DIOPHANTINE APPROXIMATION AND CONTINUED FRACTIONS IN POWER SERIES FIELDS

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Avec admiration pour Klaus Roth

1. Introduction and Notation

1.1. **The Fields of Power Series $\mathbb{F}(q)$ or $\mathbb{F}(K)$.** Let p be a prime number and $q = p^s$, where s is a positive integer. We consider the finite field $\mathbb{F}_q$ with q elements. Then we introduce with an indeterminate T, the ring of polynomials $\mathbb{F}_q[T]$ and the field of rational functions $\mathbb{F}_q(T)$. We also consider the absolute value defined on $\mathbb{F}_q(T)$ by $|P/Q| = |T|^{\deg P - \deg Q}$ for $P, Q \in \mathbb{F}_q[T]$, where $|T|$ is a fixed real number greater than 1. By completing $\mathbb{F}_q(T)$ with this absolute value, we obtain a field, denoted by $\mathbb{F}(q)$, which is the field of formal power series in $1/T$ with coefficients in $\mathbb{F}_q$. Thus, if α is a non-zero element of $\mathbb{F}(q)$, we have

$$\alpha = \sum_{k \leqslant k_0} u_k T^k,$$

where $k_0 \in \mathbb{Z}$, $u_k \in \mathbb{F}_q$, $u_{k_0} \neq 0$ and $|\alpha| = |T|^{k_0}$. Observe the analogy between the classical construction of the field of real numbers and this field of formal power series. The rôles of $\{\pm 1\}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ are played by $\mathbb{F}_q^*$, $\mathbb{F}_q[T]$, $\mathbb{F}_q(T)$ and $\mathbb{F}(q)$ respectively. Clearly the same construction as above can be made from an arbitrary base field K instead of $\mathbb{F}_q$. Then the resulting field is called the field of power series over K and will be denoted by $\mathbb{F}(K)$. We study here rational approximation to elements of $\mathbb{F}(K)$ which are algebraic over $K(T)$. We are concerned with the case of K having positive characteristic and mainly $K = \mathbb{F}_q$. For a presentation in a larger context and for more references, see [5]. Indeed the finiteness of the base field plays an essential rôle in many results and this makes the field $\mathbb{F}(q)$ particularly interesting.

1.2. **Continued Fractions in Power Series Fields.** As in the classical context of the real numbers, we have a continued fraction algorithm in $\mathbb{F}(K)$. For a general study on this subject and more references see [14]. If $\alpha \in \mathbb{F}(K)$, then we can write

$$\alpha = a_0 + \frac{1}{a_1+}\,\frac{1}{a_2+}\ldots = [a_0, a_1, a_2, \ldots],$$

where $a_i \in K[T]$. The a_i are called the partial quotients and we have $\deg a_i > 0$ for $i > 0$. This continued fraction expansion is finite if and only if $\alpha \in K(T)$. As in the classical theory, we define recursively the two sequences of polynomials $(x_n)_{n\geqslant 0}$ and $(y_n)_{n\geqslant 0}$ by

$$x_n = a_n x_{n-1} + x_{n-2} \quad \text{and} \quad y_n = a_n y_{n-1} + y_{n-2},$$

with the initial conditions $x_0 = a_0$, $x_1 = a_0 a_1 + 1$, $y_0 = 1$ and $y_1 = a_1$. We have $x_{n+1}y_n - y_{n+1}x_n = (-1)^n$, whence x_n and y_n are coprime polynomials. The rational x_n/y_n is called a convergent to α and we have $x_n/y_n = [a_0, a_1, a_2, ..., a_n]$.

Because of the ultrametric absolute value, we have

$$\left|\alpha - \frac{x_n}{y_n}\right| = \left|\frac{x_{n+1}}{y_{n+1}} - \frac{x_n}{y_n}\right| = \frac{1}{|y_n y_{n+1}|} = \frac{1}{|a_{n+1}||y_n|^2}.$$

1.3. **A Subset of Algebraic Elements in $\mathbb{F}(q)$.** Here we are concerned with elements in $\mathbb{F}(q)$ which are algebraic over $\mathbb{F}_q(T)$ and we introduce a special subset of algebraic elements. Let $r = p^t$, where $t \geqslant 0$ is an integer. We denote by $H_t(q)$ the subset of irrational elements α in $\mathbb{F}(q)$ such that there exist $A, B, C, D \in \mathbb{F}_q[T]$ with

$$\alpha = \frac{A\alpha^r + B}{C\alpha^r + D}. \tag{1}$$

We put

$$H(q) = \bigcup_{t \geqslant 0} H_t(q).$$

If t is the smallest non-negative integer such that $\alpha \in \mathbb{F}(q)$ satisfies an equation of type (1), we will say that α is a hyperquadratic element of order t. With our definition, a hyperquadratic element of order zero is simply a quadratic element. We observe that elements of $\mathbb{F}(q)$ which are quadratic or cubic over $\mathbb{F}_q(T)$ belong to $H_1(q)$, since then $1, \alpha, \alpha^p, \alpha^{p+1}$ are linked over $\mathbb{F}_q(T)$. Note that if $\alpha \in H_t(q)$, then by iteration in equation (1), $\alpha \in H_{kt}(q)$ for all positive integers k. Moreover, $H(q)$ contains also elements of arbitrarily large degree over $\mathbb{F}_q(T)$. Note that if K is a field of positive characteristic p but not necessarily finite, we will use the notation $H(K)$ for the corresponding subset of algebraic elements in $\mathbb{F}(K)$. We recall that the analogue of Lagrange's theorem on quadratic real numbers holds in the context of power series over a finite field. For complements on quadratic elements, see [14].

Theorem 1. *Let $\alpha \in \mathbb{F}(q)$ be irrational. Then the sequence of partial quotients in the continued fraction expansion of α is ultimately periodic if and only if $\alpha \in H_0(q)$.*

As we will see, the elements of the subset $H(q)$ have special properties of rational approximation due to the form of equation (1). For these elements the sequence of the degrees of the partial quotients can be bounded, as for instance in the case of quadratic elements, or unbounded. To illustrate this, given a prime p, let us consider the element θ_{1r} of $\mathbb{F}(p)$ defined by the infinite expansion

$$\theta_{1r} = [T, T^r, T^{r^2}, \ldots, T^{r^n}, \ldots],$$

where $r = p^t$ and $t \geqslant 0$. Because of the property of the Frobenius isomorphism, this element is indeed algebraic (quadratic if $r = 1$), satisfying the equation

$$\theta_{1r} = T + \frac{1}{\theta_{1r}^r},$$

and it belongs to $H(p)$.

2. General Results

2.1. **Mahler's Theorem.** Diophantine approximation in the function field case was initiated by K. Mahler [10]. The starting point in the study of rational approximation to algebraic real numbers is a famous theorem of Liouville established in 1850. This theorem has been adapted by Mahler in the fields of power series with an arbitrary base field.

Theorem 2 (Mahler, 1949)**.** *Let K be a field and $\alpha \in \mathbb{F}(K)$ be an algebraic element over $K(T)$ of degree $n > 1$. Then there is a positive real number C such that*

$$\left|\alpha - \frac{P}{Q}\right| \geqslant \frac{C}{|Q|^n} \tag{2}$$

for all $P, Q \in K[T]$ *with* $Q \neq 0$.

In the case of real numbers, it is well known that Liouville's theorem was the beginning of a long path, with the works of Thue, Siegel, Dyson and others, leading to the celebrated Roth's theorem which was established in 1955. This improvement can be transposed in fields of power series if the base field has characteristic zero, as shown by Uchiyama in 1960. In this case, the exponent n on the right hand side of the inequality (2) can be replaced by $2+\epsilon$ for all $\epsilon > 0$. But this is not the case in positive characteristic and consequently the study of rational approximation to algebraic elements becomes somewhat more complex.

2.2. **The Approximation Exponent.** Let $\alpha \in \mathbb{F}(K)$ be an irrational element. We define the *approximation exponent* of α by

$$\nu(\alpha) = \limsup_{|Q|\to\infty} \left(-\frac{\log|\alpha - P/Q|}{\log|Q|} \right),$$

where P and Q run over polynomials in $K[T]$ with $\deg(Q) > 0$. Using the continued fraction expansion $\alpha = [a_0, a_1, \ldots, a_n, \ldots]$, since the convergents are the best rational approximations to α and since we have

$$\left| \alpha - \frac{x_n}{y_n} \right| = \frac{1}{|a_{n+1}||y_n|^2},$$

it is clear that the approximation exponent can also be defined directly by

$$\nu(\alpha) = 2 + \limsup_k \frac{\deg a_{k+1}}{\deg y_k}.$$

As we have

$$\deg y_k = \sum_{1 \leqslant i \leqslant k} \deg a_i,$$

we see that $\nu(\alpha)$ is directly connected to the growth of the sequence $(\deg a_i)_{i\geqslant 1}$. Particularly if the sequence $(\deg a_i)_{i\geqslant 1}$ is bounded, then $\nu(\alpha) = 2$, but this is clearly not a necessary condition. Observe that, because of Mahler's theorem, for all $\alpha \in \mathbb{F}(q)$ algebraic over $\mathbb{F}_q(T)$ and of degree $n > 1$, we have

$$\nu(\alpha) \in [2, n].$$

In the same paper [10] mentioned above, Mahler introduced, for a prime number p, the element θ_{2r} of $\mathbb{F}(p)$, defined by

$$\theta_{2r} = \sum_{k\geqslant 0} T^{-r^k},$$

where $r = p^t$ and $t > 0$. He observes that this element satisfies

$$\theta_{2r} = \frac{1}{T} + \theta_{2r}^r \quad \text{and} \quad \nu(\theta_{2r}) = r.$$

Thus it is an algebraic element of degree r. With our notation, θ_{2r} belongs to $H(p)$. Note that according to its continued fraction expansion, the element $\theta_{1r} \in H(p)$ introduced earlier satisfies $\nu(\theta_{1r}) = r+1$ and is therefore algebraic of degree $r+1$.

2.3. **Osgood's Theorem.** At the beginning of the 1970's, C.F. Osgood [13] used differential algebra to study diophantine approximation in the function field case. We introduce on $\mathbb{F}(K)$ the ordinary formal differentiation, where $(aT^n)' = anT^{n-1}$ if $a \in K$ and $n \in \mathbb{Z}$. Observe that if K has positive characteristic p, then the subfield of constants for this differentiation is the field of power series in T^p over K. If $\alpha \in \mathbb{F}(K)$ is algebraic of degree n over $K(T)$, then we have $P(\alpha) = 0$, where $P \in K[T][X]$. Differentiating this equation, we obtain $\alpha' P'_X(\alpha) + P'_T(\alpha) = 0$. Therefore $\alpha' \in K(\alpha, T)$. Consequently there is an integer d with $0 \leqslant d \leqslant n-1$

such that $\alpha' = Q(\alpha)$, where $Q \in K(T)[X]$ and $\deg_X(Q) = d$. If $d \leqslant 2$, then we say that α satisfies a Riccati differential equation. Osgood was able to establish the following result.

Theorem 3 (Osgood, 1974)**.** *Let K be a field of positive characteristic, and let $\alpha \in \mathbb{F}(K)$ be an algebraic element over $K(T)$, of degree $n > 1$. If α does not satisfy a Riccati differential equation, then there is a positive real number C such that*

$$\left|\alpha - \frac{P}{Q}\right| \geqslant \frac{C}{|Q|^{[n/2]+1}}$$

for all $P, Q \in K[T]$ with $Q \neq 0$.

In the same paper [13], Osgood also introduced a new family of algebraic elements in $\mathbb{F}(p)$ with a maximal approximation exponent. If $n > 1$ is an integer coprime with p, then $\theta_{3n} \in \mathbb{F}(p)$, defined by

$$\theta_{3n}^n = 1 + \frac{1}{T},$$

is algebraic of degree n over $\mathbb{F}_p(T)$ and has $\nu(\theta_{3n}) = n$. Observe that, if t is the order of p in $(\mathbb{Z}/n\mathbb{Z})^*$ and $r = p^t$, then we have

$$\theta_{3n} = \left(1 + \frac{1}{T}\right)^{-(r-1)/n} \theta_{3n}^r$$

and consequently $\theta_{3n} \in H(p)$.

2.4. **Diophantine Approximation in $H(K)$.** Later J.-F. Voloch, inspired by Osgood's work on diophantine approximation in positive characteristic, pointed out the importance of the algebraic equation (1) stated above. He first observed that if $\alpha \in H(K)$, then α satisfies a Riccati differential equation. He proved the following result [16].

Theorem 4 (Voloch, 1988)**.** *Let K be a field of positive characteristic. If $\alpha \in H(K)$ and has approximation exponent $\nu(\alpha)$, then there is a positive real number C such that*

$$\left|\alpha - \frac{P}{Q}\right| \geqslant \frac{C}{|Q|^{\nu(\alpha)}}$$

for all $P, Q \in K[T]$ with $Q \neq 0$.

Corollary. *Let $\alpha \in H(K)$. Then $\nu(\alpha) = 2$ if and only if the sequence of partial quotients for α is bounded.*

Shortly afterwards, by studying more deeply rational approximation of elements in $H(K)$, B. de Mathan obtained the following result [11] which contains Theorem 4.

Theorem 5 (de Mathan, 1992)**.** *Let K be a field of positive characteristic. If $\alpha \in H(K)$ and has approximation exponent $\nu(\alpha)$, then*

$$\liminf_{|Q| \to \infty} |Q|^{\nu(\alpha)} \left|\alpha - \frac{P}{Q}\right| \neq 0, \infty \quad \textit{and} \quad \nu(\alpha) \in \mathbb{Q}.$$

2.5. **Singularity of $H(K)$.** Later, by adapting the method used by Thue on rational approximation to algebraic real numbers, we proved the following result [7].

Theorem 6 (Lasjaunias and de Mathan, 1996)**.** *Let K be a field of positive characteristic, and let $\alpha \in \mathbb{F}(K)$ be an algebraic element over $K(T)$ of degree $n > 1$. If $\alpha \notin H(K)$, then for every $\epsilon > 0$, there is a positive real number C such that*

$$\left|\alpha - \frac{P}{Q}\right| \geqslant \frac{C}{|Q|^{[n/2]+1+\epsilon}}$$

for all $P, Q \in K[T]$ with $Q \neq 0$.

Using Osgood's theorem and introducing the hypothesis of a finite base field, Theorem 6 could be improved slightly [8].

Theorem 7 (Lasjaunias and de Mathan, 1998)**.** *Let $\alpha \in \mathbb{F}(q)$ be an algebraic element over $\mathbb{F}_q(T)$ of degree $n > 1$. If $\alpha \notin H(q)$, then there is a positive real number C such that*

$$\left|\alpha - \frac{P}{Q}\right| \geqslant \frac{C}{|Q|^{[n/2]+1}}$$

for all $P, Q \in \mathbb{F}_q[T]$ with $Q \neq 0$.

3. Continued Fractions in $H(q)$

In the middle of the 1970's, L. Baum and M. Sweet studied diophantine approximation in $\mathbb{F}(2)$ by means of the continued fraction expansion. They gave different examples of algebraic continued fractions with bounded or unbounded partial quotients. Particularly, they proved the following result [2].

Theorem 8 (Baum and Sweet, 1976)**.** *Let α be the unique root in $\mathbb{F}(2)$ of the irreducible algebraic equation*

$$Tx^3 + x + T = 0.$$

Then the partial quotients of the continued fractions for α have degree 1 or 2.

Note that this element satisfies the equation $\alpha = T/(T\alpha^2 + 1)$ and therefore belongs to $H(2)$. Ten years later, this important work by Baum and Sweet on algebraic continued fractions in $\mathbb{F}(2)$ was substantially extended by W. Mills and D. Robbins [12]. They first pointed out the rôle played by the shape of the equation. In arbitrary positive characteristic, they introduced the elements of the subset which we have denoted above by $H(q)$. Moreover they developed an algorithm to obtain the continued fraction expansion for such elements. Thus they could describe explicitly the expansion for the cubic example introduced by Baum and Sweet. Also they could produce algebraic and nonquadratic elements in $\mathbb{F}(p)$ for all $p \geqslant 3$, with all partial quotients of degree 1; see Example 1 in Section 3.3 below. Finally they proved the following: If $\alpha \in H(q)$ and $\deg(AD - BC) < r - 1$ in (1), then the sequence of partial quotients is unbounded.

In spite of the attempt made by Mills and Robbins, the possibility of describing the continued fraction expansion for all the elements in $H(q)$ is still out of reach. Nevertheless an explicit description is possible for many examples and also for large subclasses.

3.1. **Elements of Class IA.** We will say that an element in $H(q)$ is of class IA if $AD - BC \in \mathbb{F}_q^*$ in (1). The number θ_{1r} introduced above belongs to this subclass. Observe that according to the property first stated by Mills and Robbins, if α is of class IA and $r > 1$, then $\deg(AD - BC) = 0 < r - 1$, and the sequence of partial quotients is unbounded. Such algebraic elements have been studied by W.M. Schmidt [14] and also independently by D. Thakur [15]. We have the following theorem and its corollary.

Theorem 9 (Schmidt, 2000)**.** *The element $\alpha \in \mathbb{F}(q)$ is algebraic of class IA if and only if there exist $k \geqslant 0$, $a_j, b_i \in \mathbb{F}_q[T]$ with $1 \leqslant j \leqslant k$ and $i \geqslant 1$, an integer $l \geqslant 1$ and $\epsilon \in \mathbb{F}_q^*$ such that*

$$\alpha = [a_1, a_2, \ldots, a_k, \beta],$$

where $\beta \in \mathbb{F}(q)$ is defined by $\beta = [b_1, \ldots, b_l, \beta_{l+1}]$ with $\beta_{l+1} \in \mathbb{F}(q)$ and $\beta^r = \epsilon\beta_{l+1}$. Consequently we have $\beta = [b_1, b_2, \ldots, b_l, \ldots, b_n, \ldots]$, where

$$b_{l+i} = \epsilon^{(-1)^i} b_i^r, \quad i \geqslant 1.$$

Observe that the expansion for such an element is determined by the first $k+l$ partial quotients. In the case $r=1$, we obtain the classical periodic expansion for quadratic power series. By choosing the integer l and the polynomials b_i for $1 \leqslant i \leqslant l$, the explicit knowledge of the continued fraction expansion for these elements implies the following important corollary.

Corollary (Thakur, Schmidt, 1999–2000). *Let μ be a rational real number with $\mu \geqslant 2$. Then there is an element $\alpha \in H(q)$ such that $\nu(\alpha) = \mu$.*

3.2. **Expansions of Type** (r, l, P, Q, R)**.** In analogy to Lagrange's theorem on continued fractions for quadratic real numbers, it is reasonable to think that the subset of hyperquadratic power series corresponds to expansions generated in a particular way. In that direction we have the following result [6].

Theorem 10 (Lasjaunias, 2006). *Let P, Q, R be polynomials in $\mathbb{F}_q[T]$, with $PR \neq 0$ and $\deg Q < \deg P < r$. Let $l \geqslant 1$ be an integer and $(a_1, \ldots, a_l) \in (\mathbb{F}_q[T])^l$ be given, with $\deg a_i > 0$ for $1 \leqslant i \leqslant l$. Then there is a unique infinite continued fraction expansion $\alpha = [a_1, \ldots, a_l, \alpha_{l+1}] \in \mathbb{F}(q)$ satisfying*

$$R\alpha^r = P\alpha_{l+1} + Q.$$

This element α is the unique root, in $\mathbb{F}(q)$ with $|\alpha| \geqslant |T|$, of the algebraic equation

$$x = \frac{Ax^r + B}{Cx^r + D},$$

where $A = Rx_l$, $B = Px_{l-1} - Qx_l$, $C = Ry_l$ and $D = Py_{l-1} - Qy_l$.

Note that the elements considered in Section 3.1 have a tail expansion corresponding to this model with $R = 1$, $P = \epsilon$ and $Q = 0$.

3.3. **Expansions of Type** $T^2 + u$**.** In this section, we derive a family of expansions generated in the same way as in Theorem 10 for $R = 1$ and a special pair (P, Q). We have the following proposition; see [6] and also Example 2 below.

Proposition 11 (Lasjaunias, 2006). *Let $r = p^t$ for some integer $t \geqslant 1$, let $u, \epsilon \in \mathbb{F}_q^*$, and let k be an integer satisfying $1 \leqslant k \leqslant (p-1)/2$. For any integer $l \geqslant 1$, let $\Lambda(l) = (\lambda_1, \ldots, \lambda_l) \in (\mathbb{F}_q^*)^l$. Then there is a unique infinite continued fraction expansion $\alpha = [a_1, \ldots, a_l, \alpha_{l+1}] \in \mathbb{F}(q)$ defined by*

$$\alpha^r = (T^2 + u)^k \alpha_{l+1} + \epsilon \int_0^T (x^2 + u)^{k-1}\, \mathrm{d}x$$

and

$$(a_1, \ldots, a_l) = (\lambda_1 T, \ldots, \lambda_l T).$$

Let $\alpha = [a_1, a_2, \ldots, a_l, \ldots, a_n, \ldots]$ be this expansion. If $\Lambda(l)$ and ϵ are well chosen, then there is a sequence $(\lambda_n)_{n \geqslant 1}$ in $\mathbb{F}_q^*$ and a sequence of positive integers $(i(n))_{n \geqslant 1}$ such that

$$a_n = \lambda_n A_{i(n)}, \quad n \geqslant 1,$$

where the sequence $(A_i)_{i \geqslant 1}$ in $\mathbb{F}_q[T]$ is defined recursively by $A_1 = T$ and

$$A_{i+1} = \left[\frac{A_i^r}{(T^2 + u)^k}\right], \quad i \geqslant 1.$$

If $1 \leqslant l \leqslant 2k$, then $i(n) = v_{2k+1}(2kn + l - 2k) + 1$, where $v_m(n)$ is the largest power of m dividing n.

Note that this proposition is given in an incomplete way. Actually the existence of the sequence $(\lambda_n)_{n \geqslant 1}$ has only been proved in some particular cases; see Examples 1 and 2 below. This sequence is defined recursively from $\Lambda(l)$ and ϵ. This recursive definition is very complex in its general form and not yet properly described. If $\Lambda(l)$ and ϵ are taken arbitrarily, then the sequence $(\lambda_n)_{n\geqslant 1}$ can only be defined up to a certain rank and consequently the expansion has the regular pattern given above only up to a certain rank. We give now three examples of expansions in $\mathbb{F}(p)$ with $p \geqslant 5$ having the pattern described in this proposition.

Example 1. Here $r = p > 3$, $k = (p-1)/2$, $u = 4$ and $l = 2$. The element α is defined by

$$\alpha^p = (T^2+4)^{(p-1)/2}\alpha_3 + 2\int_0^T (x^2+4)^{(p-3)/2}\,\mathrm{d}x$$

and

$$a_1 = aT, \quad a_2 = -\frac{aT}{2a+1},$$

where $a \in \mathbb{F}_p$ and $a(2a+1) \neq 0$. This example was introduced by Mills and Robbins [12, pages 400–401]. Observe here that the sequence $(A_i)_{i\geqslant 1}$ is constant and therefore all the partial quotients of the expansion for α are linear. We have $a_n = \lambda_n T$ with $\lambda_n \in \mathbb{F}_p^*$ for $n \geqslant 1$. More precisely, setting $u_k = 2(p^k-1)/(p-1)+1$ for $k \geqslant 0$, we have

$$\lambda_n = \begin{cases} a & \text{if } n = u_k, \\ -a/(2a+1) & \text{if } n = u_{k+1} - p^k, \\ (2a+1)^{(-1)^{n-u_k}} & \text{if } u_k < n < u_{k+1} - p^k, \\ -1 & \text{if } u_{k+1} - p^k < n < u_{k+1}. \end{cases}$$

Such hyperquadratic power series, with all partial quotients of degree 1 and in particular linear, were studied with a different approach earlier; see [9].

Example 2. Here $r = p > 3$, $k = 1$, $u = -1$ and $l = 1$. The element α is defined by

$$\alpha^p = (T^2-1)\alpha_2 + T$$

and $a_1 = 3T$. Here the expansion for α is described by $a_n = \lambda_n A_{v_3(2n-1)+1}$ with $\lambda_n \in \mathbb{F}_p^*$ for $n \geqslant 1$. More precisely, set $l_1 = 1$. For $n \geqslant 1$, set

$$l_{3n-1} = l_n - 1, \quad l_{3n} = -l_n \quad \text{and} \quad l_{3n+1} = l_n + 1,$$

as well as

$$k_n = \left[\frac{\log(2n-1)}{\log 3}\right] + 1 \quad \text{and} \quad \delta_n = (-1)^{k_n} 2^{l_n}.$$

Then $\lambda_1 = 3$ and, for $n \geqslant 1$, we have

$$\lambda_{3n-1} = \lambda_n, \quad \lambda_{3n} = -\delta_n^{-1} \quad \text{and} \quad \lambda_{3n+1} = \delta_n.$$

Example 3. This last example is only a conjecture. The first conjecture about the continued fractions for this element was made by Mills and Robbins [12, page 404]. Then ten years later a more precise one was given by Buck and Robbins [3, page 343]. Here $r = p = 13$, $k = 4$, $u = 8$ and $l = 6$. The element α is the only root in $\mathbb{F}(13)$ of the algebraic equation $x^4 - Tx^3 + x^2 + 1 = 0$. Although this element does not appear to be hyperquadratic, it is so; see [4, page 209] and also [1]. Indeed we have

$$\alpha^{13} = (T^2+8)^4\alpha_7 + 4\int_0^T (x^2+8)^3\,\mathrm{d}x$$

and

$$(a_1, \ldots, a_6) = (T, 12T, 7T, 11T, 8T, 5T).$$

Here the expansion for α should be such that $a_n = \lambda_n A_{v_9(4n-1)+1}$ with $\lambda_n \in \mathbb{F}_{13}^*$ for $n \geqslant 1$. This is verified by observing the first terms by computer, but the sequence $(\lambda_n)_{n\geqslant 1}$ remains very complicated to describe; see [3, page 343].

4. A Non-Hyperquadratic Example

We conclude this brief survey with an example which is somewhat connected to Example 3 given earlier; see [12, page 404] and [3]. Indeed the quartic equation $x^4 - Tx^3 + x^2 + 1 = 0$ has a unique root in $\mathbb{F}(p)$ for all primes p. We have discussed the case $p = 13$ but the case $p = 3$ is particularly interesting. The continued fraction expansion for the element θ_4 in $\mathbb{F}(3)$ is explicitly known. We have the following observation; see [3, 4] and [14, page 162].

Observation. *Let $\Omega_0 = \emptyset$, $\Omega_1 = T$ and, for $n \geqslant 2$, let Ω_n be defined by*

$$\Omega_n = \Omega_{n-1}, -T, \Omega_{n-2}^{(3)}, -T, \Omega_{n-1},$$

where the commas signify juxtaposition of sequences and $\Omega^{(3)}$ is obtained by cubing each element of Ω. If we define Ω_∞ to be the infinite sequence beginning with Ω_n for all $n \geqslant 1$, then θ_4 has the continued fraction expansion $\theta_4 = [\Omega_\infty]$.

The knowledge of this expansion implies easily

$$\liminf_{|Q|\to\infty} |Q|^2 \left|\theta_4 - \frac{P}{Q}\right| = 0 \quad \text{and} \quad \nu(\theta_4) = 2.$$

According to Theorem 4 above, this shows that θ_4 is not hyperquadratic. Nevertheless it is interesting to notice that, up to a change of variable, θ_4 is the square of a hyperquadratic element. Indeed we have $\theta_4(T^2) = \theta_{13}^2(T)$, where $\theta_{13} \in H(3)$ is the element introduced at the end of Section 1.3. This equality implies the regularity of the expansion described above; see [4].

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