SUBGROUPS OF SEMI-ABELIAN VARIETIES

QING LIU

Poincaré's complete reducibility theorem states that if A is an abelian variety over a field k, then any surjective morphism of abelian varieties $f : A \to B$ over k induces an isogeny $A \sim B \times C$ where C is a sub-abelian variety of A. In standard books on abelian varieties, this theorem is proved over perfect field. Here we give a proof in the general case following hints of Michel Raynaud [2]. Of course, mistakes in the proof are mine.

As over a perfect field, the complete reducibility theorem can be deduced from the following proposition.

Theorem 1. *Let A be an abelian variety. Then for any algebraic subgroup F of A, F^0_{red} is an abelian variety.*

Proof. Apply Proposition 5 to the subset F of A. Then $B = F_{\mathrm{red}}$ and B^0 is an abelian variety. □

Remark 2 In fact for any proper algebraic group F, F^0_{red} is an abelian variety. See [1], Lemma 3.3.7 for a short proof using some general theorems on algebraic groups.

Lemma 3. *Let X, Y be algebraic varieties (i.e. schemes of finite type) over a field k. Let $S \subseteq X$ and $T \subseteq Y$ be dense subsets and let p, q be the projections from $X \times_k Y$ to X and Y. Then $S \times T := p^{-1}(S) \cap q^{-1}(T)$ is dense in $X \times_k Y$.*

Proof: Let U be a non-empty open subset of $X \times_k Y$. Then $p(U) \cap S \neq \emptyset$. Let $s \in p(U) \cap S$. The projection $\{s\} \times_k Y \to Y$ is an open map, so $\{s\} \times_k T$ meets $(\{s\} \times_k Y) \cap U$. Any point of the intersection belongs to $(S \times T) \cap U$, so the latter is non-empty.

Lemma 4. (Raynaud, [2]) *Let G be a semi-abelian variety over k. Let $p = \mathrm{char}(k) \geq 0$. Let B be a connected reduced closed subscheme of G, stable by addition and inverse. For any $n \geq 1$, denote by $B[n] = G[n] \cap B$ (as a set). Then*

(1) *$\cup_{n \in \mathbb{N}, (p,n)=1} B[n]$ is dense in B.*
(2) *B is geometrically reduced.*

Proof. (1) As B contains a rational point (the unit of G), it is geometrically connected. Now notice that $B_{\bar{k}}$ is stable by addition and inverse, so $(B_{\bar{k}})_{\mathrm{red}}$ is a connected smooth algebraic subgroup of $G_{\bar{k}}$, hence a semi-abelian variety over $\bar{k}$. Let

$$Z = \overline{\cup_{(n,p)=1} B[n]}$$

denote the Zariski closure in G. By Lemma 3, Z is stable by addition and by inverse. So the neutral component $(Z_{\bar{k}})^0_{\mathrm{red}}$ is a semi-abelian variety over $\bar{k}$ contained in $(B_{\bar{k}})^0_{\mathrm{red}}$. Let a (resp. a') and t (resp. t') be the abelian and toric ranks of $(B_{\bar{k}})^0_{\mathrm{red}}$

(resp. $(Z_{\bar{k}})^0_{\rm red}$). Then $a' \leq a$ and $t' \leq t$. Let c be the number of connected components of $(Z_{\bar{k}})_{\rm red}$. Then for any natural integr n prime to pc, we have

$$n^{2a'+t'} = |(Z_{\bar{k}})^0_{\rm red}[n]| = |(Z_{\bar{k}})_{\rm red}[n]| \geq |(B_{\bar{k}})_{\rm red}[n]| = n^{2a+t}.$$

Hence $a' = a$, $t' = t$ and $\dim Z_{\bar{k}} = a'+t' = \dim B_{\bar{k}}$. Therefore $Z_{\bar{k}} = B_{\bar{k}}$ and $Z = B$.

(2) For all $n \geq 1$ prime to p, $G[n]$ is étale over k. Hence $B[n]$ is open in $G[n]$ and is geometrically reduced. Let U be any affine open subset of B. The natural map

$$\mathcal{O}_B(U) \to \prod_{(n,p)=1} \mathcal{O}_{B[n]}(B[n] \cap U)$$

is injective. Indeed, if $f \in \mathcal{O}_B(U)$ is in the kernel, then $V(f)$ contains the dense subset $(\cup_{(n,p)=1} B[n]) \cap U$ of U. Therefore $V(f) = U$ and $f = 0$ because U is reduced. Extending the above map to $\bar{k}$, we get an injective map

$$\mathcal{O}_B(U) \otimes_k \bar{k} \to \prod_{(n,p)=1} \mathcal{O}_{B[n]}(B[n] \cap U) \otimes_k \bar{k}.$$

As the right-hand side is reduced, we see that U (hence B) is geometrically reduced. □

Proposition 5. *Let G be a semi-abelian variety over a field k. Let F be a non-empty subset of G such that $-F = F$ and $F + F \subseteq F$ (as sets). Let B be the Zariski closure of F in G, endowed with the reduced structure. Then the connected component B^0 of 0 in B is a smooth algebraic subgroup of G.*

Proof. We first notice that Lemma 3 says that $F \times F$ is dense in $B \times_k B$. Hence $-B = B$ and $B + B \subseteq B$ (set-theoretically). Then B^0 is geometrically connected because it contains $0 \in B(k)$, and we have $B^0 = -B^0$ and $B^0 + B^0 \subseteq B^0$. So we can suppose that B is connected. By Lemma 4, B is geometrically reduced. As it is stable by addition (as scheme) and inverse, it is a smooth algebraic subgroup of G. □

Corollary 6. (Raynaud, [2]) *Let A be an abelian variety over a field k. Then any abelian subvariety of $A_{\bar{k}}$ is defined over k^s.*

Proof. We can suppose k is separably closed of characteristic p. Let C be an abelian subvariety of $A_{\bar{k}}$. Let $F = \cup_{(n,p)=1}(A_{\bar{k}}[n] \cap C)$. Then F is defined over k. Let B be the Zariski closure of F in A, endowed with the reduced structure. By Lemma 4, $B_{\bar{k}} = C$ set-theoretically. Hence B is connected. Using again Lemma 4, we see that this equality holds as schemes. □

Remark 7 Proposition 5 is clearly false if G is not semi-abelian. For example, let $G = \mathbb{G}_a^2 = \operatorname{Spec} k[x,y]$ over an imperfect field k of characteristic p. Let $\lambda \in k \setminus k^p$. Then $B := V(x^p - \lambda y^p)$ is a reduced subgroup of G, but it is not smooth. The intersection $B \cap V(y^p - y) = \operatorname{Spec} k[x]/(x^p(x^{p^2-p} - \lambda))$ is a finite algebraic group, whose reduced subscheme is not an algebraic group.

References

[1] Michel Brion, *Some structure theorems for algebraic groups*, To appear in Proceedings of Symposia in Pures Mathematics.

[2] Michel Raynaud, email to the author, April 2008.