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A note on hyperquadratic continued fractions in characteristic 2 with partial quotients of degree 1

by

Alain Lasjaunias (Bordeaux)

1. Introduction. Let p be a prime number, $q = p^s$ with $s \geq 1$, and let $\mathbb{F}_q$ be the finite field with q elements. We let $\mathbb{F}_q[T]$, $\mathbb{F}_q(T)$ and $\mathbb{F}(q)$ denote respectively the ring of polynomials, the field of rational functions and the field of power series in $1/T$ with coefficients in $\mathbb{F}_q$, where T is a formal indeterminate. These fields are valuated by the ultrametric absolute value (and its extension) introduced on $\mathbb{F}_q(T)$ by $|P/Q| = |T|^{\deg(P)-\deg(Q)}$, where $|T| > 1$ is a fixed real number. Hence a non-zero element of $\mathbb{F}(q)$ is written as $\alpha = \sum_{k \leq k_0} a_k T^k$ with $k_0 \in \mathbb{Z}$, $a_k \in \mathbb{F}_q$, and $a_{k_0} \neq 0$, and we have $|\alpha| = |T|^{k_0}$. The field $\mathbb{F}(q)$ is the completion of $\mathbb{F}_q(T)$ for this absolute value.

We recall that each irrational [rational] element α of $\mathbb{F}(q)$ can be expanded into an infinite [finite] continued fraction. This will be denoted $\alpha = [a_1, a_2, \dots]$ where the partial quotients a_i are in $\mathbb{F}_q[T]$, with $\deg(a_i) > 0$ for $i > 1$.

In this note we are concerned with infinite continued fractions in $\mathbb{F}(q)$ which are algebraic over $\mathbb{F}_q(T)$. Indeed, we consider algebraic elements of a particular type. Letting $r = p^t$ with $t \geq 0$, we say that α belonging to $\mathbb{F}(q)$ is *hyperquadratic* (of order t) if α is irrational and satisfies an algebraic equation

$$A\alpha^{r+1} + B\alpha^r + C\alpha + D = 0, \quad \text{where } A, B, C, D \in \mathbb{F}_q[T].$$

For more details on these particular power series, see the introduction of [BL]. Note that quadratic elements are hyperquadratic. We recall that an element in $\mathbb{F}(q)$ is quadratic if and only if its sequence of partial quotients is ultimately periodic. Many explicit continued fractions are known for nonquadratic but hyperquadratic elements; see the references below. These particular algebraic

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elements play an important role in diophantine approximation and that is why they were first considered. For a general account of continued fractions in power series fields and diophantine approximation, and for more references on this matter, consult [S, L2].

The study of continued fractions for hyperquadratic power series was started by Baum and Sweet [BS1, BS2] and developed in the 1980's by Mills and Robbins [MR]. Mills and Robbins pointed out the existence of hyperquadratic continued fractions with all partial quotients of degree one, in odd characteristic with a prime base field. Different approaches were developed to explain this phenomenon (see [L1, LR1, LR2]), first with the base field $\mathbb{F}_3$, then in all characteristics with an arbitrary base field. However, these methods left many questions unanswered. In [L3] a new approach was introduced which led to the presentation of different families of explicit hyperquadratic continued fractions, also with unbounded partial quotients. In a recent paper with Yao [LY1], with this new approach, in the case of an arbitrary base field of odd characteristic, we gave the description of a large family, including the historical examples due to Mills and Robbins, of hyperquadratic continued fractions with all partial quotients of degree one.

The case of even characteristic appears to be singular. The simplest case, where the base field is $\mathbb{F}_2$, has been fully treated by Baum and Sweet [BS2]. They described the set of all continued fractions with partial quotients of degree one (i.e., equal to T or $T+1$). Among them, some are algebraic, and for every even integer $d \geq 2$ there is, in this set, an algebraic element of degree d. However, it was proved later that none of these algebraic elements (for $d > 2$) is hyperquadratic (see [L2, p. 225]). Later, in a joint paper with Ruch, we exhibited hyperquadratic continued fractions in $\mathbb{F}(2^s)$, with partial quotients all of the form λT where $\lambda \in \mathbb{F}_{2^s}$ (see [LR1, p. 280] with $r = 2$ and also [LR2, p. 556]). Other examples were introduced by Thakur [T, p. 290]. Finally, we mention the impressive arXiv note by Robbins [R], in which several conjectures on continued fractions for particular cubic power series in $\mathbb{F}(4)$ and $\mathbb{F}(8)$, with partial quotients of bounded degree, are presented.

Here we present a family of hyperquadratic (for each order $t \geq 1$) continued fractions in $\mathbb{F}(2^s)$ of the form $\alpha = [\lambda_1 T, \lambda_2 T, \ldots]$. In the next section, we state a theorem showing how the sequence $(\lambda_n)_{n\geq 1}$ in $\mathbb{F}_{2^s}^*$ is defined recursively from an arbitrary number of initial values. Moreover, in a final remark, we comment on the properties of this sequence. However, our main goal is to show that this family has a common origin with another family of continued fractions in the case of odd characteristic, introduced earlier and fully studied in [LY1]. Indeed, even though the case of characteristic 2 is simpler, it appears that the existence of both families is directly connected to a particular universal quadratic power series $\omega = [T, T, \ldots]$ belonging to

$\mathbb{F}(p)$ for all prime numbers p. Note that ω is actually the analogue, in the formal case, of the celebrated quadratic real number $[1,1,\dots] = (1+\sqrt{5})/2$.

2. Results. Our method to obtain the explicit continued fraction for certain hyperquadratic power series was introduced in [L3]. We shall also use the notation concerning continued fractions and continuants, presented in [LY1, Section 2].

Let α be an irrational element in $\mathbb{F}(q)$ with continued fraction expansion $[a_1, a_2, \dots]$. We let $\mathbb{F}(q)^+$ denote the subset of $\mathbb{F}(q)$ containing the elements having an integral part of positive degree (i.e., with $\deg(a_1) > 0$). For all integers $n \geq 1$, we set $\alpha_n = [a_n, a_{n+1}, \dots]$ ($\alpha_1 = \alpha$), called the nth *complete quotient*. We introduce the usual continuants $x_n, y_n \in \mathbb{F}_q[T]$ such that $x_n/y_n = [a_1, \dots, a_n]$ for $n \geq 1$. As usual we extend the latter notation to $n = 0$ with $x_0 = 1$ and $y_0 = 0$. So, in what follows, with our notation [LY1, p. 266], given a vector $A = (a_1, \dots, a_n)$, the polynomials x_n and y_n are the continuants $\langle a_1, \dots, a_n \rangle$ and $\langle a_2, \dots, a_n \rangle$.

As above we set $r = p^t$, where $t \geq 0$ is an integer. Let $P, Q \in \mathbb{F}_q[T]$ be such that $\deg(Q) < \deg(P) < r$. Let $\ell \geq 1$ be an integer and $A_\ell = (a_1, \dots, a_\ell)$ a vector in $(\mathbb{F}_q[T])^\ell$ such that $\deg(a_i) > 0$ for $1 \leq i \leq \ell$. Then by [L3, Theorem 1, p. 333], there exists an infinite continued fraction in $\mathbb{F}(q)$ defined by $\alpha = [a_1, \dots, \alpha_{\ell+1}]$ such that $\alpha^r = P\alpha_{\ell+1} + Q$. We consider the four continuants x_ℓ, y_ℓ, $x_{\ell-1}$ and $y_{\ell-1}$, in $\mathbb{F}_q[T]$, built from the vector A_ℓ. The element α is hyperquadratic and it is the unique root in $\mathbb{F}(q)^+$ of the algebraic equation

$$y_\ell X^{r+1} - x_\ell X^r + (y_{\ell-1}P - y_\ell Q)X + x_\ell Q - x_{\ell-1}P = 0. \tag{1}$$

A continued fraction defined as above will be called *of type* (r, ℓ, P, Q). Note that such a continued fraction depends on the choice of the vector $A_\ell = (a_1, \dots, a_\ell)$ in $(\mathbb{F}_q[T])^\ell$.

Now, let us introduce an important sequence $(F_n)_{n\geq 0}$ of polynomials in $\mathbb{F}_p[T]$, for all prime numbers p (see [L3, p. 331]). This sequence is defined by induction as follows:

$$F_0 = 1, \quad F_1 = T, \quad F_{n+1} = TF_n + F_{n-1} \quad \text{for } n \geq 1.$$

Note the similarity with the celebrated sequence of Fibonacci numbers. With our notation, for $n \geq 1$, we clearly have

$$F_n/F_{n-1} = [T, \dots, T] \quad \text{and} \quad F_n = \langle T, \dots, T \rangle,$$

where the finite sequence of T's has length n. Then we define in $\mathbb{F}(p)$ the infinite continued fraction

$$\omega = [T, T, \dots] = \lim_{n\to\infty} F_n/F_{n-1}.$$

Note that ω is quadratic and satisfies $\omega = T + 1/\omega$.

We have the following lemma.

LEMMA 2.1. *Let p be a prime. Let the sequence $(F_n)_{n\geq 0}$ in $\mathbb{F}_p[T]$ and $\omega \in \mathbb{F}(p)$ be defined as above. Let $r = p^t$ where $t \geq 1$ is an integer. Then*

$$F_{n-1} = (\omega^n - (-\omega^{-1})^n)/(\omega + \omega^{-1}) \quad \textit{for } n \geq 1.$$

If $p = 2$ then $F_{r-1} = T^{r-1}$, and if $p > 2$ then $F_{r-1} = (T^2 + 4)^{(r-1)/2}$.

Proof. For $n \geq 0$, we set $\Omega_n = \omega^n - (-\omega^{-1})^n$. Then

$$T\Omega_n + \Omega_{n-1} = \omega^{n-1}(T\omega + 1) - (-\omega^{-1})^{n-1}(-T\omega^{-1} + 1). \tag{2}$$

Since $\omega = T + \omega^{-1}$, we get $\omega^2 = T\omega + 1$ and $\omega^{-2} = -T\omega^{-1} + 1$. Hence (2) becomes

$$T\Omega_n + \Omega_{n-1} = \omega^{n+1} - (-\omega^{-1})^{n+1} = \Omega_{n+1}. \tag{3}$$

By (3), the sequence $(\Omega_n)_{n\geq 0}$ satisfies the same recurrence as $(F_n)_{n\geq 0}$. We observe that $\Omega_1/(\omega+\omega^{-1}) = 1 = F_0$ and $\Omega_2/(\omega+\omega^{-1}) = \omega-\omega^{-1} = T = F_1$. Since the sequences $(F_n)_{n\geq 0}$ and $(\Omega_{n+1}/(\omega+\omega^{-1}))_{n\geq 0}$ satisfy the same linear recurrence relation and are equal on the first two values, they coincide and we have $F_{n-1} = \Omega_n/(\omega + \omega^{-1})$ for $n \geq 1$, as stated in the lemma.

If $p = 2$, using the Frobenius isomorphism, we have $\Omega_r = (\omega + \omega^{-1})^r$ and also $\omega + \omega^{-1} = T$. Therefore, we can write

$$F_{r-1} = (\omega + \omega^{-1})^r/(\omega + \omega^{-1}) = (\omega + \omega^{-1})^{r-1} = T^{r-1}.$$

If $p > 2$, then again using the Frobenius isomorphism, we also have $\Omega_r = (\omega + \omega^{-1})^r$. But here $\omega - \omega^{-1} = T$ and $(\omega - \omega^{-1})^2 + 4 = (\omega + \omega^{-1})^2$. Consequently, we get

$$F_{r-1} = (\omega + \omega^{-1})^{r-1} = ((\omega - \omega^{-1})^2 + 4)^{(r-1)/2} = (T^2 + 4)^{(r-1)/2}. \blacksquare$$

It is easy to check that, for $n \geq 2$, we have $\omega^n = F_{n-1}\omega + F_{n-2}$. Since $\omega = \omega_{l+1}$ for $l \geq 1$, with our terminology ω is a continued fraction of type $(r, \ell, F_{r-1}, F_{r-2})$, for all $r = p^t$ and $\ell \geq 1$. We are now interested in the family of continued fractions of type $(r, \ell, \epsilon_1 F_{r-1}, \epsilon_2 F_{r-2})$ for a pair $(\epsilon_1, \epsilon_2) \in (\mathbb{F}_q^*)^2$. Besides ω, this family contains continued fractions with all partial quotients of degree one, hence the ℓ first partial quotients must be chosen as polynomials of degree 1 in $\mathbb{F}_q[T]$. Note that in the first papers on hyperquadratic continued fractions with bounded partial quotients, the role played by the pair (F_{r-1}, F_{r-2}) was not highlighted.

There are two distinct cases according to the characteristic $p = 2$ or $p > 2$. The case of characteristic $p > 2$ was considered first. In [MR], Mills and Robbins described, in the case $q = r = p > 3$, with $\ell = 2$ and $(a_1, a_2) = (aT, bT)$, a family of such examples (see also the introduction of [L3, p. 332]). However in that case, in general, the solution of (1) will not have all partial quotients of degree 1. This will happen if a mysterious condition, relating the coefficients of the polynomials in A_ℓ and the pair (ϵ_1, ϵ_2), is satisfied.

The process of obtaining this condition has been long and complicated (see [LY1] and particularly the first section for the background information on the matter).

Here, we shall only consider the case of characteristic 2. Due to the simpler form of F_{r-1}, given in the previous lemma, we will avoid the sophistication appearing in odd characteristic.

The following general lemma on the continued fraction algorithm is the basic tool of our method. It can be found in several articles (see [L3, Lemma 3.1, p. 336] or [LY1, p. 267]).

LEMMA 2.2. *For $n \geq 2$, given $n+1$ variables $x_1, \dots, x_n$ and x, we have formally*

$$[[x_1, x_2, \dots, x_n], x] = [x_1, x_2, \dots, x_n, y],$$

where

$$y = (-1)^{n-1}\langle x_2, \dots, x_n\rangle^{-2}x - \langle x_2, \dots, x_{n-1}\rangle\langle x_2, \dots, x_n\rangle^{-1}.$$

As a consequence of Lemmas 2.1 and 2.2, we obtain the following lemma.

LEMMA 2.3. *Let $q = 2^s$ and $r = 2^t$ where $s, t \geq 1$ are integers. Let $\alpha = [a_1, a_2, \dots]$ be an infinite continued fraction in $\mathbb{F}(q)$. Assume that there is a pair $(\epsilon_1, \epsilon_2) \in (\mathbb{F}_q^*)^2$ and a pair (i, j) of integers, where $1 \leq i < j$, such that $\alpha_i^r = \epsilon_1 F_{r-1}\alpha_j + \epsilon_2 F_{r-2}$. If $a_i = \lambda_i T$ with $\lambda_i \in \mathbb{F}_q^*$, then*

$$a_j = \epsilon_1^{-1}\lambda_i^r T \quad \text{and} \quad a_{j+k} = (\epsilon_2/\epsilon_1)^{(-1)^k} T \quad \text{for } 1 \leq k \leq r-1,$$

and also

$$\alpha_{i+1}^r = \epsilon_1\epsilon_2^{-2}F_{r-1}\alpha_{j+r} + \epsilon_2^{-1}F_{r-2}.$$

Proof. From the equality $\alpha_i^r = \epsilon_1 F_{r-1}\alpha_j + \epsilon_2 F_{r-2}$, and since $\alpha_i^r = [a_i, \alpha_{i+1}]^r = [a_i^r, \alpha_{i+1}^r]$, we obtain

$$\left[\frac{a_i^r + \epsilon_2 F_{r-2}}{\epsilon_1 F_{r-1}}, \epsilon_1 F_{r-1}\alpha_{i+1}^r\right] = \alpha_j. \tag{4}$$

We denote by W_i the word $T, \dots, T$, where T is repeated i times. We have $(\epsilon_1/\epsilon_2)F_{r-1}/F_{r-2} = (\epsilon_1/\epsilon_2)[W_{r-1}]$. By Lemma 2.1, we have $F_{r-1} = T^{r-1}$, therefore (4) becomes

$$\begin{aligned}[][[\epsilon_1^{-1}\lambda_i^r T, (\epsilon_1/\epsilon_2)[W_{r-1}]], \epsilon_1 F_{r-1}\alpha_{i+1}^r] \\ = [[\epsilon_1^{-1}\lambda_i^r T, x_2, \dots, x_r], \epsilon_1 F_{r-1}\alpha_{i+1}^r] = \alpha_j,\end{aligned} \tag{5}$$

where $x_i = (\epsilon_1/\epsilon_2)^{(-1)^i} T$ for $2 \leq i \leq r$. We shall now apply Lemma 2.2. Using classical properties of continuants (see [LY1, p. 266]), we can write

$$\begin{aligned}\langle x_2, \dots, x_r\rangle &= \langle(\epsilon_1/\epsilon_2)T, (\epsilon_2/\epsilon_1)T, \dots, (\epsilon_1/\epsilon_2)T\rangle \\ &= (\epsilon_1/\epsilon_2)\langle T, T, \dots, T\rangle = (\epsilon_1/\epsilon_2)F_{r-1}\end{aligned}$$

since r is even, and also

$$\langle x_2,\ldots,x_{r-1}\rangle = \langle(\epsilon_1/\epsilon_2)T,(\epsilon_2/\epsilon_1)T,\ldots,(\epsilon_2/\epsilon_1)T\rangle = \langle T,T,\ldots,T\rangle = F_{r-2}.$$

We set $x_1=\epsilon_1^{-1}\lambda_i^rT$ and $x=\epsilon_1F_{r-1}\alpha_{i+1}^r$. Thus, according to Lemma 2.2,

$$(6)\quad [[x_1,\ldots,x_r],x] = [x_1,\ldots,x_r,((\epsilon_1/\epsilon_2)F_{r-1})^{-2}x+((\epsilon_1/\epsilon_2)F_{r-1})^{-1}F_{r-2}].$$

Combining (5) and (6), since $x=\epsilon_1F_{r-1}\alpha_{i+1}^r$, we obtain

$$(7)\quad \alpha_j = [x_1,\ldots,x_r,(\epsilon_2^2/\epsilon_1)F_{r-1}^{-1}\alpha_{i+1}^r+(\epsilon_2/\epsilon_1)F_{r-1}^{-1}F_{r-2}] = [x_1,\ldots,x_r,y].$$

Since $|\alpha_{i+1}|>1$, we observe that $|y|>1$. Comparing (7) with $\alpha_j = [a_j,a_{j+1},\ldots,a_{j+r-1},\alpha_{j+r}]$, we get $a_j=x_1$, $a_{j+1}=x_2,\ldots a_{j+r-1}=x_r$ and $\alpha_{j+r}=y$. From the values given to x_i for $1\le i\le r-1$, we obtain the values for the partial quotients from a_j to a_{j+r-1}. Finally, from the value of y in (7), we obtain the last formula of the lemma. ∎

We can now state and prove the following theorem.

THEOREM 2.4. *Let $q=2^s$ and $r=2^t$ where $s,t\ge 1$ are integers. Let $\ell\ge 1$ be an integer and let (F_{r-1},F_{r-2}) in $(\mathbb{F}_2[T])^2$ be as in Lemma 2.1. Let $\Lambda_{\ell+2}=(\lambda_1,\ldots,\lambda_\ell,\epsilon_1,\epsilon_2)\in(\mathbb{F}_q^*)^{\ell+2}$. Let x_ℓ, y_ℓ, $x_{\ell-1}$ and $y_{\ell-1}$ be the four continuants, in $\mathbb{F}_q[T]$, built from the vector $A_\ell=(\lambda_1T,\ldots,\lambda_\ell T)$. Consider the algebraic equation*

$$(\mathrm{E})\quad y_\ell X^{r+1}+x_\ell X^r+(\epsilon_1y_{\ell-1}F_{r-1}+\epsilon_2y_\ell F_{r-2})X+\epsilon_2x_\ell F_{r-2}+\epsilon_1x_{\ell-1}F_{r-1}=0.$$

Then (E) *has a unique root α in $\mathbb{F}(q)^+$ and*

$$\alpha=[\lambda_1T,\lambda_2T,\ldots]\quad \textit{where } \lambda_i\in\mathbb{F}_q^*.$$

The sequence $(\lambda_n)_{n\ge1}$ in $\mathbb{F}_q^*$ is defined recursively, from the vector $\Lambda_{\ell+2}$, as follows:

$$\lambda_{\ell+rm+1}=(\epsilon_2/\epsilon_1)\epsilon_2^{(-1)^{m+1}}\lambda_{m+1}^r,$$
$$\lambda_{\ell+rm+i}=(\epsilon_1/\epsilon_2)^{(-1)^i}\quad \textit{for } m\ge 0 \textit{ and } 2\le i\le r.$$

Proof. We observe that (E) is simply equation (1) built from the vector $A_\ell=(\lambda_1T,\ldots,\lambda_\ell T)$ and the pair $(P,Q)=(\epsilon_1F_{r-1},\epsilon_2F_{r-2})$. According to the theorem from [L3], mentioned before, (E) has a unique root α in $\mathbb{F}(q)^+$. This element is the infinite continued fraction defined by $\alpha=[\lambda_1T,\ldots,\lambda_\ell T,\alpha_{\ell+1}]$ and $\alpha^r=\epsilon_1F_{r-1}\alpha_{\ell+1}+\epsilon_2F_{r-2}$. Since $a_1=\lambda_1T$, we can apply Lemma 2.3 with $i=1$ and $j=\ell+1$. Hence,

$$(8)\qquad a_{\ell+1}=\epsilon_1^{-1}\lambda_1^rT\quad\text{and}\quad a_{\ell+i}=(\epsilon_1/\epsilon_2)^{(-1)^i}T\quad\text{for } 2\le i\le r.$$

We also have

$$\alpha_2^r=\epsilon_1\epsilon_2^{-2}F_{r-1}\alpha_{\ell+r+1}+\epsilon_2^{-1}F_{r-2}.$$

Define $f:(\mathbb{F}_q^*)^2\to(\mathbb{F}_q^*)^2$ by $f(x,y)=(xy^{-2},y^{-1})$. We observe that f is an involution. Hence, we can apply Lemma 2.3 again (note that $a_2=\lambda_2T$ even if $\ell = 1$), replacing (i, j) by $(2, l+r+1)$, and (ϵ_1, ϵ_2) by $f(\epsilon_1, \epsilon_2)$. We obtain

$$(9) \qquad a_{\ell+r+1} = \epsilon_2^2 \epsilon_1^{-1} \lambda_2^r T \quad \text{and} \quad a_{\ell+r+i} = (\epsilon_1/\epsilon_2)^{(-1)^i} T \quad \text{for } 2 \le i \le r$$

and

$$\alpha_3^r = \epsilon_1 F_{r-1} \alpha_{\ell+r+1} + \epsilon_2 F_{r-2}.$$

Hence, by repeated application of the lemma, we see that $a_i = \lambda_i T$ for $i \ge 1$. Note that $(\epsilon_2/\epsilon_1)\epsilon_2^{(-1)^{m+1}}$ is either ϵ_1^{-1} or $\epsilon_2^2 \epsilon_1^{-1}$, according to the parity of $m \ge 0$. From (8) and (9), we get the formulas for $(\lambda_n)_{n\ge1}$ in $\mathbb{F}_q^*$, stated in the theorem. ∎

Remark. In a recent joint paper with J.-Y. Yao [LY2], we have observed that the sequence of the leading coefficients of the partial quotients for a hyperquadratic continued fraction could be an automatic sequence. Naturally, the same question arises for the sequence described in this theorem. In case $r = 2$, with the terminology introduced above, the element α is of type $(2, \ell, \epsilon_1 T, \epsilon_2)$. Indeed, we have proved in [LY2, Theorem 3] that the corresponding sequence is 2-automatic.

Let us indicate another criterion for a sequence in $\mathbb{F}_q$ to be automatic (see [AS, p. 356]): $(\lambda_n)_{n\ge1}$ in $\mathbb{F}_q$ is p-automatic if the power series $\sum_{i\ge1} \lambda_i T^{-i} \in \mathbb{F}(q)$ is algebraic over $\mathbb{F}_q(T)$. Concerning the sequence $(\lambda_n)_{n\ge1}$ described in our theorem, we conjecture the following: Set $\theta = \sum_{i\ge1} \lambda_i T^{-i} \in \mathbb{F}(q)$. Then there exist $A, B \in \mathbb{F}_q(T)$, depending on r and on the vector $\Lambda_{\ell+2}$, such that $\theta^r + A\theta + B = 0$. Accordingly, by the criterion cited, the conjecture would imply the 2-automaticity of $(\lambda_n)_{n\ge1}$ for all $r = 2^t$ and $t \ge 1$. This conjecture is based on a partial study, made by the author and as yet unpublished, of the sequence $(\lambda_n)_{n\ge1}$. Indeed, in the simplest case $r = 2$, the conjecture is true and this study was complete enough to yield

$$\theta = \sum_{i=1}^{\ell} \lambda_i T^{-i} + (\epsilon_1/\epsilon_2) T^{-\ell} (T+1)^{-2} (\epsilon_2^{(-1)^\ell} T + 1) + (\epsilon_1/\epsilon_2) \epsilon_2^{(-1)^\ell} T^{\ell-1} \rho$$

and

$$\rho^2 + \rho + T^{-2\ell} \sum_{i=1}^{\ell} (1 + \lambda_i^2 (\epsilon_2/\epsilon_1)^2 \epsilon_2^{(-1)^i - (-1)^\ell}) T^{-2i+2} = 0.$$

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Alain Lasjaunias
Institut de Mathématiques de Bordeaux
CNRS-UMR 5251
Talence 33405, France
E-mail: Alain.Lasjaunias@math.u-bordeaux.fr

Abstract (will appear on the journal's web site only)

We describe a family of algebraic nonquadratic power series over an arbitrary finite field of characteristic 2, having a continued fraction expansion with all partial quotients of degree one. The main purpose is to point out a common origin with another analogous family in odd characteristic, previously studied by the author.