=\epsilon_1\lambda_nT^rx_{nr+k}. \tag{12}$$

Starting from the defining recurrence relation for $X(A)$, i.e.

$$x_{m+1}=a_{m+1}x_m+x_{m-1}\qquad\text{for } m\geq 0,$$

we see easily that there exists a double sequence $b_{m,i}\in\mathbb{F}_q[T]$ such that

$$x_m=b_{m,i}x_{m-i}+b_{m,i-1}x_{m-i-1}\qquad\text{for } m\geq 1 \text{ and } 0\leq i\leq m, \tag{13}$$

with

$$\begin{cases} b_{m,-1}=0,\ b_{m,0}=1\\ b_{m,i+1}=a_{m-i}b_{m,i}+b_{m,i-1}\end{cases} \tag{14}$$

In the same way there exists a double sequence $c_{m,i}\in\mathbb{F}_q[T]$ such that

$$x_m=c_{m,i}x_{m+i}+c_{m,i-1}x_{m+i+1}\qquad\text{for } m\geq 1 \text{ and } 0\leq i\leq m, \tag{15}$$

with

$$\begin{cases} c_{m,-1}=0,\ c_{m,0}=1\\ c_{m,i+1}=-a_{m+i+2}c_{m,i}+c_{m,i-1}\end{cases} \tag{16}$$

We turn now to the equation (12). Using the above notations we can write

$$x_{(n+1)r+k}=b_{(n+1)r+k,r-1}x_{nr+k+1}+b_{(n+1)r+k,r-2}x_{nr+k}$$

and

$$x_{(n-1)r+k}=c_{(n-1)r+k,r-1}x_{nr+k+1}+c_{(n-1)r+k,r}x_{nr+k}.$$

Thus the left hand side of (12), $x_{(n+1)r+k}-x_{(n-1)r+k}$ can be written

$$(b_{(n+1)r+k,r-1}-c_{(n-1)r+k,r-1})x_{nr+k+1}+(b_{(n+1)r+k,r-2}-c_{(n-1)r+k,r})x_{nr+k}. \tag{17}$$

We are now going to compute the coefficients $b$ and $c$ involved in the expression (17). We need the following auxiliary result.

**Lemma.** *Let $R$ be a ring with $\eta, \rho \in R$. Let $(U_n)_{n\geq -1}$ be a sequence of elements in $R$ defined by the following recurrence relation :*

$$U_{-1} = 0, \quad U_0 = 1 \quad \textit{and} \quad \forall n \geq -1,\ U_{n+2} = r_{n+2}U_{n+1} + U_n,$$

*with $r_n = \eta$ if $n = 2l+1$ and $r_n = \rho$ if $n = 2l$, for $n \geq 1$. Then*

$$U_n = \begin{cases} \displaystyle\sum_{j=0}^{l} \binom{2l-j}{j} (\eta\rho)^{l-j} & \textit{if } \ n = 2l,\ l \geq 0 \\ \displaystyle\eta \sum_{j=0}^{l} \binom{2l+1-j}{j} (\eta\rho)^{l-j} & \textit{if } \ n = 2l+1,\ l \geq 0 \end{cases}$$

The proof of this result is very easily obtained by induction and so we omit it. Observe that we may have $\eta = \rho$ and in that case $U_n = \sum_{0\leq j\leq \lfloor n/2\rfloor} \binom{n-j}{j}\eta^{n-2j}$ for $n \geq 1$.

We turn to the expansion defining $\Theta_k^t(\alpha)$. We denote $E$ the set of positive integers $n$ for which $a_n = T$. We observe that in both cases

$$n \in E \Leftrightarrow 1 \leq n \leq k \text{ or } n = (k+1) \sum_{0\leq i\leq m_1} r^i + m_2 r^{m_1+1} \tag{18}$$

where $m_1$ and $m_2$ are integers with $m_1 \geq 0$ and $0 \leq m_2 \leq k$. From this relation we deduce that for $n \geq 1$

$$\begin{cases} n \in E \Leftrightarrow nr + k + 1 \in E \\ n \in E \Rightarrow n \equiv k+1 \pmod r \end{cases} \tag{19}$$

Observe that to compute $b_{m,i}$ we need to know the partial quotients $a_j$ for $m-i+1 \leq j \leq m$. In the same way to compute $c_{m,i}$ we need to know the partial quotients $a_j$ for $m+2 \leq j \leq m+i+1$. First we want to compute $b_{(n+1)r+k,r-1}$ and $c_{(n-1)r+k,r-1}$. For $b_{(n+1)r+k,r-1}$ we have to know the $r-1$ partial quotients $a_j$ for $(n+1)r+k-r+2 \leq j \leq (n+1)r+k$. By (18), as none of the integers $j$ is congruent to $k+1$ modulo $r$, these partial quotients are alternatively $\alpha T$ and $\beta T$ (with possibly $\beta = \alpha$). We can then apply the above lemma in the ring $\mathbb{F}_q[T]$ for the sequence $b_{m,i}$. We obtain

$$b_{(n+1)r+k,r-1} = \sum_{j=0}^{\frac{r-1}{2}} \binom{r-1-j}{j} (\alpha\beta T^2)^{\frac{r-1}{2}-j} \qquad \text{if } p \neq 2 \tag{20}$$

and

$$b_{(n+1)r+k,r-1} = \sum_{j=0}^{\lfloor\frac{r-1}{2}\rfloor} \binom{r-1-j}{j} (\alpha T)^{r-1-2j} \qquad \text{if } p = 2. \tag{21}$$

The same arguments show that the $r-1$ partial quotients involved to compute $c_{(n-1)r+k,r-1}$ are alternatively $\alpha T$ and $\beta T$ (with possibly $\beta=\alpha$) and thus we can apply the lemma again. Finally, since $\alpha\beta=(-\alpha)(-\beta)$, we obtain the same formula as above in both cases, $p\neq 2$ and $p=2$. For $n\geq 1$

$$b_{(n+1)r+k,r-1}=c_{(n-1)r+k,r-1}. \tag{22}$$

Consequently (17) becomes

$$(b_{(n+1)r+k,r-2}-c_{(n-1)r+k,r})x_{nr+k}.$$

Therefore, comparing to (12), we have to prove that for $n\geq 1$

$$b_{(n+1)r+k,r-2}-c_{(n-1)r+k,r}=\epsilon_1\lambda_n T^r. \tag{23}$$

We will now distinguish two cases. First case : $p\neq 2$.
To compute $b_{(n+1)r+k,r-2}$ we need to know the $r-2$ partial quotients $a_j$ for $(n+1)r+k-r+3\leq j\leq (n+1)r+k$. According to (18) and (19) these are alternatively $\alpha T$ and $\beta T$. As $r$ is odd, and using the lemma, we have

$$b_{(n+1)r+k,r-2}=\lambda_{(n+1)r+k}T\sum_{j=0}^{\frac{r-3}{2}}\binom{r-2-j}{j}(\alpha\beta T^2)^{\frac{r-3}{2}-j} \tag{24}$$

To compute $c_{(n-1)r+k,r}$ we use first the recurrence relation on the $c_{m,i}$. By (16), we can write

$$c_{(n-1)r+k,r}=-a_{nr+k+1}c_{(n-1)r+k,r-1}+c_{(n-1)r+k,r-2}. \tag{25}$$

By (20) and (22), we know that

$$c_{(n-1)r+k,r-1}=\sum_{j=0}^{\frac{r-1}{2}}\binom{r-1-j}{j}(\alpha\beta T^2)^{\frac{r-1}{2}-j}.$$

This can be written again

$$c_{(n-1)r+k,r-1}=\sum_{j=1}^{\frac{r-1}{2}}\binom{r-1-j}{j}(\alpha\beta T^2)^{\frac{r-1}{2}-j}+(\alpha\beta T^2)^{\frac{r-1}{2}}$$

and finally

$$c_{(n-1)r+k,r-1}=\sum_{l=0}^{\frac{r-3}{2}}\binom{r-2-l}{l+1}(\alpha\beta T^2)^{\frac{r-3}{2}-l}+(\alpha\beta)^{\frac{r-1}{2}}T^{r-1}. \tag{26}$$

To compute $c_{(n-1)r+k,r-2}$ we need to know the $r-2$ partial quotients $a_j$ for $(n-1)r+k+2\leq j\leq (n-1)r+k+r-1$. According to (18) these are alternatively $\alpha T$ and $\beta T$. As $r$ is odd, using the lemma and since $\alpha\beta = (-\alpha)(-\beta)$, we have

$$c_{(n-1)r+k,r-2} = -\lambda_{nr+k-1} T \sum_{j=0}^{\frac{r-3}{2}} \binom{r-2-j}{j} (\alpha\beta T^2)^{\frac{r-3}{2}-j}. \tag{27}$$

By (26) and (27), the equality (25) becomes

$$c_{(n-1)r+k,r} = -\lambda_{nr+k+1}(\alpha\beta)^{\frac{r-1}{2}} T^r -$$

$$T \sum_{j=0}^{\frac{r-3}{2}} \left( \lambda_{nr+k-1} \binom{r-2-j}{j} + \lambda_{nr+k+1} \binom{r-2-j}{j+1} \right) (\alpha\beta T^2)^{\frac{r-3}{2}-j}.$$

We observe that, for $0 \le j \le (r-3)/2$, we have $2\binom{r-2-j}{j} + \binom{r-2-j}{j+1} = 0$ in $\mathbb{F}_q$. Hence this becomes

$$c_{(n-1)r+k,r} = -\lambda_{nr+k+1}(\alpha\beta)^{\frac{r-1}{2}} T^r -$$

$$(\lambda_{nr+k-1} - 2\lambda_{nr+k+1}) T \sum_{j=0}^{\frac{r-3}{2}} \binom{r-2-j}{j} (\alpha\beta T^2)^{\frac{r-3}{2}-j}. \tag{28}$$

Consequently, by (24) and (28) we obtain

$$b_{(n+1)r+k,r-2} - c_{(n-1)r+k,r} = \lambda_{nr+k+1}(\alpha\beta)^{\frac{r-1}{2}} T^r +$$

$$\big(\lambda_{(n+1)r+k} + \lambda_{nr+k-1} - 2\lambda_{nr+k+1}\big) T \sum_{j=0}^{\frac{r-3}{2}} \binom{r-2-j}{j} (\alpha\beta T^2)^{\frac{r-3}{2}-j}. \tag{29}$$

We will now see that, for $n \ge 1$, we have

$$\lambda_{(n+1)r+k} + \lambda_{nr+k-1} - 2\lambda_{nr+k+1} = 0. \tag{30}$$

This is implied by the property of the sequence $(\lambda_i)_{i\ge 1}$ decribed in (18) and (19). As $r$ is odd we first notice that $nr+k-1$, $nr+k+1$ and $(n+1)r+k$ have same parity. Moreover by (19), the only integer between $nr+k-1$ and $(n+1)r+k$ which could be in the set $E$ is $nr+k+1$. So if $\lambda_{nr+k+1} \ne 1$ then $\lambda_{(n+1)r+k} = \lambda_{nr+k-1} = \lambda_{nr+k+1}$ and if $\lambda_{nr+k+1} = 1$ then $\lambda_{(n+1)r+k} + \lambda_{nr+k-1} = \alpha + \beta$. This shows that (30) is fulfilled in all cases. Hence, by (29), we have for $n \ge 1$

$$b_{(n+1)r+k,r-2} - c_{(n-1)r+k,r} = \epsilon_1 \lambda_{nr+k+1} T^r. \tag{31}$$

Now we consider the case $p = 2$. According to (21), we have

$$b_{(n+1)r+k,r-1} = \sum_{j=0}^{\lfloor \frac{r-1}{2} \rfloor} \binom{r-1-j}{j} (\alpha T)^{r-1-2j}.$$

Here, for $1 \leq j \leq \lfloor (r-1)/2 \rfloor$, we have $\binom{r-1-j}{j} = 0$ in $\mathbb{F}_q$. Thus we obtain

$$b_{(n+1)r+k,r-1} = \alpha^{r-1} T^{r-1} = T^{r-1}. \tag{32}$$

Further, by the recurrence relation (16) for $c_{m,i}$, we can write

$$c_{(n-1)r+k,r} = \lambda_{nr+k+1} T c_{(n-1)r+k,r-1} + c_{(n-1)r+k,r-2}. \tag{33}$$

Using the same recurrence relation, i.e. $U_i = \alpha T U_{i-1} + U_{i-2}$, with the same initial conditions, we observe that

$$c_{(n-1)r+k,r-1} = b_{(n+1)r+k,r-1} \quad \text{and} \quad c_{(n-1)r+k,r-2} = b_{(n+1)r+k,r-2}.$$

By (32) and (33), this implies again

$$b_{(n+1)r+k,r-2} - c_{(n-1)r+k,r} = \lambda_{nr+k+1} T^r = \epsilon_1 \lambda_{nr+k+1} T^r. \tag{34}$$

Comparing (31) or (34) to (23), we see that (23) will be proved if we have for $n \geq 1$

$$\lambda_{nr+k+1} = \lambda_n. \tag{35}$$

From the definition of the sequence $(\lambda_i)_{i \geq 1}$, i.e. using (18) and distinguishing the cases $n \in E$ and $n \notin E$, we see that (35) holds for all $n \geq 1$. So the proof of the proposition is complete.

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