DIFFERENTIAL EQUATIONS AND DIOPHANTINE APPROXIMATION IN POSITIVE CHARACTERISTIC

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Abstract

We prove that the Osgood-Thue Theorem, about Diophantine Approximation in function fields, holds under a more general condition when the ground field is finite.

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1. Introduction

Let K be a field of positive characteristic p , and let $K((T^{-1}))$ be the field of formal Laurent series. We consider the field of rational functions $K(T)$ as embedded into $K((T^{-1}))$. If $\alpha = \sum_{n=-\infty}^{n_0} a_n T^n$ is an element of $K((T^{-1}))$, with $a_{n_0} \neq 0$, the integer n_0 is the *degree* of α, $n_0 = \deg \alpha$. We may put $\deg 0 = -\infty$, and we define an absolute value $|.|$ on $K((T^{-1}))$ by $|\alpha| = q^{\deg \alpha}$ where $q > 1$. When K is a finite field, we will take $q = |K|$. However, it is not necessarily so.

It is well known that Roth's Theorem fails in positive characteristic. Liouville's Theorem holds, and a celebrated example of Mahler shows that this result is the best possible. However, it is possible to obtain more precise results while excluding some exceptional cases. For instance, it has been proved by Osgood ([3]) that $|\alpha - P/Q| >> |Q|^{-[(n+3)/2]}$ when α is an algebraic element in $K((T^{-1}))$, of degree $n > 1$, which satisfies no rational Riccati differential equation (and P and Q are polynomials in $K[T]$, $Q \neq 0$). Actually, the method of Osgood leads to $|\alpha - P/Q| >> |Q|^{-([n/2]+1)}$ (under the assumption that α satisfies no rational Riccati differential equation). It was recently proved by the authors ([2]) that when α satisfies no equation of the form $\alpha = (A\alpha^{p^s} + B)/(C\alpha^{p^s} + D)$, where A, B, C and D are coefficients in $K(T)$, not all zero, then $|\alpha - P/Q| >> |Q|^{-([n/2]+1+\epsilon)}$ for every $\epsilon > 0$. Our method was close to the original Thue's method . Recently a very interesting paper of Voloch ([6]) gave a result which allows us to adapt the method of Thue-Osgood: using this result, we can give another proof of our result when the ground field K is *finite*. In this case, we can prove that the same result as the one given by Osgood, $|\alpha - P/Q| >> |Q|^{-([n/2]+1)}$, holds under the assumption that α satisfies no equation $\alpha = (A\alpha^{p^s} + B)/(C\alpha^{p^s} + D)$.

2. Osgood's Theorem

In this section, we recall briefly the proof of Osgood's Theorem, to make obvious the fact that Osgood's method actually leads to the slightly improved result:

Theorem 2.1. *(Osgood). Let α be an algebraic element in $K((T^{-1}))$, of degree $n > 1$ over $K(T)$. If α satisfies no rational Riccati differential equation, then $|\alpha - P/Q| >> |Q|^{-([n/2]+1)}$ for every pair (P, Q) of elements of $K[T]$, $Q \neq 0$.*

Proof. First notice that every algebraic element $\alpha \in K((T^{-1}))$ is separable over $K(T)$, since it is clear that if $\alpha_1, ..., \alpha_n$ are elements of $K((T^{-1}))$ which are linearly independent over $K(T)$, so are $\alpha_1^p, ..., \alpha_n^p$. Hence α^p as the same degree n as α over $K(T)$. As α is separable over $K(T)$, its derivative α' lies in $K(T)(\alpha)$: indeed if $\alpha^n + a_{n-1}\alpha^{n-1} + ... + a_0 = 0$, with coefficients $a_0, ..., a_{n-1}$, in $K(T)$, putting $F_1(X) = nX^{n-1} + ... + a_1$, and $F_2(X) = a'_{n-1}X^{n-1} + ... + a'_0$, we have $F_1(X) \neq 0$, hence $F_1(\alpha) \neq 0$, and since $\alpha' F_1(\alpha) + F_2(\alpha) = 0$, we get $\alpha' = -F_2(\alpha)/F_1(\alpha) \in K(T)(\alpha)$. Then the crucial point in the proof of Osgood's Theorem is the use of Thue's method in the following way: if d is any integer with $0 \leq d < n$, the $n+1$ elements α', $\alpha'\alpha$, ..., $\alpha'\alpha^d$, 1, α, ..., α^{n-d-1} of $K(T)(\alpha)$ are linearly dependent over $K(T)$, hence there exist polynomials $A(X)$ and $B(X)$ in $K[T][X]$, not both zero, such that $\deg_X A \leq n - d - 1$, $\deg_X B \leq d$, and $\alpha' B(\alpha) = A(\alpha)$. If $B(\alpha) = 0$, we have $A(\alpha) = 0$ and thus $A(X) = B(X) = 0$ since $\max(\deg_X A, \deg_X B) < n$. As this is impossible, we thus have $B(\alpha) \neq 0$. Moreover, we can suppose that A and B are relatively prime. Consider the following "differential polynomial" on $K((T^{-1}))$: for each $\beta \in K((T^{-1}))$, put $H(\beta) = \beta' B(\beta) - A(\beta)$. Notice that $|\beta' - \alpha'| \leq |\beta - \alpha|/|T|$. Moreover, when $|\beta - \alpha| \leq 1$, we have $|A(\beta) - A(\alpha)| \leq C_1|\alpha - \beta|$ and $|B(\beta) - B(\alpha)| \leq C_1|\alpha - \beta|$, where C_1 is a positive real constant ($C_1 = \max(|A|, |B|)\max(1, |\alpha|^{n-2})$, where $|A|$ (respectively $|B|$) denotes the maximum of the absolute values of the coefficients of A (resp. B), regarded as a polynomial with coefficients in $K[T]$). As $H(\alpha) = 0$, and $|B(\beta)| \leq \max(1, |\alpha|)C_1$, we thus have $|H(\beta)| \leq C_2|\alpha - \beta|$, with $C_2 = \max(1, |\alpha|)C_1$. Then let P and Q be elements of $K[T]$, $Q \neq 0$. As $Q^2(P/Q)'$, as well as $Q^{n-d-1}A(P/Q)$ and $Q^d B(P/Q)$, are elements of $K[T]$, then $Q^{\max(d+2, n-d-1)}H(P/Q)$ lies in $K[T]$. Choosing $d = [(n-2)/2]$, we have $d + 2 \geq n - d - 1$ since $[(n-2)/2] \geq (n-3)/2$. Hence $\max(d+2, n-d-1) = d+2 = [n/2]+1$. So if $H(P/Q) \neq 0$, we have $|H(P/Q)| \geq |Q|^{-([n/2]+1)}$. Now it is proved in [3] or [4], that if the differential equation $H(\beta) = 0$ is not a Riccati equation, that is to say, if we have not both the conditions $\deg_X B = 0$ and $\deg_X A \leq 2$, then $H(P/Q) \neq 0$ for each pair (P, Q) of coprime elements of $K[T]$, with $|Q|$ sufficiently large. So, for such (P, Q), we get $|H(P/Q)| \geq |Q|^{-([n/2]+1)}$, and thus $|\alpha - P/Q| \geq C_2^{-1}|Q|^{-([n/2]+1)}$ (the condition $|\alpha - P/Q| \leq 1$ being removed since $C_2 \geq 1$).

3. **A generalized Osgood-Thue Theorem**

In this section, we prove:

Theorem 3.1. *Suppose that K is a finite field of characteristic p. Let α be an algebraic element of $K((T^{-1}))$ of degree $n > 1$. Suppose that α satisfies no equation of the form $\alpha = (A\alpha^{p^s} + B)/(C\alpha^{p^s} + D)$, where s is a positive integer, and A, B, C and D are elements of $K[T]$, not all zero. Then there exists a positive real constant C_3 such that $|\alpha - P/Q| \geq C_3|Q|^{-([n/2]+1)}$ for every pair (P, Q) of elements of $K[T]$ with $Q \neq O$.*

Notice that we have proved in [2] by using the original method of Thue, that under the same hypotheses, but K being *any* field of positive characteristic p, we have $|\alpha - P/Q| >> |Q|^{-([n/2]+\epsilon+1)}$ for every $\epsilon > 0$. Although the case where K is finite is simpler, we cannot obtain a general effective result. The result of [2] is not effective either.

Proof. We use the following lemma, which is due to Voloch ([6]). This lemma holds for any field of positive characteristic p.

Lemma 3.2. *(Voloch). Let α be an algebraic element of $K((T^{-1}))$, of degree $n \geq 4$ over $K(T)$. Suppose that α does not satisfy the conclusion of Theorem 3.1. Let $(\alpha_i)_{1 \leq i \leq 4}$ be four distinct conjugates of α in an algebraic closure Ω of $K(T)$. The cross ratio $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ is algebraic over K.*

Proof. As we make a slight change because of a minor error in Voloch's paper, let us recall the proof of this Lemma. We note $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (\alpha_3 - \alpha_1)(\alpha_4 - \alpha_2)/((\alpha_3 - \alpha_2)(\alpha_4 - \alpha_1))$. We may suppose K perfect, since the statement is unchanged while replacing K by any algebraic extension K' of K. Indeed an algebraic element $\alpha \in K((T^{-1}))$ of degree n over $K(T)$ has the same degree n over $K'(T)$, when it is regarded as an element of $K'((T^{-1}))$. Moreover if P and Q are coprime polynomials in $K'[T]$, Q monic, with $|\alpha - P/Q| < |Q|^{-2}$, then P and Q lie in $K[T]$ since P/Q is a convergent in the continued fraction expansion of α .

First, if the inequality $|\alpha - P/Q| >> |Q|^{-([n/2]+1)}$ does not hold, there follows from Osgood's Theorem that α satisfies a rational Riccati differential equation (E). Moreover, there are rational solutions P/Q of this differential equation, where P and Q are coprime in $K[T]$, $Q \neq 0$, with deg Q unbounded ([4]). Indeed, the same method as in the proof of Osgood's Theorem shows that, if α satisfies a rational Riccati differential equation, then $|\alpha - P/Q| >> |Q|^{-2}$ for all the pairs $(P, Q) \in K[T] \times (K[T]\backslash\{0\})$ such that P/Q is *not* a solution of this Riccati equation. Then the equation (E) has at least three rational solutions R_1, R_2, R_3. The other solutions of (E) in $K((T^{-1}))$ are the elements $z \in K((T^{-1}))$ for which the cross ratio (R_1, R_2, R_3, z) lies in $K((T^{-p}))$. Indeed, as in the classical theory, the differential equation which must be satisfied by $y = (R_1, R_2, R_3, z)$ so that z satisfies (E), is $y' = 0$. Then there exists $\beta \in K((T^{-1}))$ such that $(R_1, R_2, R_3, \alpha) = \beta^p$, that is to say that $\alpha = (U_1\beta^p + V_1)/(W_1\beta^p + X_1)$, where U_1, V_1, W_1, and X_1 are elements of $K[T]$, with $U_1X_1 - V_1W_1 \neq 0$. Clearly β is an element of $K(T)(\alpha)$ of degree n, since, β being separable over $K(T)$, one has $K(T)(\beta) = K(T)(\beta^p)$. Moreover, as α does not satisfy the condition $|\alpha - P/Q| >> |Q|^{-([n/2]+1)}$, neither does β. Indeed the condition is invariant by rational homography, and if we had $|\beta - P/Q| >> |Q|^{-([n/2]+1)}$, we would also have $|\beta^p - P/Q| >> |Q|^{-([n/2]+1)}$ since a rational function P/Q with $|\beta^p - P/Q| < |Q|^{-2}$ is a convergent in the continued fraction expansion of β^p, that is to say $P/Q = (R/S)^p$ where R and S are polynomials in $K[T]$, $S \neq 0$. So we construct inductively a sequence $(\beta_s)_{s\in\mathbf{N}}$ of elements of $K(T)(\alpha)$ and sequences of polynomials U_s, V_s, W_s, and X_s in $K[T]$ with $U_sX_s - V_sW_s \neq 0$ such that $\alpha = (U_s\beta^{p^s} + V_s)/(W_s\beta^{p^s} + X_s)$. Therefore, if σ_i $(1 \leq i \leq 4)$ are four distinct $K(T)$-isomorphisms from $K(T)(\alpha)$ into an algebraic closure Ω of $K(T)$, the cross ratio $(\sigma_1(\alpha), \sigma_2(\alpha), \sigma_3(\alpha), \sigma_4(\alpha))$, being equal to $(\sigma_1(\beta), \sigma_2(\beta), \sigma_3(\beta), \sigma_4(\beta))^{p^s}$, belongs to $(K(T)(\alpha_1, \alpha_2, \alpha_3, \alpha_4))^{p^s}$ for each s. Now any element γ of $\bigcap_s (K(T)(\alpha_1, \alpha_2, \alpha_3, \alpha_4))^{p^s}$ is algebraic over K since $K(T)(\alpha_1, ..., \alpha_n)$ being separable over $K(T)$, the monic irreducible polynomial in $K(T)[X]$ which vanishes at γ has coefficients lying in $K(T^{p^s})$ for each positive integer s. These coefficients thus lie in K.

We can now prove Theorem 3.1. We establish that if K is a finite field $\mathbf{F}_q$, and if α *does not* satisfy $|\alpha - P/Q| >> |Q|^{-([n/2]+1)}$, then there exists a positive integer s and coefficients A, B, C and D in $K[T]$, with $AD - BC \neq 0$ such that $\alpha = (A\alpha^{p^s} + B)/(C\alpha^{p^s} + D)$. We can suppose that $n \geq 4$ since any α of degree less than 4 satisfies an equation $\alpha = (A\alpha^p + B)/(C\alpha^p + D)$, where A, B, C and D are coefficients in $K[T]$, $AD - BC \neq 0$. Let σ_i $(1 \leq i \leq n)$ be the n distinct $\mathbf{F}_q(T)$-isomorphisms from $\mathbf{F}_q(T)(\alpha)$ into Ω (an algebraic closure of $\mathbf{F}_q(T)$). As the cross ratio $(\sigma_1(\alpha), \sigma_2(\alpha), \sigma_3(\alpha), \sigma_4(\alpha))$ is algebraic over $\mathbf{F}_q$, its degree over $\mathbf{F}_q$ is the same as over $\mathbf{F}_q(T)$, and this degree thus divides $n!$. So the cross ratio $(\sigma_1(\alpha), \sigma_2(\alpha), \sigma_3(\alpha), \sigma_4(\alpha))$ belongs to the field $\mathbf{F}_{q^{n!}}$. If we put $q^{n!} = p^s$, we thus have $(\sigma_1(\alpha^{p^s}), \sigma_2(\alpha^{p^s}), \sigma_3(\alpha^{p^s}), \sigma_4(\alpha^{p^s})) = (\sigma_1(\alpha), \sigma_2(\alpha), \sigma_3(\alpha), \sigma_4(\alpha))$. Then we will complete the proof by applying the following lemma to α and α^{p^s}:

Lemma 3.3. *Let α be an algebraic element in $K((T^{-1}))$ of degree at least 4 over $K(T)$. Let $\beta \in K(T)(\alpha)$. Suppose that for each system $(\sigma_1, ..., \sigma_4)$ of four distinct $K(T)$-isomorphisms from $K(T)(\alpha)$ into an algebraic closure, the cross ratios $(\sigma_1(\alpha), \sigma_2(\alpha), \sigma_3(\alpha), \sigma_4(\alpha))$ and $(\sigma_1(\beta), \sigma_2(\beta),$*

$\sigma_3(\beta), \sigma_4(\beta))$ are equal. Then there exist coefficients A, B, C and D in $K[T]$, with $AD - BC \neq 0$, such that $\alpha = (A\beta + B)/(C\beta + D)$.

Proof. If we consider the equality $(\sigma_1(\alpha), \sigma_2(\alpha), \sigma_3(\alpha), \sigma(\alpha)) = (\sigma_1(\beta), \sigma_2(\beta), \sigma_3(\beta), \sigma(\beta)$ while fixing three distinct $K(T)$-isomorphisms σ_1, σ_2, and σ_3, from $K(T)(\alpha)$, and letting $\sigma_4 = \sigma$ run through the $K(T)$-isomorphisms from $K(T)(\alpha)$, we obtain a relation $\sigma(\alpha) = (A\sigma(\beta)+B)/(C\sigma(\beta)+D)$, where A, B, C and D are coefficients in $K(T)((\sigma_i(\alpha))_{1\leq i\leq n})$, with $AD - BC \neq 0$, and independent from σ. One of these coefficients is not zero, for instance $A \neq 0$, then taking $A = 1$, the coefficients A, B, C and D in $K(T)((\sigma_i(\alpha))_{1\leq i\leq n})$, such that $\sigma(\alpha) = (A\sigma(\beta) + B)/(C\sigma(\beta) + D)$ for each $K(T)$-isomorphism σ from $K(T)(\alpha)$, become unique. Now for each $K(T)$-automorphism τ of $K(T)((\sigma_i(\alpha))_{1\leq i\leq n})$, the coefficients $\tau(A)$, $\tau(B)$, $\tau(C)$ and $\tau(D)$ satisfy the same condition, hence $\tau(A) = A$, $\tau(B) = B$, $\tau(C) = C$ and $\tau(D) = D$. As $K(T)((\sigma_i(\alpha))_{1\leq i\leq n})$ is a Galois extension of $K(T)$, then A, B, C and D lie in $K(T)$, and in particular we have $\alpha = (A\beta + B)/(C\beta + D)$. We make A, B, C and D polynomials, by multiplying by a non-zero polynomial. So the proof is complete.

4. Rational solutions of a Riccati equation

In the previous paragraph, we have seen that if a rational Riccati equation has no rational solution, then a solution $\alpha \in K((T^{-1}))$ of this equation satisfies $|\alpha - P/Q| >> |Q|^{-2}$. Therefore, one can ask whether there actually exist algebraic elements α satisfying such an equation. There are examples in characteristic 2. For instance L. E. Baum and M. M. Sweet have proved that an element $\alpha \in \mathbf{F}_2((T^{-1}))$ satisfies $|\alpha - P/Q| \geq |Q|^{-2}/2$ for every pair (P, Q) of polynomials in $\mathbf{F}_2[T]$, $Q \neq 0$, if and only if $\alpha^2 + T\alpha + 1 = (1 + T)\gamma^2$, with $\gamma \in \mathbf{F}_2((T^{-1}))$, $|\gamma| \leq 1$ ([1]). We notice that:

Theorem 4.1. *Every element $\alpha \in \mathbf{F}_2((T^{-1}))$ such that $\alpha^2 + T\alpha + 1 = (1 + T)\gamma^2$ where $\gamma \in \mathbf{F}_2((T^{-1}))$, $|\gamma| \leq 1$, is a solution of a Riccati rational equation which has no rational solution.*

Proof. For such an α, we have $T\alpha' + \alpha = \gamma^2 = (\alpha^2 + T\alpha + 1)/(1 + T)$, hence we get $\alpha' = \alpha^2/(T^2 + T) + \alpha/(T^2 + T) + 1/(T^2 + T)$. Putting $\alpha/(T^2 + T) = \beta$, we obtain the equation $\beta' = \beta^2 + 1/(T^2 + T)^2$. Now the equation $G' = G^2 + 1/(T^2 + T)^2$ has no solution $G \in \mathbf{F}_2(T)$ since it has no solution $G \in \mathbf{F}_2((T))$. Indeed, if $G = \sum_{n=d}^{+\infty} g_n T^n$, with $d \in \mathbf{Z}$, were a solution of this equation, putting $g_n = 0$ when $n < d$ and writing $G' = G^2 + \sum_{n=-1}^{+\infty} T^{2n}$, we would have $g_{2n+1} = g_n + 1$ for each $n \geq -1$, which is impossible for $n = -1$.

Notice that for each $\gamma \in \mathbf{F}_2((T^{-1}))$, $|\gamma| \leq 1$, there is an element $\alpha \in \mathbf{F}_2((T^{-1}))$ such that $\alpha^2 + T\alpha + 1 = (1 + T)\gamma^2$ (indeed $T^{-1}\alpha$ satisfies an equation $x^2 + x = y$ with $|y| < 1$; such an equation has roots in $\alpha \in \mathbf{F}_2((T^{-1}))$ by Hensel's Lemma). All these elements α satisfy the same Riccati equation without rational solution. If γ is algebraic, so is α, and the degree of α is $[K(T)(\gamma) : K(T)]$ or $2[K(T)(\gamma) : K(T)]$. This degree thus is arbitrarily great.

Also notice that we obtain another proof of the result of Baum and Sweet ([1]): if α is an element of $\mathbf{F}_2((T^{-1}))$ with $\alpha^2 + T\alpha + 1 = (1 + T)\gamma^2$ and $\gamma \in \mathbf{F}_2((T^{-1}))$, $|\gamma| \leq 1$, then $|\alpha - P/Q| \geq |Q|^{-2}/2$ for every pair (P, Q) of polynomials in $\mathbf{F}_2[T]$, $Q \neq 0$. Indeed, putting for $z \in K((T^{-1}))$, $H(z) = z' + z^2/(T^2 + T) + z/(T^2 + T) + 1/(T^2 + T)$, as $|\alpha' - (P/Q)'| \leq |\alpha - P/Q|/2$, we have $|H(P/Q)| \leq |\alpha - P/Q|/2$. As $|H(P/Q)| \geq |Q|^{-2}/4$, we thus get $|\alpha - P/Q| \geq |Q|^{-2}/2$.

However we are able to prove that the situation is different in characteristic 3:

Theorem 4.2. *Let K be a field of characteristic 3. If a rational Riccati equation has a solution in $K((T^{-1}))$, which is neither quadratic nor rational, then it has infinitely many rational solutions.*

Proof. Consider first a linear equation $\alpha' = V\alpha$, where $V \in K(T)$. We can suppose that $V \neq 0$ since the result is trivial if $V = 0$. A non-zero solution in $K((T^{-1}))$ satisfies $\alpha^{(k)} = V_k\alpha$ for each positive integer k, where V_k is given inductively by $V_1 = V$ and for $k > 1$, $V_k = V'_{k-1} + V_{k-1}V$. As $\alpha^{(3)} = 0$, there exists an integer k with $1 < k \leq 3$ such that $V_k = 0$ and $V_{k-1} \neq 0$. We see that for every $\beta \in K((T^{-1}))$ with $\beta' = 0$, β/V_{k-1} is a solution of the above equation. Taking $\beta \in K(T^3)$, we so get infinitely many rational solutions.

Then consider a Riccati equation $\alpha' = U\alpha^2 + V\alpha + W$, where U, V and W are rational coefficients, with $U \neq 0$ or $W \neq 0$. Eventually replacing α by $1/\alpha$, we can suppose $U \neq 0$. Now by replacing α by $U\alpha$, we can suppose $U = 1$, and then, replacing α by $\alpha + V/2$ in the equation $\alpha' = \alpha^2 + V\alpha + W$, we may work with an equation of the form $\alpha' = \alpha^2 + W$. A solution α of this equation in $K((T^{-1}))$ shall satisfy $\alpha'' = 2\alpha\alpha' + W' = 2\alpha^3 + 2\alpha W + W'$ and $\alpha^{(3)} = 2\alpha' W + 2\alpha W' + W'' = 2W\alpha^2 + 2\alpha W' + 2W^2 + W'' = 0$. As there is a solution which is neither rational, nor quadratic, the coefficients of the last polynomial equation in α are zero. Thus $W = 0$. Now the equation $\alpha' = \alpha^2$ has infinitely many rational solutions, since, for $\alpha \neq 0$, this equation is $(1/\alpha)' = -1$, and has the rational solutions $\alpha = -1/(T + \beta)$, where $\beta \in K(T^3)$.

One may think that Theorem 4.2 holds for any odd characteristic p, but we are unable to prove that when $p > 3$.

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