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Hyperquadratic continued fractions in odd characteristic with partial quotients of degree one

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In 1986, some examples of algebraic, and nonquadratic, power series over a finite prime field, having a continued fraction expansion with partial quotients all of degree 1 were discovered by W. Mills and D. Robbins. In this note we show that these few examples belong to a very large family of continued fractions for certain algebraic power series over an arbitrary finite field of odd characteristic.

1. Introduction and results

This note deals with continued fractions in fields of power series. For a general account on this matter, the reader can consult W. Schmidt's article [14]. For a wider survey on

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Diophantine approximation in the function field case and full references, the reader may also consult D. Thakur's book [15, Chap. 9]. Let us recall that the pioneer work on the matter treated here, i.e., algebraic continued fractions in power series fields over a finite field, is due to L. Baum and M. Sweet [2].

Let p be a prime number, $q = p^s$ with $s \geq 1$, and let $\mathbb{F}_q$ be the finite field with q elements. We let $\mathbb{F}_q[T]$, $\mathbb{F}_q(T)$ and $\mathbb{F}(q)$ respectively denote the ring of polynomials, the field of rational functions and the field of power series in $1/T$ over $\mathbb{F}_q$, where T is a formal indeterminate. These fields are equipped with the ultrametric absolute value defined by its restriction to $\mathbb{F}_q(T)$: $|P/Q| = |T|^{\deg(P)-\deg(Q)}$, where $|T| > 1$ is a fixed real number. We recall that each irrational (rational) element α of $\mathbb{F}(q)$ can be expanded as an infinite (finite) continued fraction. This will be denoted $\alpha = [a_1, a_2, \dots, a_n, \dots]$ where the $a_i \in \mathbb{F}_q[T]$, with $\deg(a_i) > 0$ for $i > 1$, are the partial quotients and the tail $\alpha_i = [a_i, a_{i+1}, \dots] \in \mathbb{F}(q)$ is the complete quotient. We shall be concerned with infinite continued fractions in $\mathbb{F}(q)$ which are algebraic over $\mathbb{F}_q(T)$.

Regarding Diophantine approximation and continued fractions, a particular subset of elements in $\mathbb{F}(q)$, algebraic over $\mathbb{F}_q(T)$, must be considered. Let $r = p^t$ with $t \geq 0$, we let $\mathcal{H}(r, q)$ denote the subset of irrationals α belonging to $\mathbb{F}(q)$ and satisfying an algebraic equation of the particular form $A\alpha^{r+1} + B\alpha^r + C\alpha + D = 0$, where A, B, C and D belong to $\mathbb{F}_q[T]$. Note that $\mathcal{H}(1, q)$ is simply the set of quadratic irrational elements in $\mathbb{F}(q)$. The union of the subsets $\mathcal{H}(p^t, q)$, for $t \geq 0$, denoted by $\mathcal{H}(q)$, is the set of hyperquadratic power series. For more details and references, the reader may see the introduction of [4]. Even though it contains algebraic elements of arbitrary large degree, this subset $\mathcal{H}(q)$ should be regarded as an analogue, in the formal case, of the subset of quadratic numbers, in the real case. An old and famous theorem, due to Lagrange, gives a characterization of quadratic real numbers as ultimately periodic continued fractions. It is an open problem to know whether another characterization, as particular continued fractions, would be possible for hyperquadratic power series.

The origin of this work is certainly due to a famous example of a cubic power series over $\mathbb{F}_2$, having partial quotients of bounded degrees (1 or 2), introduced in [2]. In a second article [3], Baum and Sweet could characterize all power series in $\mathbb{F}(2)$ having all partial quotients of degree 1 and, among them, those which are algebraic. Underlining the singularity of this context, in [9, p. 5], a different approach could allow to rediscover these particular power series in $\mathbb{F}(2)$. Also in characteristic 2, other algebraic power series over a finite extension of $\mathbb{F}_2$, having all partial quotients of degree 1, were presented (see for instance [10, p. 280]). The case of even characteristic appears to be singular for different reasons. In this note we only consider the case of odd characteristic. Our aim is to show the existence of hyperquadratic continued fractions, in all $\mathbb{F}(q)$'s with odd q, having all partial quotients of degree 1. In $\mathbb{F}(p)$, the first examples were given by Mills and Robbins [12].

Before developing the background of the work presented in this article, we first give an example of such algebraic continued fractions with the purpose of illustrating the subject discussed here. The following result is derived from an elementary and particular case of the theorem which is stated at the end of this section.

Example. Let p be an odd prime number. Let $\epsilon \neq 0, 1$ in $\mathbb{F}_p$. Let us consider the algebraic equation, with coefficients in $\mathbb{F}_p[T]$:

$$X^{p+1} - TX^p + \epsilon T\big(\big(T^2-1\big)^{(p-1)/2} - T^{p-1}\big)X + \epsilon\big(T^{p+1} - \big(T^2-1\big)^{(p-1)/2}\big(T^2+\epsilon-1\big)\big) = 0.$$

This equation has a unique root α in $\mathbb{F}(p)$, with $|\alpha| \geq |T|$, which can be expanded as the following infinite continued fraction

$$\alpha = \big[T, \big(\epsilon(\epsilon-1)\big)^{-1}T, \big(2\epsilon T, -2\epsilon^{-1}T\big)^{(p-1)/2}, \ldots, \big(\epsilon(\epsilon-1)\big)^{u_m}T, \big(2v_mT, -2v_m^{-1}T\big)^{(p^m-1)/2}, \ldots\big],$$

where $(a, b)^k$ denotes the finite sequence $a, b, a, \ldots, b$ of length $2k$, the pair a, b being repeated k times, with $u_m = -1$ if m is odd and $u_m = 0$ if m is even, while $v_m = \epsilon$ if m is odd and $v_m = (\epsilon-1)^{-1}$ if m is even.

To explain the existence of such continued fractions, our method is based on the following statement, proved by the first author [7, pp. 332–333].

Given an integer $l \geq 1$, an l-tuple $(a_1, a_2, \ldots, a_l) \in (\mathbb{F}_q[T])^l$, with $\deg(a_i) \geq 1$ for $1 \leq i \leq l$, and a pair $(P, Q) \in (\mathbb{F}_q[T])^2$ with $\deg(Q) < \deg(P) < r$, there exists a unique infinite continued fraction $\alpha \in \mathbb{F}(q)$ satisfying

$$\alpha = [a_1, \ldots, a_l, \alpha_{l+1}] \quad \text{and} \quad \alpha^r = P\alpha_{l+1} + Q. \tag{$*$}$$

Note that in the degenerated case, $r = 1$, consequently $\deg(P) = 0$ and $Q = 0$, we simply have $\alpha = \epsilon\alpha_{l+1}$, where $\epsilon \in \mathbb{F}_q^*$. This implies the (pure) periodicity of the continued fraction, with a period of length multiple of l. In general, the continued fraction α is algebraic over $\mathbb{F}_q(T)$ of degree d, with $1 < d \leq r+1$. Indeed, from the continued fraction algorithm, we know that there is a linear fractional transformation f_l, having coefficients in $\mathbb{F}_q[T]$, built from the first l partial quotients, such that $\alpha = f_l(\alpha_{l+1})$ (see the end of Section 2). Consequently, by $(*)$ we have $\alpha = f_l((\alpha^r - Q)/P) = f(\alpha^r)$ where f is a linear fractional transformation with integer (polynomial) coefficients. Hence, α is solution of the following algebraic equation of degree $r+1$:

$$y_lX^{r+1} - x_lX^r + (Py_{l-1} - Qy_l)X + Qx_l - Px_{l-1} = 0, \tag{$**$}$$

where the polynomials x_l, x_{l-1}, y_l and y_{l-1} are the continuants built from the l first partial quotients (see the end of Section 2). Thus, α is hyperquadratic. Moreover, it is also true that α is the unique root in $\mathbb{F}(q)$, satisfying $|\alpha| \geq |T|$, of Eq. $(**)$.

In this note, we shall consider continued fractions in $\mathbb{F}(q)$ defined by $(*)$, for a particular choice of the polynomials $(a_1, a_2, \dots, a_l, P, Q)$. Here we consider $p > 2$, $q = p^s$ and $r = p^t$ as above. In the sequel a is given in $\mathbb{F}_q^*$. We consider the following pair of polynomials in $\mathbb{F}_q[T]$:

$$P_a(T) = \left(T^2 + a\right)^{(r-1)/2} \quad \text{and} \quad Q_a(T) = a^{-1}\left(TP_a(T) - T^r\right).$$

We have $\deg(P_a) = r - 1 > \deg(Q_a) = r - 2$. For an integer $l \geq 1$, we let $\mathcal{E}(r, l, a, q)$ denote the subset of infinite continued fraction expansions $\alpha \in \mathbb{F}(q)$ satisfying

$$\alpha = [a_1, \dots, a_l, \alpha_{l+1}] \quad \text{and} \quad \alpha^r = \epsilon_1 P_a \alpha_{l+1} + \epsilon_2 Q_a,$$

where $a_i = \lambda_i T + \mu_i$, $\lambda_i \in \mathbb{F}_q^*$, $\mu_i \in \mathbb{F}_q$, for $1 \leq i \leq l$ and $(\epsilon_1, \epsilon_2) \in (\mathbb{F}_q^*)^2$ are arbitrarily given. Note that in the extremal case, $r = 1$, the pair of polynomials would be $P_a = 1$ and $Q_a = 0$ and $\mathcal{E}(1, l, a, q)$ would be a subset of quadratic power series, corresponding to purely periodic continued fractions. In the sequel we assume $r > 1$. We observe that α in $\mathcal{E}(r, l, a, q)$ is defined by the $(2l + 2)$-tuple $(\lambda_1, \dots, \lambda_l, \mu_1, \dots, \mu_l, \epsilon_1, \epsilon_2)$. Consequently $\mathcal{E}(r, l, a, q)$ has $q^l(q - 1)^{l+2}$ elements.

Our aim is to show that, under a particular choice of the l-tuple $(a_1, \dots, a_l) \in (\mathbb{F}_q[T])^l$ and of the pair $(\epsilon_1, \epsilon_2) \in (\mathbb{F}_q^*)^2$, the element α defined as above will satisfy $\deg(a_n) = 1$, for all the partial quotients a_n in its continued fraction expansion. These particular expansions are said perfect and form a subset of $\mathcal{E}(r, l, a, q)$, which will be denoted by $\mathcal{E}^*(r, l, a, q)$.

A particular and simpler case of this situation can be considered. Let us denote by $\mathcal{E}_0(r, l, a, q)$ the subset of $\mathcal{E}(r, l, a, q)$ where $a_i = \lambda_i T$, for $1 \leq i \leq l$, and also $\mathcal{E}_0^*(r, l, a, q) = \mathcal{E}^*(r, l, a, q) \cap \mathcal{E}_0(r, l, a, q)$. Considering the algebraic equations which they satisfy, it can be observed that the continued fractions belonging to $\mathcal{E}_0(r, l, a, q)$ are odd functions of T and therefore the partial quotients must be odd polynomials of the indeterminate T. Consequently, the elements of $\mathcal{E}_0^*(r, l, a, q)$ have all partial quotients of the form $a_n = \lambda_n T$, for $n \geq 1$, where $\lambda_n \in \mathbb{F}_q^*$. It can be observed that the example introduced above actually belongs to $\mathcal{E}_0^*(p, 1, -1, p)$, and that it is defined by the triple $(\lambda_1, \epsilon_1, \epsilon_2) = (1, \epsilon(\epsilon - 1), \epsilon)$.

In Mills and Robbins article [12], several examples of algebraic continued fractions are presented, some of them with all partial quotients of degree 1. The first examples, [12, pp. 400–401], belong to $\mathcal{E}_0^*(p, 2, 4, p)$, for all primes $p \geq 5$ (see [7, p. 332]). Also in [12, pp. 401–402], we have an example belonging to $\mathcal{E}^*(3, 7, 1, 3)$. In this last case, the partial quotients are not linear. Inspired by this example and using a new approach, in an earlier work [6], the first author could present a particular family of such hyperquadratic continued fractions, all in $\mathcal{E}^*(3, l, 1, 3)$, for all $l \geq 3$. In a joint work with J.-J. Ruch [10], a generalization of the approach introduced in [6] was developed, for all characteristics; however, this led to unsolved questions.

Yet, other examples of algebraic continued fractions, with partial quotients of unbounded degrees, were also presented by Mills and Robbins [12]. One could observe that all these continued fractions are generated as indicated above, in connection with a polynomial of the form $(T^2+a)^k$, for different values of an integer k. Following this, in a larger context than the one we consider here (i.e. for different values of k, see [7] and [8]), the first author could develop a method showing the link between all these algebraic continued fractions. In a particular case, this method can be used to describe continued fractions in $\mathcal{E}_0^*(r,l,a,q)$, including the previously mentioned examples [12, pp. 400–401]. However this method (introduced in [7] and developed in [8]) concerned only elements in $\mathcal{E}_0(r,l,a,q)$. For the simplest case, $p=q=r=3$, in a recent joint work with D. Gomez [5], a modification of this approach has allowed to obtain a large extension of the results presented in [6]. The aim of the present note is to give a full description of these particular algebraic continued fractions for all $p>2$, r and q.

Before stating our result, it is pertinent, just for the sake of completeness, to recall what is already known in this area. Indeed, the following could be proved [8, p. 256]:

If $\alpha \in \mathcal{E}_0(r,l,-1,q)$ and if we have (C_0): $[\lambda_1,\lambda_2,\dots,\lambda_l+\epsilon_1/\epsilon_2]^r=\epsilon_2$, then $\alpha \in \mathcal{E}_0^*(r,l,-1,q)$.

This condition must be understood as a set of several conditions implying the existence of the square bracket on the left. As an illustration, for $l=1$, (C_0) is simply $(\lambda_1+\epsilon_1/\epsilon_2)^r=\epsilon_2$. There are $q-1$ choices for ϵ_2 and, for each one, $q-2$ choices for λ_1, since $\lambda_1^r \neq 0, \epsilon_2$, while ϵ_1 is fixed by $\epsilon_1^r=(\epsilon_2-\lambda_1^r)\epsilon_2^r$. More generally, we can observe that there are $(q-1)^{l+2}$ elements in $\mathcal{E}_0(r,l,-1,q)$ and a basic computation shows that, among them, only $(q-1)(q-2)^l$ satisfy condition (C_0).

In this statement, note that we only consider the case $a=-1$. The general case is derived from the following argument. Let $\alpha \in \mathcal{E}_0(r,l,a,q)$ be defined by

$$\alpha=[\lambda_1 T,\dots,\lambda_l T,\alpha_{l+1}] \quad \text{and} \quad \alpha^r=\epsilon_1 P_a\alpha_{l+1}+\epsilon_2 Q_a.$$

Let v, in $\mathbb{F}_q$ or $\mathbb{F}_{q^2}$, be such that $v^2=-a$, then define $\beta(T)=v\alpha(vT)$. One can show that β belongs to $\mathbb{F}(q)$ and that it is defined by

$$\beta=\left[-a\lambda_1 T,\lambda_2 T,-a\lambda_3 T,\dots,a(l)\lambda_l T,\beta_{l+1}\right] \quad \text{and} \quad \beta^r=\epsilon_1' P_{-1}\beta_{l+1}+\epsilon_2' Q_{-1},$$

where $\epsilon_2'=a^{r-1}\epsilon_2$, $\epsilon_1'=a^{r-1}a(l)\epsilon_1$ and $a(l)=1$ if l is even or $-a$ if l is odd. Consequently, we have $\beta \in \mathcal{E}_0(r,l,-1,q)$ and there is a one-to-one correspondence between the sets $\mathcal{E}_0(r,l,-1,q)$ and $\mathcal{E}_0(r,l,a,q)$. Accordingly, condition (C_0) can easily be generalized, and we have:

If $\alpha \in \mathcal{E}_0(r,l,a,q)$ and (C_0): $[-a\lambda_1,\lambda_2,\dots,a(l)(\lambda_l+\epsilon_1/\epsilon_2)]^r=a^{r-1}\epsilon_2$, then $\alpha \in \mathcal{E}_0^*(r,l,a,q)$.

The present work is organized as follows. In the next section, we introduce the basic results concerning continued fractions, used in this note. In Section 3, we establish an important property of the pair (P_a,Q_a) which plays a key role in the description of our particular algebraic continued fractions. In Section 4, we give the proof of the theorem which is stated here below. In a short and last section, we make further comments and we give an orientation toward further studies.

Theorem. *Let p be an odd prime number, $q = p^s$, $r = p^t$, with integers $s, t \geq 1$. Let $l \geq 1$ be an integer. Let $(a, \epsilon_1, \epsilon_2) \in (\mathbb{F}_q^*)^3$ be given. Let $P_a, Q_a \in \mathbb{F}_q[T]$ be defined as above. Let $\alpha = [a_1, a_2, \ldots, a_n, \ldots] \in \mathbb{F}(q)$ be the infinite continued fraction defined by*

$$(a_1, \ldots, a_l) = (\lambda_1 T + \mu_1, \ldots, \lambda_l T + \mu_l), \quad \text{where } (\lambda_i, \mu_i) \in \mathbb{F}_q^* \times \mathbb{F}_q \text{ for } 1 \leq i \leq l,$$

and

$$\alpha^r = \epsilon_1 P_a \alpha_{l+1} + \epsilon_2 Q_a.$$

We assume that the $(2l+1)$-tuple $(\lambda_1, \ldots, \lambda_l, \mu_1, \ldots, \mu_{l-1}, a, \epsilon_2)$ is such that, for $1 \leq i \leq l$, we can define the pair $(\delta_i, \nu_i) \in \mathbb{F}_q^* \times \mathbb{F}_q$ in the following way:

$$\delta_1 = a\lambda_1^r + \epsilon_2, \qquad \nu_1 = 0, \quad \text{and} \quad \text{for } 1 \leq i \leq l-1$$

$$(C_1) \quad \delta_{i+1} = a\lambda_{i+1}^r - \frac{\delta_i}{a^{r-2}\delta_i^2 + (\nu_i - \mu_i^r)^2} \quad \text{and} \quad \nu_{i+1} = \frac{(\nu_i - \mu_i^r)}{a^{r-2}\delta_i^2 + (\nu_i - \mu_i^r)^2}.$$

We also assume that the pair (ϵ_1, μ_l) is such that we have

$$(C_2) \quad \delta_l = -a(\epsilon_1/\epsilon_2)^r \quad \text{and} \quad (C_3) \quad \mu_l^r = \nu_l.$$

Then we have $\alpha \in \mathcal{E}^*(r, l, a, q)$. Moreover the sequence of partial quotients, defined by $a_n = \lambda_n T + \mu_n$ for $n \geq 1$, is described as follows.

For $n \in \mathbb{N}^*$, we set $f(n) = nr + l + 1 - r$ and $g(n) = nr + l + (1 - r)/2$. We introduce the following subsets of $\mathbb{N}^*$: $I = \{i \in \mathbb{N} \mid 1 \leq i \leq l\}$, $I^* = \{i \in I \mid \nu_i - \mu_i^r \neq 0\}$, $F = \{f^m(i) \mid m \geq 1 \text{ and } i \in I\}$ and $G = \{g^m(i) \mid m \geq 1 \text{ and } i \in I^*\}$. Note that the subsets I^* and G may both be empty: namely if $\mu_i = 0$ for $1 \leq i \leq l$.

For $n > l + 1$, we define $C(n)$ by:

$$C(n) = 4a^{-1} \quad \text{if } n \notin F \cup (F+1) \cup (G+1) \quad \text{and}$$

$$C(n) = 2a^{-1}\left(1 - a^{-1}\lambda_i^{-r}\delta_i\right)^{-r^{m-1}} \quad \text{if } n = f^m(i) \text{ for } m \geq 1 \text{ and } i \in I \ \left(n \neq f(1)\right),$$

$$C(n) = 2a^{-1}\left(a\lambda_i^r\delta_i^{-1}\right)^{r^{m-1}} \quad \text{if } n = f^m(i) + 1 \text{ for } m \geq 1 \text{ and } i \in I,$$

$$C(n) = 4a^{-1}\left(1 + a^{2-r}\left(\nu_i - \mu_i^r\right)^2\delta_i^{-2}\right)^{-r^{m-1}} \quad \text{if } n = g^m(i) + 1 \text{ for } m \geq 1 \text{ and } i \in I^*.$$

Then the sequence $(\lambda_n)_{n\geq1}$ in $\mathbb{F}_q^*$ is defined recursively, for $n \geq l + 1$, by

$$\lambda_{l+1} = \lambda_1^r\epsilon_1^{-1} \quad \text{and} \quad \lambda_n = C(n)\lambda_{n-1}^{-1} \quad \text{for } n > l + 1.$$

While the sequence $(\mu_n)_{n\geq1}$ in $\mathbb{F}_q$ is defined, for $n \geq l + 1$, as follows.

If $n \notin G \cup (G+1)$, then $\mu_n = 0$. If $n = g^m(i)$ for $m \geq 1$ and $i \in I^*$, then

$$\mu_n \lambda_n^{-1} = -\mu_{n+1}\lambda_{n+1}^{-1} = (-a)^{(r^{m-1}(2-r)+1)/2}\big(\big(\nu_i - \mu_i^r\big)\delta_i^{-1}\big)^{r^{m-1}}/2.$$

Remark. If $l = 1$, we observe that conditions (C_1), (C_2) and (C_3) reduce to $a\lambda_1^r + \epsilon_2 = -a(\epsilon_1/\epsilon_2)^r$ with $\mu_1 = 0$. Consequently, the corresponding continued fraction belongs to $\mathcal{E}_0^*(r, 1, a, q)$. As we already observed, the example introduced at the beginning of this section belongs to $\mathcal{E}_0^*(p, 1, -1, p)$ and we have the desired condition $\delta_1 = -1 + \epsilon_2 = \epsilon_1/\epsilon_2$. Besides, the reader may check that the description of the continued fraction given in this example can be derived from the formulas stated in this theorem.

Also, in the simplest case $(\mu_1, \mu_2, \ldots, \mu_l) = (0, 0, \ldots, 0)$, i.e. $\alpha \in \mathcal{E}_0(r, l, a, q)$, (C_1) implies inductively $\nu_i = 0$ for $1 \leq i \leq l$. Consequently, (C_1) reduces to $\delta_{i+1} = a\lambda_{i+1}^r - a^{2-r}\delta_i^{-1}$ for $1 \leq i \leq l-1$. One can check that conditions (C_1), (C_2) and (C_3) then reduce to the sufficient condition (C_0), already stated above, in order to have $\alpha \in \mathcal{E}_0^*(r, l, a, q)$.

Before concluding this section, we make a more general comment in order to underline the place taken by the family of power series described here among the hyperquadratic power series. All the known examples of algebraic continued fractions, in odd characteristic, having all partial quotients of degree 1, are related to continued fractions generated in the way presented above. To be more precise, let us denote by $\mathcal{E}^*(q)$ the union of the sets $\mathcal{E}^*(r, l, a, q)$, for all $l \geq 1$, all $a \in \mathbb{F}_q^*$ and all $r = p^t$, with $t \geq 0$. Then we conjecture that, if $\alpha \in \mathcal{H}(q)$ (q odd) and if all its partial quotients are of degree 1, then there is a linear fractional transformation $f(x) = (ax+b)/(cx+d)$, with $(a, b, c, d) \in \mathbb{F}_q[T]^4$ and $ad - bc \in \mathbb{F}_q^*$, and $\beta \in \mathcal{E}^*(q)$, such that $\alpha(T) = f(\beta(\lambda T + \mu))$ where $(\lambda, \mu) \in \mathbb{F}_q^* \times \mathbb{F}_q$.

Also, it is a known fact that if $\alpha \in \mathcal{E}^*(q)$ and $f(x) = (ax+b)/(cx+d)$, with $(a, b, c, d) \in \mathbb{F}_q[T]^4$ is a linear fractional transformation, then $f(\alpha)$ belongs to $\mathcal{H}(q)$. Moreover α and $f(\alpha)$ have the same algebraic degree, and $f(\alpha)$ has also bounded partial quotients. Hence the set $\mathcal{E}^*(q)$ generates a subset of elements in $\mathcal{H}(q)$ with bounded partial quotients. However, most elements in $\mathcal{H}(q)$ have unbounded partial quotients. The reader may see the introduction of [6] for more information on this matter.

Finally, thinking of a famous conjecture in number theory in the classical context of real numbers, we ask the following question: are there algebraic irrational power series, in odd characteristic, which are not hyperquadratic and which have partial quotients of bounded degrees?

2. Notation and basic formulas for continued fractions

Let $W = w_1, w_2, \ldots, w_n$ be a sequence of variables over a ring $\mathbb{A}$. We set $|W| = n$ for the length of the word W. We define the following operators for the word W.

$$W' = w_2, w_3, \ldots, w_n \quad \text{or} \quad W' = \emptyset \quad \text{if } |W| = 1.$$
$$W'' = w_1, w_2, \ldots, w_{n-1} \quad \text{or} \quad W'' = \emptyset \quad \text{if } |W| = 1.$$
$$W^* = w_n, w_{n-1}, \ldots, w_1.$$

We consider the finite continued fraction associated to W to be

$$[W] = [w_1, w_2, \dots, w_n] = w_1 + \cfrac{1}{w_2 + \cfrac{1}{\ddots + \frac{1}{w_n}}}.$$

This continued fraction is a quotient of multivariate polynomials, usually called continuants, built upon the variables $w_1, w_2, \dots, w_n$. More details about these polynomials can be found, for example in [13], and also in [7] (although here, trying to simplify, we adopt different notation). The continuant built on W will be denoted $\langle W \rangle$. We now recall the definition of this sequence of multivariate polynomials.

Set $\langle \emptyset \rangle = 1$. If the sequence W has only one element, then we have $\langle W \rangle = W$. Hence, with the above notation, the continuants can be computed, recursively on the length $|W|$, by the following formula

$$\langle W \rangle = w_1 \langle W' \rangle + \langle (W')' \rangle \quad \text{for } |W| \geq 2. \tag{1}$$

Thus, with this notation, for any finite word W, the finite continued fraction $[W]$ satisfies

$$[W] = \frac{\langle W \rangle}{\langle W' \rangle}.$$

It is easy to check that the polynomial $\langle W \rangle$ is, in a certain sense, symmetric in the variables $w_1, w_2, \dots, w_n$. Hence we have $\langle W^* \rangle = \langle W \rangle$ and this symmetry implies the classical formula

$$[W^*] = \frac{\langle W \rangle}{\langle W'' \rangle}.$$

The continuants satisfy a number of useful identities. First we will need a generalization of (1). For any finite sequences A and B, of variables over $\mathbb{A}$, defining A, B as the concatenation of sequences A and B, we have

$$\langle A, B \rangle = \langle A \rangle \langle B \rangle + \langle A'' \rangle \langle B' \rangle. \tag{2}$$

Secondly, using induction on $|W|$, we have the following classical identity

$$\langle W \rangle \langle (W')'' \rangle - \langle W' \rangle \langle W'' \rangle = (-1)^{|W|} \quad \text{for } |W| \geq 2. \tag{3}$$

Now, let y be an invertible element of $\mathbb{A}$, then we define $y \cdot W$ as the following sequence

$$y \cdot W = yw_1, y^{-1}w_2, \dots, y^{(-1)^{n-1}} w_n.$$

With these notations, it is easy to check that we have $y[W] = [y \cdot W]$ and more precisely

$$\langle y \cdot W \rangle = \langle W \rangle \quad \text{if } |W| \text{ is even} \quad \text{and} \quad \langle y \cdot W \rangle = y \langle W \rangle \quad \text{if } |W| \text{ is odd}. \tag{4}$$

Let us come back to the notation used in the introduction. If $\alpha \in \mathbb{F}(q)$ is irrational (rational), then it can be expanded as an infinite (finite) continued fraction $\alpha = [a_1, a_2, \ldots, a_n, \ldots]$, where the $a_n \in \mathbb{F}_q[T]$ are called the partial quotients. We have $\deg(a_n) > 0$ for $n > 1$. For $n \geq 1$, we set $x_n = \langle a_1, a_2, \ldots, a_n\rangle$ and $y_n = \langle a_2, \ldots, a_n\rangle$, with $x_0 = 1$ and $y_0 = 0$. The rational $x_n/y_n = [a_1, a_2, \ldots, a_n]$ is called a convergent to α. The continued fraction expansion of an irrational element measures the quality of its rational approximation. The convergents of α are the best rational approximations and we have $|\alpha - x_n/y_n| = |a_{n+1}|^{-1}|y_n|^{-2}$. If the partial quotients have bounded degrees then the element is said to be badly approximable. Let us recall that we also have $\alpha = (x_n\alpha_{n+1} + x_{n-1})/(y_n\alpha_{n+1} + y_{n-1})$ for $n \geq 1$, where $\alpha_{n+1} = [a_{n+1}, a_{n+2}, \ldots]$ is the complete quotient.

We will also make use of the following general and basic lemma.

Lemma 0. *Let W be a finite word, with $|W| \geq 2$. Let a be a variable over $\mathbb{A}$, then we have*

$$[W] + a = [W, b],$$

where $b = (-1)^{|W|-1}\langle W'\rangle^{-2}a^{-1} - \langle (W')''\rangle\langle W'\rangle^{-1}$.

Indeed, we have

$$[w_1, w_2, \ldots, w_n, b] = \frac{x_n b + x_{n-1}}{y_n b + y_{n-1}} = \frac{x_n}{y_n} + \frac{(-1)^{n-1}}{y_n(y_n b + y_{n-1})} = [w_1, \ldots, w_n] + a.$$

An early publication of the idea under this statement is due to M. Mendès France (see [11, p. 209]).

3. A finite continued fraction in $\mathbb{F}_q(T)$

We recall that p is an odd prime number, $q = p^s$ and $r = p^t$ where s and t are positive integers. For $a \in \mathbb{F}_q^*$, we consider the polynomials P_a and Q_a in $\mathbb{F}_q[T]$ defined by:

$$P_a(T) = \left(T^2 + a\right)^{(r-1)/2} \quad \text{and} \quad Q_a(T) = a^{-1}\left(TP_a(T) - T^r\right). \tag{5}$$

The following sequence $(F_n)_{n\geq 0}$ of polynomials in $\mathbb{F}_p[T]$ was introduced by Mills and Robbins [12, p. 400] (see also [7, p. 331]). This sequence is defined recursively by

$$F_0 = 1, \qquad F_1 = T \quad \text{and} \quad F_n = TF_{n-1} + F_{n-2} \quad \text{for } n \geq 2. \tag{6}$$

From (6), we clearly have the finite continued fraction expansion

$$F_n/F_{n-1} = [T, T, \ldots, T] \quad (n \text{ terms}).$$

This sequence can be regarded as the analogue in the function field case of the Fibonacci sequence of integers. By elementary computations (see [7, pp. 331–332]), one can check that the following formulas hold in $\mathbb{F}_p[T]$:

$$F_{r-1} = P_4 \quad \text{and} \quad F_{r-2} = -2Q_4.$$

Consequently, with (4), we can write

$$P_4/Q_4 = -2F_{r-1}/F_{r-2} = [-2T, -T/2, \dots, -2T, -T/2]. \tag{7}$$

Now, let $v \in \mathbb{F}_{q^2}$ be such that $v^2 = a/4$. From (5) we get

$$P_a(T) = (a/4)^{(r-1)/2} P_4(T/v) \quad \text{and} \quad Q_a(T) = v(a/4)^{(r-3)/2} Q_4(T/v). \tag{8}$$

Therefore, by (7) and (8), we have

$$P_a/Q_a = (a/4v)(P_4/Q_4)(T/v) = v[-2T/v, -T/2v, \dots, -2T/v, -T/2v]$$

and, with (4), finally

$$P_a/Q_a = [-2T, -2T/a, \dots, -2T, -2T/a] \quad (r-1 \text{ terms}). \tag{9}$$

Let us make a remark on the infinite continued fraction $\omega = [T, T, \dots, T, \dots]$ in $\mathbb{F}(p)$. This element is quadratic and it clearly satisfies $\omega^2 = T\omega + 1$ (it is an analogue of the golden mean in the case of real numbers). One can prove, for all $n \geq 1$, the equality $\omega^{n+1} = F_n\omega + F_{n-1}$. Consequently, we obtain $\omega^r = P_4\omega - 2Q_4$. Since we have $\omega = \omega_{l+1}$, it follows that ω belongs to $\mathcal{E}_0^*(r, l, 4, p)$, for all r, all $p > 2$ and all $l \geq 1$. It is well known that, also in the case of power series over a finite field, quadratic continued fractions are characterized by an ultimately periodic sequence of partial quotients. For a general element in $\mathcal{E}^*(r, l, a, q)$, this sequence is not so and therefore this element is not quadratic. However the precise algebraic degree of such an element is generally unknown. Concerning this matter, we can make an observation about the example introduced at the beginning of this article. Indeed, one could check that, for the particular value $\epsilon = 1/2$, this element α is actually quadratic and we have, for all $p \geq 3$, $\alpha(T) = (\sqrt{-1}/2)\omega((2/\sqrt{-1})T)$. While if $\epsilon \neq 1/2$ the algebraic degree of the corresponding element might well be $p + 1$.

The aim of the following proposition is to give a generalization for the continued fraction expansion (9) concerning the pair (P_a, Q_a).

Proposition 1. *Let r, q, a, P_a and Q_a be defined as above. We set $k = (r-1)/2$. Let $x \in \mathbb{F}_q$, we set $\omega = 1 + a^{2-r}x^2$. Then P_a and $Q_a + x$ are coprime polynomials in $\mathbb{F}_q[T]$ if and only if $\omega \neq 0$. We assume that $\omega \neq 0$. We define $2k$ polynomials in $\mathbb{F}_q[T]$:*

$$v_i = -2T \quad \textit{and} \quad v_{k+1+i} = -\omega^{(-1)^{i+1}} 2T \quad \textit{for } 1 \le i \le k-1,$$
$$v_k = -2T - (-a)^{1-k} x \quad \textit{and} \quad v_{k+1}(x) = \omega^{-1}\bigl(-2T + (-a)^{1-k} x\bigr).$$

We set $W(a, x) = v_1, v_2/a, v_3, v_4/a, \dots, v_{2k-1}, v_{2k}/a$. Then we have the following equalities in $\mathbb{F}_q(T)$:

$$P_a(Q_a + x)^{-1} = \bigl[W(a, x)\bigr], \qquad \bigl\langle W(a, x)\bigr\rangle = \omega(k)a^{-k} P_a,$$
$$\bigl\langle W'(a, x)\bigr\rangle = \omega(k)a^{-k}(Q_a + x)$$

and also

$$\bigl\langle W''(a, x)\bigr\rangle = \omega(k)\omega^{(-1)^{k+1}} a^{1-k}(Q_a - x),$$

where $\omega(k) = 1$ if k is even and $\omega(k) = \omega^{-1}$ if k is odd.

Proof. We denote by $v \in \mathbb{F}_{q^2}$ a square root of $-a$, so that $\pm v$ are the only roots of P_a. Hence we see that P_a and $Q_a + x$ are coprime if and only if $(Q_a(v) + x)(Q_a(-v) + x) \ne 0$. From (5), we obtain $Q_a(\pm v) = \mp a^{-1} v^r$. Therefore, this becomes $x^2 - a^{-2} v^{2r} \ne 0$. Observing that $v^{2r} = -a^r$, we obtain the desired condition.

First, since $\omega = 1$ if $x = 0$, we observe that $W(a, 0) = -2T, -2T/a, -2T, \dots, -2T/a$. Hence, with our notations, equality (9) can be written as

$$\frac{P_a}{Q_a} = \frac{\langle W(a, 0)\rangle}{\langle W'(a, 0)\rangle}.$$

Since the numerators of both fractions above have the same degree, $r - 1$ in T, it follows that there exists $u \in \mathbb{F}_q^*$ such that

$$\bigl\langle W(a, 0)\bigr\rangle = uP_a \quad \text{and} \quad \bigl\langle W'(a, 0)\bigr\rangle = uQ_a. \tag{10}$$

For a continuant built from polynomials in T, the leading coefficient is obtained as the leading coefficient of the product of its terms. Consequently, the leading coefficient of $\langle W(a, 0)\rangle$ is $(-2)^{r-1} a^{-k} = a^{-k}$ while P_a is unitary. This implies $u = a^{-k}$. Now we observe that k can be even or odd. If $p = 4m+1$ then k is always even, while if $p = 4m+3$ then k has the same parity as t if $r = p^t$.

We set $W(a, 0) = a_1, a_2, \dots, a_{r-1}$ and $W(a, x) = b_1, b_2, \dots, b_{r-1}$. We will use the notation $a(i) = 1$ if i is odd and $a(i) = a^{-1}$ if i is even. Hence we have $a_i = -2a(i)T$ and $b_i = a(i)v_i$ for $1 \le i \le 2k$. To shorten the writing, we denote $\langle a_i, \dots, a_j\rangle$ by $A_{i,j}$ and similarly $\langle b_i, \dots, b_j\rangle$ by $B_{i,j}$. According to these notations we have $\langle W(a, 0)\rangle = A_{1,2k}$ and $\langle W(a, x)\rangle = B_{1,2k}$. From the definition of both sequences $W(a, x)$ and $W(a, 0)$, we have $b_i = a_i$ and $b_{k+1+i} = \omega^{(-1)^{i+1}} a_{k+1+i}$ for $1 \le i \le k-1$. Consequently, we get

$$B_{1,k-1} = A_{1,k-1} \quad \text{and} \quad B_{2,k-1} = A_{2,k-1}. \tag{11}$$

But also, by (4), according to the parity of k,

$$B_{k+2,2k} = \omega\omega(k)A_{k+2,2k} \quad \text{and} \quad B_{k+3,2k} = \omega(k)A_{k+3,2k}. \tag{12}$$

Applying (2), by (11) and (12), we can write

$$B_{1,2k} = B_{1,k}B_{k+1,2k} + B_{1,k-1}B_{k+2,2k} = B_{1,k}B_{k+1,2k} + \omega\omega(k)A_{1,k-1}A_{k+2,2k} \tag{13}$$

$$B_{2,2k} = B_{2,k}B_{k+1,2k} + B_{2,k-1}B_{k+2,2k} = B_{2,k}B_{k+1,2k} + \omega\omega(k)A_{2,k-1}A_{k+2,2k}. \tag{14}$$

For notational convenience define $x_k = (-a)^{1-k}x$. From the definition of v_k and v_{k+1}, we have

$$b_k = a_k - a(k)x_k \quad \text{and} \quad b_{k+1} = \omega^{-1}\big(a_{k+1} + a(k+1)x_k\big). \tag{15}$$

By (2), we have $B_{1,k} = B_{1,k-1}b_k + B_{1,k-2}$. Again by (2), (11) and (15), this becomes

$$B_{1,k} = A_{1,k-1}b_k + A_{1,k-2} = A_{1,k} - a(k)x_kA_{1,k-1}. \tag{16}$$

In the same way, by (2), (11) and (15), we get

$$B_{2,k} = A_{2,k-1}b_k + A_{2,k-2} = A_{2,k} - a(k)x_kA_{2,k-1}. \tag{17}$$

By (2), (12) and (15), since $B_{k+1,2k} = B_{k+2,2k}b_{k+1} + B_{k+3,2k}$, we also get

$$B_{k+1,2k} = \omega(k)A_{k+2,2k}\big(a_{k+1} + a(k+1)x_k\big) + \omega(k)A_{k+3,2k}$$

and this becomes

$$B_{k+1,2k} = \omega(k)\big(A_{k+1,2k} + a(k+1)x_kA_{k+2,2k}\big). \tag{18}$$

By (4), according to the parity of k, we also obtain

$$A_{k+1,2k} = a(k+1)A_{1,k} \quad \text{and} \quad A_{k+2,2k} = a(k)A_{1,k-1}. \tag{19}$$

From (19), we have $a(k)A_{1,k-1}A_{k+1,2k} - a(k+1)A_{1,k}A_{k+2,2k} = 0$. We also have $a(k)a(k+1)x_k^2 = a^{-1}a^{2-2k}x^2 = a^{2-r}x^2$. Consequently, by multiplication, from (16) and (18), we get

$$B_{1,k}B_{k+1,2k} = \omega(k)\big(A_{1,k}A_{k+1,2k} - a^{2-r}x^2A_{1,k-1}A_{k+2,2k}\big). \tag{20}$$

In the same way, by multiplication, from (17) and (18), we get

$$B_{2,k}B_{k+1,2k} = \omega(k)\big(A_{2,k}A_{k+1,2k} - a^{2-r}x^2A_{2,k-1}A_{k+2,2k} + X\big), \tag{21}$$

where, according to (19), using (3) and $a(k)a(k+1) = a^{-1}$, we have

$$X = a(k)a(k+1)x_k(A_{1,k-1}A_{2,k} - A_{2,k-1}A_{1,k}) = a^{-1}x_k(-1)^{k-1} = a^{-k}x. \tag{22}$$

Combining (13) and (20), using (2) and since $\omega = 1 + a^{2-r}x^2$, we get

$$B_{1,2k} = \omega(k)\big(A_{1,k}A_{k+1,2k} + \big(\omega - a^{2-r}x^2\big)A_{1,k-1}A_{k+2,2k}\big) = \omega(k)A_{1,2k}.$$

By (10), recalling that $u = a^{-k}$, this becomes

$$\big\langle W(a,x)\big\rangle = B_{1,2k} = \omega(k)A_{1,2k} = \omega(k)\big\langle W(a,0)\big\rangle = \omega(k)a^{-k}P_a.$$

In the same way, combining (14), (21) and (22), we obtain

$$\begin{aligned} B_{2,2k} &= \omega(k)\big(A_{2,k}A_{k+1,2k} + \big(\omega - a^{2-r}x^2\big)A_{2,k-1}A_{k+1,2k} + a^{-k}x\big) \\ &= \omega(k)\big(A_{2,2k} + a^{-k}x\big). \end{aligned}$$

By (10), with $u = a^{-k}$, this becomes

$$\begin{aligned} \big\langle W'(a,x)\big\rangle = B_{2,2k} = \omega(k)\big(A_{2,2k} + a^{-k}x\big) &= \omega(k)\big(\big\langle W'(a,0)\big\rangle + a^{-k}x\big) \\ &= \omega(k)a^{-k}(Q_a + x). \end{aligned}$$

Consequently, we get

$$\big[W(a,x)\big] = \frac{\langle W(a,x)\rangle}{\langle W'(a,x)\rangle} = \frac{\omega(k)a^{-k}P_a}{\omega(k)a^{-k}(Q_a + x)} = P_a(Q_a + x)^{-1}.$$

Moreover, from the definition of the sequence $W(a,x)$, we observe the "pseudo-symmetry" between $W(a,x)$ and $W(a,-x)$, i.e. $W(a,x) = a\omega^{(-1)^{k+1}} \cdot W^*(a,-x)$. Finally, using this equality, by (4) and since $|W'(a,-x)| = 2k-1$ is odd, we obtain

$$\big\langle W''(a,x)\big\rangle = \big\langle W''^*(a,x)\big\rangle = \big\langle a\omega^{(-1)^{k+1}} \cdot W'(a,-x)\big\rangle = \omega(k)\omega^{(-1)^{k+1}}a^{1-k}(Q_a - x).$$

So the proof of Proposition 1 is complete. □

4. Proof of the theorem

Throughout this section, the integers p, q and r, as well as $a \in \mathbb{F}_q^*$ and P_a, Q_a in $\mathbb{F}_q[T]$, are defined as above. Moreover, as above, we set $k = (r-1)/2$. We need the following lemma, which is a straightforward consequence of Lemma 0 from Section 2 and of Proposition 1 from Section 3.

Lemma 1. *Let $b_0 \in \mathbb{F}_q[T]$ and $y \in \mathbb{F}_q^*$. For $a \in \mathbb{F}_q^*$ and $x \in \mathbb{F}_q$, assuming that $\omega = 1 + a^{2-r}x^2 \neq 0$, as above we denote by $W(a,x)$ the sequence of the $r-1$ partial quotients of the rational function $P_a(Q_a + x)^{-1}$. Then, for $X \in \mathbb{F}(q)$, we have the formal identity:*

$$\big[b_0, y \cdot W(a,x)\big] + X = \big[b_0, y \cdot W(a,x), Y\big],$$

where

$$Y = \omega^{(-1)^{k+1}}\left(\omega a^{r-1} P_a^{-2} X^{-1} - ya(Q_a - x)P_a^{-1}\right).$$

Proof. According to Lemma 0 in Section 2, we can write

$$\left[b_0, y \cdot W(a,x)\right] + X = \left[b_0, y \cdot W(a,x), Y\right],$$

where Y is linked to X as follows

$$Y = (-1)^{r-1}\left\langle y \cdot W(a,x)\right\rangle^{-2} X^{-1} - \left\langle \left(y \cdot W(a,x)\right)''\right\rangle \left\langle y \cdot W(a,x)\right\rangle^{-1}. \tag{23}$$

We recall that $\langle y \cdot W(a,x)\rangle = \langle W(a,x)\rangle$, since the sequence of terms is of even length $r-1$. In the same way, since $r-2$ is odd, we also have $\langle (y \cdot W(a,x))''\rangle = y\langle W''(a,x)\rangle$. Applying Proposition 1, we have $\langle W(a,x)\rangle = \omega(k)a^{-k}P_a$ and $\langle W''(a,x)\rangle = \omega(k)\omega^{(-1)^{k+1}} a^{1-k}(Q_a - x)$. Consequently, (23) becomes

$$Y = \omega(k)^{-2} a^{2k} P_a^{-2} X^{-1} - ya\omega^{(-1)^{k+1}}(Q_a - x)P_a^{-1}. \tag{24}$$

Since $\omega(k)^{-2} = \omega^{1+(-1)^{k+1}}$ and $2k = r-1$, (24) implies the conclusion of this lemma. □

The proof of the theorem relies on the following proposition.

Proposition 2. *Let p, q and r be as above. Let $\alpha = [a_1, a_2, \ldots, a_n, \ldots]$ be an irrational element of $\mathbb{F}(q)$. For an integer $n \geq 1$, we set $f(n) = (n-1)r + l + 1$. For an index $n \geq 1$, we assume that $a_n = \lambda_n T + \mu_n$, where $(\lambda_n, \mu_n) \in \mathbb{F}_q^* \times \mathbb{F}_q$ and that α_n and $\alpha_{f(n)}$ are linked by the following equality*

$$\alpha_n^r = \epsilon_{1,n} P_a \alpha_{f(n)} + \epsilon_{2,n} Q_a + \nu_n,$$

where $(\epsilon_{1,n}, \epsilon_{2,n}, \nu_n) \in (\mathbb{F}_q^*)^2 \times \mathbb{F}_q$. We set $\delta_n = a\lambda_n^r + \epsilon_{2,n}$.

First we assume that $\delta_n \neq 0$. We set $\pi_n = (\nu_n - \mu_n^r)\delta_n^{-1}$ and $\omega_n = 1 + a^{2-r}\pi_n^2$. We assume that $\omega_n \neq 0$. The word $W(a, \pi_n)$ is defined in Proposition 1. Then we have

$$a_{f(n)} = \epsilon_{1,n}^{-1}\lambda_n^r T \quad \textit{and} \quad a_{f(n)+1}, \ldots, a_{f(n)+r-1} = \left(-\epsilon_{1,n}\delta_n^{-1}\right) \cdot W(a, \pi_n).$$

Moreover we have $\alpha_{n+1}^r = \epsilon_{1,n+1} P_a \alpha_{f(n+1)} + \epsilon_{2,n+1} Q_a + \nu_{n+1}$, where

$$\epsilon_{1,n+1} = a^{1-r}\epsilon_{1,n}^{-1}\omega_n^{(-1)^k - 1}$$
$$\epsilon_{2,n+1} = -a^{2-r}(\omega_n \delta_n)^{-1}$$
$$\nu_{n+1} = a^{2-r}\left(\nu_n - \mu_n^r\right)\omega_n^{-1}\delta_n^{-2}.$$

Finally, if $\delta_n = 0$ then we have $a_{f(n)} = \epsilon_{1,n}^{-1}\lambda_n^r T$, but $\deg(a_{f(n)+1}) > 1$.

Proof. By hypothesis, we have $\alpha_n = [\lambda_n T + \mu_n, \alpha_{n+1}]$ and also

$$\alpha_n^r = \epsilon_{1,n} P_a \alpha_{f(n)} + \epsilon_{2,n} Q_a + \nu_n. \tag{25}$$

Therefore, combining the first equality with (25), and since $\alpha_n^r = [a_n^r, \alpha_{n+1}^r]$, we can write

$$\left[\lambda_n^r T^r + \mu_n^r - \epsilon_{2,n} Q_a - \nu_n, \alpha_{n+1}^r\right] = \epsilon_{1,n} P_a \alpha_{f(n)}. \tag{26}$$

Recalling that P_a and Q_a satisfy the equality $T^r = TP_a - aQ_a$, we obtain, with our notation,

$$\lambda_n^r T^r - \epsilon_{2,n} Q_a = \lambda_n^r T P_a - \delta_n Q_a. \tag{27}$$

Combining (26) and (27), we get

$$\left[\frac{\lambda_n^r T P_a - \delta_n Q_a + \mu_n^r - \nu_n}{\epsilon_{1,n} P_a}, \epsilon_{1,n} P_a \alpha_{n+1}^r\right] = \alpha_{f(n)}. \tag{28}$$

Assuming that $\delta_n \neq 0$, with our notation, since $\pi_n = (\nu_n - \mu_n^r)\delta_n^{-1}$, (28) can be written as

$$\epsilon_{1,n}^{-1} \lambda_n^r T - \epsilon_{1,n}^{-1} \delta_n P_a^{-1}(Q_a + \pi_n) + \epsilon_{1,n}^{-1} P_a^{-1} \alpha_{n+1}^{-r} = \alpha_{f(n)}. \tag{29}$$

Applying Proposition 1, we have $P_a(Q_a + \pi_n)^{-1} = [W(a, \pi_n)]$. We set $y = -\epsilon_{1,n}\delta_n^{-1}$. Then we have

$$-\epsilon_{1,n}\delta_n^{-1} P_a (Q_a + \pi_n)^{-1} = y\left[W(a, \pi_n)\right] = \left[y \cdot W(a, \pi_n)\right].$$

Consequently, if we set $b_0 = \epsilon_{1,n}^{-1} \lambda_n^r T$ and $X = \epsilon_{1,n}^{-1} P_a^{-1} \alpha_{n+1}^{-r}$, (29) becomes

$$b_0 + \frac{1}{[y \cdot W(a, \pi_n)]} + X = \alpha_{f(n)}, \tag{30}$$

which is $[b_0, y \cdot W(a, \pi_n)] + X = \alpha_{f(n)}$. Since $\omega_n = 1 + a^{2-r}\pi_n^2 \neq 0$, we can apply Lemma 1 above. We get

$$\left[\epsilon_{1,n}^{-1} \lambda_n^r T, \left(-\epsilon_{1,n}\delta_n^{-1}\right) \cdot W(a, \pi_n), Y\right] = \alpha_{f(n)}. \tag{31}$$

This lemma gives

$$Y = \omega_n^{(-1)^{k+1}} \epsilon_{1,n} P_a^{-1} \left(\omega_n a^{r-1} \alpha_{n+1}^r + a\delta_n^{-1}(Q_a - \pi_n)\right). \tag{32}$$

We have $|Y| = |P_a^{-1} \alpha_{n+1}^r| > 1$, consequently (31) implies

$$a_{f(n)} = \epsilon_{1,n}^{-1}\lambda_n^r T \quad \text{and} \quad a_{f(n)+1}, \dots, a_{f(n)+r-1} = \left(-\epsilon_{1,n}\delta_n^{-1}\right) \cdot W(a, \pi_n).$$

But also $Y = \alpha_{f(n)+r} = \alpha_{f(n+1)}$. Hence, by (32), we get

$$\omega_n^{(-1)^k}\epsilon_{1,n}^{-1}P_a\alpha_{f(n+1)} = \omega_n a^{r-1}\alpha_{n+1}^r + a\delta_n^{-1}(Q_a - \pi_n) \tag{33}$$

and (33) becomes

$$\alpha_{n+1}^r = a^{1-r}\epsilon_{1,n}^{-1}\omega_n^{(-1)^k-1}P_a\alpha_{f(n+1)} - a^{2-r}\delta_n^{-1}\omega_n^{-1}Q_a + a^{2-r}\pi_n\delta_n^{-1}\omega_n^{-1}. \tag{34}$$

From (34), we obtain the desired formulas for $\epsilon_{1,n+1}$, $\epsilon_{2,n+1}$ and ν_{n+1}.

Finally if $\delta_n = 0$, from (28), we get

$$\alpha_{f(n)} = \epsilon_{1,n}^{-1}\lambda_n^r T + \frac{(\mu_n^r - \nu_n)\alpha_{n+1}^r + 1}{\epsilon_{1,n}P_a\alpha_{n+1}^r} = \epsilon_{1,n}^{-1}\lambda_n^r T + Z.$$

We have $|Z| < |T|^{-1}$, consequently we get $a_{f(n)} = \epsilon_{1,n}^{-1}\lambda_n^r T$ and $\deg(a_{f(n)+1}) = \deg(Z^{-1}) > 1$.

So the proof of Proposition 2 is complete. □

Proof of the theorem. We start from $\alpha \in \mathcal{E}(r, l, a, q)$, satisfying

$$\alpha^r = \epsilon_1 P_a\alpha_{l+1} + \epsilon_2 Q_a. \tag{I_1}$$

Recalling that $a_1 = \lambda_1 T + \mu_1$, we set

$$(DL_1) \quad \delta_1 = a\lambda_1^r + \epsilon_2 \quad \text{and} \quad (N_1) \quad \nu_1 = 0. \tag{36}$$

In the sequel from the triple (δ_n, ν_n, μ_n) in $\mathbb{F}_q^3$, if $\delta_n \neq 0$, as above, we define $\pi_n = (\nu_n - \mu_n^r)\delta_n^{-1}$ and $\omega_n = 1 + a^{2-r}\pi_n^2$. By (C_1), or by (C_2) and (C_3) if $l = 1$, we have $\delta_1 \neq 0$ and $a^{r-2}\delta_1^2 + (\nu_1 - \mu_1^r)^2 \neq 0$. Therefore we have $\delta_1\omega_1 \neq 0$ and we can apply Proposition 2. Hence, with $f(1) = l + 1$, we get r partial quotients, from a_{l+1} to a_{l+r}, all of degree 1. The following equality holds

$$\alpha_2^r = \epsilon_{1,2}P_a\alpha_{f(2)} + \epsilon_{2,2}Q_a + \nu_2, \tag{I_2}$$

where $\epsilon_{1,2}$, $\epsilon_{2,2}$ and ν_2 are as stated in Proposition 2. Observe that $a_2 = \lambda_2 T + \mu_2$ if $l > 1$, but also if $l = 1$. Indeed, if $l = 1$, then $a_2 = a_{f(1)} = \epsilon_1^{-1}\lambda_1^r T$. Consequently, we can consider δ_2 and we have

$$(DL_2) \quad \delta_2 = a\lambda_2^r - a^{2-r}(\omega_1\delta_1)^{-1} \quad \text{and} \quad (N_2) \quad \nu_2 = a^{2-r}\left(\nu_1 - \mu_1^r\right)\omega_1^{-1}\delta_1^{-2}.$$

If $l = 1$, we might have $\delta_2 = 0$ and this would imply $\alpha \notin \mathcal{E}^*(r, l, a, q)$. However, if $l = 1$ we will see here below that $\delta_2 = a^{1-r}(\delta_1\epsilon_1^{-1})^r \neq 0$. If $l > 1$, again by (C_1), or by (C_2)

and (C_3) if $l = 2$, we have $\delta_2 \neq 0$ and $\omega_2 = 1 + a^{2-r}\pi_2^2 \neq 0$. Consequently, Proposition 2 can be applied again. This process can be carried on as long as we have $a_n = \lambda_n T + \mu_n$ and $\delta_n \omega_n \neq 0$. As long as this process carries on, δ_n and ω_n are defined by means of the following recursive formulas:

$$\delta_n = a\lambda_n^r - a^{2-r}(\omega_{n-1}\delta_{n-1})^{-1} \tag{DL_n}$$

and for ω_n, since $\omega_n = 1 + a^{2-r}(\nu_n - \mu_n^r)^2\delta_n^{-2}$, via

$$\nu_n = a^{2-r}\big(\nu_{n-1} - \mu_{n-1}^r\big)\omega_{n-1}^{-1}\delta_{n-1}^{-2} = a^{2-r}\pi_{n-1}(\omega_{n-1}\delta_{n-1})^{-1}. \tag{N_n}$$

At each stage, we have

$$\alpha_n^r = \epsilon_{1,n}P_a\alpha_{f(n)} + \epsilon_{2,n}Q_a + \nu_n, \tag{I_n}$$

where $\epsilon_{2,n} = \delta_n - a\lambda_n^r$. While $\epsilon_{1,n}$ is defined recursively by $\epsilon_{1,1} = \epsilon_1$ and

$$\epsilon_{1,n} = a^{1-r}\epsilon_{1,n-1}^{-1}\omega_{n-1}^{(-1)^k-1}. \tag{E_n}$$

Moreover, also by Proposition 2, we have $a_m = \lambda_m T + \mu_m$, for $f(n) \leq m\langle f(n+1)$. Let us describe these partial quotients. We recall the notation $a(i) = 1$ if i is odd and $a(i) = a^{-1}$ if i is even. Combining Proposition 1 and Proposition 2, the following equalities hold:

$$\lambda_{f(n)} = \epsilon_{1,n}^{-1}\lambda_n^r \tag{$L_{n,0}$}$$

$$\lambda_{f(n)+i} = 2a(i)\big(\delta_n\epsilon_{1,n}^{-1}\big)^{(-1)^i} \quad \text{for } 1 \leq i \leq k \tag{$L_{n,i}$}$$

and

$$\lambda_{f(n)+i} = 2a(i)\big(\delta_n\epsilon_{1,n}^{-1}\big)^{(-1)^i}\omega_n^{(-1)^{k-i}} \quad \text{for } k+1 \leq i \leq r-1. \tag{$L_{n,i}$}$$

And also

$$\mu_{f(n)+i} = 0 \quad \text{for } 0 \leq i \leq r-1 \text{ and } i \neq k, k+1. \tag{$M_{n,i}$}$$

$$\mu_{f(n)+k} = (-a)^{1-k}\lambda_{f(n)+k}\pi_n/2. \tag{$M_{n,k}$}$$

$$\mu_{f(n)+k+1} = -\mu_{f(n)+k}\lambda_{f(n)+k+1}\lambda_{f(n)+k}^{-1}. \tag{$M_{n,k+1}$}$$

From the equalities $(L_{n,i})$, by multiplication, we easily get the following equalities for $1 \leq i \leq r-2$

$$\lambda_{f(n)+k}\lambda_{f(n)+k+1} = 4a^{-1}\omega_n^{-1} \tag{$X_{n,k}$}$$

and for $i \neq k$

$$\lambda_{f(n)+i}\lambda_{f(n)+i+1} = 4a^{-1}. \tag{$X_{n,i}$}$$

Our aim is to show that the quantities δ_n and ω_n can be defined, through the recursive formulas (DL_n) and (N_n), up to infinity. That is to say that we have $\delta_n\omega_n \neq 0$ at each stage. The first hypothesis of the theorem, namely (C_1), implies that we can define recursively, by the above formulas (DL_i) and (N_i), δ_i and ω_i in $\mathbb{F}_q^*$, for $i = 1, \dots, l-1$. Conditions (C_2) and (C_3), will turn out to be important to keep the process going on. Now, by (C_2) and (C_3), we observe that we also have $\delta_l \neq 0$ and $\omega_l = 1$. Consequently, the hypotheses (C_1), (C_2) and (C_3) imply that Proposition 2 can be applied repeatedly at least l times. It follows that we have $a_m = \lambda_m T + \mu_m$, for $1 \leq m < f(l+1)$. In order to have all the partial quotients of degree 1, up to infinity, we shall prove that $\delta_m\omega_m \neq 0$ for all $m \geq l+1 = f(1)$. For $m \geq f(1)$, we can write $m = f(n) + i$ where $n \geq 1$ and $0 \leq i \leq r-1$. Therefore we want to prove that $\omega_{f(n)+i}\delta_{f(n)+i} \neq 0$ for $n \geq 1$ and for $0 \leq i \leq r-1$. To prove this, we shall show by induction that, for $n \geq 1$ and for $0 \leq i \leq r-1$, the following equalities hold:

$$\delta_{f(n)} = a^{1-r}\left(\epsilon_{1,n}^{-1}\delta_n\right)^r \quad \text{and} \qquad (D_{n,0})$$

$$\delta_{f(n)+i} = (a/2)\lambda_{f(n)+i}^r \quad \text{for } 1 \leq i \leq r-1, \qquad (D_{n,i})$$

together with

$$\omega_{f(n)+k} = \omega_n^r \quad \text{and} \qquad (O_{n,k})$$

$$\omega_{f(n)+i} = 1 \quad \text{for } 0 \leq i \leq r-1 \text{ and } i \neq k. \qquad (O_{n,i})$$

Note that if $\delta_j\omega_j \neq 0$ for $j < m = f(n) + i$, and $(D_{n,i})$ and $(O_{n,i})$ hold, then we have $\delta_m\omega_m \neq 0$.

The proof of the equalities $(D_{n,i})$ and $(O_{n,i})$ will follow by induction from (DL_n), (E_n), (N_n), $(L_{n,i})$, $(M_{n,i})$ and $(X_{n,i})$.

First, we prove that $(D_{1,0})$ and $(O_{1,0})$ hold. Using (C_2), (C_3), (DL_1) and $(L_{1,0})$, we have $\omega_l = 1$, $\delta_l = -a\epsilon_1^r\epsilon_2^{-r}$, $\delta_1 = a\lambda_1^r + \epsilon_2$ and $\lambda_{l+1} = \epsilon_1^{-1}\lambda_1^r$. Hence, from (DL_{l+1}), we get

$$\delta_{f(1)} = \delta_{l+1} = a\lambda_{l+1}^r - a^{2-r}(\omega_l\delta_l)^{-1} = a\lambda_1^{r^2}\epsilon_1^{-r} + a^{1-r}\epsilon_2^r\epsilon_1^{-r} = a^{1-r}\left(\delta_1\epsilon_1^{-1}\right)^r.$$

Besides, since $\omega_l = 1$, we have $\pi_l = 0$. Consequently, by (N_{l+1}), we get $\nu_{l+1} = 0$. By $(M_{1,0})$, we have $\mu_{l+1} = 0$. It follows that $\pi_{l+1} = 0$ and $\omega_{f(1)} = \omega_{l+1} = 1$.

Let $n \geq 1$ and $0 \leq i \leq r-1$. First, we shall prove that for $0 \leq i \leq r-2$, $(D_{n,i})$ and $(O_{n,i})$ imply $(D_{n,i+1})$ and $(O_{n,i+1})$. Secondly, we shall prove that $(D_{n,r-1})$ and $(O_{n,r-1})$ imply $(D_{n+1,0})$ and $(O_{n+1,0})$. The proof is divided into five cases, each comprising two parts.

• *Case 1:* $i = 0$. By $(D_{n,0})$, we have $\delta_{f(n)} = a^{-r+1}(\epsilon_{1,n}^{-1}\delta_n)^r$. Furthermore, by $(L_{n,1})$, we have $\lambda_{f(n)+1} = 2(\delta_n\epsilon_{1,n}^{-1})^{-1}$. Therefore, with $\omega_{f(n)} = 1$, from $(DL_{f(n)+1})$, we get:

$$\begin{aligned}\delta_{f(n)+1} &= a\lambda^r_{f(n)+1} - a^{2-r}(\omega_{f(n)}\delta_{f(n)})^{-1}\\ &= a\lambda^r_{f(n)+1} - a^{2-r}a^{r-1}\left(\delta_n\epsilon^{-1}_{1,n}\right)^{-r}\\ &= a\lambda^r_{f(n)+1} - (a/2)\lambda^r_{f(n)+1} = (a/2)\lambda^r_{f(n)+1}.\end{aligned}$$

Hence $(D_{n,1})$ holds.

By $(O_{n,0})$, we have $\omega_{f(n)} = 1$ and, consequently, $\pi_{f(n)} = 0$. By $(N_{f(n)+1})$, we obtain $\nu_{f(n)+1} = 0$. If $k > 1$, $(M_{n,1})$ implies $\mu_{f(n)+1} = 0$. Therefore, $\nu_{f(n)+1} = \mu^r_{f(n)+1}$ and $\omega_{f(n)+1} = 1$. Thus $(O_{n,1})$ holds. If $k = 1$ (i.e. $r = 3$), the second part of Case 3 below must be applied.

- *Case 2:* $i \neq 0, k-1, k, r-1$. By $(D_{n,i})$, we have $\delta_{f(n)+i} = (a/2)\lambda^r_{f(n)+i}$ and, by $(O_{n,i})$, $\omega_{f(n)+i} = 1$. Furthermore, by $(X_{n,i})$, we have $\lambda_{f(n)+i} = 4a^{-1}\lambda^{-1}_{f(n)+i+1}$. Therefore, from $(DL_{f(n)+i+1})$, we get:

$$\begin{aligned}\delta_{f(n)+i+1} &= a\lambda^r_{f(n)+i+1} - a^{2-r}(\omega_{f(n)+i}\delta_{f(n)+i})^{-1}\\ &= a\lambda^r_{f(n)+i+1} - a^{2-r}\left(2a^{-1}\lambda^{-r}_{f(n)+i}\right)\\ &= a\lambda^r_{f(n)+i+1} - 2a^{1-r}\left(a^r/4\right)\lambda^r_{f(n)+i+1} = (a/2)\lambda^r_{f(n)+i+1}.\end{aligned}$$

Hence $(D_{n,i+1})$ holds.

By $(O_{n,i})$, we have $\omega_{f(n)+i} = 1$ and consequently $\pi_{f(n)+i} = 0$. By $(N_{f(n)+i+1})$, we obtain $\nu_{f(n)+i+1} = 0$. Since $i+1 \neq k, k+1$, $(M_{n,i+1})$ implies $\mu_{f(n)+i+1} = 0$. Therefore $\nu_{f(n)+i+1} = \mu^r_{f(n)+i+1}$ and $\omega_{f(n)+i+1} = 1$. Thus $(O_{n,i+1})$ hold.

- *Case 3:* $i = k-1$. If $k > 1$, by the same arguments as in the first part of the previous case, since $i \neq k$, we see that $(D_{n,i})$ and $(O_{n,i})$ imply $(D_{n,i+1})$. Hence $(D_{n,k})$ holds. If $k = 1$ (i.e. $r = 3$), the first part of Case 1 must be applied.

By $(D_{n,k})$, we have $\delta_{f(n)+k} = (a/2)\lambda^r_{f(n)+k}$. By $(O_{n,k-1})$, we have $\omega_{f(n)+k-1} = 1$ and $\pi_{f(n)+k-1} = 0$. Hence, from $(N_{f(n)+k})$, we obtain $\nu_{f(n)+k} = 0$. Therefore, using $(M_{n,k})$, we get:

$$\begin{aligned}\omega_{f(n)+k} &= 1 + a^{2-r}\left(\nu_{f(n)+k} - \mu^r_{f(n)+k}\right)^2\delta^{-2}_{f(n)+k}\\ &= 1 + a^{2-r}4a^{-2}\left(\mu_{f(n)+k}\lambda^{-1}_{f(n)+k}\right)^{2r}\\ &= 1 + 4a^{-r}\left((-a)^{1-k}\pi_n/2\right)^{2r}\\ &= 1 + a^{2r-r^2}\pi_n^{2r} = \left(1 + a^{2-r}\pi_n^2\right)^r = \omega_n^r.\end{aligned}$$

Hence $(O_{n,k})$ holds.

- *Case 4:* $i = k$. By $(D_{n,k})$, we have $\delta_{f(n)+k} = (a/2)\lambda^r_{f(n)+k}$ and, by $(O_{n,k})$, $\omega_{f(n)+k} = \omega_n^r$. Furthermore, by $(X_{n,k})$, we have $\omega_n\lambda_{f(n)+k} = 4a^{-1}\lambda^{-1}_{f(n)+k+1}$. Therefore, from $(DL_{f(n)+k+1})$, we get:

$$\begin{aligned}\delta_{f(n)+k+1} &= a\lambda^r_{f(n)+k+1} - a^{2-r}(\omega_{f(n)+k}\delta_{f(n)+k})^{-1}\\ &= a\lambda^r_{f(n)+k+1} - a^{2-r}\bigl(2a^{-1}\omega_n^{-r}\lambda^{-r}_{f(n)+k}\bigr)\\ &= a\lambda^r_{f(n)+k+1} - 2a^{1-r}\bigl(a^r/4\bigr)\lambda^r_{f(n)+k+1} = (a/2)\lambda^r_{f(n)+k+1}.\end{aligned}$$

Hence $(D_{n,k+1})$ holds.

By $(O_{n,k-1})$, we have $\pi_{f(n)+k-1} = 0$ and $\nu_{f(n)+k} = 0$. By $(X_{n,k})$, we have $\omega_n\lambda_{f(n)+k} = 4a^{-1}\lambda^{-1}_{f(n)+k+1}$. By $(D_{n,k})$, we have $\delta_{f(n)+k} = (a/2)\lambda^r_{f(n)+k}$. Using $(M_{n,k+1})$, from $(N_{f(n)+k+1})$, we get:

$$\begin{aligned}\nu_{f(n)+k+1} &= a^{2-r}\bigl(\nu_{f(n)+k} - \mu^r_{f(n)+k}\bigr)\omega^{-1}_{f(n)+k}\delta^{-2}_{f(n)+k}\\ &= -a^{2-r}\mu^r_{f(n)+k}\omega_n^{-r}\bigl((a/2)\lambda^r_{f(n)+k}\bigr)^{-2}\\ &= -4a^{-r}\mu^r_{f(n)+k}(\omega_n\lambda_{f(n)+k})^{-r}\lambda^{-r}_{f(n)+k}\\ &= -4a^{-r}\mu^r_{f(n)+k}\bigl(4a^{-1}\lambda^{-1}_{f(n)+k+1}\bigr)^{-r}\lambda^{-r}_{f(n)+k}\\ &= -\bigl(\mu_{f(n)+k}\lambda_{f(n)+k+1}\lambda^{-1}_{f(n)+k}\bigr)^r = \mu^r_{f(n)+k+1}.\end{aligned}$$

Consequently, $\pi_{f(n)+k+1} = 0$ and $\omega_{f(n)+k+1} = 1$. Hence $(O_{n,k+1})$ holds.

• *Case 5:* $i = r-1$. Recall that $f(n+1) = f(n)+r$. By $(D_{n,r-1})$ and $(O_{n,r-1})$, we have $\delta_{f(n+1)-1} = \delta_{f(n)+r-1} = (a/2)\lambda^r_{f(n)+r-1}$ and $\omega_{f(n+1)-1} = \omega_{f(n)+r-1} = 1$. By $(L_{n,r-1})$, we have $\lambda_{f(n)+r-1} = 2a^{-1}\delta_n\epsilon^{-1}_{1,n}\omega_n^{(-1)^k}$ and also, by $(L_{n+1,0})$, $\lambda_{f(n+1)} = \lambda^r_{n+1}\epsilon^{-1}_{1,n+1}$. By (E_{n+1}), we also have $\epsilon^{-1}_{1,n}\omega_n^{(-1)^k} = a^{r-1}\epsilon_{1,n+1}\omega_n$. Moreover, by (DL_{n+1}), we have $(\omega_n\delta_n)^{-1} = a^{r-2}(a\lambda^r_{n+1} - \delta_{n+1})$. Therefore, from $(DL_{f(n+1)})$, we get:

$$\begin{aligned}\delta_{f(n+1)} &= a\lambda^r_{f(n+1)} - a^{2-r}(\omega_{f(n+1)-1}\delta_{f(n+1)-1})^{-1}\\ &= a\lambda^r_{f(n+1)} - a^{2-r}\bigl(2a^{-1}\lambda^{-r}_{f(n)+r-1}\bigr)\\ &= a\lambda^r_{f(n+1)} - 2a^{1-r}\bigl(2a^{-1}\delta_n\epsilon^{-1}_{1,n}\omega_n^{(-1)^k}\bigr)^{-r}\\ &= a\lambda^r_{f(n+1)} - a\bigl(a^{r-1}\epsilon_{1,n+1}\omega_n\delta_n\bigr)^{-r}\\ &= a\lambda^{r^2}_{n+1}\epsilon^{-r}_{1,n+1} - a^{-r^2+r+1}\epsilon^{-r}_{1,n+1}\bigl(a^{r-2}\bigl(a\lambda^r_{n+1} - \delta_{n+1}\bigr)\bigr)^r\\ &= \epsilon^{-r}_{1,n+1}\bigl(a\lambda^{r^2}_{n+1} - a^{1-r}\bigl(a^r\lambda^{r^2}_{n+1} - \delta^r_{n+1}\bigr)\bigr) = a^{1-r}\bigl(\epsilon^{-1}_{1,n+1}\delta_{n+1}\bigr)^r.\end{aligned}$$

Consequently $(D_{n+1,0})$ holds.

By $(O_{n,r-1})$, we have $\pi_{f(n)+r-1} = 0$. Thus, from $(N_{f(n+1)})$, we get $\nu_{f(n+1)} = 0$. By $(M_{n+1,0})$, we have $\mu_{f(n+1)} = 0$. Therefore, $\pi_{f(n+1)} = 0$ and $\omega_{f(n+1)} = 1$. Consequently, $(O_{n+1,0})$ holds.

Thus we have proved that $\delta_m\omega_m \neq 0$, for $m \geq 1$ and this implies that $\alpha \in \mathcal{E}^*(r,l,a,q)$.

Now we turn to the description of the sequence of partial quotients $a_n = \lambda_n T + \mu_n$. The first l values of both sequences $(\lambda_n)_{n\geq1}$ and $(\mu_n)_{n\geq1}$ are given, as well as δ_i, ν_i and

ω_i for $1 \le i \le l$. We recall that, for $n \ge 1$, we have $g(n) = nr + l - k = f(n) + k$. According to $(O_{n,k})$, we have $\omega_{g(n)} = \omega_n^r$. According to $(O_{n,i})$ for $i \neq k$, if $m > l$ and $m \neq g(n)$, we have $\omega_m = 1$. For $i \in I$, we have $\omega_i = 1$ if and only if $i \notin I^*$. Consequently, by iteration, we obtain

$$\omega_n = 1 \quad \text{if } n \notin G \cup I^* \quad \text{and} \quad \omega_n = \omega_i^{r^m} \quad \text{if } n = g^m(i) \text{ for } m \ge 1 \text{ and } i \in I^*. \tag{35}$$

We start by the description of the sequence $(\mu_n)_{n \ge l+1}$. For $n \ge l+1$, if $n \notin G$, we have $\omega_n = 1$ and therefore $\pi_n = 0$. This implies, according to $(M_{n,i})$ for $0 \le i \le r-1$, that $\mu_n = 0$ if $n \notin G \cup (G+1)$. Let $i \in I^*$ and $m \ge 1$, since $\nu_{n+1} = a^{2-r}\pi_n(\omega_n\delta_n)^{-1}$ and $\pi_{g^m(i)-1} = 0$, we get $\nu_{g^m(i)} = 0$, and we have

$$\pi_{g^m(i)} = -\mu_{g^m(i)}^r \delta_{g^m(i)}^{-1} \quad \text{for } m \ge 1 \text{ and } i \in I^*. \tag{36}$$

We set $n' = g^{m-1}(i)$, then $g^m(i) = f(n') + k$. Applying $(M_{n',k})$ and $(D_{n',k})$, we have

$$\left(\mu_{g^m(i)}\lambda_{g^m(i)}^{-1}\right)^r = (-a)^{(1-k)r}\pi_{n'}^r/2 \quad \text{and} \quad \delta_{g^m(i)}^{-1} = 2a^{-1}\lambda_{g^m(i)}^{-r}. \tag{37}$$

From (36) and (37), we get

$$\pi_{g^m(i)} = -2a^{-1}\left(\mu_{g^m(i)}\lambda_{g^m(i)}^{-1}\right)^r = (-a)^{(1-k)r-1}\pi_{n'}^r. \tag{38}$$

We set $A = (-a)^{(1-k)r-1}$. Then (38) becomes $\pi_{g^m(i)} = A\pi_{g^{m-1}(i)}^r$. By iteration, we get

$$\pi_{g^m(i)} = A^{u_m}\pi_i^{r^m} \quad \text{for } m \ge 1 \text{ and } i \in I^*, \text{ where } u_m = \left(r^m - 1\right)/(r-1). \tag{39}$$

Hence, if $n = g^m(i)$ for $i \in I^*$ and $m \ge 1$, by $(M_{n',k})$ and (39), we have

$$\begin{aligned} \mu_n\lambda_n^{-1} &= (-a)^{1-k}\pi_{g^{m-1}(i)}/2 = (-a)^{1-k}A^{u_{m-1}}\pi_i^{r^{m-1}}/2 \\ &= (-a)^{v_m}\left(\left(\nu_i - \mu_i^r\right)\delta_i^{-1}\right)^{r^{m-1}}/2, \end{aligned} \tag{40}$$

where $v_m = (1-k) + ((1-k)r - 1)u_{m-1}$. It is elementary to check that $v_m = (r^{m-1}(2-r)+1)/2$. Recalling that, by $(M_{n',k+1})$, we also have $\mu_n\lambda_n^{-1} = -\mu_{n+1}\lambda_{n+1}^{-1}$, with (40) we have completed the description of the sequence $(\mu_n)_{n \ge 1}$.

Finally we turn to the description of the sequence $(\lambda_n)_{n \ge l+1}$. We recall that, according to $(L_{1,0})$, we have $\lambda_{l+1} = \epsilon_1^{-1}\lambda_1^r$. Hence, this description will follow from computing $C(n) = \lambda_{n-1}\lambda_n$ for $n > l+1$. If $n = g^m(i)+1$ for $i \in I^*$ and $m \ge 1$, then $n-1 = f(n')+k$, where $n' = g^{m-1}(i)$. Consequently, applying $(X_{n',k})$ and (35), we have

$$\begin{aligned} \lambda_n\lambda_{n-1} &= \lambda_{f(n')+k+1}\lambda_{f(n')+k} = 4a^{-1}\omega_{n'}^{-1} = 4a^{-1}\omega_i^{-r^{m-1}} \\ &= 4a^{-1}\left(1 + a^{2-r}\left(\nu_i - \mu_i^r\right)^2\delta_i^{-2}\right)^{-r^{m-1}}. \end{aligned} \tag{41}$$

According to $(L_{n,0})$ and $(D_{n,0})$, we have $\lambda_{f(n)} = \epsilon_{1,n}^{-1}\lambda_n^r$ and $\delta_{f(n)} = a^{1-r}(\epsilon_{1,n}^{-1}\delta_n)^r$. Hence, we obtain directly

$$\delta_{f(n)}^{-1}\lambda_{f(n)}^r = a^{r-1}\left(\delta_n^{-1}\lambda_n^r\right)^r. \tag{42}$$

For $i \in I$ and $m \geq 1$, from (42), we get $\delta_{f^m(i)}^{-1}\lambda_{f^m(i)}^r = a^{r-1}(\delta_{f^{m-1}(i)}^{-1}\lambda_{f^{m-1}(i)}^r)^r$. Consequently, by iteration, we get

$$\delta_n^{-1}\lambda_n^r = a^{r^m-1}\left(\delta_i^{-1}\lambda_i^r\right)^{r^m} \quad \text{if } n = f^m(i) \text{ for } m \geq 1 \text{ and } i \in I. \tag{43}$$

We set $n' = f^{m-1}(i)$. By $(L_{n',0})$ and $(L_{n',1})$, since $\lambda_{f(n')+1} = 2(\epsilon_{1,n'}^{-1}\delta_{n'})^{-1}$, from (43), we obtain

$$\lambda_{f(n')+1}\lambda_{f(n')} = 2\delta_{n'}^{-1}\lambda_{n'}^r = 2a^{r^{m-1}-1}\left(\delta_i^{-1}\lambda_i^r\right)^{r^{m-1}}. \tag{44}$$

Hence, by (44), we have

$$\lambda_n\lambda_{n-1} = 2a^{-1}\left(a\lambda_i^r\delta_i^{-1}\right)^{r^{m-1}} \quad \text{if } n = f^m(i) + 1 \text{ for } m \geq 1 \text{ and } i \in I. \tag{45}$$

We set $n = f^m(i)$ for $m \geq 1$ and $i \in I$, with $n \neq f(1)$. We have $n = f(n')$ with $n' > 1$. Consequently, by $(D_{n'-1,r-1})$, we get

$$\delta_{n-1} = \delta_{f(n')-1} = \delta_{f(n'-1)+r-1} = (a/2)\lambda_{f(n'-1)+r-1} = (a/2)\lambda_{n-1}^r. \tag{46}$$

As $n > l+1$, then $n-1 \geq l+1$ and $n-1 \notin G$. Therefore, by (35), we have $\omega_{n-1} = 1$. Consequently, by $(L_{n,0})$ and by (E_n), since $\omega_{n-1} = 1$, we have

$$\lambda_n^r = \epsilon_{1,n}\lambda_{f(n)} = a^{1-r}\epsilon_{1,n-1}^{-1}\lambda_{f(n)}. \tag{47}$$

Combining (46) and (47), we obtain

$$(a\lambda_{n-1}\lambda_n)^r = a^r\left(2a^{-1}\delta_{n-1}\right)a^{1-r}\epsilon_{1,n-1}^{-1}\lambda_{f(n)} = 2\delta_{n-1}\epsilon_{1,n-1}^{-1}\lambda_{f(n)}. \tag{48}$$

Also, by $(L_{n-1,r-1})$, since $\omega_{n-1} = 1$, we have

$$\lambda_{f(n)-1} = \lambda_{f(n-1)+r-1} = 2a^{-1}\delta_{n-1}\epsilon_{1,n-1}^{-1}. \tag{49}$$

Combining (48) and (49), we obtain

$$(a\lambda_{n-1}\lambda_n)^r = a\lambda_{f(n)-1}\lambda_{f(n)}. \tag{50}$$

Hence, by iteration from (50), we get

$$a\lambda_{f^m(i)-1}\lambda_{f^m(i)} = \left(a\lambda_{f(i)-1}\lambda_{f(i)}\right)^{r^{m-1}} \quad \text{for } m \geq 1 \text{ and } i \neq 1 \in I \tag{51}$$

and also

$$a\lambda_{f^m(1)-1}\lambda_{f^m(1)} = \left(a\lambda_{f^2(1)-1}\lambda_{f^2(1)}\right)^{r^{m-2}} \quad \text{for } m \geq 2. \tag{52}$$

Now, let $n \geq 2$, as above we have

$$\lambda_{f(n)} = \epsilon_{1,n}^{-1}\lambda_n^r \quad \text{and} \quad \lambda_{f(n)-1} = 2a^{-1}\delta_{n-1}\epsilon_{1,n-1}^{-1}\omega_{n-1}^{(-1)^k}. \tag{53}$$

By (E_n), we have

$$\omega_{n-1}^{(-1)^k} = a^{r-1}\omega_{n-1}\epsilon_{1,n-1}\epsilon_{1,n}. \tag{54}$$

From (53) and (54), we obtain

$$\lambda_{f(n)-1}\lambda_{f(n)} = 2a^{r-2}\delta_{n-1}\omega_{n-1}\lambda_n^r. \tag{55}$$

Recalling that, by (DL_n), we have $a^{2-r}(\delta_{n-1}\omega_{n-1})^{-1} = a\lambda_n^r - \delta_n$, we obtain

$$\lambda_{f(n)-1}\lambda_{f(n)} = 2\lambda_n^r\left(a\lambda_n^r - \delta_n\right)^{-1} = 2a^{-1}\left(1 - a^{-1}\lambda_n^{-r}\delta_n\right)^{-1}. \tag{56}$$

Combining (51) and (56), we get

$$\lambda_{f^m(i)-1}\lambda_{f^m(i)} = 2a^{-1}\left(1 - a^{-1}\lambda_i^{-r}\delta_i\right)^{-r^{m-1}} \quad \text{for } m \geq 1 \text{ and } i \neq 1 \in I. \tag{57}$$

Besides, according to (56), we have

$$a\lambda_{f^2(1)-1}\lambda_{f^2(1)} = 2\left(1 - a^{-1}\lambda_{f(1)}^{-r}\delta_{f(1)}\right)^{-1}. \tag{58}$$

From (42), we get

$$a^{-1}\lambda_{f(1)}^{-r}\delta_{f(1)} = \left(a^{-1}\lambda_1^{-r}\delta_1\right)^r. \tag{59}$$

Hence, by (58) and (59), we have

$$a\lambda_{f^2(1)-1}\lambda_{f^2(1)} = 2\left(1 - a^{-1}\lambda_1^{-r}\delta_1\right)^{-r}. \tag{60}$$

Combining (52) and (60), we get

$$\lambda_{f^m(1)-1}\lambda_{f^m(1)} = 2a^{-1}\left(1 - a^{-1}\lambda_1^{-r}\delta_1\right)^{-r^{m-1}} \quad \text{for } m \geq 2. \tag{61}$$

In summary, according to (41), (45), (57) and (61), we have obtained the values stated in the theorem for $C(n)$ if $n \in F \cup (F+1) \cup (G+1)$. Let us turn to the last case $n \notin F \cup (F+1) \cup (G+1)$. We have $n > l+1$, consequently there exist $n_1 \geq 1$ and

$i = 0, \dots, r-1$ such that $n-1 = f(n_1) + i$ and $n-1 \notin (F-1) \cup F \cup G$. There will be four cases according to the value of i.

- *Case 1:* $i \neq 0, k, r-1$. According to $(X_{n_1,i})$, we have

$$\lambda_{n-1}\lambda_n = \lambda_{f(n_1)+i}\lambda_{f(n_1)+i+1} = 4a^{-1}. \tag{62}$$

- *Case 2:* $i = k$. Here we have $n-1 = g(n_1)$ and, since $n-1 \notin G$, we have $n_1 \notin G \cup I^*$. Therefore, by (35), we have $\omega_{n_1} = 1$. Consequently, according to $(X_{n_1,k})$, we have

$$\lambda_{n-1}\lambda_n = \lambda_{f(n_1)+k}\lambda_{f(n_1)+k+1} = 4a^{-1}\omega_{n_1}^{-1} = 4a^{-1}. \tag{63}$$

For the last two cases, $i = 0$ or $i = r-1$, we need the following. If $n \notin F \cup I$ then there are three integers $m \geq 0$, $n_2 \geq 1$ and $j = 1, \dots, r-1$ such that $n = f^m(f(n_2) + j)$. Set $n_3 = f(n_2) + j$. By (42) and by iteration, since $n = f^m(n_3)$, we can write

$$\delta_n^{-1}\lambda_n^r = a^{r^m-1}\left(\delta_{n_3}^{-1}\lambda_{n_3}^r\right)^{r^m}. \tag{64}$$

By $(D_{n_2,j})$, with $j \neq 0$, we get $\delta_{n_3}^{-1}\lambda_{n_3}^r = 2a^{-1}$. Consequently, by (64), the previous argument implies:

$$n \notin F \cup I \quad \Rightarrow \quad \delta_n^{-1}\lambda_n^r = 2a^{-1}. \tag{65}$$

- *Case 3:* $i = 0$. Here we have $n-1 = f(n_1)$. Hence, according to (44), we have

$$\lambda_{n-1}\lambda_n = \lambda_{f(n_1)}\lambda_{f(n_1)+1} = 2\delta_{n_1}^{-1}\lambda_{n_1}^r. \tag{66}$$

Since $n-1 \notin F$ and $n-1 = f(n_1)$, we have $n_1 \notin F \cup I$. Therefore, by (65), we have $\delta_{n_1}^{-1}\lambda_{n_1}^r = 2a^{-1}$. Consequently, (66) becomes $\lambda_{n-1}\lambda_n = 4a^{-1}$.

- *Case 4:* $i = r-1$. Here we have $n = f(n_1) + r = f(n_1 + 1)$. According to (56), we have

$$\lambda_{n-1}\lambda_n = \lambda_{f(n_1+1)-1}\lambda_{f(n_1+1)} = 2a^{-1}\left(1 - a^{-1}\lambda_{n_1+1}^{-r}\delta_{n_1+1}\right)^{-1}. \tag{67}$$

Since $n \notin F$ and $n = f(n_1 + 1)$, we have $n_1 + 1 \notin F \cup I$. Therefore, by (65) we have $\lambda_{n_1+1}^{-r}\delta_{n_1+1} = a/2$. Hence, from (67), we get

$$\lambda_{n-1}\lambda_n = 2a^{-1}\left(1 - a^{-1}\lambda_{n_1+1}^{-r}\delta_{n_1+1}\right)^{-1} = 2a^{-1}\left(1 - a^{-1}a/2\right)^{-1} = 4a^{-1}. \tag{68}$$

In summary, according to (62), (63), (66) and (68), if $n \notin F \cup (F+1) \cup (G+1)$, we have obtained $C(n) = 4a^{-1}$. Hence the description of the sequence $(\lambda_n)_{n\geq 1}$ is over.

So the proof of the theorem is complete. □

5. Last comments

We want to come back to the statement of the theorem presented in this note. Starting from $\alpha \in \mathcal{E}(r,l,a,q)$, we have proved that the three conditions (C_1), (C_2) and (C_3) are sufficient to have $\alpha \in \mathcal{E}^*(r,l,a,q)$. However, condition (C_1) is particular and clearly necessary. Indeed, by repeated application of Proposition 2, it allows to have the first (up to the rank $l+r(l-1)$) partial quotients of degree 1. While conditions (C_2) and (C_3) are useful to keep this repetition of Proposition 2 up to infinity and to obtain a sequence of partial quotients having a relatively simple pattern. Then it is natural to ask whether conditions (C_2) and (C_3) are also necessary to have $\alpha \in \mathcal{E}^*(r,l,a,q)$. We know that the subset $\mathcal{E}(r,l,a,q)$ is finite, consequently all the elements can be tested by computer. We have done so for $r = q = p$ and for small values of p and l. We have observed that, for $\alpha \in \mathcal{E}(p,l,a,p)$, if (C_1), (C_2) or (C_3) is not satisfied then $\alpha \notin \mathcal{E}^*(p,l,a,p)$. Consequently we conjecture that this set of conditions is not only sufficient but also necessary when the finite base field is prime. However, this is not generally so if the base field is not prime. Indeed, it was a surprise to discover in $\mathcal{E}_0(r,l,a,q)$, with $q > p$, certain continued fractions belonging to $\mathcal{E}_0^*(r,l,a,q)$ but for which the condition (C_2) of the theorem stated here is not satisfied. This phenomenon has been explained in [8] and an example in $\mathcal{E}_0^*(3,1,2,27)$ has been given there [8, p. 258]. Note that for this type of examples the sequence of partial quotients has a much more complex pattern than the ones presented in this note.

With the conditions of our theorem, let us give a short and global description of the structure of this sequence of partial quotients. We can write this infinite continued fraction expansion $\alpha = [a_1, a_2, \ldots, a_n, \ldots]$ in the following way:

$$\alpha = [W_\infty] = [W_0, W_1, \ldots, W_m, \ldots],$$

where, for $m \geq 0$, we define the finite word

$$W_m = a_{f^m(1)}, a_{f^m(1)+1}, \ldots, a_{f^{m+1}(1)-1}.$$

Note that we have $W_0 = a_1, \ldots, a_l$, which plays the role of the "basis" of the expansion. We also have $W_1 = a_{l+1}, \ldots, a_{lr+l}$ and so on. We can check that $|W_m| = lr^m$ for $m \geq 0$. Define, for $n \geq 1$, the following word of length r:

$$W^f(a_n) = a_{f(n)}, a_{f(n)+1}, \ldots, a_{f(n)+r-1}.$$

Then with this notation, we see that W_{m+1} is built upon W_m, by concatenation, in the following way

$$W_{m+1} = W^f(a_{f^m(1)}), W^f(a_{f^m(1)+1}), \ldots, W^f(a_{f^{m+1}(1)-1}).$$

We see that the pattern of these sequences is somehow very regular. Let us come back to the original examples, due to Mills and Robbins [12, p. 400]. They belong to $\mathcal{E}_0^*(p, 2, 4, p)$ for all $p > 3$ and they are defined by $(\lambda_1, \lambda_2, \epsilon_1, \epsilon_2) = (\lambda_1, -\lambda_1(1+2\lambda_1)^{-1}, 1, 2)$, with $\lambda_1 \in \mathbb{F}_p^*$ and $\lambda_1 \neq -1/2$. Here, we have $G = \emptyset$, $\omega_n = 1$ and $\mu_n = 0$ for $n \geq 1$. The conditions of the theorem are satisfied and the description of the sequence $(\lambda_n)_{n\geq 1}$ follows. In 1988, shortly after the publication of these examples, J.-P. Allouche [1] could show the regularity of this sequence $(\lambda_n)_{n\geq 1}$, by proving that it is p-automatic. It should probably be possible to extend this type of result to all the sequences presented in this note.

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